UNIVERSITY OF CAPE COAST

OPTIMIZING THE THERMO-PHYSICAL PROPERTIES OF FLUORINATED CARBON NANOTUBE FOR THERMOELECTRIC APPLICATIONS

DANIEL SEKYI-ARTHUR

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BY

DANIEL SEKYI-ARTHUR

Thesis submitted to the Department of Physics of the School of Physical Sciences, College of Agriculture and Natural Sciences, University of Cape Coast, in partial fulfilment of the requirements for the award of Doctor of Philosophy degree in Physics

JULY 2020

DECLARATION

Candidate's Declaration

I hereby declare that this thesis is the result of my own original research and that no part of it has been presented for another degree in this university or elsewhere.

Candidate's Signature........................ Date:........................

Name: Daniel Sekyi-Arthur

Supervisors' Declaration

We hereby declare that the preparation and presentation of the thesis were supervised in accordance with the guidelines on supervision of thesis laid down by the University of Cape Coast.

Principal Supervisor's Signature.................................. Date:........................

Name: Prof. Samuel Yeboah Mensah

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Name: Prof. Raymond Edziah

ABSTRACT

In a nondegenerate fluorinated single-walled carbon nanotube (FSWCNT) in the hypersound domain, $q\ell \gg 1$, where q is the acoustic wavenumber and ℓ is the carrier mean free path, a theoretical study of semiclassical carrier miniband transport across a periodic potential was carried out. First, the effect of an acoustic wave on FSWCNT was investigated, and it was discovered that high-frequency carrier dynamics can be created, though the wavenumber or wave amplitude is critical. Depending on the wave amplitude and the carrier's initial position in the acoustic wave, there were two dynamical regimes. Bloch-like oscillations could be induced by applying a large enough potential amplitude/wavenumber, resulting in ultra-high negative differential velocity, or the carrier could be dragged through the FSWCNT and permitted to drift in periodic orbits with frequencies far above the gigahertz frequencies (GHz) of the acoustic wave. A high negative differential velocity induces charge domains in FSWCNT at transitions between these two carrier dynamic regimes, which generated extra features in the current oscillations. Secondly, invoking an analytical technique which is traceable and the phonon LBM, the dimensionless figure of merit (ZT) for FSWCNT was explored. The ZT was found to be substantially influenced by the FSWCNT parameters Δ_s, Δ_z, E_o and n_o. At room temperatures and beyond, optimizing Δ_s, Δ_z and n_o resulted in a ZT greater than 6 (i.e., $ZT > 6$). As a result of the high ZT achieved, the FSWCNT can be considered a good thermoelement material.

KEY WORDS

Acoustoelectric

Amplification

Figure of merit

Hall-like current

Temperature gradient

Thermoelectric effect

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DEDICATION

To my family and memory of my father

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Velocity of Light	$c = 2.999 \times 10^8$ ms^{-1}
Velocity of Sound	$V_s = 2.5 \times 10^3$ ms^{-1}
Planck's constant	$\hbar = 6.582 \times 10^{-16}$ eVs
Boltzmann's coonstant	$k_B = 8.617 \times 10^{-5}$ eVK^{-1}
Carrier charge	$e = 1.6 \times 10^{-19}$ J
Deformation potential	$\Lambda = 9$ eV
Relaxation time	$\tau = 10^{-12}$ s
Carrier mobility	$\mu = 10^5$ cm$^2/v.s$
Nearest-neighbor hopping energy	$\gamma_o = 2.6$ eV
Acoustic wave frequency	$\omega_q = 10^{12}$ s^{-1}
Surface carrier density	$n_o = 10^{18}$ cm^{-3}
Fermi Velocity	$V_F = 10^8$ cms^{-1}
C-C bond length	$b = 0.142$ nm

LIST OF SYMBOLS

ω_q	Acoustic phonon frequency (s^{-1})
p_z	Quasi-momentum ($eVsm^{-1}$)
T	Temperature (K)
q	Acoustic wavenumber (m^{-1})
ℓ	Mean free path (m)
$\Gamma_q/\Gamma(q)$	Hypersound absorption coefficient (cm^{-1})
$\varepsilon(p)$	Energy dispersion relation (eV)
$\mathbf{H}$	Magnetic field (T)
E	Electric Field (V/m)
A	Vector potential
n_o	Surface concentration of electrons (cm^{-3})
$\delta(x-a)$	Dirac delta function
$\vec{\Phi}$	Sound flux density (Wb/m^2)
λ	wavelength (m)
Λ	Deformation potential constant (eV)
$G(p,p')$	Electron-phonon coefficient
$\vec{k}$	Wavevector
$c^\dagger$	Electron creation operator
c	Electron annihilation operator

$b^{\dagger}$	Phonon creation operator
b	Phonon annihilation operator
$\Theta(1-\alpha)$	Heaviside step function
$f_o(r,k,t)$	Equilibrium electron distribution function
$N(r,q,t)$	Phonon distribution function
U	Acoustic wave amplitude
$\Psi(r,t)$	Wavefunction
$\vec{j}^{AE}$	Acoustoelectric current (Am^{-1})
$\vec{j}^{AME}$	Acoustomagnetoelectric current (Am^{-1})
α_{zz}/α_{cz}	axial/circumferential thermopower of FSWCNT
α^{SL}	axial thermopower of superlattice
$\mathscr{P}_{zz}/\mathscr{P}_{cz}$	axial/circumferential power factor
κ	Thermal conductivity
κ_e/κ_ℓ	Electron/Lattice thermal conductivity
ZT	Figure of merit
Π_{zz}/Π_{cz}	axial/circumferential Peltier coefficient
L_{zz}/L_{cz}	axial/circumferential Lorentz number
Δ_z/Δ_s	axial/circumferential overlapping integral for jumps

LIST OF ABBREVIATIONS

SWCNT	Single-Walled Carbon Nanotube
FSWCNT	Fluorinated Single-walled Carbon Nanotube
FMWCNT	Fluorinated Multiple-walled Carbon Nanotube
AE	Acoustoelectric Effect
AME	Acoustomagnetoelectric Effect
QAME	Quantum Acoustomagnetoelectric Effect
QW	Quantum Wires
CQW	Cylinderical Quantum Wires
RQW	Rectangular Quantum Wires
SL	Superlattice
GNR	Graphene Nanoribbon
AGNR	Armchair Graphene Nanoribbon
ACNT	Armchair Carbon Nanotube
ZGNR	Zigzag Graphene Nanoribbon
LA	Longitudinal Acoustic Wave
TA	Transverse Acoustic Wave
BTE	Boltzmann Transport Equation
MOCVD	Metal-Organic Chemical Vapour Desposition
CdS	Cadmium Sulphide

FET	Field Effect Transistor
SASER	Sound Amplification by Stimulated Emission of Radiation
LCAO	Linear Combination of Atomic Orbitals
LBM	Lattice Boltzmann Method
NSE	Navier-Stoke Equation
DOS	Density of State
LDOS	Local Density of State
ZT	Dimensionless Figure of Merit
STM	Scanning Tunneling Microscopy
MO	Molecular Orbital

CHAPTER ONE

INTRODUCTION

In view of current global energy and environmental problems, the need to use energy resources more efficiently has become critical. A considerable quantity of renewable energy remains untapped because a large amount of energy is still dumped into the environment as waste heat. Regardless of the magnitude of the source, thermoelectric power generating systems have the ability to convert this available heat energy directly into electrical energy. However, due to the poisonous, limited, and poor chemical stability of today's semiconductor materials at high temperatures, their incorporation into functional thermoelements has been impeded. If their thermal conductivity can be lowered, the utilization of carbon-based compounds as prospective thermoelectric materials offers immense potential to address the difficulties outlined above. The acoustoelectric, electric, thermal, and thermoelectric metrics of a fluorine doped carbon based semiconductor will be computed in this work by an analytical analysis.

Background to the Study

Thermoelectricity is the direct energy conversion in devices that uses the Seebeck effect to convert thermal energy to electrical energy. When an electric current is conducted between two different wires, the Peltier effect cools the junction, resulting in similar energy conversion processes. The thermal fluctuations of carriers on the hot region of the material are higher than those on the cold region. For n-type materials, this results in carrier diffusion to the cold region. On the cold region, the increased carrier concentration generates an electric field that counteracts the diffusive force. The Seebeck coefficient describes the strength of the effect: $\alpha = -(\Delta V/\Delta T)$, with ΔV denoting the potential difference generated by the temperature gradient, ΔT. The negative sign of α depends on whether the predominant carriers are carriers or holes. However, thermoelectricity has miriads of applications in the semiconductor sector, including chip level carrier cooling, remote telecommunication power production, temperature control technologies in solid state lasers, and so on.

However, in the near future, the introduction of cheap, high-efficient thermopower generators will assist to minimize reliance on non-renewable energy supplies such as fossil fuels. The low efficiency and performance of thermoelectrics has limited their commercial uses, as defined by the figure of merit, $ZT = \alpha^2\sigma T/\kappa$, where α is the Seebeck coefficient, σ is the carrier conductivity, and κ is the thermal conductivity [1]. The Seebeck coefficient and carrier conductivity are solely determined by the system's carrier characteristics, but the thermal conductivity is influenced by both carrier and lattice vibrations. $ZT \approx 0.99$ is the best commercially available thermoelectric device [2]. The goal of extensive thermoelectric research has been to find materials with a higher thermoelectric power but lower thermal conductivity, yielding a higher ZT.

Based on these number of advantages, the introduction of nanofilm and superlattice nanowire structures in the 1970's significantly improved the ZT of thermoelectric devices, and altered the emphasis away from bulk materials and toward understanding the dynamics of carrier transport in nanodevices. The carrier local density of state around the Fermi level is increased by quantum confinement of carriers in nanostructures [2]. Phonon confinement [3, 4] and phonon scattering at the material interfaces in SLs [4-6], increases the thermoelectric power [2], but the thermal conductivity decreases. The huge bandgap of semiconductors and discrepancies between the phonon and carrier mean free pathways have little effect on electrical conductivity [7, 8].

In comparison to bulk structures, impact of combined low thermal conductivity and increased thermoelectric power shows a theoretical large ZT. SLs have a lower thermal conductivity than their bulk equivalents, despite the experimentally observed reductions in thermal conductivity [2, 9], experimental investigations, especially when it comes to $SiGe$ SLs, have not achieved the presumed benefits of SL thermoelectric devices. As a result, a deeper understanding of the impact of all the key components that contribute to ZT in nanoscale devices is required. Two primary phenomena that affects carrier transport in nanostructures are (i) carrier confinement and (ii) carrier transport in nanostructures including carrier-phonon, carrier-impurity, phonon-phonon, and other carrier scattering effects.

The semiclassical BTE is proposed as a tool for studying thermoelectric metrics in FSWCNTs with optimized doping and cylindrical structure, taking into account the effect of carrier scattering and intraminiband transition to obtain the best ZT. The semiclassical BTE will serve as a foundation for analyzing other new technologies in the field of solid-state energy conversion devices where temperature effects on carrier transport are quite severe, in addition to researching thermoelectric transport at the nanoscale. The demand for mesocopic modeling has never been greater, because of the expanding impact of nanotechnology in a wide range of industries such as microelectronics, medical imaging, and nanocomposite materials. As a result, using the semiclassical approach to explore carrier and phonon transport in nanoscale devices shows a paradigm change in carrier transport modeling.

One of the new and efficient procedures for chemical activation and functionalization of carbon nanotubes is the modification of SWCNT with fluorine dopants [10, 11]. Fluorination of MWCNTs was done decades ago, while SWCNTs were fluorinated in recent years [12, 13]. Fluorination is important in the functionalization process because it allows for a high surface concentration of functional groups, up to C_2F, without destroying the physical structure of the tube. The nanotube is debundled by the repulsive interactions of the fluorine atoms on the surface, which improves carrier dispersion [13]. Functionalization is a simple, fast exothermic reaction, and the repulsive interactions of the fluorine atoms on the surface debundle the nanotube, enhancing their carrier dispersion. A one-dimensional SWCNT doped with fluorine atoms is shown in Figure 1 [10]. Hydrogen bonds existing between

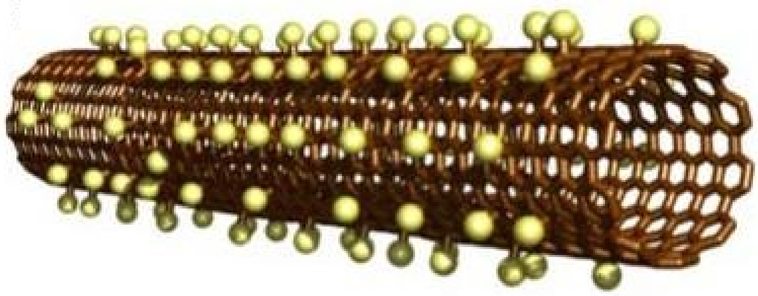

Figure 1: Fluorine-doped SWCNT with Fluorine Atoms as Yellow Balls [13].

fluorine atoms and protons are polar and thus, results in high solvent solubility [13, 14] such as alkanol, which enables solution phase chemistry, purification and processing of the nanotube. Fluorine modified nanotube has proven to be a better carrier acceptor than pristine SWCNT, aiding positive interaction between nucleophilic and non-nucleophilic reagents [15]. Fluorine modified nanotubes are used as precursors for further functionalization by nucleophilic substitution [13, 15]. By using anhydrous hydrazine reagents, defluorination of the FSWCNT can also be achieved by the method of sonication [13, 14]. It has been observed experimentally that thermal defluorination of FSWCNTs contrary to FMWCNTs is impossible without partial deformation of the hexagonal honey-comb structure of the FSWCNT.

Regardless of the observed decreased conductivity for fluorinated nanotubes [16- 18], axial fluorine addition pattern predicts the band gap not to vanish for metallic nanotubes [13, 14]. Potential applications of FSWCNT include its use in supercapacitors [15- 22], Lithium-ion batteries [23-26] and as lubricants [27-31]. Recent investigations for FSWCNT as a modern approach for polymer reinforcement, predicts it to improve the physical properties of polymers in comparison to undoped nanotube fillers [31, 32]. Theoretical studies on addition patterns have dwelled on maximum C_2F fluorination [15-17, 33-34]. Contiguous fluorine addition was used to explain the mismatch between the axial fluorine pattern and the circumferential banding, as the fluorinated axial addition pattern was projected to generate sharp circumferential margins with unfluorinated nanotube portions [35-39].

Consider a fluorine doped SWCNT (n, n) where the fluorine atoms forms a one-dimensional chain. A SWCNT of this nature is equivalent to a band with unit cell as shown in Figure 2, where b is the bond length $(c-c)$ [40].

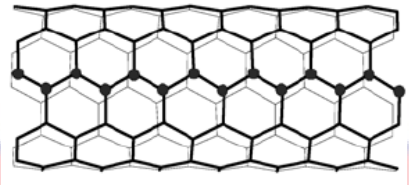

Figure 2: Fluorine-doped Nanotube F-(n,n) where Black Dots Denotes the Positions of Fluorine Atoms [40].

The width for the F-(n,n) nanotube equals n periods with a periodic length of $3b$, and this unit cell contains $N = 4n-2$ carbon atoms which is shown in Figure 3 [40].

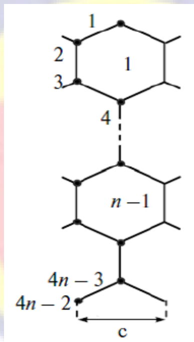

Figure 3: FSWCNT Unit Cells [40].

The Hückel matrix is used to find the carrier energy band for a conjugated $\pi-$ system with an alternation of single and double bonds along a linear chain. In the molecular orbital theory, the wavefunction $\psi_k(1)$ is a linear combination of atomic orbitals χ_i written as:

$$\psi_k(1) = \sum_{i=1}^{n} C_i \chi_i(1). \tag{1}$$

The corresponding orbital energies W, may also be written as:

$$W(C_1, C_2, C_3...C_n) = \frac{\int \psi_k(1)\hat{H}\psi_k(1)d\tau}{\int \psi_k(1)\psi_k(1)d\tau} = \frac{\langle \psi_k(1)|\hat{H}|\psi_k(1)\rangle}{\langle \psi_k(1)|\psi_k(1)\rangle}, \tag{2}$$

which can be optimized with respect to $\{C_1, C_2, C_3...C_n\}$ as:

$\delta W/\delta C_i = 0$; where C_i is the normalization constant and $i = 1,2,3...n$, leading to a set of homogenous equations, for which the secular determinant must be zero. These set of secular equations are given as:

$$\sum_{i=1}^{n} C_k |H_{ik} - S_{ik}| = 0 \tag{3}$$

$$\sum_{i=1}^{n} C_k |H_{ik} - WS_{ik}| = 0 \tag{4}$$

H_{ik} is the matrix element of the Hamiltonian, S_{ik} is the matrix element of the eigen values. For non-trivial solutions, a hexagonal honey-comb structure in the framework of benzene formed by the overlap of sp^2 hybridized orbital, which fits without strain into a hexagonal arrangement is employed as:

$$\begin{bmatrix} \alpha_\pi - \varepsilon & \beta & 0 & 0 & 0 & \beta \\ \beta & \alpha_\pi - \varepsilon & \beta & 0 & 0 & 0 \\ \beta & 0 & \alpha_\pi - \varepsilon & \beta & 0 & 0 \\ 0 & 0 & \beta & \alpha_\pi - \varepsilon & \beta & 0 \\ 0 & 0 & 0 & \beta & \alpha_\pi - \varepsilon & \beta \\ \beta & 0 & 0 & \beta & \beta & \alpha_\pi - \varepsilon \end{bmatrix} = 0, \tag{5}$$

where $\varepsilon = \alpha \pm 2\beta$. ε denotes the energy band and α_π is the minimum energy within the first Brillouin zone. The Hückel matrix theory was developed to allow qualitative analysis of π-carrier system in planar, conjugated hydrocarbon molecules in a flat C-H molecules with mirror symmetry. When the bonding is depicted in a localized fashion, carbon atoms that make up carbon skeleton are linked by alternating double and single carbon-carbon bonds. As a result, it is ideal for molecules such as SWCNT and FSWCNT, although the technique and concepts are applicable to a broader range of molecules. Similarly, by applying the following basic assumptions, Huckel matrix which sets carrier dispersion rules in account of translational symmetry, can be applied to a chemically modified nanotube F-(n,n);

- In a planar molecule, the atomic orbitals that contribute to π-bonding are antisymmetric with regard to reflection in the molecular plane. They differ in symmetry from atomic orbitals that contribute to σ-bonding and can be treated separately.
- All of the carbon atoms' Coulombic integrals are considered the same.
- All resonance integrals between directly-bonded atoms are considered to be the same, whereas those between non-directly-bonded atoms are ignored. $\int \phi_i \hat{H} \phi_j d\tau = \beta$: if atoms i and j are directly σ-bonded. $\int \phi_i \hat{H} \phi_j d\tau = 0$:if atoms i and j are non-bonded.
- All overlap integrals expressing atomic orbital overlap between separate atoms are ignored.

Following Sadykov et al. [40-42], a unit cell for the fluorine modified tube

F-(n,n) with N periods may be expressed as:

$$\begin{vmatrix} \alpha_\pi-\varepsilon & \beta & 0 & 0 & \ldots 0 & 0 & 0 \\ \beta & \alpha_\pi-\varepsilon & \gamma_o & 0 & \ldots 0 & 0 & 0 \\ 0 & \gamma_o & \alpha_\pi-\varepsilon & \beta & \ldots 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots \gamma_o & \alpha_\pi-\varepsilon & \beta \\ 0 & 0 & 0 & 0 & \ldots 0 & \beta & \alpha_\pi-\varepsilon \end{vmatrix} = 0. \qquad (6)$$

$\beta = \gamma_o(1+e^{ik_z\sqrt{3}b})$ and Eq. (6) contains $(4n-2)$ rows, k_z is carrier wavevector and $\sqrt{3}b$ is the distance between carbon atoms in the F-(n,n). Moreover, a primitive cell containing $N = 4n-2$ carbon atoms with a band having a width of two periods can be considered. Deleting the first row $(0 < k < N)$ and the first k column which can be denoted by $|N-k, N-k|$, the energy dispersion in the Hückel matrix forms where translation symmetry is accounted for in Ref.[40] will now be written as:

$$\left\| \begin{matrix} \alpha_\pi-\varepsilon & \beta & 0 & 0 & 0 & 0 \\ \beta & \alpha_\pi-\varepsilon & \gamma_o & 0 & & 0 \\ 0 & \gamma_o & \alpha_\pi-\varepsilon & \beta & 0 & 0 \\ 0 & 0 & \beta & \alpha_\pi-\varepsilon & \gamma_o & 0 \\ 0 & 0 & 0 & 0 & \alpha_\pi-\varepsilon & \beta \\ 0 & 0 & 0 & 0 & \beta & \alpha_\pi-\varepsilon \end{matrix} \right\| = 0. \qquad (7)$$

Solving the secular equation, the determinant from Eq. (7) must be zero,

$$|N,N| = \{(\alpha_\pi - \varepsilon)^2 - 4\gamma_o^2 \cos^2 ap_z\}|N-2,N-2|$$
$$-(\alpha_\pi - \varepsilon)\gamma_o^2|N-3,N-3|$$

$$|N-1,N-1| = (\alpha_\pi - \varepsilon)|N-2,N-2| - \gamma_o^2|N-3,N-3| \tag{8}$$

which results in a recurrence relation. So from Eq. (7) the energy dispersion law for $N = 6$ is:

$$|6,6| = \{(\alpha_\pi - \varepsilon)^2 - 4\gamma_o^2 \cos^2 ap_z\}|4,4| - (\alpha_\pi - \varepsilon)\gamma_o^2|3,3|$$

then;

$$|4,4| = \{(\alpha_\pi - \varepsilon)^2 - 4\gamma_o^2 \cos^2 ap_z\}|2,2| - (\alpha_\pi - \varepsilon)\gamma_o^2|1,1|.$$

From the recurring relations:

$$|6,6| = \{(\alpha_\pi - \varepsilon)^2 - 4\gamma_o^2 \cos^2 ap_z\}^2$$
$$-2(\alpha_\pi - \varepsilon)^2\gamma_o^2\{(\alpha_\pi - \varepsilon)^2 - 4\gamma_o^2 \cos^2 ap_z\}$$
$$+(\alpha_\pi - \varepsilon)^2\gamma_o^4 = 0. \tag{9}$$

The dispersion curve at the Fermi surface is given in the boundary of the BZ, as $\alpha_\pi - \varepsilon \to 0$ and $\cos ap_z \to 0$. Thus, the second term in Eq. (9) is far lesser than the third term and can be neglected. Letting $t = (\alpha_\pi - \varepsilon)^2/\gamma_o^2$ and $A^2 = 4\cos^2 ap_z$

$$t - 2t(t - A^2) = (A^2 - t)^3 \tag{10}$$

To solve Eq. (9), the following conditions are imposed; $t \ll A^2 \ll 1$.

Let $t = t_o + t_1, \quad t_1 \ll t_o$ then from Eq. (9)

$$t_o = A^6, t_1 = -2A^8$$

$$t = t_o + t_1 = A^6(1 - 2A^2)$$

As $A^8 \ll A^6$, the assupmtion that $t_1 \ll t_o$ is valid, and $t \ll A^2$ is also valid. Thus, from Eq. (9):

$$(\varepsilon - \alpha_\pi)^2 = 64\gamma_o^2 \cos^6 ap_z\{1 - 8\cos^4 ap_z\},$$

$$\varepsilon - \alpha_\pi = \pm 8\gamma_o \cos^3 ap_z\{1 - 8\cos^4 ap_z\}^{1/2},$$

$$\varepsilon = \alpha_\pi \pm 8\gamma_o \cos^3 ap_z\{1 - 8\cos^4 ap_z\}^{1/2}. \tag{11}$$

When $\cos(ap_z) \ll 1$,

$$\varepsilon(p_z) = \alpha_\pi \pm 8\gamma_o \cos^3(ap_z). \tag{12}$$

For cyclic conjugated systems, the folllowing assumptions are made [43-49]:

- The lowest-energy molecular orbital (MO) *i.e* conduction band is non-degenerate.
- The highest-energy MO *i.e* valence band could be non-degenerate for even

n or degenerate for odd n.

- All remaining solutions form pairs of degenerate MOs.

From the recurrence relation established in Eq. (8), a generalized energy band for the FSWCNT is obtained as:

$$\varepsilon(p_z) = \alpha_\pi + \Xi_n \gamma_o \cos^{2n-1}(ap_z). \tag{13}$$

Ξ_n is a constant and $a = \sqrt{3}b/2\hbar$ is the lattice constant and b is the interatomic distance. From Eq. (13), at the boundary of the BZ the derivatives of $\varepsilon(p_z)$ with respect to the quasi-momenta are zero to the order of $(2n-1)th$ order.

The unique structure of carbon nanotubes makes it one of the materials with miriads of applications. The electrical, mechanical, optical, and thermal properties of carbon nanotubes are exceptional. The band gap of semiconducting tubes can be controlled by varying the diameter of the cylinderical tube. The diameter of the nanotube has an inverse relationship with the band gap size [50, 51]. Molecular field-effect transistors (FETs) are created from an array of semiconducting SWCNTs, whereas single-carrier transistors [51] are made from metallic nanotubes. Field Emission Display (FED) in Samsung technologies [52-54], lightening element [55], nanotube sensors [52], hydrogen storage [55, 56], and memory device [57] are some of the other applications of carbon nanotubes.

Statement of the Problem

The need to use energy resources more efficiently has grown crucial in light of current global energy and environmental concerns. A considerable quantity of renewable energy remains untapped because a large amount of energy is still dumped into the environment as waste heat. Regardless of the magnitude of the source, thermoelectric power generating systems have the ability to convert this available heat energy directly into electrical energy. However, because of the toxicity, unavailability, and poor chemical stability of today's semiconductor materials at high temperatures, their use in practical thermoelements has been restricted. As a result, the goal of this research is to optimize the thermo-physical metrics of FSWCNT for acoustoelectric and thermoelectric applications computationally.

Purpose of the Study

The study aims to find the acoustoelectric and thermoelectric metrics of FSWCNT for thermo-physical applications in contemporary micro- and nanoelectronics industry.

Research Objectives

The general objective of the research work is:

- To study the thermo-physical properties of FSWCNT for Bloch oscillator and thermoelectric applications.

The research is guided by the following specific goals. To study:

- the acoustodynamics in FSWCNT.
- the induced Hall-like current by acoustic phonons in FSWCNT.
- FSWCNT as a low voltage current amplifier acoustic device.
- FSWCNT as a SASER device.
- the differential thermoelectric power in FSWCNT.
- the tunable differential electrical power factor in FSWCNT.
- the lattice conductivity of FSWCNT using the LBM.
- the thermal conductivity of FSWCNT.
- the figure of merit in FSWCNT.

Significance of the Study

This research's findings will guide material scientists to fabricate FSWCNTs of optimized thermo-physical metrics for thermoelectric and acoustoelectric applications.

Scope of the Study

The primary objective of this thesis is to investigate how to improve the thermo-physical properties of FSWCNT. The thermoelectric metrics of the FSWCNT will be studied, including carrier current density $\vec{J}$, electrical conductivity σ, resistivity $\vec{\rho}$, thermoelectric power α, electrical power factor $\mathscr{P}$, and lattice conductivity κ_ℓ. Temperature T, real overlapping integral for jumps (Δ_s and Δ_z), and carrier concentration n_o are all used to study the metrics. Prior to that, the research will focus on acoustoelectric metrics such as attenuation, acoustoelectric, and acoustomagnetoelectric currents in the hypersound regime, for nondegenerate carrier gas ($k_B T \ll 1$), which are crucial in the study of thermoelectric phenomena.

Delimitations and Limitations

The BTE is solved using a semiclassical approach to yield acoustoelectric and thermoelectric metrics for the FSWCNT in the context of continuous relaxation time. Quantum mechanical phenomena such as interband transition, quantum mechanical correction to intraband motion, correlation energy, Coulombic correction, and Coulombic interaction will be excluded from this study for mathematical simplicity, as did Ktitorov [52], Slepyan [56], Mensah [58], Maksimenko [59], Apostolaki s[60], and Abukari et al. [61]. The research will be purely semiclassical, with the acoustoelectric and thermoelectric metrics in FSWCNT being added.

Organization of the Study

The thesis is composed of Five sections. Chapter One provides background information and an overview of FSWCNTs, general and specific objectives, as well as the scope and organization of the thesis. The literature on carrier miniband transport, thermoelectricity, and acoustoelectric transport, as well as the principle underlying the theory employed in this study, are reviewed in Chapter Two. The acoustoelectric and thermoelectric metrics are obtained using a tractable mathematical model presented in Chapter Three. It also explains how the phonon lattice Boltzmann transport model (LBM), a computer tool for calculating lattice thermal conductivity, came to be. The model's results, analyses, and discussions are graphically presented in Chapter Four. Conclusions are formed and suitable recommendations are made in Chapter Five to facilitate future research in this field of study.

Chapter Summary

A review on FSWCNT was presented in this chapter. Works done by researchers in this field as well as related fields were discussed. Theoretical studies on fluorine doping of armchair SWCNT which changes the bandstructure from metallic to semiconducting using the Huckel matrix approach was also presented.

CHAPTER TWO

LITERATURE REVIEW

Introduction

In this chapter, the dynamics of carrier miniband transport leading to the derivation of the acoustoelectric and thermoelectric metrics of the FSWCNT in the semiclassical regime are reviewed. Acoustoelectric metrics such as gain and loss as well as acoustoelectric current density in the absence and presence of external fields will be reviewed. In addition to this carriers, phonons and carrier-phonon dynamics yielding phenomena such as the Seebeck effect, and other interesting phenomena will be reviewed as well.

Carrier tranpsort in semiconductors

Several previous treatments of semiconductor electrodynamics have focused on parabolic band structures, such as those found in *Si* and *Ge*. The focus of this thesis will be on semiconductors with non-parabolic band structures and how they relate to the FSWCNT under investigation. Recognizing the dynamics significant to the periodic potential's translation symmetry, elicits the physical interpretation of carrier dynamics in a crystalline lattice. Using the tight-binding technique, the band relation for the sample semiconductor is:

$$\varepsilon(p_x) = \frac{p_\perp^2}{2m^*} + \frac{\Delta}{2}[1 - \cos(\frac{p_x d}{\hbar})], \quad (14)$$

where p_x is the carrier miniband quasi-momentum, Δ is the miniband width, and d is the periodicity, in this case a superlattice (SL). Bloch proposed that a wave packet is made up of states with a single energy band that peaks around a crystal momentum ($\hbar\vec{k}$) and propagates with a group velocity specified by the slope of the band relation with respect to $\vec{k}$ [62]. The change in quasi-momentum with respect to time is equal to force imposed on the carrier by an external electric field ($\vec{E}$).

Bloch states which are eigenstates of the field-free Hamiltonian, allow carriers in the lower energy band to be represented and move freely through the potential field created by the ion cores [62]. In fact, the invariance qualities of the lattice periodicity lead to the formation of these localized wave packets. Impurity scattering, on the other hand, affects the dynamics of carriers in a solid, causing substantial decoherence of time-periodic oscillations. The traditional way to characterize solid-state transport is encompassed by this semiclassical reasoning [63]. However, in SL structures with negligible interminiband tunneling processes, carrier localization occurs in the higher range of constant electric field amplitudes, accompanied by Bloch oscillations, the same formulation holds.

When electric forces of finite magnitude are applied to the semiconductor, a basis consisting of Bloch states is usually useless [60]. To represent the occupation of the Bloch states at different times, it is preferable to use a semiclassical probability distribution function, $f(\vec{r},\vec{p})$. In the ”real” case, when Bloch state oscillations are dampened by scattering processes in the diffusive regime, the later dynamics description is preferable. Carrier dynamics in the semiconductor's miniband obeys $\vec{k}$-space version of Newton's second law for Bloch waves in the absence of scattering [64-66], with the carrier velocity equal to the group velocity of the wave packet to which it belongs. Thus:

$$\hbar \frac{d\vec{k}}{dt} = \vec{E}, \tag{15}$$

$$\vec{v}(\vec{k}) = \frac{1}{\hbar} \nabla_{\vec{k}} \varepsilon(\vec{k}). \tag{16}$$

An arbitrary energy potential causes $\vec{E}$ to be a constant field. The miniband velocity for the different directions is derived using the tight-binding approach in Eq. (14):

$$v_x(k_x) = \frac{\Delta d}{2\hbar} \sin(k_x d), \tag{17}$$

$$v_y(k_y) = \frac{\hbar k_y}{m^*}, \tag{18}$$

$$v_y(k_z) = \frac{\hbar k_z}{m^*}, \tag{19}$$

where the carrier effective mass in a path parallel to the SL layers is represented by m^*. Differentiating Eq. (17) with respect to t yields:

$$\frac{v(\vec{k})}{dt} = \hbar^{-1} \frac{dk}{dt} \frac{d^2 \varepsilon(\vec{k})}{dk^2}. \tag{20}$$

The effective mass, taking into consideration an isotropic energy surface, for an external field applied along one of the SL's directions ($i = x, y, z$) is defined as:

$$\frac{1}{m^*} = \frac{1}{\hbar^2} \frac{d^2 \varepsilon(k_i)}{dk_i^2}. \tag{21}$$

The reciprocal mass, $1/m^*$, determines directly the curvature of $\varepsilon(k)$ versus $\vec{k}$ in the y, z directionss, according to Eq. (21), implying that the carrier is accelerated relative to the lattice in an applied field but with a constant mass. For a *GaAs* SL, the conduction band effective mass is a scalar with a magnitude of $0.067m_e$ for parabolic band approximation. The effective mass in the growth direction, on the other hand, is given as:

$$\frac{1}{m^*} = \frac{1}{m_o}\cos(k_x d), \tag{22}$$

where $m_o = 2\hbar^2/\Delta d^2$ is the effective mass at the bottom of the miniband. For a large range of k_x, the carrier acts as if it has a mass in the BZ's center, as seen in Figure 4(a). At the upper end of the miniband, the effective mass becomes negative, causing carrier velocity to be suppressed (see Figure 4(b)).

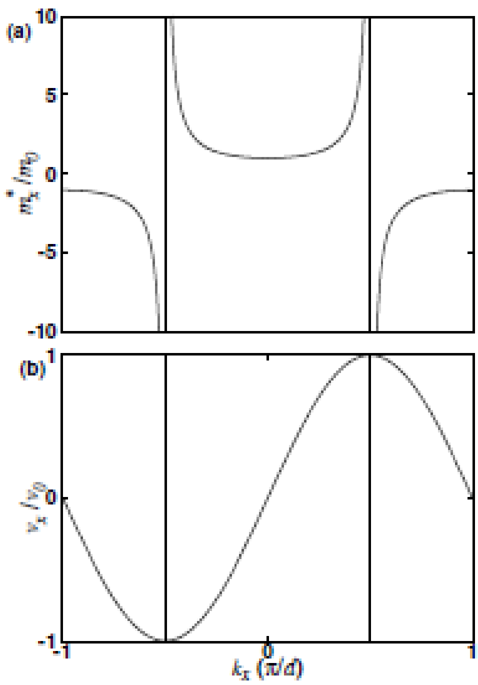

Figure 4: (a) Effective Mass Localized in the First Minizone (b) Miniband Velocity as a Function of Wavevector [60].

Thus, the effect of various external field configurations on carrier dynamics provides some physical insight into phenomena like Bloch oscillations and Landau level spacings. The following sections of this chapter will go through these phenomena in respect to cyclotron frequencies, and dynamic localization.

Bloch oscillations

Consider the effect of a static electric field applied in the axial direction of SL. The Hamiltonian of the system for carriers in the first miniband with a constant electric field E_o is:

$$\mathscr{H} = \varepsilon(p_x) - eE_o x. \tag{23}$$

When the electric field is present, the semi-classical equations of motion for the miniband velocity v_x and the crystal momentum of a wave packet, $p_x = \hbar k_x$, produce the following set of equations for single-particle dynamics:

$$\frac{p_x d}{\hbar} = \omega_B + \frac{p_x^o d}{\hbar}, \tag{24}$$

and

$$v_x(t) = \sum_{\ell=1}^{\infty} v_\ell \sin(\frac{\ell p_x d}{\hbar} + \ell\omega_B t). \tag{25}$$

The crystal momentum varies linearly with time, as seen by Eq. (24). Assume the carrier is initially at $k_x = 0$. Applying a steady electric bias causes a positive change in k_x, and the Bloch carrier within the reduced-zone approaches the BZ's edge, $k_x = \pi/d$. Halfway along the first BZ (vertical line in Figure 4(a) corresponding to $k_x = \pi/2d$), a carrier traverses the SL's minizone to attain the maximum miniband velocity. As a result, $d\varepsilon/dk_x$ reaches its maximum value, but at the BZ border, $d\varepsilon/dk_x = 0$, meaning that the carrier's miniband velocity is zero. The particle at $k_x = \pi/d$ experiences Bragg reflection, which results in the carrier reappearing at the opposite border of the minizone $k_x = -\pi/d$, and the crystal momentum continues to build from there.

When the norm of $d\varepsilon_x/dk_x$ is maximal [60], the carrier track in real space exhibits a negative drift with maximum negative velocity. The carrier now begins to decelerate until it reaches its initial point ($k_x = 0$), in quasi-momentum space, where it is briefly stalled. Bloch oscillation with frequency (ω_B) is a whole cycle of motion in real space that is defined by a single Bloch period (T_B) with frequency:

$$\omega_B = \frac{eE_o d}{\hbar}. \tag{26}$$

These periodic oscillations can be easily obtained in real space by integrating Eq. (25) as:

$$x(t) - x(0) = \sum_{\ell=1}^{\infty} x_\ell [\cos\left(\frac{\ell p_x d}{\hbar} + \ell\omega_B t\right) - \cos\left(\frac{\ell p_x^o d}{\hbar}\right)]. \tag{27}$$

The initial position is $x(0)$, and the span of oscillations in real space is determined by $x_\ell = 2T_\ell/eE_o$. The sum of the contributions of oscillations occurring in neighbouring wells can also be regarded as the spatial expansion of oscillations. This physical picture matches the position of the wave packet predicted analytically by solving the Schrodinger equation [67, 68], and more recently, the spatial displacement of Bloch-oscillating carriers has been determined directly in experiments measuring a dipole field caused by the wave packet's optical excitation [69]. This was Zener's initial prediction, based on Bloch's [62] work, implying that carrier wave packets do not delocalize but oscillate at high frequencies. Nonetheless, the period of these oscillations in normal crystals is substantially longer than ordinary relaxation. Delocalization of carrier paths is caused by the passage of time. The increased length of its lattice period d [62], is a countereffect of SL structures that permits Bloch oscillations to be realized. Bloch oscillations were confirmed experimentally using a brief four-wave mixing signal that showed a few Bloch cycles before the oscillations were suppressed due to scattering events [71].

Alternatively, it has been observed that coherent electromagnetic (EM) radiation is directly linked to charge carriers executing Bloch oscillations [72]. The SL and FSWCNT structures, on the other hand, are not the only ones that could lead to Bloch state coherent oscillations. In the absence of scattering events, an atom in an optical lattice can be efficiently adjusted to exhibit numerous cycles of Bloch oscillations [73-75].

Esaki-Tsu formulation of drift velocity

The relaxation time approximation is used to analyze how semiclassical carrier transport along the x-axis of the semiconductor would be impacted by scattering events. A carrier after a scattering event has no 'memory' of its behaviour prior to the event, and the scattering duration is constant in space and time, according to this approximation. Starting by determining the amount of carriers that remain unscattered at time t, $N(t)$, to compute the carrier drift velocity. In time dt, the chance of a carrier scattering is dt/τ, where the scattering time is τ. As a result, the number of carriers scattered over time dt equals;

$$N(t)\frac{dt}{\tau}. \tag{28}$$

As a result, the number of carriers that have not been scattered at time, $t+dt$ is;

$$N(t+dt) = N(t) - N(t)\frac{dt}{\tau}. \tag{29}$$

The change in unscattered carriers with respect to time can be calculated as follows:

$$\frac{N(t)}{dt} = \frac{N(t+dt) - N(t)}{dt} = -\frac{N(t)}{\tau}. \tag{30}$$

The number of unscattered carriers at time t is determined by integrating Eq. (30);

$$N(t) = N_o e^{-t/\tau}, \tag{31}$$

where the number of carriers at $t=0$ is $N_o = N(t=0)$. The number of carriers that scatter in time dt in Eq. (28) divided by the total number of carriers gives probability of carriers scattering in time dt:

$$P(t)dt = \frac{N(t)dt}{\tau}\frac{1}{N_o}. \tag{32}$$

Substituting Eq. (31) into Eq. (32) results in:

$$P(t)dt = \frac{1}{\tau}e^{-t/\tau}dt. \tag{33}$$

It is worth noting that scattered carriers do not ‘remember’ their behaviour prior to the scattering event. Thus, the average carrier velocity at time t is determined only by the behaviour of carriers after their scattering event at time t. As a result, the component of carrier drift velocity at time t is as follows:

$$v_d(t) = v_x(t)P(t)dt, \tag{34}$$

i.e. the proportion of carriers scattered at time t multiplied by the velocity of the carriers at time t. The system’s total drift velocity is determined by integrating Eq. (34) across all t to find v_d as follows:

$$v_d = \int_0^\infty v_x(t)P(t)dt. \tag{35}$$

Plugging Eq. (33) into Eq. (35) gives;

$$v_d = \frac{1}{\tau}\int_0^\infty v_x(t)e^{-t/\tau}dt, \tag{36}$$

which is the general form for the drift velocity of a carrier in a semiconductor. An analytical forecast for the drift velocity of a carrier in a semiconductor can be made using the derived drift velocity relation. The carrier velocity in the x

direction of a one-dimensional semiconductor is known from Eq. (34);

$$v_x = \frac{\partial \varepsilon(p_x)}{\partial p_x}. \tag{37}$$

Thus, the carrier drift velocity is given as;

$$v_d = \frac{\Delta d}{2\hbar}\frac{1}{\tau}\int_0^\infty \sin\left(\frac{p_x d}{\hbar}\right) e^{-t/\tau} dt. \tag{38}$$

Assume a carrier is initially at the bottom of the lowest miniband, i.e. $p_x(0) = 0$, and that an electric field ($\vec{E} = (-E, 0, 0)$) is applied along the semiconductor as;

$$p_x = eEt. \tag{39}$$

Substituting Eq. (39) into Eq. (38) and employing Laplace transform yields;

$$v_d = \frac{\Delta d}{2\hbar}\left(\frac{\omega_B \tau}{1+\omega_B^2\tau^2}\right), \tag{40}$$

where $\omega_B = eEd/\hbar$ is the frequency of the Bloch oscillations. This drift velocity-field relation is known as the Esaki-Tsu curve and plotted in Figure 5.

As the Esaki-Tsu equation is differentiated, the drift velocity reaches its maximum when $\omega_B \tau = 1$. The drift velocity curve is linear in the range of $\omega_B \tau \ll 1$, which is typical of an Ohmic I-V characteristic. The carriers scatter long before they can move very far down the dispersion curve in this domain, and the semiconductor behaves like a typical conductor. The carrier is permitted to cross about half of the BZ before scattering at $\omega_B \tau = 1$, and so the drift velocity is at its highest. When $\omega_B \tau > 1$, however, a strange occurrence

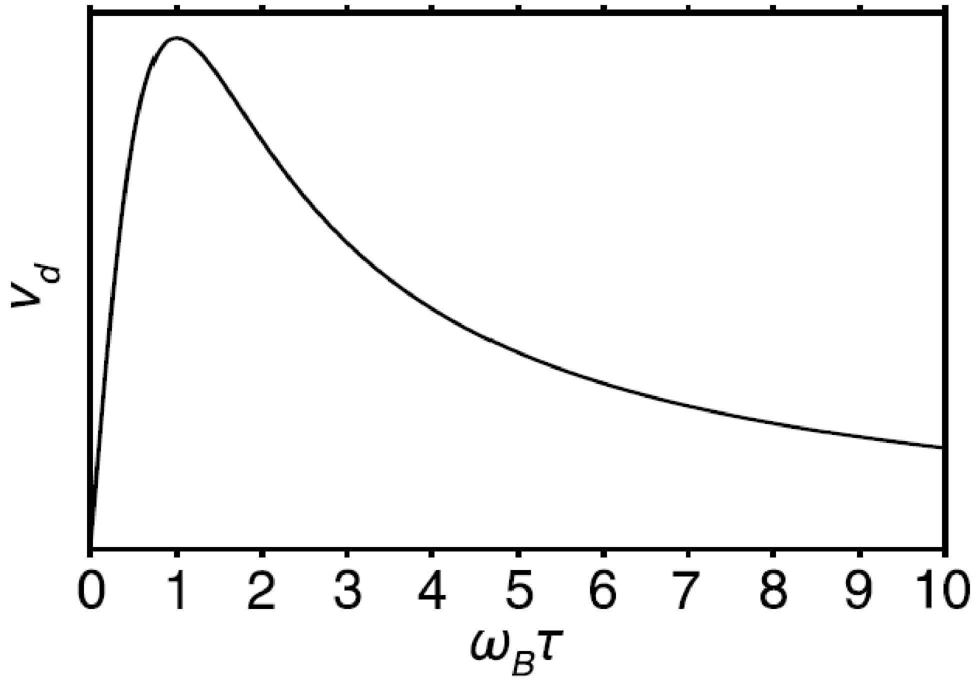

Figure 5: Esaki-Tsu Curve, Calculated using Eq. (40).

occurs, in which the carrier's transport appears to be impeded as the field is increased - this is known as a region of negative differential velocity or negative differential conductance. Many carriers are allowed to reach the BZ boundary in this region, where they are Bragg reflected and allowed to Bloch oscillate. While enduring Bloch oscillations, the carriers are localized, preventing transport.

As the electric field is raised, ω_B rises, suppressing carrier transport and resulting in a negative differential velocity, as seen in Figure 5. Negative differential conductance in SL was initially reported in an experiment by Esaki and Chang in 1974 [61], who showed that as the external bias voltage was increased, the conductance of the superlattice displayed a series of dips, eventually reaching negative values. The superlattice utilized in their investigation, however, was weakly linked, and the dips are due to resonant tunnelling between the wells. Bloch oscillation produced negative differential conductance was first demonstrated in an experiment by F. Beltram et al. in 1990 [61], two decades after Esaki and Tsu proposed it in 1970. They discovered direct evidence of negative differential conductance due to electric-field-induced localization of the carrier wavefunction in their experiment, and established that it was physically identical to the process outlined before for a tightly connected SL. It is worth noting that the Esaki-Tsu relation's drift velocity characteristic is valid for carriers in the lowest or first miniband. When $\omega_B \tau \gg 1/$, however, the carrier can tunnel into higher energy minibands, causing the carriers' velocity to increase, resulting in a substantial increase in drift velocity. An N-shaped drift velocity describes this type of behaviour.

The semiclassical model of carrier transport provided here is useful for understanding the behaviour of a single carrier in an indefinitely long semiconductor when subjected to electric and magnetic forces. The drift velocity analysis accounts for scattering within a relaxation time approximation, and thermal broadening can be accounted for by averaging over several initial circumstances. As the electric field increases, the localization of carrier trajectories due to Bloch oscillations generates an area of negative differential velocity, which induces charge and field domains in the device. In several devices, such as the Gunn diode [74, 75], negative differential velocity has been proven to produce charge and field domain creation.

Acoustic phonons in semiconductors

In this study, it is assumed that a strain pulse comparable to that created in the intraminiband population inversion induces semiclassical carrier miniband transport in a semiconductor's spatio-temporal potential. Furthermore, the acoustoelectric interaction is considered in the context of the deformation potential. The atomic displacement within the unit cell is represented by the acoustic wave as:

$$\vec{u}(\vec{r},t) = \frac{u_o}{2}\hat{e}_r[e^{i(\vec{k}\vec{r}-\omega_s t)} - e^{-i(\vec{k}\vec{r}-\omega_s t)}],$$
$$= u_o\hat{e}_r\cos(\vec{k}\vec{r}-\omega_s t), \tag{41}$$

where u_o represents the largest displacement and $\hat{e}_r$ is the unit vector along the primary axis [1 0 0]. The strain tensor's generic form as differential displacement of the atomic site is:

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{42}$$

$x_j(x_i)$ signifies the components of the vector $\vec{r}$ that determines the location of a lattice point. x_j represents the cartesian components of the vector $\vec{r}$, and the subscript i has values of 1, 2, or 3 [76] yields a change in total energy of the carrier generated by elastic strain:

$$\Delta\varepsilon = \sum_{ij}\Lambda_{ij}S_{ij}, \tag{43}$$

where Λ_{ij} is the tensor of deformation potential. Reducing summing to a single component, $\Delta\varepsilon = \Lambda_{ij}Tr(S)$, where Λ is the hydrostatic deformation potential indicating the unit cell's pure dilation. One of the major concerns is the possibility of charge polarization as a result of stress-related effects. A linear relationship between a second order tensor and a polarization vector [77] gives the piezoelectric coupling:

$$\delta P_i = \sum_j e_{ij}\varepsilon_j. \tag{44}$$

According to Voigt notation, e_{ij} determines the constituents of the piezoelectric tensor. There is only one non-vanishing element e_{14} that does not contribute to piezoelectricity in the absence of shear tension, such as in a GaAs crystal [78, 79]. The deformation potential, without interference from piezoelectric effects, is thus the primary process of the effective carrier-phonon interaction for a biaxial strain wave [80, 81]. The potential energy due to strain is directly proportional to the differential displacement as follows:

$$V_s(\vec{r},t) = \Lambda \cdot \nabla \vec{u} = -u_o \Lambda \hat{e}_r \vec{k} \sin(\vec{k}\vec{r} - \omega_s t). \tag{45}$$

Taking into account the one-dimensional nature of carrier miniband transport in the semiconductor and the longitudinal mode of phonon, $\hat{e}_r \vec{k} = k_s$ from Eq. (43) yields:

$$V_s(x,t) = -u_o \Lambda k_s \sin(k_s x - \omega_s t) = -U \sin(k_s x - \omega_s t), \tag{46}$$

where the wave amplitude is $U = k_s u_o \Lambda$, and the maximum strain caused by the lattice vibration is, $S_o = k_s u_o$. The wavevector is orthogonal to the other two transverse modes. The later equations show how S_o is affected by the phonon frequency, $\omega_s = v_s k_s$, where $v_s = 2.5 \times 10^3 m/s$ is the sound velocity in the medium. The linear dispersion indicates a tiny k group velocity, which is taken into account for LA phonons propagating in the x-direction, as in zincblende structures [82-84]. The highest wave amplitude in this study is $9meV$, based on S_o, which has been measured to be $\approx 10^{-4} - 10^{-2}$ and $\Lambda = 10eV$ [76, 85, 86], denoting the matrix element of the carrier-phonon coupling constant. However, it would be advisable to consider an enhanced high-field strain. The technique requires an ultrafast laser stimulation of a separate thin metal film formed on the substrate side to be considered for this assumption to hold [87, 88].

Effects of sound on intraminiband transport

A deformation potential method is used to describe the effect of a plane wave travelling down the semiconductor's axis in the x-direction. As a result, the moving potential can be written as:

$$V_s(x,t) = -U \sin[(k_s(x+x_o) - \omega_s t)], \tag{47}$$

where the initial phase of the driving wave is denoted by the displacement x_o. The carrier is a subatomic particle and the carrier's travel is considered to be within the lowest miniband, and interminiband transport is ignored. The semiclassical Hamiltonian, $H(x, p_x) = \varepsilon(p_x) + V_s(x,t)$, yields the following equations of motion from Eq. (40):

$$v_x = \frac{dx}{dt} = \frac{\partial H}{\partial p_x} = \frac{\Delta d}{2\hbar} \sin(\frac{p_x d}{\hbar}), \tag{48}$$

$$\frac{\partial p_x}{\partial t} = -\frac{\partial H}{\partial x} = k_s U \cos[(k_s(x+x_o) - \omega_s t)]. \tag{49}$$

Eq. (48) and Eq. (49) can be integrated numerically using a 4th order Runge-Kutta method to obtain the carrier trajectories.

Drift and mean velocity characteristics

The well-known miniband transport model based on the BTE [89, 90] is used to characterize dissipative carrier dynamics in the presence of a spatial periodic potential. The carrier drift velocity has been calculated using a different method based on the Esaki-Tsu formalism [92]. The electric field distribution along the SL is treated as almost uniform and constant in time in this study's methodology. It is worth noting that this formula produces a drift velocity wave amplitude dependence that behaves differently depending on the starting conditions.

In addition, this discovery demonstrates a link between ballistic transport and conditions for substantial v_d suppression. The time-dependent route integral formulation [93-96] as a steady solution of the time-dependent BTE is used to improve the prior analysis and account for the variable initial conditions. The drift velocity is calculated as follows given a constant relaxation time, τ, and a more general excitation, such as that defined by Eq. (48) and Eq. (49):

$$v_d = \int_0^T \frac{dt}{T} \int_{-\infty}^{t} \exp\left(\frac{-(t-t_o)}{\tau}\right) v_x(t,t_o) \frac{dt_o}{\tau}, \tag{50}$$

where t_o is the time at which the carrier is at point x_o, and $T = 2\pi/\omega_s$ is the acoustic wave's period. Eq. (37) demonstrates that drift velocity can be estimated without using the BTE, but its formal solution necessitates the explicit description of nonlinear carrier transport required by Eq. (48) and

Eq. (49). The velocity of a single carrier averaged across all beginning moments can thus be used to represent the steady-state time-dependent drift velocity, and t_o after accounting for the likelihood of carrier scattering in the time interval between $t - t_o$ and $t - t_o + dt$. To calculate the average drift velocity, it must be averaged over a time period that corresponds to the acoustic wave period. The concept of drift velocity calculation can be explored by using a new approach that involves averaging carrier velocities over the starting positions, x_o, or across the early phases of the acoustic wave, rather than over the beginning time, t_o. In this manner, the Fourier series [93] may be enlarged to ensure that the Hamiltonian H is periodic in time and that the velocity of the carriers in the presence of scattering events $v(x,t)$ is:

$$v(x,t) = \sum_n v_n e^{ink_s(x - v_s t)} \tag{51}$$

with

$$v_n = \frac{1}{\lambda} \int_0^{\lambda} dx_o \int_0^{\infty} \exp(\frac{-t'}{\tau}) v_x(x_o, t') e^{-ink_s(x_o + v_s t')} \frac{dt'}{\tau}, \tag{52}$$

with $\lambda = 2\pi/k_s$ denoting the wavelength of the acoustic wave. Applying the Jacobian, $J = \partial(tv_s, t - t_o)/\partial(t, t_o)$, the group of variables (x_o, t') can be substituted with (t_o, t), for which Eq. (50) assumes the form:

$$v_n = \frac{1}{T} \int_0^{T} dt \int_{-\infty}^{t} \exp(\frac{-(t - t_o)}{\tau}) v_x(t, t_o) e^{in\omega_s t} \frac{dt_o}{\tau} \tag{53}$$

The velocity's zero order component of Fourier analysis (for $n = 0$) in Eq. (53) is particularly important, as it ensures that v_o is identical to the drift velocity v_d

determined with the time dependent path integral in Eq. (51). Integration over all initial beginning times t_o in Eq. (51) is comparable to integration over all carrier initial positions x_o realized in Eq. (52), according to the link between these calculations.

It is feasible to use this equivalence to analyze the effect of carrier trajectories with varied dynamic qualities on kinetic transport in this example with the sound wave travelling through the SL. Furthermore, because direct averaging over the initial positions is a simpler numerical task, the latter approach is used for efficient v_d computation optimization. This method can be thought of as a viable alternative to the path-integral BTE solution for arbitrary time-dependent electric and static magnetic fields in semiconductors [97-99]. A heavily linked SL is considered in the simulations for $v_o > v_s$ with a set of realistic parameters as follows: $\Delta = 7meV$, $d = 12.5nm$, $\omega_s = 4 \times 10^{11} rad/s$, $v_s = 5000m/s$ and $\tau = 250fs$. Assuming the band relation in Eq. (14), the maximum miniband velocity v_o and maximum average miniband energy W_o that the carrier can achieve are as follows:

$$\vec{v}(t) = \frac{2}{(2\pi\hbar)^3 N_o \tau} \int d^3 p f_o(\vec{p}_s) \int_{-\infty}^{t} ds e^{\frac{t-s}{\tau}} \vec{v}(\vec{p}_s^{\,t}) \tag{54}$$

$$W(t) = \frac{2}{(2\pi\hbar)^3 N_o \tau} \int d^3 p f_o(\vec{p}_s) \int_{-\infty}^{t} ds e^{\frac{t-s}{\tau}} \varepsilon(\vec{p}_s^{\,t}) \tag{55}$$

The solution of Eq. (54) and Eq. (55) is quoted in the form:

$$v_o = \frac{2}{(2\pi\hbar)^3 N_o} \frac{\Delta d}{2\hbar} \int d^3 p f_o(\vec{p}_s) \cos(p_x d/\hbar), \tag{56}$$

$$W_o = \frac{2}{(2\pi\hbar)^3 N_o} \frac{\Delta d}{2} \int d^3 p f_o(\vec{p}_s) \cos(p_x d/\hbar), \tag{57}$$

where the integration limits for p_z and p_y in both equations are $\pm\infty$, whereas p_x is integrated over the BZ. Eq. (56) and Eq. (57) for non-degenerate carrier gas can indicate the temperature dependence of the kinetic transport as:

$$v_o = \frac{\Delta d}{2\hbar} \frac{I_1(\Delta/2k_B T_e)}{I_o(\Delta/2k_B T_e)}, \tag{58}$$

$$W_o = \frac{\Delta d}{2} \frac{I_1(\Delta/2k_B T_e)}{I_o(\Delta/2k_B T_e)}, \tag{59}$$

where the lattice temperature is T_e, and the Boltzmann constant is k_B. The peak current density can thus be calculated with as:

$$j_p = N_o e v_o. \tag{60}$$

In SL's with small band widths, this type of temperature dependency of current density has been confirmed experimentally [100-102]. Even at higher temperatures, our measurements confirmed that carrier dynamics are dominated by miniband transport. The SL parameter $v_o = 6.65 \times 10^4 m/s$ ($v_o \approx 13.3 v_s$) is found assuming v_o is determined from Eq. (58). SL structures with broader minibands or periods can achieve higher maximum miniband velocities. Figures 6 and 7(a) show how drift velocity v_d changes as acoustic wave amplitude, U, increases.

The path-integral formulation based on Eq. (51) and the drift velocity obtained (see Eq. (52)) by averaging the carrier velocities throughout the four beginning phases x_o of the driving wave exhibit excellent agreement in Figure 6.

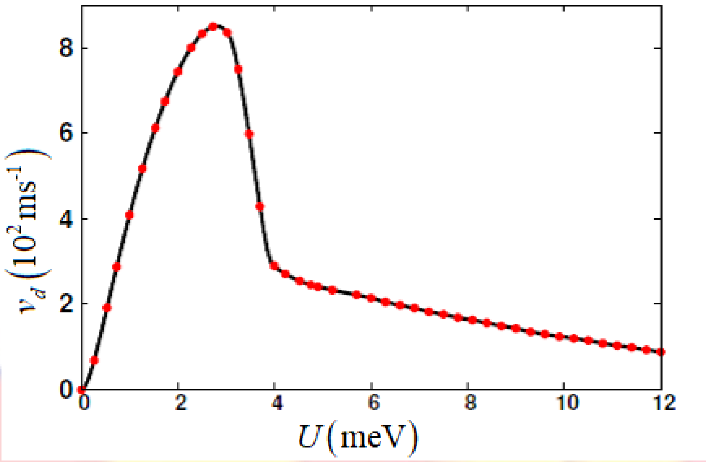

Figure 6: Drift Velocity against Acoustic Amplitude Calculated using Eq. (41) shown by Solid Curve and the Red Dots show the Calculation of v_d obtained with Eq. (52) [60].

Although the conduction carrier temperature influences carrier transport [103], carriers with a starting momentum of $p_o = 0$, which corresponds to $T_e \to 0$, are considered for simplicity. One can see that the $v_d(U)$ characteristic has a strong non-monotonic dependence, which presents itself as a drastic suppression of drift velocity at a characteristic value U_{cr1}, as in Figure 7 (a).

Following that, at U_{cr2} the curve shows an observable change in slope. Esaki-Tsu $v_d(E_o)$ curve [92] mimics the appearance of a significant peak for the former value of wave amplitude. The carriers can reach the BZ edge before scattering if the electric field is big enough ($E_o > E_{cr}$), and the drift velocity reduces as the electric field increases. When U passes a crucial threshold, however, driving carriers with sub-THz waves can cause an abrupt commencement of Bloch-type oscillations, resulting in negative differential velocity [91, 103]. Figure 7(a) shows a non-monotonic dependence that can be linked to either changes in dynamics or scattering events. To determine which process is dominant, the mean carrier velocity, v_m, is determined and averaged across time as follows:

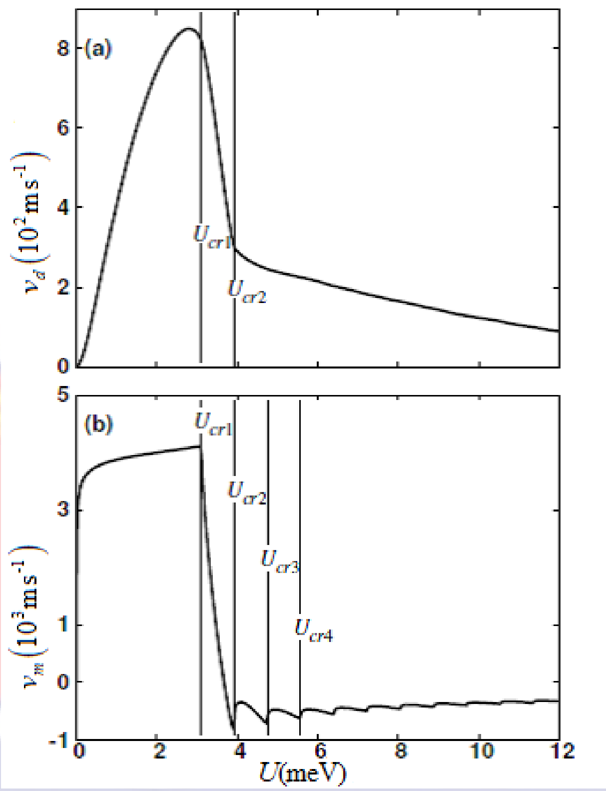

Figure 7: (a) Drift Velocity v_d against Wave Amplitude U; (b) Dependency of Average-time Carrier Velocity v_m on U. The Vertical Lines Correspond to the Critical Values of U [60].

$$v_m = \frac{1}{\lambda}\int_0^{\lambda} dx_o \int_0^{\Delta t} v_x(t+t_o,t_o)\frac{dt}{\Delta t} \tag{61}$$

The mean velocity can be regarded of as a finite-time version of Eq. (52), where the scattering effect is insignificant ($\tau \to \infty$). v_m is averaging over a period of $\Delta t = 2ns$, which is equal to a significant number of times of acoustic wave oscillations in these computations. The reliance of $v_m(U)$ is seen in Figure 7(b). The curve has a non-monotonic feature here as well. Surprisingly, the drift velocity wave amplitude characteristic (see U_{cr1},U_{cr2}-dashed lines in Figure 7(b)) exhibits its significant features at nearly the same values (U_{cr1},U_{cr2}-dashed lines in Figure 7(b)). In contrast to the latter, Figure 7(b) shows the establishment of multiple maxima and the carrier's ability to access a negative velocity zone. Bloch oscillations, on the other hand, would have resulted in the localization of carriers in the absence of scattering, and hence the suppression of the mean velocity [104] if only a *dc* electric field had been applied. Because the carrier periodically returns to its starting condition after a single Bloch period, dynamic localization occurs.

Acoustic Phonon Amplification

The concept of acoustic phonon emission in bulk materials was first theorized by Tolpygo and Uritskii [105], Weinreich [106] and experimentally observed by Pomerantz [107] for *n*-Ge and for CdS by Hutson et al. [108], who amplified a radio-frequency ultrasound in CdS piezoelectric material. The effect was achieved by a strong coupling between the carrier waves and acoustic waves in the piezoelectric material (CdS). They were of the view that, the coupling between the phonons and the carriers leads to the exchange of energy and momentum.

Spector, numerically modeled the carrier-phonon energy transfer analytically [109]. The model showed that sound wave could be gained when the carrier drift velocity superceeds the sound speed which leads to a switch from net acoustic loss to gain [110]. Since the advent of nanoscience, much work has been conducted and reported by scientists on heterogenous structures such as SLs [111-114], cylindrical quantum well (CQW) [114,115-121] and SWCNTs [122], in relation to acoustic phonon amplification and absorption.

Shmelev and Mensah [123] theoretically studied the acoustic wave amplification in SLs under the action of carrier transport making use of the semiclassical BTE in the presence of an external source of non-equilibrium phonons. The amplifcation of the acoustic wave in a crossed electric and magnetic fields was studied theoretically by Dompreh et al. [122] employing the quantum kinetic equation for carrier-phonon interactions in AGNR. Analytical expressions for the amplification under different conditions was numerically analysed. The general amplification (Γ/Γ_o) was obtained in the region of $ql \gg 1$. Their findings revealed a linear relationship for Γ/Γ_o with constant electric field but nonlinear for Γ/Γ_o with q. SASER in AGNR was reported in a case where an increase in acoustic wavenumber (q) causes intraband transition which modulates the gain [124-126].

Acoustoelectric Effect (AE)

The acoustoelectric effect (AE) is commonly linked to the attenuation of propagating acoustic waves [127]. In 1941, Shaposhnikov [110] was the first to propose that phonon energy may be absorbed by carriers in a substance. Parmenter [128] confirmed this result experimentally by measuring a dc electric current due to carrier drag in the direction of the sonic waves. Weinrich [106] performed simulations of wave-particle migration in the direction of acoustic waves, which he detailed in his classic work. When acoustic waves pass through a piezoelectric material, a dc electric field is created, causing charge carriers to redistribute spatially [127]. There is a phase change in the particle drag and acoustic wave velocities. This phase difference causes energy to be transferred from sound waves to carriers, resulting in acoustic phonon attenuation (acoustoelectric attenuation) [117]. SWCNT's huge carrier densities and high drift velocities, as well as its perculiar band structure with carrier mobility, $\mu = 10^5\ cm^2$/Vs at 300K, show how carrier control approaches can be utilized instead of direct electrical control. Carrier-phonon interactions in SWCNT in the nonlinear regime at low temperatures, i.e., $k_BT \ll 1$, reveal the gain or attenuation of a quite number of coherent acoustic phonons [129-131]. As conducting carriers absorb the momentum of acoustic phonons, the Acoustoelectric Effect (AE) [138, 139] appears as a dc electric field [110-123, 132-139].

Studies of AE in semiconductors with parabolic energy bands, bulk semiconductors like Gallium Nitride (GaN) [139, 140] with applications in Indium Antimonide [141], $GaAs/LiNbO_3$ [142] and GaN film acoustic resonators [142] have been reported over the years. Thus, the advent of low-dimensional devices has seen AE getting much attention in quantum wells [143] for producing quantized current in one-dimensional channels [144] and induced carrier pumping in nanotube quantum dots [145]. There have been studies of AE in superlattices [146, 147], quantum wires [148-150], and Zinc Oxide (ZnO) Nanowires [148-150]. A small amount of experimental work in SWCNT has been reported in Refs. [151,152], whilst the effect of hypersound in SWCNT has been explored in Ref. [122] analytically.

Acoustomagnetoelectric Effect (AME)

There is a transfer of energy and momentum from the acoustic wave to the conducting carriers when it propagates across a material medium, and acoustoelectric effect is what happens as a result of this. A lot of attention has been paid to the acoustoelecric effect in bulk semiconductors [128, 153]. Mensah et al. theoretically analyzed this effect in superlattices [146], and the observation of this phenomenon in mesoscopic structures [146, 154] drew a lot of attention. The AME is produced for audio wave propagating in a conductor under external magnetic field. As a sample put under magnetic field $\vec{H}$ carries an acoustic wave propagating in a path perpendicular to $\vec{H}$, AME develops a j^{AME} when the sample is short-circuited in the Hall direction, or an AME field if the sample is open-circuited in the Hall direction.

Grinberg predicted the AME for bipolar semiconductors theoretically, and Yamada saw it empirically in Bismuth [155]. The discovery of this effect in monopolar semiconductors [156] and Kane semiconductors [157] drew a lot of attention at the time. Quantum Acoustomagnetoelectric Effect (QAME) caused by Rayleigh sound waves has been explored by Margulis and Margulis [158] and Minneichi [159]. AME is synonymous to Hall effect in conventional semiconductors, where the sound flux $\vec{\Phi}$ plays the role of electric current $\vec{j}$. AME in bulk semiconductors for the condition $q\ell \gg 1$, has been investigated in Ref. [160].

Thermoelectric Effect

The search for efficient thermoelements have attracted numerous attention owing to potential applications in refrigeration and power generation [161]. The performance of a thermoelement is defined by its figure of merit, $ZT = \alpha^2\sigma T/\kappa$, where α, σ, T, κ are the Seebeck coefficient, carrier conductivity, absolute temperature, and thermal conductivity, which includes the carrier and lattice components. High performance thermoelectric material requires a large electrical power factor $\alpha^2\sigma$, but low thermal conductivity. Rapid development of thermoelectrics has been witnessed for the past two decades, where various contemporary techniques have been applied to improve the efficiency of thermoelement, such as band convergence [162, 163], resonance energy level [164], nanostructuring [165] and all-scale hierarchical architectures [166].

Thermal Conductivity

Thermal transport in solids is usually governed by electric carriers, lattice vibrations, electromagnetic waves, spin waves and other excitations [167]. Electric carriers transmit the overwhelming amount of the heat in metals, whereas lattice vibration is the principal heat transporter in semiconductors and insulators. Thermal conductivity of a solid is calculated as the total of all components representing the various excitations: $\kappa = \sum_\beta k_\beta$, where β signifies an excitation [167].

Thermal conductivity of solids in general is composed of the charge carrier component κ_e and the lattice component, κ_ℓ. κ_e is reliant on the carrier band structure, effective mass, carrier scattering, carrier-ion and carrier-phonon interactions, whereas the lattice component κ_ℓ mainly depends on phonon-phonon scattering or Umklampp processes [168, 169]. In dielectrics, $\kappa_\ell \gg \kappa_e$ whilst in metals, $\kappa_\ell \ll \kappa_e$. In semiconductors, κ is strongly reliant on the semiconductor composition and so κ_ℓ is far higher than κ_e [170, 171]. Acoustic phonons carry out the majority of thermal conductivity in semiconductors due to their higher group velocity than optical phonons. The BTE is used to create analytic models for thermal conductivity using the single relaxation time approach published by Bhatnager-Groos-Krook (BGK) in 1954 [168-173].

In semiconductors, the most prevalent scattering processes includes mass-difference scattering, three-phonon Umklapp scattering, boundary scattering, phonon-carrier scattering, and so on [174]. To add up all of the individual scattering rates, Matthiessen's rule is used. In a nutshell, two border effects have been considered. The first is alteration of phonon dispersion owing to spatial confinement, and the second is the change in nonequilibrium phonon probability distribution owing to partially diffuse border scattering [175, 176]. The former causes a decrease in both group and phase velocity and the latter corrects the formula for determining the lattice conductivity [167, 176].

Limitations of Classical Heat Transfer

The movement of thermal energy in a structure is caused by a temperature gradient, ∇T, according to classical studies of conductive heat transfer. The Fourier law has been used to characterize this flow as a dispersion of energy within the structure:

$$q = -\kappa \nabla T \tag{62}$$

This rule asserts that heat flux through a structure is proportional to the temperature gradient, with the thermal conductivity, κ, as the proportionality constant. Thermal conductivity is a microscopic property that varies by material. Heat conduction in bulk structures is also effectively described by this law. Heat transfer in nanostructures, on the other hand, has been found to have a poorer thermal conductivity than that of bulk materials [177]. Thermal conductivity tests of silicon nanowires show that, depending on the diameter of the nanowires, κ can be lowered by one to three orders of magnitude compared to Si [177]. The mechanisms of heat transport in materials must be thoroughly explored in order to comprehend the impact of nanostructuring on thermal conductivity.

Heat Carriers

The movement of heat via heat carriers and transporters is the mechanism for heat conduction. In conductive heat transport, two types of carriers have been identified: (i) carriers and (ii) phonons. In a crystal, weakly bound valence carriers travel freely and serve as energy carriers [178-180]. Phonons are crystal lattice vibrations caused by thermal energy given to the lattice, which causes excitations in the lattice structure. Phonons, like carriers, exhibit wave-like and particle-like properties (wave-particle duality). Thermal energy excites carriers and phonons, which then ”transport” the surplus energy through the material [177, 180].

A simple harmonic oscillator, i.e. mass and spring analogies, can be used to visualize the wave-like features of phonons [178- 181]. Assume that the atoms in a crystal lattice are spherical masses connected by length a springs (crystal lattice length). For the sake of simplicity, assume a 1D array of masses and springs, as shown in Figure 8, with all spring lengths between the masses being the same.

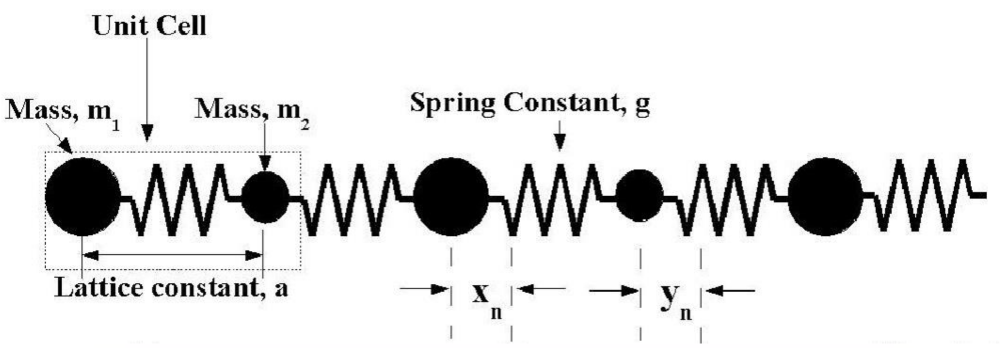

Figure 8: A One-dimensional Spring-mass System with Two Masses of Differing Mass Composing each Primitive Cell [182].

If the n-th atom is displaced by a distance x_n, then from Newton's law, the equation of motion is given as:

$$m^* \left(\frac{d^2x}{dt^2} \right) = g \cdot (x_{n+1} + x_{n-1} - 2x_n) \tag{63}$$

In Eq. (63), the force on the n-th atom is assumed to be contributed only by its nearest neighbouring atoms. The solution to Eq. (63) have the form:

$$x_n = x_o e^{-i\omega t} e^{inKa}. \tag{64}$$

Plugging Eq. (64) into Eq. (63) yields:

$$\omega^2 m^* = g[2 - e^{-inKa} - e^{inKa}] = 2g(1 - \cos Ka) \tag{65}$$

$$\omega = \sqrt{\frac{2g}{m^*}(1 - \cos Ka)} \tag{66}$$

Eq. (66) is known as the dispersion relation which can be plotted as $\omega - K$ plot in Figure 9 to show the phonon dispersion. The plot of Eq. (66) represents the acoustic and optical branches of phonons. The lower frequency spectrum or

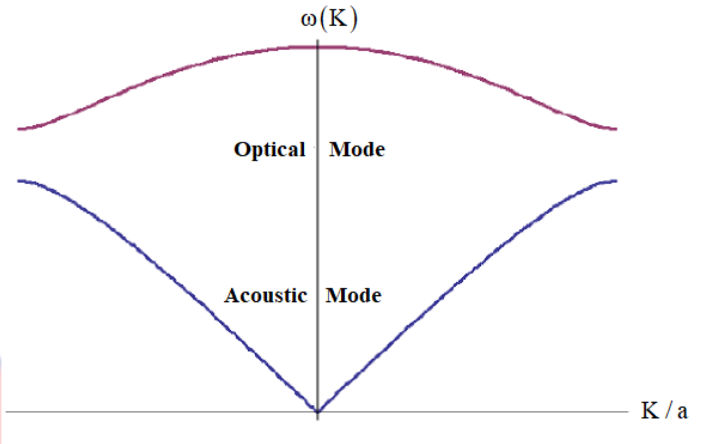

Figure 9: The Optical and Acoustic Vibrational Modes Associated with a One-dimensional Spring-mass System with two Atoms of Differing Mass within each Primitive Cell [182].

polarisation resembles sound wave in that the frequencies are small at longer wavelenghts. Thus, this branch is termed the “acoustic branch”. Acoustic waves are associated with the motion of the center of mass of primitive cell. The higher polarisation is associated with the movement of atoms within the primitive cell. If both atoms carried opposite charge, this type of $\omega - K$ relation could be produced by the electric field of a light wave [183]. This branch is termed the “optical branch”. Within a three dimensional crystal, each atom then has three degrees of freedom. If the molecule has an FCC lattice, then the three atoms lead to nine equations of motion. There are nine polarisations, three acoustic and six optical branches. The branches are in sets of threes; one longitudinal and two transverse and the transverse is often times degenerate due to symmetries within the crystal.

The minimum permissible wavelength is $2a$ since the lattice is a periodic array of atoms with periodicity of a. This suggests that K has a range of $-\pi/a$ to π/a. The group velocity of phonons, v_g, determines the slope of the phonon dispersion. Furthermore, the slope of the curve in the phonon dispersion as $K \to 0$ is constant [178]. This is the phonons' long-wavelength limit, and the dispersion relationship (Eq. (66)) in this limit is as follows:

$$\omega = \sqrt{\frac{g}{m}} K a = v_g K, \tag{67}$$

which is the Debye approximation. The acoustic branch is named from the fact that the v_g of phonons at the long wavelength limit is equal to the solid medium's speed of sound. Phonons propagating at low temperatures frequently exhibit the long-wavelength limit behaviour. Another finding is that when $K \to \pi/a$ increases, so does $v_g \to \pi/a$, and phonons propagating with greater K (shorter wavelengths) contribute very little to heat conduction. The Debye cut-off frequency, ω_D, is the maximum frequency of the phonon dispersion [178]. It occurs at the limit $K \to \pi/a$. The optical branch contains phonons, which are propagated when photons or electromagnetic waves collide with the solid's surface. The optical branch phonons, on the other hand, have very low group velocities, therefore they don't contribute much to heat conduction. As a result of the phonons' transverse and longitudinal motion, there are three polarisations of the acoustic branch (one LA and two TA) and two polarisations of the optical branch (one LA and two TA) (LO and TO). Depending on the crystal orientation of the material, each of these polarisations or modes of phonon branching can be unique.

Nanoscale Heat Transfer Processes

The study of the interactions of carriers within a material at the nanoscale is called nanoscale heat transfer. In nanoscale heat transfer, two types of interactions are involved: (i) carrier-carrier interactions and (ii) carrier-defect interactions. These interactions are also referred to as scattering mechanisms because they can cause changes in the frequency or direction of the heat carriers. There are two types of scattering mechanisms: (i) elastic scattering and (ii) inelastic scattering. Inelastic scattering alters the frequencies as well as the direction of the heat carriers [178]. Elastic scattering changes the direction of the heat carriers without changing their frequencies.

Carriers dominate the heat transmission process in metals, and carrier-carrier, carrier-phonon, and carrier-defect scattering are the principal scattering mechanisms. Because carriers are rarely scattered by other carriers, carrier-carrier scattering is thought to be insignificant in heat transmission [179]. Between carriers-acoustic phonons and carriers-optical phonons, carrier-phonon scattering occurs. carriers' frequencies are not greatly modified when they are dispersed by acoustic phonons, but their direction and momentum are (elastic scattering) [178]. When carriers are dispersed by optical phonons, however, the energy exchanged during the process dramatically affects their frequencies as well as their direction and momentum (inelastic scattering) [179]. carrier-defect scattering is an elastic mechanism that causes the carriers' direction and momentum to shift but not their frequencies to change [178].

Three types of phonon scattering mechanisms are identified as: phonon-defect, carrier-phonon, and phonon-phonon scattering. Phonon-impurity, phonon-dislocation, and phonon-boundary interactions are all examples of phonon-defect scattering (see Figure 10). The phonon frequencies are unaffected by this scattering process (elastic scattering) [178].

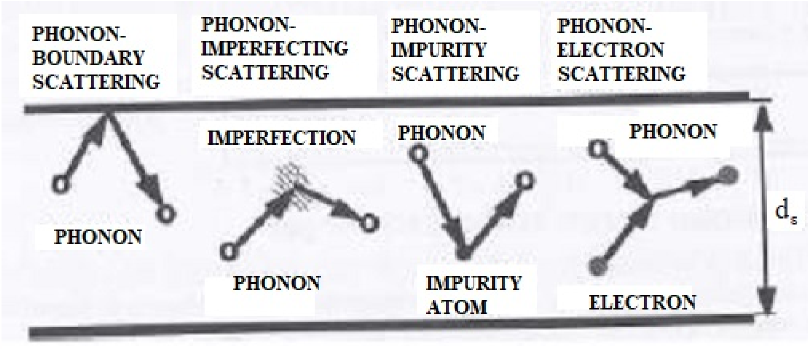

Figure 10: Elastic Scattering Mechanisms of Phonons [178].

Phonon-phonon scattering on the other hand is an inelastic process that changes the frequency and momentum of the phonons. Two types of phonon-phonon scattering have been identified as: normal and Umklapp processes. The normal process which occurs when two phonons merge to form a single phonon (case A) or when a single phonon splits into two phonons (case B). In both cases the frequencies of the phonon changes but not the momentum, which results in almost no thermal resistance in the solid [181]. The Umklapp process occurs when two phonons collide and produce a phonon that has a lower frequency and wave vector, *K* (case C). The momentum of the phonon is not conserved and contributes to thermal resistance in the solid [181] (see Figure 11).

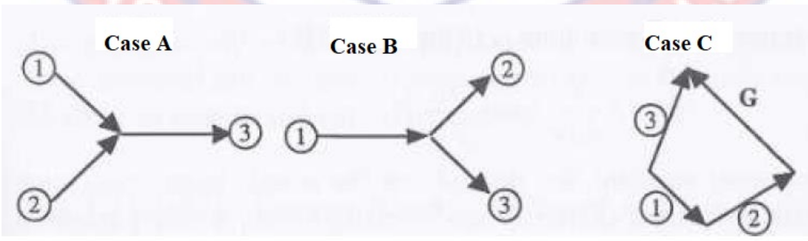

Figure 11: Inelastic Scattering Mechanisms of Phonons [273].

The aforementioned scattering mechanisms are crucial for understanding heat conduction within nanostructures since they provide the foundation for heat transfer within a solid. Although both elastic and inelastic scattering can help with thermal resistance, the amount they help depends on how they affect the heat carriers. Thermal resistance can be increased via scattering mechanisms that shift the direction of the heat carriers against the heat flow or change the momentum of the heat carriers.

The relaxation time, τ, which is defined as the average time taken by the system to return to equilibrium, can be used to quantify scattering mechanisms. Depending on the sort of scattering that occurs, many models have been devised to determine τ. It is worth to note that the effective relaxation time, τ_{eff}, is defined as the sum of all scattering mechanisms in a solid. The Matthiessen's rule, which adds up the scattering rates or the reciprocal of the relaxation time(s) (defect:τ_D; Umklapp:τ_U; boundary:τ_B) for various scattering processes, is a commonly used assumption.

$$\frac{1}{\tau_{eff}} = \frac{1}{\tau_D} + \frac{1}{\tau_U} + \frac{1}{\tau_B} \tag{68}$$

The thermal resistances posed by each scattering mechanism are effectively added in this equation. The reduction in heat conductivity of nanoscale structures is linked to thermal resistance. As a result, increasing the scattering of heat carriers in the solid is the key to lowering conductivity.

Classical Size-Effect of Nanostructures on Heat Transfer

As stated previously, the basis of heat conduction in a medium is the movement and interactions of carriers and phonons. One characteristic length scale used to describe the movement of carriers and phonons is the mean free path, Λ [182- 184]. The mean free path is defined as the average distance a heat carrier travels before transferring its excess energy [184]. Thus, the average distance that a carrier or phonon can travel before colliding with another particle is related to the relaxation time τ, as $\Lambda = v_g\tau$. Λ is in turn related to the thermal conductivity, κ, of a material by the following equation based on the kinetic theory.

Heat conduction in a medium is based on the movement and interactions of carriers and phonons, as previously indicated. The mean free path, Λ [182-184], is a common length scale used to characterize the passage of carriers and phonons. The mean free path [184] is the average distance a heat carrier travels before transmitting its excess energy. As a result, $\Lambda = v_g\tau$, the average distance that a carrier or phonon can travel before colliding with another particle is related to the relaxation time τ. Based on kinetic theory, Λ is connected to a material's thermal conductivity, κ, by the following equation:

$$\kappa = \frac{1}{3}\sum\int C_v v_g \Lambda d\omega, \tag{69}$$

where C_v is the solid's heat capacity. This is a generalised thermal conductivity equation that takes into account the frequency dependence of C_v, v_g and Λ, as well as the varied values of C_v, v_g and Λ for each polarisation of phonons. The thermal conductivity is affected differentially by different phonon frequencies. The thermal conductivity accumulation with respect to the mean free path plot [185] can be used to visualize the contribution of phonons at different frequencies to thermal conductivity (Figure 12).

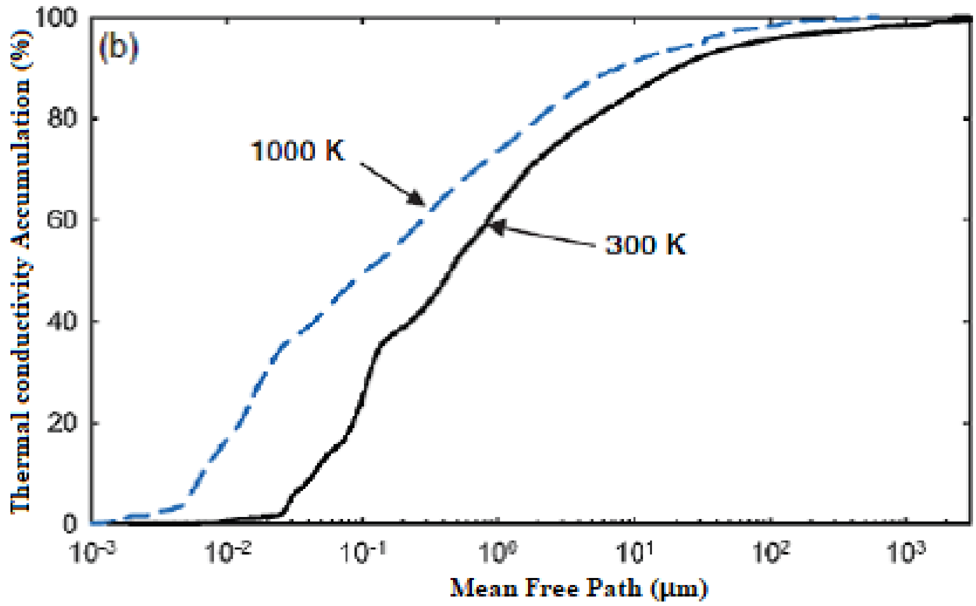

Figure 12: Thermal Conduction Accumulation in Percentage against Mean Free Path of Bulk Silicon [186].

Figure 13 indicates that phonons with mean free pathways on the scale of 100nm to 1μm play a significant role in bulk silicon's thermal conductivity. This implies that phonons at certain frequencies contribute more to thermal conductivity than phonons at others. Because of their large group velocity, low frequency phonons contribute the most to thermal conductivity. The device dimension, h, which can be either the diameter or the film thickness depending on the device structure [184], is another significant characteristic length in heat conduction.

In bulk solids, $h \gg \Lambda$ denotes that scattering occurs deep inside the solid and that a wide range of phonons contribute to the total thermal conductivity [185]. When a nanostructure is made with $h \gg \Lambda$ for low frequency phonons, however, the device's boundary confines the phonons to mean free routes that are smaller than the low frequency phonons. As a result, the bulk of low frequency phonons are screened out, lowering their contribution to thermal conductivity and, as a result, the thermal conductivity. This is referred to as the ”traditional size effect.”

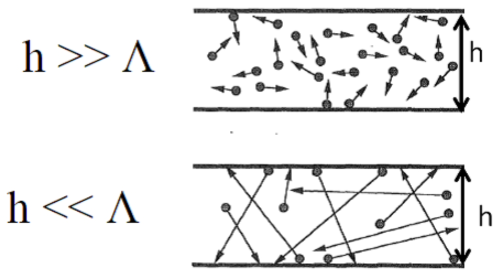

Figure 13: Phonon Dynamics in Bulk Material (see top) and Thin Film (see bottom) [185].

Chapter Summary

This chapter looked at the mechanism of carrier miniband transport in a semiconductor caused by a plane acoustic wave. In order to characterize the transport of carriers in the semiconductor, the carrier drift velocity v_d and the time averaged velocity v_m were also calculated. Average velocities were found to have strong maxima followed by a sharp decline. Theoretical and experimental evidence strongly suggests that acoustically activated semiconductors could be exploited for tunable electromagnetic wave generation. Furthermore, the carrier-phonon dynamics that contribute to phenomena like the thermopower, power factor, carrier thermal conductivity, and other intriguing phenomena studied theoretically and experimentally by other researchers were highlighted.

CHAPTER THREE

METHODOLOGY

Introduction

This chapter explored the theories, concepts, models and phenomena used in studying the acoustoelectric metrics in the FSWCNT in the hypersound regime, $q\ell \gg 1$. The chapter also elaborated the theories developed to study the thermoelectric metrics such as the carrier current density $\vec{J}$, electrical conductivity σ, thermoelectric power α, electrical power factor $\mathscr{P}$, carrier thermal current density $\vec{q}$ and the carrier thermal conductivity κ of the FSWCNT. Moreover, the chapter brought to light the phonon lattice Boltzmann method (LBM) to simulate the lattice conductivity.

Acoustic, electric and thermal properties of FSWCNT

The acoustoelectric current density, the Hall-like current generated by acoustic phonons, FSWCNT as a low voltage and SASER device were studied in the hypersound regime employing a traceable analytical approach developed from the Boltzmann transport equation. The acoustic wave considered had a wavelength, $\lambda = 2\pi/q$, and was smaller than the mean-free path of the FSWCNT carriers in the hypersound regime. The acoustic wave was then treated as packets of coherent phonons (monochromatic phonons) with a δ-function distribution as:

$$N(\vec{k}) = \frac{(2\pi)^3}{\hbar\omega_q v_s}\vec{\Phi}\delta(\vec{k}-\vec{q}), \tag{70}$$

where $\vec{k}$ is the carrier wavevector, $\hbar$ is the reduced Planck's constant, $\vec{\Phi}$ is the sound flux density and ω_q and v_s are respectively, the frequency and the group velocity of the sound wave with wavevector, $\vec{q}$.

Moreover, the thermoeletric metrics such as the carrier current density $\vec{J}$, electrical conductivity σ, thermoelectric power α, electrical power factor $\mathscr{P}$, and the carrier thermal conductivity of FSWCNT were calculated as a functional of temperature T, carrier-phonon interactions Δ_s and Δ_z, and the carrier concentration, n_o. The calculations were done using the theoretical approach developed by Mensah et al. [187] based on the phenomenological model of SWCNT developed in Ref. [188].

Boltzmann Transport Equation (BTE)

BTE is the basic model that semiclassical transport processes are based on. Even though carriers are considered as classical particles, the BTE incorporates an carrier band structure, effective mass, and scattering mechanisms that apply the Fermi golden rule, all of which are fully quantum mechanical phenomena. As a result, the BTE is a mesoscopic process that determines particle statistical distribution in solids, liquids, and gases. It is a significant equation in non-equilibrium statistical mechanics, which is frequently used to investigate heat, mass, charge, and spin transport. In modern new structures, it is employed to calculate metrics like lattice conductivity, mass conductivity, carrier conductivity, spin conductivity, and Hall conductivity. By establishing a parameter called a probability distribution function and then analyzing its variation over time, the BTE may be calculated. The distribution function represents the distribution of the particles of interest in real and momentum spaces, as well as how it changes over time. Because transportation is intrinsically non-equilibrium, a non-equilibrium distribution function must be considered for the situation under consideration.

Assume an external potential is applied to a system of N non-interacting particles, with carrier potential energy at $\vec{r}$ provided by $V(\vec{r},t)$ at time, t. The system's Hamiltonian will be defined as:

$$H = \sum_{j=1}^{N} \left[\frac{\vec{p}_j^2}{2m} + V(\vec{r}_j,t) \right]. \tag{71}$$

For non-interacting particles, $D(\vec{r}_1,\vec{p}_1;...;\vec{r}_N,\vec{p}_N,t)$ could be factorized into a product of N single-carrier distribution functions. Thus, the single-carrier probability distribution function $f(\vec{r},\vec{p},t)$, could be used without introducing any approximations. Invoking Liouville's theorem, the phase space volume, $d\vec{r}d\vec{p}$, as well as the number of carriers in it do not vary with time. Thus:

$$\frac{df(\vec{r},\vec{p},t)}{dt} = \frac{\partial f}{\partial t} + \sum_{\alpha=1}^{D} \left(\frac{\partial f}{\partial x_\alpha} \cdot \frac{\partial x_\alpha}{\partial t} + \frac{\partial f}{\partial \vec{p}_\alpha} \cdot \frac{\partial \vec{p}_\alpha}{\partial t} \right) = 0, \tag{72}$$

where D is the dimension of real space. Making use of ∇_r and ∇_p for the gradient operators in real and momentum spaces, the equation above yields:

$$\frac{\partial f(\vec{r},\vec{p},t)}{\partial t} + \frac{\vec{p}}{m} \cdot \nabla_r f(\vec{r},\vec{p},t) + \frac{d\vec{p}}{dt} \cdot \nabla_p f(\vec{r},\vec{p},t) = 0, \tag{73}$$

where $\vec{v} = d\vec{r}/dt = \vec{p}/m$ was utilised for the single-carrier Hamiltonian, $H = \vec{p}^2/2m + V(\vec{r},t)$. Classically, the Hamilton's equations of motion were quoted as: $d\vec{r}/dt = \nabla_p H = \vec{p}/m$ and $d\vec{p}/dt = -\nabla_r H = -\nabla_r V(\vec{r},t)$. Inserting the second Hamilton's equation of motion into Eq. (73) yielded:

$$\nabla_t \partial f(\vec{r},\vec{p},t) + \frac{\vec{p}}{m} \cdot \nabla_r f(\vec{r},\vec{p},t) - \nabla_r V(r,p,t) \cdot \nabla_p f(\vec{r},\vec{p},t) = 0. \tag{74}$$

In addition to the external potential discussed above, carriers in actual solids are susceptible to a variety of interactions and excitations. These interactions or excitations include carrier-carrier interactions, carrier-phonon interactions, carrier-impurity interactions, and so on. Through scattering processes with carriers scattered in and out of the phase-space volume, $d\vec{r}d\vec{p}$, these interactions affect the momenta of carriers. As a result, the distribution function $f(\vec{r},\vec{p},t)$ is no longer a conserved quantity, and $df/dt \neq 0$ is no longer true. The collision integral, represented by $I[f]$, takes into account the change in the carrier distribution function owing to various scattering processes. Thus:

$$\frac{df(\vec{r},\vec{p},t)}{dt} = \left[\frac{\partial f(\vec{r},\vec{p},t)}{\partial t}\right]_{col} = I[f]. \tag{75}$$

The $I[f]$ is a distribution function functional and the BTE is written as follows, taking collisions into account:

$$\frac{\partial f(\vec{r},\vec{p},t)}{\partial t} + \frac{\vec{p}}{m} \cdot \nabla_r f(\vec{r},\vec{p},t) + F \cdot \nabla_p f(\vec{r},\vec{p},t) = I[f], \tag{76}$$

where $F = -\nabla_r f(\vec{r},\vec{p},t)$ is the force due to the external potential. It is noteworthy that, the BTE is in general a nonlinear integro-differential equation for $f(\vec{r},\vec{p},t)$. Practically, the BTE is usually solved for small deviations from thermodynamic equilibrium. However, for an equilibrium distribution function, the collision integral vanishes.

Solution to Boltzmann transport equation

carriers in equilibrium in the FSWCNT are defined as:

$$f_o(\vec{p},\vec{r},t) = \frac{1}{e^{\theta}+1}. \tag{77}$$

Let θ be a dimensionless parameter denoted as:

$$\theta = [\varepsilon_c(\vec{r},t) + \varepsilon(\vec{p}) - F_n(\vec{r},t)]/k_B T, \tag{78}$$

where $\varepsilon_c(\vec{r},t)$ is the conduction band edge, $\varepsilon(\vec{p})$ is the carrier band structure and $F_n(\vec{r},t)$ is the quasi Fermi level. Suppose that, $f(\vec{r},\vec{p},t) = f_o(\vec{r},\vec{p},t) + f_1(\vec{p},\vec{r},t)$, where $f_1(\vec{p},\vec{r},t)$ is a small perturbation. The BTE under constant relaxation time is expressed as:

$$\frac{\partial f(\vec{r},\vec{p},t)}{\partial t} + \vec{v}\cdot\frac{\partial f(\vec{r},\vec{p},t)}{\partial \vec{r}} + F\cdot\frac{\partial f(\vec{r},\vec{p},t)}{\partial \vec{p}} = -\frac{f(\vec{r},\vec{p},t) - f_o(\vec{p})}{\tau}. \tag{79}$$

Here $f(\vec{r},\vec{p},t)$ is the non-equilibrium distribution function, $f_o(\vec{p})$ is the equilibrium distribution function, $v(\vec{p})$ is the carrier miniband velocity, F is a weak constant applied field, and $\vec{r}$ is the carrier position at any time t. The collision integral is assumed to be constant using the well known Bhatnagar-Gross-Krook (BGK) approximation. Substituting a non-equilibrium distribution, $f(\vec{r},\vec{p},t) = f_o(\vec{r},\vec{p},t) + f_1(\vec{p},\vec{r},t)$ into Eq. (79) and after some cumbersome calculations, the BTE in the presence of temperature gradient ∇T, has a solution of the form:

$$f(\vec{r},\vec{p},t) = \tau^{-1}\int_0^{-\infty} dt\exp(-t/\tau)f_o(\vec{p}) + \tau\int_0^{-\infty} dt\exp(-t/\tau)\left\{[\varepsilon(\vec{p})-\mu]\frac{}{T} - \nabla\mu\right\}\vec{v}\cdot\frac{\partial f_o(\vec{p})}{\partial\varepsilon(\vec{p})} \tag{80}$$

Making the transformation $\vec{p} \to \vec{p} - e\int_{t-t'}^{t}[\vec{E}_o + \vec{E}cos\omega t']dt'$

$$f(\vec{r},\vec{p},t) = \tau^{-1}\int_0^{-\infty} dt\exp(-t/\tau)f_o\left(\vec{p} - e\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right) + \tau\int_0^{-\infty} dt\exp(-t/\tau)\left\{[\varepsilon(\vec{p})-\mu]\frac{}{T} - \nabla\mu\right\} \times\vec{v}(\vec{p})\cdot\frac{\partial f_o}{\partial\varepsilon}(\vec{p} - e\int_{t-t'}^{t}[\vec{E}_o + \vec{E}cos\omega t']dt') \tag{81}$$

Acoustodynamics in FSWCNT

The dispersion relation for a chemically modified SWCNT with fluorine, where the fluorine atoms form a one-dimensional chain, was calculated using the Huckel matrix method, with translational symmetry taken into account in Ref. [40] is given as:

$$\varepsilon(\vec{p}_z) = \varepsilon_o + \Xi_n\Delta\cos^{2N-1}(a\vec{p}_z). \tag{82}$$

Here, $a = \sqrt{3}b/(2\hbar)$, Ξ is a constant, Δ is the overlapping integral, $\vec{p}_z$ is the quasi-momentum of an carrier along the axial direction (z-axis), N is an integer and ε_o is the carrier energy in the first Brillouin zone with momentum p_o, i.e $-\pi/a \leq p_o \leq \pi/a$. For $N = 2$, the energy dispersion for FSWCNT at the Fermi surface at the edge of the Brillouin zone is:

$$\varepsilon(\vec{p}_z) = \varepsilon_o + 8\Delta\cos^3(a\vec{p}_z), \quad (83)$$

Expanding Eq. (83) yields:

$$\varepsilon(\vec{p}_z) = \varepsilon_o + \Delta_1\cos(3a\vec{p}_z) + \Delta_2\cos(a\vec{p}_z), \quad (84)$$

where $\Delta_1 = 2\Delta$ and $\Delta_2 = 6\Delta$. Consider a single longitudinal acoustic wave traveling down a uniform FSWCNT tube with electrically insulated ends. The travelling acoustic wave is created by vibrating one end of the tube while matching the other end to a proper acoustic impedance to prevent the wave from reflecting at the termination. The acoustic wave would drag conduction carriers to one end of the tube because the ends were electrically isolated, causing a shortfall of carriers at the other end. The resulting electric field along the tube generates a conventional electric current that cancels out the acoustoelectric effect current perfectly. As a result, the acoustoelectric effect can be assessed by calculating the pd between the tube's tyro ends. Surprisingly, this process is similar to the voltage generation in an open circuit due to a temperature differential in the thermoelectric effect. The net flow of phonons along a temperature gradient can be thought of as the net flow of traveling acoustic waves.

Assume the sound wave and the constant electric field travel along the z-axis of the FSWCNT, then the AE current density in the FSWCNT is defined to be:

$$\vec{j}^{AE} = -e\sum_{n,n'}\int U^{ac}_{n,n'}\Psi_i(\vec{p}_z)d^2\vec{p}_z, \tag{85}$$

where $\Psi_i(\vec{p}_z)$ is the solution to the Boltzmann kinetic equation in the absence of a magnetic field. $\vec{p}_z$ is the carrier momentum along the axial direction of the FSWCNT and $U^{ac}_{n,n'}$ in is the carrier-phonon transition rate expressed as:

$$U^{ac}_{n,n'} = \frac{2\pi\vec{\Phi}}{\omega_q v_s}\sum_{n,n'}\{|G_{\vec{p}_z-\hbar\vec{q},\vec{p}_z}|^2[f(\varepsilon_n(\vec{p}_z-\hbar\vec{q}))-f(\varepsilon_n(\vec{p}_z))]\delta(\varepsilon_n(\vec{p}_z-\hbar\vec{q})-\varepsilon_n(\vec{p}_z)+\hbar\omega_q)$$
$$+|G_{\vec{p}_z+\hbar\vec{q},\vec{p}_z}|^2[f(\varepsilon_{n'}(\vec{p}_z+\hbar\vec{q}))-f(\varepsilon_{n'}(\vec{p}_z))]\delta(\varepsilon_{n'}(\vec{p}_z+\hbar\vec{q})-\varepsilon_{n'}(\vec{p}_z)-\hbar\omega_q)\} \tag{86}$$

where $f(\vec{p}_z)$ is the carrier distribution function, $\varepsilon_{n,n'}(\vec{p}_z)$ is the carrier energy band, $\vec{\Phi}$ is the sound flux density and n and n' denote the quantisation of the energy band induced by the wavevector as a result of the periodic boundary conditions imposed on the wavefunction in the circumferential direction. $G(\vec{p}_z \pm \hbar\vec{q}, \vec{p}_z)$ is the matrix element of the carrier-phonon interaction. Employing the principle of detailed balance and denoting $\vec{p}_z^{'} = \vec{p}_z \pm \hbar\vec{q}$, yields the condition:

$$|G_{p',p}|^2 = |G_{p,p'}|^2 \tag{87}$$

and the matrix element of the carrier-phonon interaction is given as:

$$|G_{p',p}| = \frac{4\pi e\mathscr{K}}{\sqrt{2\rho\omega_q\varepsilon}}. \tag{88}$$

Here, $\mathscr{K}$ is the piezoelectric modulus, ε is the lattice dielectric constant, ρ is the density of the FSWCNT. The AE current density then takes the form:

$$\vec{j}_z^{AE} = -\frac{2e}{(2\pi\hbar)^2}\frac{2\pi\vec{\Phi}}{\omega_q v_s}\sum_{n,n'}|G_{p'_z,p_z}|^2[f(\varepsilon_{n'}(\vec{p}_z)) - f(\varepsilon_{n'}(\vec{p}_z+\hbar\vec{q}))]$$
$$\times[\Psi_i(\vec{p}_z+\hbar\vec{q}) - \Psi_i(\vec{p}_z)]\delta(\varepsilon_{n'}(\vec{p}_z+\hbar\vec{q}) - \varepsilon_{n'}(\vec{p}_z) - \hbar\omega_q)d\vec{p}_z \tag{89}$$

and $\Psi_i(\vec{p}_z) = l_i(\vec{p}_z)$ is the mean free path given as $l_z = \tau v_z$ where v_z is expressed as:

$$\vec{v}_z = \frac{\partial\varepsilon(\vec{p}_z)}{\partial\vec{p}_z} \tag{90}$$

Substituting l_z, Eq. (88) and Eq. (90) into Eq. (89), yields:

$$\vec{j}_z^{AE} = -\frac{16e^3\pi\vec{\Phi}\mathscr{K}^2}{2\varepsilon^2\omega_q^2 v_s\rho\hbar^2}\sum_{n,n'}[f(\varepsilon_{n'}(\vec{p}_z)) - f(\varepsilon_{n'}(\vec{p}_z+\hbar\vec{q}))]$$
$$\times[\vec{v}_z(\vec{p}_z+\hbar\vec{q}) - \vec{v}_z(\vec{p}_z)]\delta(\varepsilon_{n'}(\vec{p}_z+\hbar\vec{q}) - \varepsilon_{n'}(\vec{p}_z) - \hbar\omega_q)d\vec{p}_z \tag{91}$$

The carrier distribution function is given by the Fermi-Dirac distribution defined as:

$$f_o(\vec{p}) = \frac{1}{1+\exp[(\varepsilon(\vec{p}_z) - \mu)/k_B T]} \tag{92}$$

where k_B is Boltzmann's constant, T is the absolute temperature in energy units, and μ is the electrochemical potential and ensures the conservation of carriers. Substituting Eq. (92) into Eq. (91) yields an equation with the term $\mathscr{F}_{1/2}$, which denotes Fermi-Dirac integral of the order $1/2$, which is defined as:

$$\mathscr{F}_{1/2}(\eta_f) = \frac{1}{\Gamma(1/2)} \int_0^\infty \frac{\eta_f^{1/2} d\eta}{1+\exp(\eta - \eta_f)} \tag{93}$$

where $(\mu - \varepsilon_c)/k_B T \equiv \eta_f$ and $\Gamma(1/2)$ is the gamma function of $1/2$. For nondegenerate carrier gas, where the Fermi level is several $k_B T$ below the conduction band edge ε_c, (i.e $k_B T \ll \varepsilon_c$), the integral in Eq. (93) approaches $\sqrt{\pi}/2 \exp(\eta_f)$ and so Eq. (92) simplifies to:

$$f_o(\vec{p}) = A^\dagger \exp\left(-\frac{\varepsilon(\vec{p}) - \mu}{k_B T}\right) \tag{94}$$

The carrier distribution function in the presence of the applied constant electric field, $\vec{E}(t)$ is obtained by solving the BTE using the BGK approximation as:

$$f(\vec{p}_z) = \int_0^\infty \frac{dt}{\tau} \exp(-t/\tau) f_o(\vec{p}_z - \vec{p}') \tag{95}$$

and $A^\dagger$ is the normalisation constant to be determined from the normalisation condition $\int f(\vec{p}) d\vec{p} = n_o$ as:

$$A^\dagger = \frac{3 n_o a^2}{2 I_o(\Delta_1^*) I_o(\Delta_2^*)} \exp\left(\frac{\varepsilon_o - \mu}{k_B T}\right) \tag{96}$$

where n_o is the carrier concentration and $I_o(x)$ is the modified bessel function of zero order and all other parameters are as defined previously. Assuming the carriers are confined to the lowest miniband, then $n = n' = 1$. The carrier velocity obtained from the dispersion relation is given as:

$$v_z(\vec{p}_z) = -[3a\Delta_1 \sin(3a\vec{p}_z) + a\Delta_2 \sin(a\vec{p}_z)] \tag{97}$$

Substituting Eq. (94)-Eq. (97) into Eq. (91), the current density is obtained as:

$$\begin{aligned}
\vec{j}_z^{AE} &= \frac{4A^\dagger \pi \vec{\Phi} e^3 \mathscr{K}^2 \Theta(1-\alpha^2)}{\hbar^3 \omega_q^2 \varepsilon^2 v_s \rho a q \sqrt{1-\alpha^2}} \int_0^\infty \exp\left(\frac{dt'}{\tau}\right) \\
&\times \left\{ \sinh\left[\Delta_1^* \cos(3ea\vec{E}t') \sin A \sin\left(\frac{3}{2}a\hbar q\right) + \Delta_2^* \cos(ea\vec{E}t') \sin B \sin\left(\frac{a}{2}\hbar q\right)\right] \right. \\
&\times \sinh\left[\Delta_1^* \cos(3ea\vec{E}t') \cos A \cos\left(\frac{3}{2}a\hbar q\right) + \Delta_2^* \cos(ea\vec{E}t') \cos B \cos\left(\frac{a}{2}\hbar q\right)\right] \\
&- 4\left(\Delta_2^* \sin(ea\vec{E}t') \cos B \sin\left(\frac{a}{2}\hbar q\right) + \Delta_1^* \cos A \sin(3ea\vec{E}t') \sin\left(\frac{3}{2}a\hbar q\right)\right. \\
&\left. + \Delta_1^*\Delta_2^* \sin(ea\vec{E}t') \sin(3ea\vec{E}t') \cos A \cos B \sin\left(\frac{a}{2}\hbar q\right) \sin\left(\frac{3}{2}a\hbar q\right)\right) \\
&\times \cosh\left[\Delta_1^* \cos(3ea\vec{E}t') \cos A \cos\left(\frac{3}{2}a\hbar q\right) + \Delta_2^* \cos(ea\vec{E}t') \cos B \cos\left(\frac{a}{2}\hbar q\right)\right] \\
&\left. \times \cosh\left[\Delta_1^* \cos(3ea\vec{E}t') \sin A \sin\left(\frac{3}{2}a\hbar q\right) + \Delta_2^* \cos(ea\vec{E}t') \sin B \sin\left(\frac{a}{2}\hbar q\right)\right] \right\}
\end{aligned} \tag{98}$$

where $\Delta_1^* = \Delta_1/k_BT$ and $\Delta_2^* = \Delta_2/k_BT$. Switching off the external electric field ($\vec{E} = 0$), Eq. (98) reduces to:

$$\begin{aligned}
\vec{j}_z^{FSWCNT} &= \frac{4A^\dagger \pi \vec{\Phi} e^3 \mathscr{K}^2 \tau \Theta(1-\alpha^2)}{\hbar^3 \omega_q^2 \varepsilon^2 v_s \rho a q \sqrt{1-\alpha^2}} \\
&\times \left[\sinh\left\{\Delta_1^* \sin\left(\frac{3}{2}a\hbar q\right) \sin A + \Delta_2^* \sin\left(\frac{a}{2}\hbar q\right) \sin B\right\} \right. \\
&\left. \times \sinh\left\{\Delta_1^* \cos\left(\frac{3}{2}a\hbar q\right) \cos A + \Delta_2^* \cos\left(\frac{a}{2}\hbar q\right) \cos B\right\} \right]
\end{aligned} \tag{99}$$

Simplifying Eq. (99), yields:

$$\vec{j}_z^{FSWCNT} = \vec{j}_o \left[\sinh\left\{ \Delta_1^* \sin\left(\frac{3}{2}a\hbar q\right) \sin A + \Delta_2^* \sin\left(\frac{a}{2}\hbar q\right) \sin B \right\} \right.$$
$$\left. \times \sinh\left\{ \Delta_1^* \cos\left(\frac{3}{2}a\hbar q\right) \cos A + \Delta_2^* \cos\left(\frac{a}{2}\hbar q\right) \cos B \right\} \right] \quad (100)$$

where

$$\vec{j}_o = \frac{6A^\dagger \pi \vec{\Phi} e^3 \mathscr{K}^2 a \tau \Theta(1-\alpha^2)}{\hbar^3 \omega_q^2 \varepsilon v_s \rho q \sqrt{1-\alpha^2}}, \qquad \alpha = \frac{\omega_q}{12\Delta a q} \quad (101)$$

$$A = \frac{3}{4}\sin^{-1}\left(\frac{\omega_q}{12\Delta a q}\right), \qquad B = \frac{1}{4}\sin^{-1}\left(\frac{\omega_q}{12\Delta a q}\right)$$

and Θ is a Heaviside step function. To compare the results obtained with that of undoped SWCNT, the same procedure for the FSWCNT is followed. Using the tight-binding energy dispersion of the $\vec{p}_z$ orbital which is given as:

$$\varepsilon(\vec{p}_z) = \pm\gamma_o\left(1 - 2\cos\left(\frac{\vec{p}_z\sqrt{3}b}{2\hbar}\right)\right) \quad (102)$$

From Eq. (102), the acoustocurrent density in undoped SWCNT obtained via deformation potential is expressed as:

$$\vec{j}_z^{SWCNT} = \vec{j}_o^{SWCNT} \sinh\left\{\frac{\hbar\omega_q}{k_B T}\right\} \sinh\left\{\frac{4\gamma_o}{k_B T}\sqrt{1-\alpha^2}\cos\left(\frac{a\hbar q}{2}\right)\right\} \quad (103)$$

where

$$\vec{j}_o^{SWCNT} = \frac{\Lambda^2 \vec{\Phi} q^2 e \tau n_o a \Theta(1-\alpha_o^2)}{\pi\hbar^2\omega_q^2 v_s \rho \sin(a\hbar q/2) I_o(\Delta^*)\sqrt{1-\alpha_o^2}} \quad (104)$$

$$\alpha_o = \frac{\hbar\omega_q}{4\gamma_o \sin(a\hbar q/2)}$$

FSWCNT as a low voltage, current amplifier acoustic device

From Eq. (98), in the presence of the external electric field yields:

$$\begin{aligned}
\vec{j}_z^{AE} &= \frac{4A^\dagger \pi \vec{\Phi} e^3 \mathscr{K}^2 \Theta(1-\alpha^2)}{\hbar^3 \omega_q^2 \varepsilon^2 \rho a q \sqrt{1-\alpha^2}} \int_0^\infty \exp\left(-\frac{dt'}{\tau}\right) \\
&\times \left\{ \sinh\left[\Delta_1^* \cos(3ea\vec{E}t') \sin A \sin\left(\frac{3}{2}a\hbar q\right) + \Delta_2^* \cos(ea\vec{E}t') \sin B \sin\left(\frac{a}{2}\hbar q\right)\right]\right. \\
&\times \sinh\left[\Delta_1^* \cos(3ea\vec{E}t') \cos A \cos\left(\frac{3}{2}a\hbar q\right) + \Delta_2^* \cos(ea\vec{E}t') \cos B \cos\left(\frac{a}{2}\hbar q\right)\right] \\
&\quad - 4\left(\Delta_2^* \sin(ea\vec{E}t') \cos B \sin\left(\frac{a}{2}\hbar q\right) + \Delta_1^* \cos A \sin(3ea\vec{E}t') \sin\left(\frac{3}{2}a\hbar q\right)\right. \\
&\qquad \left. + \Delta_1^* \Delta_2^* \sin(p'a) \sin(3ea\vec{E}t') \cos A \cos B \sin\left(\frac{a}{2}\hbar q\right) \sin\left(\frac{3}{2}a\hbar q\right)\right) \\
&\times \cosh\left[\Delta_1^* \cos(3ea\vec{E}t') \cos A \cos\left(\frac{3}{2}a\hbar q\right) + \Delta_2^* \cos(ea\vec{E}t') \cos B \cos\left(\frac{a}{2}\hbar q\right)\right] \\
&\times \left.\cosh\left[\Delta_1^* \cos(3ea\vec{E}t') \sin A \sin\left(\frac{3}{2}a\hbar q\right) + \Delta_2^* \cos(ea\vec{E}t') \sin B \sin\left(\frac{a}{2}\hbar q\right)\right]\right\}
\end{aligned} \tag{105}$$

Simplifying Eq. (105) explicitly results in:

$$\begin{aligned}
\vec{j}_z^{AE} = \vec{j}_z^{AE}(0) &\left\{1 - 4\left(\Delta_2^* \sin\left(\chi'\left(1 - \frac{v_d}{v_s}\right)\right) \cos B \sin\left(\frac{a}{2}\hbar q\right)\right.\right. \\
&\quad \left. + \Delta_1^* \cos A \sin\left(3\chi'\left(1 - \frac{v_d}{v_s}\right)\right) \sin\left(\frac{3}{2}a\hbar q\right)\right) \\
&\quad \times \coth\left[\Delta_1^* \cos\left(3\chi'\left(1 - \frac{v_d}{v_s}\right)\right) \cos A \cos\left(\frac{3}{2}a\hbar q\right)\right. \\
&\quad \left. + \Delta_2^* \cos\left(\chi'\left(1 - \frac{v_d}{v_s}\right)\right) \cos B \cos\left(\frac{a}{2}\hbar q\right)\right] \\
&\quad \times \coth\left[\Delta_1^* \cos\left(3\chi'\left(1 - \frac{v_d}{v_s}\right)\right) \sin A \sin\left(\frac{3}{2}a\hbar q\right)\right. \\
&\quad \left.\left. + \Delta_2^* \cos\left(\chi'\left(1 - \frac{v_d}{v_s}\right)\right) \sin B \sin\left(\frac{a}{2}\hbar q\right)\right]\right\}
\end{aligned} \tag{106}$$

where $\vec{j}_z^{AE}(0)$ is given as:

$$\vec{j}_z^{AE}(0) = \vec{j}_o \left[\sinh\left\{\Delta_1^* \sin\left(\frac{3}{2}a\hbar q\right)\sin A + \Delta_2^* \sin\left(\frac{a}{2}\hbar q\right)\sin B\right\} \times \sinh\left\{\Delta_1^* \cos\left(\frac{3}{2}a\hbar q\right)\cos A + \Delta_2^* \cos\left(\frac{a}{2}\hbar q\right)\cos B\right\}\right] \quad (107)$$

with $\alpha = \omega_q(1 - v_d/v_s)/12\Delta aq$ and $\vec{j}_o$ is given as:

$$\vec{j}_o = \frac{4A^\dagger \pi \vec{\Phi} e^3 \mathscr{K}^2 \tau \Theta(1-\alpha^2)}{\hbar^3 \omega_q^2 \varepsilon^2 \rho a q \sqrt{1-\alpha^2}} \qquad \chi' = \hbar\omega_q(1 - v_d/v_s)a/v_s \quad (108)$$

Hall-like current induced by acoustic phonons in FSWCNT

As discussed previously in acoustodynamics (Eq. (100)), the phenomenon is similar to the thermoelectric effect where the net flow of phonons along a temperature gradient is considered as a net flow of travelling acoustic waves along the gradient. The combined AE-AME current density in FSWCNT is calculated using a tractable analytic model. Assume an array of FSWCNT forms a two-dimensional structure. The sound wave and external electric field $\vec{E}(t)$ propagate along the FSWCNT axis (z-axis), and the magnetic field is directed parallel to the x-axis and the Hall-like (AME) current appears in the y-axis. The current density for such an orientation is given as:

$$\vec{j} = \frac{2e}{(2\pi\hbar)^2} \sum_{n,n'} \int U_{n,n'}^{ac} \Psi_i(\vec{p}, \vec{H}) d^2\vec{p}_z \quad (109)$$

where $\Psi_i(\vec{p}_z)$ is solution to BTE under zero magnetic field, $\vec{p}_z$ is the carrier momentum along the axial direction of the FSWCNT and $U_{n,n'}^{ac}$ is the acoustic distribution function which was obtained using the Fermi golden rule and defined previously. Ψ_i is the root of the kinetic equation given as:

$$-\frac{e}{c}\left(\vec{v}\times\vec{H}\right)\frac{\partial\Psi_i}{\partial\vec{p}}+\vec{W}_p\{\Psi\}=\vec{v}_i. \tag{110}$$

Here, $\vec{v}_i$ is carrier velocity and $\vec{W}_p\{...\} = (\partial f/\partial\varepsilon...)^{-1}\vec{W}_p(\partial f/\partial\varepsilon)$. The operator $\vec{W}_{\vec{p}}$ is a Hermitian and it is the collision operator describing the relaxation of non-equilibrium distribution of the carrier. Assuming τ to be constant, the collision operator has the form, $\vec{W}_{\vec{p}} = 1/\tau$. Assuming a solution to Eq. (110) is of the form:

$$\Psi_i = \Psi_i^0+\Psi_i^1+\Psi_i^2+.... \tag{111}$$

Following Mensah et al. [189], Eq. (111) is substituted into Eq. (110) and solved by the method of iteration to obtain the zero approximation in the absence of a magnetic field ($\vec{H}=0$) as:

$$\Psi_i^0 = v_i\tau \tag{112}$$

Similarly, the first approximation yields:

$$\Psi_i^1 = -\frac{\tau^2 e}{mc}(\vec{v}\times\vec{H})_i \tag{113}$$

and $i = x,y,z$. Substituting Eq. (112) and Eq. (113) into Eq. (109) and using the principle of detailed balance, $|G_{p',p}|^2 = |G_{p,p'}|^2$, the net current density is obtained as:

$$\vec{j}_i = \frac{2e}{(2\pi\hbar)^2}\frac{2\pi\Phi}{\omega_q v_s}\sum_{n,n'}\int|G_{\vec{p}_z+\hbar q,\vec{p}_z}|^2[f(\varepsilon_{n'}(\vec{p}_z))-f(\varepsilon_{n'}(\vec{p}_z+\hbar\vec{q}))]$$

$$\times [\Psi_i(\vec{p}_z+\hbar\vec{q})-\Psi_i(\vec{p}_z)]\delta(\varepsilon_{n'}(\vec{p}_z+\hbar\vec{q})-\varepsilon_{n'}(\vec{p}_z)-\hbar\omega_q)d^2\vec{p}_z$$

$$-\frac{2e}{(2\pi\hbar)^2}\frac{2\pi\Phi}{\omega_q v_s}\frac{e\tau^2}{mc}\sum_{n,n'}\int |G_{\vec{p}_z+\hbar\vec{q},\vec{p}_z}|^2[f(\varepsilon_{n'}(\vec{p}_z))-f(\varepsilon_{n'}(\vec{p}_z+\hbar\vec{q}))]$$

$$\times [\Psi_i(\vec{p}_z+\hbar\vec{q})-\Psi_i(\vec{p}_z)]\delta(\varepsilon_{n'}(\vec{p}_z+\hbar\vec{q})-\varepsilon_{n'}(\vec{p}_z)-\hbar\omega_q)d^2\vec{p}_z \quad (114)$$

Under the orientation considered, the current ($\vec{j}_z^{AE}$) is given by the first term and its solution is found in Eq. (106). The second term gives the Hall-like current ($\vec{j}_z^{AME}$). Substituting Eq. (94)-Eq. (97) into Eq. (114), in the absence of external magnetic field, the current density in the first term of Eq. (114) yields:

$$\begin{aligned}
\vec{j}_z^{AE} = \vec{j}_z^{AE}(0)\Bigg\{1-4\Bigg(&\Delta_z^*\sin\left(\chi'\left(1-\frac{v_d}{v_s}\right)\right)\cos B\sin\left(\frac{a}{2}\hbar q\right)\\
&+\Delta_s^*\cos A\sin\left(3\chi'\left(1-\frac{v_d}{v_s}\right)\right)\sin\left(\frac{3}{2}a\hbar q\right)\Bigg)\\
&\times\coth\Bigg[\Delta_s^*\cos\left(3\chi'\left(1-\frac{v_d}{v_s}\right)\right)\cos A\cos\left(\frac{3}{2}a\hbar q\right)\\
&+\Delta_z^*\cos\left(\chi'\left(1-\frac{v_d}{v_s}\right)\right)\cos B\cos\left(\frac{a}{2}\hbar q\right)\Bigg]\\
&\times\coth\Bigg[\Delta_s^*\cos\left(3\chi'\left(1-\frac{v_d}{v_s}\right)\right)\sin A\sin\left(\frac{3}{2}a\hbar q\right)\\
&+\Delta_z^*\cos\left(\chi'\left(1-\frac{v_d}{v_s}\right)\right)\sin B\sin\left(\frac{a}{2}\hbar q\right)\Bigg]\Bigg\} \quad (115)
\end{aligned}$$

Switching off the external electric field, Eq. (115) simplifies Eq. (116) as:

$$\begin{aligned}
\vec{j}_z^{AE}(0) = \vec{j}_o\Bigg[&\sinh\left\{\Delta_s^*\sin\left(\frac{3}{2}a\hbar q\right)\sin A+\Delta_z^*\sin\left(\frac{a}{2}\hbar q\right)\sin B\right\}\\
&\times\sinh\left\{\Delta_s^*\cos\left(\frac{3}{2}a\hbar q\right)\cos A+\Delta_z^*\cos\left(\frac{a}{2}\hbar q\right)\cos B\right\}\Bigg] \quad (116)
\end{aligned}$$

Similarly, the Hall-like current density j_y^{AME}, as in Eq. (114) is obtained after some cumbersome calculations as:

$$\vec{j}_y^{AME} = -\frac{2A^{\dagger}\mathscr{K}^2\pi\Phi e^3\tau^2\Theta(1-\alpha^2)\vec{\Omega}}{\hbar^3\omega_q^2\varepsilon\rho a\sqrt{1-\alpha^2}}\int_0^{\infty}\exp\left(-\frac{dt'}{\tau}\right)$$
$$\times\left\{\sinh\left[\Delta_1^*\cos(3ea\vec{E}t')\sin A\sin\left(\frac{3}{2}a\hbar q\right)+\Delta_2^*\cos(ea\vec{E}t')\sin B\sin\left(\frac{a}{2}\hbar q\right)\right]\right.$$
$$\times\sinh\left[\Delta_1^*\cos(3ea\vec{E}t')\cos A\cos\left(\frac{3}{2}a\hbar q\right)+\Delta_2^*\cos(ea\vec{E}t')\cos B\cos\left(\frac{a}{2}\hbar q\right)\right]$$
$$-4\left(\Delta_2^*\sin(ea\vec{E}t')\cos B\sin\left(\frac{a}{2}\hbar q\right)+\Delta_1^*\cos A\sin(3ea\vec{E}t')\sin\left(\frac{3}{2}a\hbar q\right)\right.$$
$$\left.+\Delta_1^*\Delta_2^*\sin(ea\vec{E}t')\sin(3ea\vec{E}t')\cos A\cos B\sin\left(\frac{a}{2}\hbar q\right)\sin\left(\frac{3}{2}a\hbar q\right)\right)$$
$$\times\cosh\left[\Delta_1^*\cos(3ea\vec{E}t')\cos A\cos\left(\frac{3}{2}a\hbar q\right)+\Delta_2^*\cos(ea\vec{E}t')\cos B\cos\left(\frac{a}{2}\hbar q\right)\right]$$
$$\left.\times\cosh\left[\Delta_1^*\cos(3ea\vec{E}t')\sin A\sin\left(\frac{3}{2}a\hbar q\right)+\Delta_2^*\cos(ea\vec{E}t')\sin B\sin\left(\frac{a}{2}\hbar q\right)\right]\right\} \tag{117}$$

where $\vec{\Omega}=\mu\vec{H}/\hbar c$. Simplifying further, yields:

$$\vec{j}_y^{AME} = \vec{j}_{oy}\left[\sinh\left\{\Delta_1^*\sin\left(\frac{3}{2}a\hbar q\right)\sin A+\Delta_2^*\sin\left(\frac{a}{2}\hbar q\right)\sin B\right\}\right.$$
$$\left.\times\sinh\left\{\Delta_1^*\cos\left(\frac{3}{2}a\hbar q\right)\cos A+\Delta_2^*\cos\left(\frac{a}{2}\hbar q\right)\cos B\right\}\right]$$
$$\times\left\{1-4\left(\Delta_z^*\sin\left(\chi'\left(1-\frac{v_d}{v_s}\right)\right)\cos B\sin\left(\frac{a}{2}\hbar q\right)\right.\right.$$
$$\left.+\Delta_s^*\cos A\sin\left(3\chi'\left(1-\frac{v_d}{v_s}\right)\right)\sin\left(\frac{3}{2}a\hbar q\right)\right)$$
$$\times\coth\left[\Delta_s^*\cos\left(3\chi'\left(1-\frac{v_d}{v_s}\right)\right)\cos A\cos\left(\frac{3}{2}a\hbar q\right)\right.$$
$$\left.+\Delta_z^*\cos\left(\chi'\left(1-\frac{v_d}{v_s}\right)\right)\cos B\cos\left(\frac{a}{2}\hbar q\right)\right]$$
$$\times\coth\left[\Delta_s^*\cos\left(3\chi'\left(1-\frac{v_d}{v_s}\right)\right)\sin A\sin\left(\frac{3}{2}a\hbar q\right)\right.$$

$$+ \Delta_z^* \cos\left(\chi' \left(1 - \frac{v_d}{v_s}\right)\right) \sin B \sin\left(\frac{a}{2}\hbar q\right)\Big]\Big\} \quad (118)$$

Switching off the external electric field from Eq. (118), produces:

$$\vec{j}_y^{AME} = \vec{j}_{oy} \left[\sinh\left\{\Delta_1^* \sin\left(\frac{3}{2}a\hbar q\right) \sin A + \Delta_2^* \sin\left(\frac{a}{2}\hbar q\right) \sin B\right\} \times \sinh\left\{\Delta_1^* \cos\left(\frac{3}{2}a\hbar q\right) \cos A + \Delta_2^* \cos\left(\frac{a}{2}\hbar q\right) \cos B\right\}\right] \quad (119)$$

where

$$\vec{j}_{oy} = -\frac{4e^3 A^\dagger \mathscr{K}^2 \pi \Phi \tau^2 \Theta(1-\alpha^2)\vec{\Omega}}{\hbar^3 \omega_q^2 v_s \varepsilon \rho a q \sqrt{1-\alpha^2}} \quad (120)$$

The dependence of AE and AME current densities on ω_q, q and T as expressed above are highly nonlinear. Yamada [190, 191] in his phenomenological work on AME deduced that the following relation exists between the attenuation coefficient Γ_{abs} and the Hall-like field $\vec{E}_{SAME}$:

$$\Gamma_{abs}\Phi = \frac{n_o e \vec{E}_{SAME}}{\mu \vec{H}/\hbar c} \quad (121)$$

Thus, the Hall-like field, $\vec{E}_{SAME}$, becomes:

$$\vec{E}_{SAME} = \frac{2A^\dagger \pi \Phi^2 \mathscr{K}^2 \Theta(1-\alpha^2)}{3\hbar^2 \omega_q^2 \rho v_s \Delta \varepsilon a q \sqrt{1-\alpha^2} n_o e}\left(\frac{\mu \vec{H}}{\hbar c}\right) \times \left[\sinh\left\{\Delta_1^* \sin\left(\frac{3}{2}a\hbar q\right) \sin A + \Delta_2^* \sin\left(\frac{a}{2}\hbar q\right) \sin B\right\} \times \cosh\left\{\Delta_1^* \cos\left(\frac{3}{2}a\hbar q\right) \cos A + \Delta_2^* \cos\left(\frac{a}{2}\hbar q\right) \cos B\right\}\right] \quad (122)$$

The drift velocity of the carriers yields:

$$\vec{v}_d = \mu \vec{E}_{SAME} = \frac{2A^{\dagger}\pi\Phi^2\mathscr{K}^2\Theta(1-\alpha^2)}{3\hbar^2\omega_q^2\rho v_s\Delta\varepsilon aq\sqrt{1-\alpha^2}n_o e}\left(\frac{\mu^2\vec{H}}{\hbar c}\right)$$
$$\times\left[\sinh\left\{\Delta_1^*\sin\left(\frac{3}{2}a\hbar q\right)\sin A+\Delta_2^*\sin\left(\frac{a}{2}\hbar q\right)\sin B\right\}\right.$$
$$\left.\times\cosh\left\{\Delta_1^*\cos\left(\frac{3}{2}a\hbar q\right)\cos A+\Delta_2^*\cos\left(\frac{a}{2}\hbar q\right)\cos B\right\}\right] \quad (123)$$

FSWCNT as a SASER device

Following the steps in Ref. [192], the carrier-phonon system's Hamiltonian for FSWCNT is employed in the second quantization formalism as:

$$H = \sum_{p,\nu}\varepsilon^{(\nu)}(p)\left(p-\frac{e}{c}A(t)\right)a_p^{(\nu+)}a_\nu^{(\nu)} + \sum_k \omega_k b_k^+ b_k \ldots$$
$$+\frac{1}{\sqrt{N}}\sum_{p,k}\sum_{\nu\nu' n} c_k m_{\nu\nu'}(k_z)a_p^{(\nu+)}a_{p-k+ng}^{\nu'}(b_k^+ + b_{-k}) \quad (124)$$

where $\nu = 1, 2\ldots$

$A(t)$ is the vector potential and is related to the external electric field of the electromagnetic wave $\vec{E}(t) = \vec{E}_o \sin\omega t$ by the relation $\vec{E} = -(1/c)(\partial\vec{A}/\partial t)$ and is directed along the FSWCNT tubular axis. $a_p^{(+)}$ and a_p are the creation and annihilation operators of an carrier with quasi-momentum p in the νth miniband respectively, and b_k^+ and b_k are the phonon creation and annihilation operators, respectively. N is the number of FSWCNT periods, $g = (0, 0, 2\pi/b)$

is the FSWCNT reciprocal vector, and $m_{\nu\nu'}$ is given by:

$$m_{\nu\nu'}(k_z) = \int \varphi^*_{\nu'}(z)\varphi_\nu(z)e^{ik_z}dz, \tag{125}$$

where, $\varphi_\nu(z)$ is the wavefunction of the νth state in one of the one-dimensional mini band from which the FSWCNT potential is formed. The EM wave frequency is assumed to be large compared with the inverse of the carrier mean-free time $(1/\tau)$ and the wavelength is taken to be large in comparison with the FSWCNT period, carrier mean-free path and the de Broglie wavelength [192]. Moreover, the plane electromagnetic wave of frequency ω satisfies $\omega/\omega_p > 1$, where ω_p is the plasma frequency. In the case of the phonons, the study is confined to those for which the wavevector $\vec{q}$ satisfies the conditions $ql \gg 1$ where l is the carrier mean free path in FSWCNT. Such phonons constitute a well-defined elementary excitations of the system.

Assuming $\omega\tau \gg 1$ and $\omega > \omega_p$, then the EM wave should penetrate well into the sample and the condition $ql \gg 1$ implies the hypersound wavelength is far smaller than the carrier mean-free path. Proceeding to calculate the attenuation (or gain) coefficient, the equation of motion for the phonons based on the Heisenberg formalism is given as:

$$i\frac{\partial}{\partial t}\langle b_q\rangle_t = \langle[b_q, H]\rangle_t = \omega_q\langle b_q\rangle_t + \frac{1}{\sqrt{N}}G_{-q}\sum_p m_{\nu\nu'}(-q)\langle a_p^{(\nu+)} a_{p+qng}^{(\nu)}\rangle_t \tag{126}$$

Again for $\langle a_p^{(\nu+)} a_{p+qng}^{(\nu)}\rangle_t$ yields:

$$i\frac{\partial}{\partial t}\langle a_p^{(\nu+)} a_{p+qng}^{(\nu)}\rangle_t = (\varepsilon^{\nu'}_{p+q} - \varepsilon^{\nu}_p)\langle a_p^{(\nu+)} a_{p+qng}^{(\nu')}\rangle_t$$

$$+\frac{1}{N}\sum_{p}\sum_{\nu'\nu n} G_{k'k}\Big[M_{\nu'\nu''}(k_z)\langle a_p^{(\nu+)}a_{p+q-k+(n+n')g}^{(\nu')}(b_k+b_{-k}^{+})_t\rangle_t$$

$$-M_{\nu''\nu}(q_z)\langle a_{p+k-n'g}^{(\nu)}a_{p+q+ng}^{(\nu')}(b_k+b_{-k}^{+})_t\rangle_t\Big] \quad (127)$$

Solving Eq. (127) within the initial condition $\langle a_p^{(\nu)}a_{p+qng}^{(\nu+)}\rangle_{t=\infty} = 0$ and substituting into Eq. (126) yields:

$$i\frac{\partial}{\partial t}\langle b_q\rangle_t = \omega_q\langle b_q\rangle_t - I\sum_{p}\sum_{\nu'\nu n} G_{-q}M_{\nu\nu'}(-q)$$

$$\times\int_{-\infty}^{t} e^{\int_{t'}^{t}(\varepsilon_{p+q}-\varepsilon_p)dt''}dt'\sum_{\nu'\nu''} G_k[M_{\nu'\nu''}(k_z)\langle a_p^{(\nu+)}a_{p+q-k+(n'+n)g}^{(\nu'')}$$

$$\times(b_k+b_{-k}^{+})_t\rangle\tau' - M_{\nu''\nu}(k_z)\langle a_{p+k-ng}^{(\nu+)}a_p+q+(n'+n)g^{(\nu'')}(b_k+b_{-k}^{+}))\rangle_{t'}]$$

$$(128)$$

For weak carrier-phonon interaction the RHS of Eq. (128) is decoupled using:

$$\langle a_p^{(\nu+)}a_{p'}^{(\nu')}b_k\rangle = \delta_{kk'}\delta_{\nu\nu'}\langle b_k\rangle_k n_p^{\nu} \quad (129)$$

where $n_p^{\nu} = \langle a_p^{(\nu+)}a_{p'}^{(\nu')}\rangle_t$ is the carrier distribution function. This further yields:

$$\frac{\partial}{\partial t}\langle b_q\rangle_t + i\omega_q\langle b_q\rangle_t = \sum_{p}\sum_{\nu'\nu n} G_{-q}G_{q+nq}M_{\nu'\nu}(-q_z)M_{\nu'\nu}(-q_z+ng_z)[n_p^{\nu'}-n_{p+q}^{\nu'}]$$

$$\times\int_{-\infty}^{t} dt'(\langle b_{q+ng}\rangle_{t'}+\langle b_{-q-ng}^{+}\rangle_{t'})\exp\Big[i(\varepsilon_p^{\nu}-\varepsilon_{p+q}^{\nu'})(t-t')$$

$$-\frac{3e\vec{E}_o a\Delta_1}{\Omega}\{\sin(p_s+q)a-\sin p_s a\}(\sin\Omega t-\sin\Omega t')$$

$$-\frac{e\vec{E}_o a\Delta_2}{\Omega}\{\sin(p_z+q)a-\sin p_z a\}(\sin\Omega t-\sin\Omega t')\Big] \quad (130)$$

Employing planar conditions:

$$\frac{\partial}{\partial t}\langle b_q\rangle_t + i\omega_q\langle b_q\rangle_t = \sum_p \sum_{\nu'\nu n} G_{-q}G_{q+nq}M_{\nu'\nu}(-q_z)M_{\nu'\nu}(-q_z+ng_z)[n_p^{\nu'} - n_{p+q}^{\nu'}]$$

$$\times \int_{-\infty}^{t} dt'(\langle b_{q+ng}\rangle_{t'} + \langle b^+_{-q-ng}\rangle_{t'})\exp\Big[i(\varepsilon_p^{\nu} - \varepsilon_{p+q}^{\nu'})(t-t')$$

$$-\frac{e\vec{E}_o a\Delta_2}{\Omega}\{\sin(p_z+q)a - \sin p_z a\}(\sin\Omega t - \sin\Omega t')\Big] \quad (131)$$

Eliminating $\langle b^+_{-q-ng}\rangle_t$ with the help of the conjugate equation and taking the Fourier transform of the component:

$$B_q(\omega) = \int_{-\infty}^{\infty} \langle b_q\rangle_t \exp(i\omega t)dt \quad (132)$$

then yields:

$$(\omega - \omega_q)B_q(\omega) = \sum_{k=-\infty} \sum_{\nu'\nu} G_{-q}G_{q+ng}M_{\nu\nu'}(-q_z)M_{\nu'\nu}(q_z - ng_z)$$

$$\times \frac{2\omega_{q+ng}}{\omega + \omega_{q+ng} - \Omega k} B_q(\omega - k)M_k(q,\omega) \quad (133)$$

where

$$M_k(q,\omega) = \sum_{\ell=-\infty} J_\ell(\xi)J_{\ell+k}(\xi)\Pi(q,\omega+\ell\Omega) \quad (134)$$

$$\Pi_{\nu\nu'}(q,\omega+\ell\Omega) = \sum_p \frac{n_p^{\nu} - n_{p+q}^{\nu'}}{\varepsilon_{p+q}^{\nu'} - \varepsilon_p^{\nu} - \omega_q} \quad (135)$$

and

$$\xi = \frac{e\vec{E}_o a\Delta_2}{\Omega^2}[\sin(p_z+q)a - \sin(p_z a)] \quad (136)$$

Subsequently, considering non-degenerate conditions i.e., $n_p^{\nu} = n_p$. Limiting to the condition, $k = 0$, since when $k \neq 0$, the sum gives terms of higher perturbation. For $n = 0$ Eq. (133) yields:

$$\omega^2 - \omega_q^2 - 2\omega_q^2 G_q^2 M_o(q,\omega) = 0 \tag{137}$$

This simplifies to:

$$\omega - \omega_q - G_q^2 \sum_{\ell=-\infty}^{\infty} J_\ell^2(\xi)\Pi_q^0(\omega_q - \Omega\ell) = 0 \tag{138}$$

The gain in phonons in the presence of the EM wave will be given as:

$$\Gamma(\omega) = -Im\omega = \sum_{\ell=-\infty}^{\infty} J_\ell^2(\xi)\Gamma_q^0(\omega_q + \ell\Omega) \tag{139}$$

where

$$\Gamma_q^0(\omega) = \pi G_q^2 \sum_p U_{\nu,\nu'}^{ac} \tag{140}$$

$U_{\nu,\nu'}^{ac}$ is the acoustic distribution and is expressed as:

$$U_{\nu,\nu'}^{ac} = \sum_{\nu,\nu'} \{|G_{\vec{p}-\hbar\vec{q},\vec{p}}|^2[n(\vec{p}-\hbar\vec{q}) - n(\vec{p})]\delta(\varepsilon_\nu(\vec{p}-\hbar\vec{q}) - \varepsilon_\nu(\vec{p}) + \hbar\omega_q - \xi)$$
$$+|G_{\vec{p}+\hbar\vec{q},\vec{p}}|^2[n(\vec{p}+\hbar\vec{q}) - n(\vec{p})]\delta(\varepsilon_{\nu'}(\vec{p}+\hbar\vec{q}) - \varepsilon_{\nu'}(\vec{p}) - \hbar\omega_q + \xi)\} \tag{141}$$

i.e., the imaginary part of Eq. (139) is the polarization vector. In Eq. (139), $J_\ell(x)$ is the Bessel function of order ℓ and argument x. The expression for $\Gamma_q^0(\omega)$ is the same as that obtained in the absence of an external electric field. The difference being the shift in the argument of Γ_q^0 by $\ell\Omega$ which is the possibility for absorption of laser field quanta by conduction carriers upon interaction with phonons. The case $\ell = 0$ has been discussed in [193]. It follows from Eq. (139)

that if $\Gamma(\omega) > 0$ we have hypersound attenuation [192], whereas if $\Gamma(\omega) < 0$ corresponds to hypersound amplification (negative attenuation) as a result of the absorption $\Gamma(\omega) > 0$ and emission $\Gamma(\omega) < 0$ of $|\ell|$ photons from the intensified laser field.

The acoustic coefficient Γ_q, is calculated in the intense laser field region and under certain conditions the hypersound switches sign from absorption to amplification is established. In the region of an intense laser field, i.e. $\xi \gg \Omega$, only the carrier-phonon collisions with the absorption or emission of $\ell \gg 1$ photons are significant. Accordingly, in the case of $\xi \gg \Omega$ the argument of the Bessel function $J_\ell(\xi)$ is large. For large values, the Bessel function $J_\ell(\xi)$ is small except when the order is equal to the argument. From Eq. (139), this yields:

$$\xi = \frac{eE_o a^2 \Delta_2 q}{\Omega^2} \tag{142}$$

Taking the sum over $|\ell|$ and invoking the approximation in Ref. [192] yields:

$$\sum_{\ell=-\infty} J_\ell^2(\xi)\delta(E-\ell\Omega) \approx \frac{1}{2}[\delta(E-\xi)+\delta(E+\xi)], \tag{143}$$

where $E = \varepsilon(\vec{p}+\hbar\vec{q}) - \varepsilon(\vec{p}) - \hbar\omega_q$. $G(\vec{p}\pm\hbar\vec{q},\vec{p})$ is the matrix element of the carrier-phonon interaction. The attenuation/amplification coefficient becomes:

$$\Gamma_q(\omega) = \frac{\pi}{2}\sum_p |G_{\vec{p}',\vec{p}}|^2[n(\vec{p}) - n(\vec{p}+\hbar\vec{q})]$$
$$\times\left\{\delta(\varepsilon_\nu(\vec{p}+\hbar\vec{q}) - \varepsilon_\nu(\vec{p}) - \hbar\omega_q - \xi) + \delta(\varepsilon_\nu(\vec{p}+\hbar\vec{q}) - \varepsilon_\nu(\vec{p}) + \hbar\omega_q + \xi)\right\} \tag{144}$$

The first $\delta-$function in Eq. (144) corresponds to the emission and the second to the absorption of phonons. The number of photons absorbed or emitted is the same order of magnitude as the ratio of the classical oscillatory energy of the carrier to that of the phonon [193]:

$$\ell = \frac{2e^2E_o^2/(e\tau\Omega^2/\mu)}{\Omega} \tag{145}$$

Multiple photon absorption or emission processes, $\ell \gg 1$, are valid for laser fields where:

$$E_o \gg \left(\frac{\tau\Omega^3}{2e\mu}\right)^{1/2} \tag{146}$$

For low carrier temperature, and for $k_BT \ll \xi$ the emission term is negligible compared to the absorption term. This is justified provided $\Delta_2 \gg k_BT$. Thus, Eq. (139) becomes:

$$\Gamma_q(\omega) = \frac{\pi|G_{\vec{p}',\vec{p}}|^2}{2}\sum_n [n(\vec{p}) - n(\vec{p}+\hbar\vec{q})]$$
$$\times\,\delta(\varepsilon_\nu(\vec{p}+\hbar\vec{q}) - \varepsilon_\nu(\vec{p}) - \hbar\omega_q + \xi)d^2p \tag{147}$$

Changing the summation to an integral due to the continuity of the carrier states as a function of the momentum $\vec{p}$, yields:

$$\Gamma_q(\omega) = \frac{2\pi\vec{\Phi}|G_{p',p}|^2}{\omega_q v_s}\int [n(\vec{p}) - n(\vec{p}+\hbar\vec{q})]$$
$$\times\,\delta(\varepsilon_\nu(\vec{p}+\hbar\vec{q}) - \varepsilon_\nu(\vec{p}) - \hbar\omega_q + \xi)d^2p \tag{148}$$

where $n(\vec{p})$ is the unperturbed distribution function, $\varepsilon_\nu(\vec{p})$ is the energy band, ν

denotes the quantisation of the energy band. Substituting $|G_{p',p}|$ into Eq. (148), yields:

$$\Gamma(\omega) = \frac{2\pi\vec{\Phi}}{\omega_q v_s}\left(\frac{4\pi e\mathscr{K}}{\sqrt{2\rho\omega_q\varepsilon}}\right)^2 \sum_{\nu}[n(\vec{p}) - n(\vec{p}+\hbar\vec{q})]$$
$$\times\,\delta(\varepsilon_\nu(\vec{p}+\hbar\vec{q}) - \varepsilon_\nu(\vec{p}) - \hbar\omega_q + \xi)d^2p \quad (149)$$

Substituting Eq. (94)−Eq. (97) into Eq. (149) and employing cylindrical coordinates in relation to the tubular geometry of FSWCNT results in:

$$\Gamma_q(\omega) = \frac{8\pi^3 n^*\Phi e^2\mathscr{K}^2\Theta(1-\alpha^2)}{\hbar\omega_q^2\varepsilon^2\rho q v_s \Delta_1 I_o(\Delta_1^*)I_o(\Delta_2^*)\sqrt{1-\alpha^2}}\frac{2}{\sqrt{\pi}}\int_0^\infty \exp\left(-\frac{dt'}{\tau}\right)$$
$$\times\left\{\sinh\left[\Delta_1^*\cos(3ea\vec{E}t')\sin A\sin\left(\frac{3}{2}a\hbar q\right) + \Delta_2^*\cos(ea\vec{E}t')\sin B\sin\left(\frac{a}{2}\hbar q\right)\right]\right.$$
$$\times\cosh\left[\Delta_1^*\cos(3ea\vec{E}t')\cos A\cos\left(\frac{3}{2}a\hbar q\right) + \Delta_2^*\cos(ea\vec{E}t')\cos B\cos\left(\frac{a}{2}\hbar q\right)\right]$$
$$-\,4\left(\Delta_2^*\sin(ea\vec{E}t')\cos B\sin\left(\frac{a}{2}\hbar q\right) + \Delta_1^*\cos A\sin(3ea\vec{E}t')\sin\left(\frac{3}{2}a\hbar q\right)\right.$$
$$\left.+\Delta_1^*\Delta_2^*\sin(ea\vec{E}t')\sin(3ea\vec{E}t')\cos A\cos B\sin\left(\frac{a}{2}\hbar q\right)\sin\left(\frac{3}{2}a\hbar q\right)\right)$$
$$\times\sinh\left[\Delta_1^*\cos(3ea\vec{E}t')\cos A\cos\left(\frac{3}{2}a\hbar q\right) + \Delta_2^*\cos(ea\vec{E}t')\cos B\cos\left(\frac{a}{2}\hbar q\right)\right]$$
$$\left.\times\cosh\left[\Delta_1^*\cos(3ea\vec{E}t')\sin A\sin\left(\frac{3}{2}a\hbar q\right) + \Delta_2^*\cos(ea\vec{E}t')\sin B\sin\left(\frac{a}{2}\hbar q\right)\right]\right\}$$
$$(150)$$

Simplifying further yields:

$$\Gamma_q(\omega) = \Gamma(\omega,0)\left\{1 - 4\left(\Delta_z^*\sin\left(\chi'\left(1-\frac{v_d}{v_s}\right)\right)\cos B\sin\left(\frac{a}{2}\hbar q\right)\right.\right.$$
$$\left.+\;\Delta_s^*\cos A\sin\left(3\chi'\left(1-\frac{v_d}{v_s}\right)\right)\sin\left(\frac{3}{2}a\hbar q\right)\right)$$
$$\times\tanh\left[\Delta_s^*\cos\left(3\chi'\left(1-\frac{v_d}{v_s}\right)\right)\cos A\cos\left(\frac{3}{2}a\hbar q\right)\right.$$

$$+ \Delta_z^* \cos\left(\chi'\left(1 - \frac{v_d}{v_s}\right)\right) \cos B \cos\left(\frac{a}{2}\hbar q\right)\Bigg]$$
$$\times \coth\left[\Delta_s^* \cos\left(3\chi'\left(1 - \frac{v_d}{v_s}\right)\right) \sin A \sin\left(\frac{3}{2}a\hbar q\right)\right.$$
$$\left. + \Delta_z^* \cos\left(\chi'\left(1 - \frac{v_d}{v_s}\right)\right) \sin B \sin\left(\frac{a}{2}\hbar q\right)\right]\Bigg\} \quad (151)$$

Switching off the external electric field yields:

$$\Gamma_q(\omega, 0) = \Gamma_o \left[\sinh\left\{\Delta_s^* \sin\left(\frac{3}{2}a\hbar q\right) \sin A + \Delta_z^* \sin\left(\frac{a}{2}\hbar q\right) \sin B\right\}\right.$$
$$\left. \times \cosh\left\{\Delta_s^* \cos\left(\frac{3}{2}a\hbar q\right) \cos A + \Delta_z^* \cos\left(\frac{a}{2}\hbar q\right) \cos B\right\}\right] \quad (152)$$

and

$$\Gamma_o = \frac{16\pi^3 n^* \vec{\Phi} e^2 \mathscr{K}^2 \tau \Theta(1 - \alpha^2)}{\sqrt{\pi}\hbar\omega_q^2 \varepsilon^2 \rho q v_s \Delta_1 I_o(\Delta_s^*) I_o(\Delta_z^*)\sqrt{1 - \alpha^2}}. \quad (153)$$

where

$$v_d = \frac{3eE_o\Delta_1 a^2 q}{\Omega}, \quad \chi' = \hbar\omega_q a / v_s, \quad \alpha = \frac{\omega_q(1 - v_d/v_s)}{6\Delta_1 a q} \quad (154)$$

Giant differential thermoelectric power in FSWCNT

The problem is considered in the semiclassical regime using the BTE and is written as:

$$\frac{\partial f(\vec{r}, \vec{p}, t)}{\partial t} + v(\vec{p}).\nabla_r f(\vec{r}, \vec{p}, t) + e\vec{E}\nabla_p f(\vec{r}, \vec{p}, t) = -\frac{f(\vec{r}, \vec{p}, t) - f_o(\vec{p})}{\tau}. \quad (155)$$

Eq. (155) is solved by treating the second term on the left hand side as weak perturbation. Linearizing with respect to ∇T and $\nabla\mu$ the solution to the BTE reads as:

$$f(\vec{p}) = \tau^{-1}\int_0^\infty \exp(-t/\tau) f_o(\vec{p} - e\int_0^\infty [\vec{E}_o + \vec{E}\cos(\omega t')]dt')dt$$
$$+ \int_0^\infty \exp(-t/\tau)dt \left\{ [\varepsilon(\vec{p} - e\int_0^\infty [\vec{E}_o + \vec{E}\cos(\omega t') - \mu]\frac{\nabla T}{T} + \nabla\mu \right\}$$
$$\times v(\vec{p} - e\int_0^\infty [\vec{E}_o + \vec{E}\cos(\omega t'))\frac{\partial f_o}{\partial \varepsilon}(\vec{p} - e\int_0^\infty [\vec{E}_o + \vec{E}\cos(\omega t')]), \quad (156)$$

where ∇T is the temperature gradient, $\nabla\mu$ is the quasi-fermi level and μ is the electrochemical potential. Consider a long chain of fluorine doped SWCNT with a THz field directed its axis. For a p$-$type band having double periods, the energy band relation is expressed as:

$$\varepsilon(\vec{p}) = \varepsilon_o - \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} - \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar}, \quad (157)$$

where $u = \sqrt{3}/2$ and $w = 3\sqrt{3}/2$. ε_o is the energy of an outer-shell carrier in an isolated carbon atom, Δ_s and Δ_z are the real overlapping integral for jumps. The carrier distribution function is given by the shifted Fermi-Dirac distribution as:

$$f_o(\vec{p}_z) = \frac{1}{1 + \exp[(\varepsilon(\vec{p}_z) - \mu)/k_B T]} \quad (158)$$

where k_B is Boltzmann's constant, T is the absolute temperature in energy units, and μ is the electrochemical potential. Substituting Eq. (158) into Eq. (157) yields an equation with the term $\mathscr{F}_{1/2}$, representing Fermi-Dirac integral, which is expressed as:

$$\mathscr{F}_{1/2}(\eta_f) = \frac{1}{\Gamma(1/2)}\int_0^\infty \frac{\eta_f^{1/2} d\eta}{1 + \exp(\eta - \eta_f)}, \quad (159)$$

where $(\mu - \varepsilon_c)/k_B T \equiv \eta_f$. For non-degenerate carrier gas, Eq. (159) can be expressed as:

$$f_o(\vec{p}) = A^{\dagger} \exp\left(-\frac{\varepsilon(\vec{p}) - \mu}{k_B T}\right) \tag{160}$$

The carrier probability distribution function can further be expressed as:

$$f_o(\vec{p}) = A^{\dagger} \exp\left(\frac{[\Delta_s \cos(u\vec{p}_s b_s/\hbar) + \Delta_z \cos(w\vec{p}_z b_z/\hbar) + \mu - \varepsilon_o]}{k_B T}\right) \tag{161}$$

where $A^{\dagger}$ is the normalisation constant and is expressed as:

$$A^{\dagger} = \frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)} \exp\left(-\frac{\mu - \varepsilon_o}{k_B T}\right) \tag{162}$$

The miniband velocities of the FSWCNT carrier along the s and z coordinates are obtained as:

$$v_s(\vec{p}_s) = \frac{u\Delta_s b_s}{\hbar} \sin\frac{\vec{p}_s b_s}{\hbar} \qquad v_z(\vec{p}_z) = \frac{w\Delta_z b_z}{\hbar} \sin\frac{\vec{p}_z b_z}{\hbar} \tag{163}$$

and the carrier current densities along these coordinates are obtained as:

$$\begin{aligned}\vec{Z} = e\tau^{-1} \int_0^{\infty} \exp\left(\frac{-t}{\tau}\right) dt \sum_p v_z(\vec{p} - e\int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos(\omega t')]dt') f_o(\vec{p}) \\ + e\int_0^{\infty} \exp\left(\frac{-t}{\tau}\right) dt \sum_p \left\{[\varepsilon(\vec{p}) - \mu]\frac{\nabla_z T}{T} + \nabla_z \mu\right\} \\ \times v_z(\vec{p}) \frac{\partial f_o(\vec{p})}{\partial \varepsilon} v_z(\vec{p} - e\int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos(\omega t')]dt')\end{aligned} \tag{164}$$

and

$$\vec{S} = e\tau^{-1}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt\sum_p v_s(\vec{p} - e\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos(\omega t')]dt')f_o(\vec{p})$$
$$+e\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt\sum_p \left\{[\varepsilon(\vec{p}) - \mu]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times v_s(\vec{p})\frac{\partial f_o(\vec{p})}{\partial \varepsilon}v_s(\vec{p} - e\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos(\omega t')]dt') \quad (165)$$

Making the transformation

$$\sum_p \rightarrow \frac{2}{(2\pi\hbar)^2}\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s$$

Furthermore, the current densities can be expressed as:

$$\vec{Z} = \frac{2e\tau^{-1}}{(2\pi\hbar)^2}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s$$
$$\times v_z(\vec{p} - e\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos(\omega t')]dt')f_o(\vec{p})$$
$$+\frac{2e\tau^{-1}}{(2\pi\hbar)^2}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s\left\{[\varepsilon(\vec{p}) - \mu]\frac{\nabla_z T}{T} + \nabla_z\mu\right\}$$
$$\times v_z(\vec{p})\frac{\partial f_o(\vec{p})}{\partial \varepsilon}v_z(\vec{p} - e\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos(\omega t')]dt') \quad (166)$$

and similarly,

$$\vec{S} = \frac{2e\tau^{-1}}{(2\pi\hbar)^2}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s$$
$$\times v_s(\vec{p} - e\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos(\omega t')]dt')f_o(\vec{p})$$
$$+\frac{2e\tau^{-1}}{(2\pi\hbar)^2}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s\left\{[\varepsilon(\vec{p}) - \mu]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times v_s(\vec{p})\frac{\partial f_o(\vec{p})}{\partial \varepsilon}v_s(\vec{p} - e\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos(\omega t')]dt') \quad (167)$$

where the integration is carried out within the first Brillouin zone $-\pi\hbar/b_z \leq \vec{p}_z \leq \pi\hbar/b_z$ and $-\pi\hbar/b_s \leq \vec{p}_s \leq \pi\hbar/b_s$, respectively. Solving Eq. (166) and Eq. (167) explicitly, the carrier current density along s and z coordinates are obtained as:

$$\vec{S} = -\sigma_s(\vec{E})\left(\vec{E}_{sn} + \nabla_s \frac{\mu}{e}\right) - \sigma_s(\vec{E})\frac{k_B}{e}\left[\left(\frac{\varepsilon_o - \mu}{k_B T}\right) - \Delta_s^* \frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} + 2 - \Delta_z^* \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right]\nabla_s T \quad (168)$$

where $\Delta_s^* = \Delta_s/k_B T$ and $\Delta_z^* = \Delta_z/k_B T$. Then

$$\vec{S} = -\sigma_s(\vec{E})\vec{E}_{sn}^* - \sigma_s(\vec{E})\frac{k_B}{e}\left[\left(\frac{\varepsilon_o - \mu}{k_B T}\right) - \Delta_s^* \frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} + 2 - \Delta_z^* \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right]\nabla_s T \quad (169)$$

where $\vec{E}_{sn}^*$ is the external electric field along the s direction given as $\vec{E}_{sn}^* = \vec{E}_{sn} + \nabla_s \mu/e$. Similarly;

$$\vec{Z} = -\sigma_z(\vec{E})\vec{E}_{zn}^* - \sigma_z(\vec{E})\frac{k_B}{e}\left[\left(\frac{\varepsilon_o - \mu}{k_B T}\right) - \Delta_z \frac{I_o(\Delta_z^*)}{I_1(\Delta_z)} + 2 - \Delta_s \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right]\nabla_z T \quad (170)$$

Resolving the current densities into axial and circumferential components; $\vec{J}_z = \vec{Z} + \vec{S}\sin\theta_h$ and $\vec{J}_c = \vec{S}\cos\theta_h$, respectively. The axial component reads as:

$$\begin{aligned}\vec{J}_z = &-\sigma_z(\vec{E})\vec{E}_{zn}^* - \sigma_s(\vec{E})\sin\theta_h \vec{E}_{sn}^* \\ &- \sigma_z(\vec{E})\frac{k_B}{e}\left[\left(\frac{\varepsilon_o - \mu}{k_B T}\right) - \Delta_z^* \frac{I_o(\Delta_z^*)}{I_1(\Delta_z)} + 2 - \Delta_s^* \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right]\nabla_z T \\ &- \sigma_s(\vec{E})\frac{k_B}{e}\left[\left(\frac{\varepsilon_o - \mu}{k_B T}\right) - \Delta_s^* \frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} + 2 - \Delta_z^* \frac{I_1(\Delta_z)}{I_o(\Delta_z^*)}\right]\sin\theta_h \nabla_s T \quad (171)\end{aligned}$$

Making use of the following relations $\vec{E}_s = \vec{E}\sin\theta_h$, $\nabla_s T = \nabla_z T \sin\theta_h$ and $\vec{E}^*_{sn} = \vec{E}^*_{zn}\sin\theta_h$, then:

$$\begin{aligned}\vec{J}_z = &-(\sigma_z(\vec{E}) + \sigma_s(\vec{E})\sin^2\theta_h)\vec{E}^*_{zn} \\ &- \sigma_z(\vec{E})\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right) - \Delta^*_z\frac{I_o(\Delta^*_z)}{I_1(\Delta_z)} + 2 - \Delta^*_s\frac{I_1(\Delta^*_s)}{I_o(\Delta^*_s)}\right]\nabla_z T \\ &- \sigma_s(\vec{E})\frac{k_B}{e}\sin^2\theta_h\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right) - \Delta^*_s\frac{I_o(\Delta^*_s)}{I_1(\Delta^*_s)} + 2 - \Delta^*_z\frac{I_1(\Delta_z)}{I_o(\Delta^*_z)}\right]\nabla_z T, \end{aligned} \quad (172)$$

where

$$\sigma_z(\vec{E}) = -\frac{e^2\tau n_o w^2\Delta_z b_z^2}{\hbar^2}\frac{I_1(\Delta^*_z)}{I_o(\Delta^*_z)}\sum_{n=-\infty}^{\infty}\left[\frac{J_n^2(\chi)}{1+(\Omega_z+n\omega)^2\tau^2}\right]. \quad (173)$$

Ω_s, Ω_z are the Bloch frequencies along s and z directions respectively, and are given as $\Omega_s = ueb_sE_o/\hbar$, $\Omega_z = web_zE_o/\hbar$ and $\chi = ueb_sE_s/\hbar\omega = web_zE_z/\hbar\omega$. The circumferential component is given as:

$$\begin{aligned}\vec{J}_c = &-\sigma_s(\vec{E})\sin\theta_h\cos\theta_h\vec{E}^*_{zn} \\ &- \sigma_s(\vec{E})\frac{k_B}{e}\sin\theta_h\cos\theta_h\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right) - \Delta^*_s\frac{I_o(\Delta^*_s)}{I_1(\Delta^*_s)} + 2 - \Delta^*_z\frac{I_1(\Delta_z)}{I_o(\Delta^*_z)}\right]\nabla_z T\end{aligned} \quad (174)$$

where

$$\sigma_s(\vec{E}) = -\frac{e^2\tau n_o u^2\Delta_s b_s^2}{\hbar^2}\frac{I_1(\Delta^*_s)}{I_o(\Delta^*_s)}\sum_{n=-\infty}^{\infty}\left[\frac{J_n^2(\chi)}{1+(\Omega_s+n\omega)^2\tau^2}\right] \quad (175)$$

Thus, the circumferential (σ_{cs}) and axial (σ_{zz}) components of the carrier conductivities are given by the coefficients of the driving electric field $\vec{E}_{s,z}$ as:

$$\sigma_{cs} = \sigma_s \sin\theta_h \cos\theta_h \qquad \sigma_{zz} = \sigma_z + \sigma_s \sin^2\theta_h \tag{176}$$

The thermopower is defined as $\left|\vec{E}_n^*/\nabla T\right|$ in an open circuit. Setting $\vec{J}_c = 0$ the thermoelectric power along the circumferential direction α_{cz}, is obtained as :

$$\alpha_{cz} = \left|\frac{\vec{E}_{zn}^*}{\nabla_z T}\right| = -\frac{k_B}{e}\left[\left(\frac{\varepsilon_o - \mu}{k_B T}\right) - \Delta_s^* \frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} + 2 - \Delta_z^* \frac{I_1(\Delta_z)}{I_o(\Delta_z^*)}\right] \tag{177}$$

Similarly, the axial thermoelectric power α_{zz} is also obtained as :

$$\alpha_{zz} = -\frac{\sigma_z(\vec{E})}{(\sigma_z(\vec{E}) + \sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o - \mu}{k_B T}\right) - \Delta_z^* \frac{I_o(\Delta_z^*)}{I_1(\Delta_z)} + 2 - \Delta_s^* \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right]$$
$$- \frac{\sigma_s(\vec{E})\sin^2\theta_h}{(\sigma_z(\vec{E}) + \sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o - \mu}{k_B T}\right) - \Delta_s^* \frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} + 2 - \Delta_z^* \frac{I_1(\Delta_z)}{I_o(\Delta_z^*)}\right] \tag{178}$$

When the *ac* source is switched off i.e. $\vec{E}_z = 0$ and $\omega = 0$, the axial thermopower relation in Eq. (178) reduces to:

$$\alpha_{zz} = \left|\frac{\vec{E}_{zn}^*}{\nabla_z T}\right| =$$
$$- \frac{\sigma_z(\vec{E})}{(\sigma_z(\vec{E}) + \sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o - \mu}{k_B T}\right) - \Delta_z^* \frac{I_o(\Delta_z^*)}{I_1(\Delta_z)} + 2 - \Delta_s^* \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right]$$
$$- \frac{\sigma_s(\vec{E})\sin^2\theta_h}{(\sigma_z(\vec{E}) + \sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o - \mu}{k_B T}\right) - \Delta_s^* \frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} + 2 - \Delta_z^* \frac{I_1(\Delta_z)}{I_o(\Delta_z^*)}\right] \tag{179}$$

where

$$\sigma_z(\vec{E}) = -\frac{e^2\tau n_o w^2 \Delta_z b_z^2}{\hbar^2}\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left[\frac{1}{1+\Omega_z^2\tau^2}\right]$$

$$\sigma_s(\vec{E}) = -\frac{e^2\tau n_o u^2 \Delta_s b_s^2}{\hbar^2}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\left[\frac{1}{1+\Omega_s^2\tau^2}\right]$$

Tunable electrical power factor in FSWCNT

The electrical power factor of a thermoelement is the ratio of real power supplied to a load or use to do work to the apparent power supplied to the circuit. In *ac* systems the real power is complex and so is a vector quantity. The success in reducing κ probably yields an increased power factor, $\mathscr{P} = \alpha^2\sigma$. The importance of maximizing $\mathscr{P}$ can be recognized from the fact when the heat source is unlimited, the ZT value is no longer the only one metric to evaluate the efficiency. As a result, it is worthy to compute the output power density $\mathscr{Q}$ of the device. $\mathscr{P}$ occurred in the definition of $\mathscr{Q}$, particularly for the maximum value, $\mathscr{Q}_{max} = \mathscr{P}(T_h - T_c)/4h_\ell$, where T_h, T_c, and h_ℓ are the hot, and cold contact temperatures and the length between the hot and the cold contacts, respectively. However, since $(T_h - T_c)/4h_\ell$ is given by the boundary condition, $\mathscr{Q}$ is mostly affected by the $\mathscr{P}$. Thus, increase in $\mathscr{P}$ is significant to enhance not only ZT but also $\mathscr{Q}$ for electrical power generation purposes. However, maximizing $\mathscr{P}$ is of major importance in this study.

The electrical power factor along the circumferential direction is given as:

$$\begin{aligned}\mathscr{P}_{cs} &= \alpha_{cz}^2\sigma_{cz}\\ &= \left\{\left(\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z)}{I_o(\Delta_z^*)}\right]\right)^2\right.\\ &\quad\left.\times\left(\frac{e^2\tau n_o u^2\Delta_s b_s^2}{\hbar^2}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\left[\frac{1}{1+\Omega_s^2\tau^2}\right]\right)\sin\theta_h\cos\theta_h\right\} \end{aligned} \tag{180}$$

where the output power density along the circumferential direction becomes;

$$\begin{aligned}\mathscr{Q}_{cs}^{max} &= \left\{\left(\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z)}{I_o(\Delta_z^*)}\right]\right)^2\right.\\ &\quad\left.\times\left(\frac{e^2\tau n_o u^2\Delta_s b_s^2}{\hbar^2}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\left[\frac{1}{1+\Omega_s^2\tau^2}\right]\right)\frac{\sin\theta_h\cos\theta_h(T_h-T_c)}{4h_\ell}\right\}\end{aligned}$$

Moreover, the axial component of the power factor yields:

$$\begin{aligned}\mathscr{P}_{zz} &= \alpha_{zz}^2\sigma_{zz}\\ &= \left\{\left(\frac{\sigma_z(\vec{E})}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_z^*\frac{I_o(\Delta_z^*)}{I_1(\Delta_z)}+2-\Delta_s^*\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right]\right.\right.\\ &\quad+\left.\frac{\sigma_s(\vec{E})\sin^2\theta_h}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z)}{I_o(\Delta_z^*)}\right]\right)^2\\ &\quad\left.\times\frac{e^2\tau n_o}{\hbar^2}\left(w^2\Delta_z b_z^2\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left[\frac{1}{1+\Omega_z^2\tau^2}\right]+u^2\Delta_s b_s^2\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\left[\frac{1}{1+\Omega_s^2\tau^2}\right]\sin^2\theta_h\right)\right\}\end{aligned} \tag{181}$$

and similarly the output power density also gives;

$$\begin{aligned}\mathscr{Q}_{zz}^{max} &= \left\{\left(\frac{\sigma_z(\vec{E})}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_z^*\frac{I_o(\Delta_z^*)}{I_1(\Delta_z)}+2-\Delta_s^*\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right]\right.\right.\\ &\quad+\left.\frac{\sigma_s(\vec{E})\sin^2\theta_h}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z)}{I_o(\Delta_z^*)}\right]\right)^2\\ &\quad\left.\times\frac{e^2\tau n_o(T_h-T_c)}{\hbar^2 4h_\ell}\left(w^2\Delta_z b_z^2\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left[\frac{1}{1+\Omega_z^2\tau^2}\right]+u^2\Delta_s b_s^2\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\left[\frac{1}{1+\Omega_s^2\tau^2}\right]\sin^2\theta_h\right)\right\}\end{aligned}$$

Carrier Thermal Conductivity in FSWCNT

The problem is considered in the semiclassical regime using the BTE:

$$\nabla_t f(\vec{r},\vec{p},t) + v(\vec{p}).\nabla_r f(\vec{r},\vec{p},t) + e\vec{E}\nabla_p f(\vec{r},\vec{p},t) = -\frac{f(\vec{r},\vec{p},t) - f_o(\vec{p})}{\tau} \quad (182)$$

Linearizing with ∇T and $\nabla\mu$, the solution to the BTE reads as:

$$\begin{aligned} f(\vec{p}) &= \tau^{-1}\int_0^\infty \exp(-t/\tau) f_o(\vec{p} - e\int_0^\infty [\vec{E}_o + \vec{E}\cos(\omega t^{'})]dt^{'})dt \\ &+ \int_0^\infty \exp(-t/\tau)dt \left\{ [\varepsilon(\vec{p} - e\int_0^\infty [\vec{E}_o + \vec{E}\cos(\omega t^{'}) - \mu]\frac{\nabla T}{T} + \nabla\mu \right\} \\ &\times v(\vec{p} - e\int_0^\infty [\vec{E}_o + \vec{E}\cos(\omega t^{'}))\frac{\partial f_o}{\partial\varepsilon}(\vec{p} - e\int_0^\infty [\vec{E}_o + \vec{E}\cos(\omega t^{'})]) \end{aligned} \quad (183)$$

For a band having double periods with a periodicity of $3b$, the dispersion law in Eq. (157). The electon thermal current density $\vec{q}$ is defined as:

$$\vec{q} = \sum_p [\varepsilon(\vec{p}) - \mu]v(\vec{p})f(\vec{p}) \quad (184)$$

Substituting Eq. (183) into Eq. (184) yields:

$$\begin{aligned} \vec{q} &= \tau^{-1}\int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt \sum_p [\varepsilon(p) - \mu]v(\vec{p}) f_o\left(\vec{p} - e\int_{t-t^{'}}^{t} [\vec{E}_o + \vec{E}cos\omega t^{'}]dt^{'}\right) \\ &+ \int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt \sum_p [\varepsilon(\vec{p}) - \mu]v(\vec{p}) \left\{ [\varepsilon(p) - \mu]\frac{\nabla T}{T} - \nabla\mu \right\} \\ &\times v(p)\cdot\frac{\partial f_o(\vec{p})}{\partial\varepsilon(\vec{p})}(\vec{p} - e\int_{t-t^{'}}^{t} [\vec{E}_o + \vec{E}cos\omega t^{'}]dt^{'}) \end{aligned} \quad (185)$$

Using the transformation $\vec{p} - e\int_0^\infty [\vec{E}_o + \vec{E}\cos\omega t^{'}]dt^{'} \rightarrow \vec{p}$ the Eq. (185) becomes:

$$\vec{q} = \tau^{-1}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\sum_p\left[\varepsilon\left(\vec{p}-e\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)-\mu\right]$$
$$\times v\left(\vec{p}-e\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)f_o(\vec{p})$$
$$+\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\sum_p\left[\varepsilon\left(\vec{p}-e\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)-\mu\right]$$
$$\times\left\{[\varepsilon(\vec{p})-\mu]\frac{\nabla T}{T}+\nabla\mu\right\}v\left(\vec{p}-e\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\cdot\frac{\partial f_o(\vec{p})}{\partial\varepsilon(p)}v(\vec{p}) \quad (186)$$

Resolving the thermal current densities along the $\vec{S}^*$ and $\vec{Z}^*$ components yields:

$$\vec{S}^* = \tau^{-1}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\sum_p\left[\varepsilon\left(\vec{p}-e\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s cos\omega t']dt'\right)-\mu\right]$$
$$\times v_s\left(\vec{p}-e\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s cos\omega t']dt'\right)f_o(\vec{p})$$
$$+\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\sum_p\left[\varepsilon\left(\vec{p}-e\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)-\mu\right]$$
$$\times\left\{[\varepsilon(\vec{p})-\mu]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}v_s\left(\vec{p}-e\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\cdot\frac{\partial f_o(\vec{p})}{\partial\varepsilon(p)}v_s(\vec{p}) \quad (187)$$

and

$$\vec{Z}^* = \tau^{-1}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\sum_p\left[\varepsilon\left(\vec{p}-e\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_z cos\omega t']dt'\right)-\mu\right]$$
$$\times v_z\left(\vec{p}-e\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_z cos\omega t']dt'\right)f_o(\vec{p})$$
$$+\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\sum_p\left[\varepsilon\left(\vec{p}-e\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_z\cos\omega t']dt'\right)-\mu\right]$$
$$\times\left\{[\varepsilon(p)-\mu]\frac{\nabla_z T}{T}+\nabla_z\mu\right\}v_z\left(\vec{p}-e\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_z\cos\omega t']dt'\right)\cdot\frac{\partial f_o(\vec{p})}{\partial\varepsilon(\vec{p})}v_z(\vec{p}) \quad (188)$$

Making the transformation;

$$\sum_{p} \rightarrow \frac{2}{(2\pi\hbar)^2} \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z$$

Eq. (187) and Eq. (188) become:

$$\begin{aligned}
\vec{S}^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2} \int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z f_o(\vec{p}) \\
\times \left[\varepsilon\left(\vec{p} - e\int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right) - \mu\right] v_s\left(\vec{p} - e\int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right) \\
+ \frac{2}{(2\pi\hbar)^2} \int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \\
\times \left[\varepsilon\left(\vec{p} - e\int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right) - \mu\right] \left\{[\varepsilon(\vec{p}) - \mu]\frac{\nabla_s T}{T} + \nabla_s \mu\right\} \\
\times v_s\left(\vec{p} - e\int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right) \cdot \frac{\partial f_o(\vec{p})}{\partial \varepsilon(\vec{p})} v_s(\vec{p})
\end{aligned} \quad (189)$$

and

$$\begin{aligned}
\vec{Z}^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2} \int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z f_o(\vec{p}) \\
\times \left[\varepsilon\left(\vec{p} - e\int_{t-t'}^{t} [E_o + E_z\cos\omega t']dt'\right) - \mu\right] v_z\left(\vec{p} - e\int_{t-t'}^{t} [\vec{E}_o + \vec{E}_z\cos\omega t']dt'\right) \\
+ \frac{2}{(2\pi\hbar)^2} \int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \\
\times \left[\varepsilon\left(\vec{p} - e\int_{t-t'}^{t} [\vec{E}_o + \vec{E}_z\cos\omega t']dt'\right) - \mu\right] \left\{[\varepsilon(p) - \mu]\frac{\nabla_z T}{T} + \nabla_z \mu\right\} \\
\times v_z\left(\vec{p} - e\int_{t-t'}^{t} [\vec{E}_o + \vec{E}_z cos\omega t']dt'\right) \cdot \frac{\partial f_o(\vec{p})}{\partial \varepsilon(\vec{p})} v_z(\vec{p})
\end{aligned} \quad (190)$$

Solving Eq. (189) explicitly, the carrier thermal current for the S^* component is obtained as:

$$
\begin{aligned}
\vec{S}^{*} = -\sigma_s(\vec{E})\frac{1}{e}\Bigg\{(\varepsilon_o-\mu)\sum_{n=-\infty}^{\infty}J_n^2(\chi)\\
-\frac{\Delta_s}{2}\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)-\Delta_z\sum_{n=-\infty}^{\infty}J_n^2(\chi)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\Bigg\}\vec{E}_{sn}^{*}\\
-\sigma_s(\vec{E})\frac{k_B}{e^2}\Bigg\{\frac{(\varepsilon_o-\mu)^2}{k_BT}\sum_{n=-\infty}^{\infty}J_n^2(\chi)\\
-\frac{\Delta_s}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)\\
-2\Delta_z\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty}^{\infty}J_n^2(\chi)\\
+\frac{\Delta_s\Delta_s^*}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^*I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)\\
+\frac{\Delta_z\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)\\
+\Delta_z\Delta_z^*\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^*I_o(\Delta_z^*)}\right)\sum_{n=-\infty}^{\infty}J_n^2(\chi)\Bigg\}\nabla_sT \quad (191)
\end{aligned}
$$

where $\vec{E}_{sn}^{*}=\vec{E}_{sn}+\nabla_s\mu/e$. Similarly;

$$
\begin{aligned}
\vec{Z}^{*} = -\sigma_z(\vec{E})\frac{1}{e}\Bigg\{(\varepsilon_o-\mu)\sum_{n=-\infty}^{\infty}J_n^2(\chi)\\
-\frac{\Delta_z}{2}\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)-\Delta_s\sum_{n=-\infty}^{\infty}J_n^2(\chi)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\Bigg\}\vec{E}_{zn}^{*}\\
-\sigma_z(\vec{E})\frac{k_B}{e^2}\Bigg\{\frac{(\varepsilon_o-\mu)^2}{k_BT}\sum_{n=-\infty}^{\infty}J_n^2(\chi)\\
-\frac{\Delta_z}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)\\
-2\Delta_s\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\sum_{n=-\infty}^{\infty}J_n^2(\chi)\\
+\frac{\Delta_z\Delta_z^*}{2}\left(1-\frac{3I_o(\Delta_z^*)}{\Delta_z^*I_1(\Delta_z^*)}+\frac{6}{\Delta_z^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)\\
+\frac{\Delta_s\Delta_z^*}{2}\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)\\
+\Delta_s\Delta_s^*\left(1-\frac{I_1(\Delta_s^*)}{\Delta_s^*I_o(\Delta_s^*)}\right)\sum_{n=-\infty}^{\infty}J_n^2(\chi)\Bigg\}\nabla_zT \quad (192)
\end{aligned}
$$

The axial and circumferential components of the thermal current density are respectively given as: $\vec{q}_z = \vec{Z}^* + \vec{S}^* \sin\theta_h$ and $\vec{q}_c = \vec{S}^* \cos\theta_h$, respectively. The axial current density is given as:

$$
\begin{aligned}
\vec{q}_z = -\sigma_z(\vec{E})\frac{1}{e}\Bigg\{&(\varepsilon_o-\mu)\sum_{n=-\infty}^{\infty} J_n^2(\chi) \\
&-\frac{\Delta_z}{2}\left(1+3\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)-\Delta_s\sum_{n=-\infty}^{\infty} J_n^2(\chi)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\Bigg\}\vec{E}_{zn}^* \\
&-\sigma_z(\vec{E})\frac{k_B}{e^2}\Bigg\{\frac{(\varepsilon_o-\mu)^2}{k_BT}\sum_{n=-\infty}^{\infty} J_n^2(\chi) \\
&-\frac{\Delta_z}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\left(1+3\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right) \\
&-2\Delta_s\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\sum_{n=-\infty}^{\infty} J_n^2(\chi) \\
&+\frac{\Delta_z\Delta_z^*}{2}\left(1-\frac{3I_o(\Delta_z^*)}{\Delta_z^* I_1(\Delta_z^*)}+\frac{6}{\Delta_z^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right) \\
&+\frac{\Delta_s\Delta_z^*}{2}\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\left(1+3\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right) \\
&+\Delta_s\Delta_s^*\left(1-\frac{I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)}\right)\sum_{n=-\infty}^{\infty} J_n^2(\chi)\Bigg\}\nabla_z T \\
-\sigma_s(\vec{E})\frac{1}{e}\Bigg\{&(\varepsilon_o-\mu)\sum_{n=-\infty}^{\infty} J_n^2(\chi)-\frac{\Delta_s}{2}\left(1+3\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right) \\
&-\Delta_z\sum_{n=-\infty}^{\infty} J_n^2(\chi)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\Bigg\}\vec{E}_{sn}^*\sin\theta_h \\
&-\sigma_s(\vec{E})\frac{k_B}{e^2}\Bigg\{\frac{(\varepsilon_o-\mu)^2}{k_BT}\sum_{n=-\infty}^{\infty} J_n^2(\chi) \\
&-\frac{\Delta_s}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right) \\
&-2\Delta_z\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty}^{\infty} J_n^2(\chi) \\
&+\frac{\Delta_s\Delta_s^*}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right) \\
&+\frac{\Delta_z\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right)
\end{aligned}
$$

$$+ \Delta_z \Delta_z^* \left(1 - \frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right) \sum_{n=-\infty}^{\infty} J_n^2(\chi) \Bigg\} \nabla_s T \sin\theta_h \quad (193)$$

and similarly the circumferential component is given by:

$$\begin{aligned}
\vec{q}_c = -\sigma_s(\vec{E}) \frac{1}{e} \Bigg\{ & (\varepsilon_o - \mu) \sum_{n=-\infty}^{\infty} J_n^2(\chi) \\
& - \frac{\Delta_s}{2}\left(1 + 3\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*}\right) - \Delta_z \sum_{n=-\infty}^{\infty} J_n^2(\chi) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \Bigg\} \vec{E}_{sn}^* \\
& - \sigma_s(\vec{E}) \frac{k_B}{e^2} \Bigg\{ \frac{(\varepsilon_o - \mu)^2}{k_B T} \sum_{n=-\infty}^{\infty} J_n^2(\chi) \\
& - \frac{\Delta_s}{2}\left(\frac{\varepsilon_o - \mu}{k_B T}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*}\right)\left(1 + 3\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right) \\
& - 2\Delta_z \left(\frac{\varepsilon_o - \mu}{k_B T}\right) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \sum_{n=-\infty}^{\infty} J_n^2(\chi) \\
& + \frac{\Delta_s \Delta_s^*}{2}\left(1 - \frac{3 I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)} + \frac{6}{\Delta_s^{*2}}\right)\left(1 + \sum_{n=-\infty}^{\infty} J_n^2(\chi)\right) \\
& + \frac{\Delta_z \Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1 + 3\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right) \\
& + \Delta_z \Delta_z^* \left(1 - \frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right) \sum_{n=-\infty}^{\infty} J_n^2(\chi) \Bigg\} \nabla_s T \cos\theta_h \quad (194)
\end{aligned}$$

Utilizing the relations, $\vec{E}_s = \vec{E}_z \sin\theta_h$ and $\nabla_s T = \nabla_z T \sin\theta_h$, the axial carrier thermal current density yields:

$$\begin{aligned}
\vec{q}_z = -\frac{k_B T}{e} \Bigg\{ & \sigma_z(\vec{E}) \Bigg[\left(\frac{\varepsilon_o - \mu}{k_B T}\right) \sum_{n=-\infty}^{\infty} J_n^2(\chi) \\
& - \frac{\Delta_z^*}{2}\left(1 + 3\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)} - \frac{2}{\Delta_z^*}\right) - \Delta_s^* \sum_{n=-\infty}^{\infty} J_n^2(\chi) \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} \Bigg] \\
+ \sigma_s(\vec{E}) \sin^2\theta_h & \Bigg[\left(\frac{\varepsilon_o - \mu}{k_B T}\right) \sum_{n=-\infty}^{\infty} J_n^2(\chi) - \frac{\Delta_s^*}{2}\left(1 + 3\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*}\right) \\
& - \Delta_z^* \sum_{n=-\infty}^{\infty} J_n^2(\chi) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \Bigg] \Bigg\} \vec{E}_{zn}^*
\end{aligned}$$

$$
\begin{aligned}
& - \frac{k_B^2 T}{e^2} \Bigg\{ \sigma_z(\vec{E}) \Bigg[\left(\frac{\varepsilon_o - \mu}{k_B T} \right)^2 \sum_{n=-\infty}^{\infty} J_n^2(\chi) \\
& - \frac{\Delta_z^*}{2} \left(\frac{\varepsilon_o - \mu}{k_B T} \right) \left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)} - \frac{2}{\Delta_z^*} \right) \left(1 + 3 \sum_{n=-\infty}^{\infty} J_n^2(\chi) \right) \\
& - 2\Delta_s^* \left(\frac{\varepsilon_o - \mu}{k_B T} \right) \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} \sum_{n=-\infty}^{\infty} J_n^2(\chi) \\
& + \frac{\Delta_z^{*2}}{2} \left(1 - \frac{3 I_o(\Delta_z^*)}{\Delta_z^* I_1(\Delta_z^*)} + \frac{6}{\Delta_z^{*2}} \right) \left(1 + \sum_{n=-\infty}^{\infty} J_n^2(\chi) \right) \\
& + \frac{\Delta_s^* \Delta_z^*}{2} \left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)} - \frac{2}{\Delta_z^*} \right) \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} \left(1 + 3 \sum_{n=-\infty}^{\infty} J_n^2(\chi) \right) \\
& + \Delta_s^{*2} \left(1 - \frac{I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)} \right) \sum_{n=-\infty}^{\infty} J_n^2(\chi) \Bigg] \\
& + \sigma_s(\vec{E}) \sin^2 \theta_h \Bigg[\left(\frac{\varepsilon_o - \mu}{k_B T} \right)^2 \sum_{n=-\infty}^{\infty} J_n^2(\chi) \\
& - \frac{\Delta_s^*}{2} \left(\frac{\varepsilon_o - \mu}{k_B T} \right) \left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*} \right) \left(1 + 3 \sum_{n=-\infty}^{\infty} J_n^2(\chi) \right) \\
& - 2\Delta_z^* \left(\frac{\varepsilon_o - \mu}{k_B T} \right) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \sum_{n=-\infty}^{\infty} J_n^2(\chi) \\
& + \frac{\Delta_s^{*2}}{2} \left(1 - \frac{3 I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)} + \frac{6}{\Delta_s^{*2}} \right) \left(1 + \sum_{n=-\infty}^{\infty} J_n^2(\chi) \right) \\
& + \frac{\Delta_z^* \Delta_s^*}{2} \left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*} \right) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \left(1 + 3 \sum_{n=-\infty}^{\infty} J_n^2(\chi) \right) \\
& + \Delta_z^{*2} \left(1 - \frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)} \right) \sum_{n=-\infty}^{\infty} J_n^2(\chi) \Bigg] \Bigg\} \nabla_z T \qquad (195)
\end{aligned}
$$

Similarly, the circumferential component is given as:

$$
\begin{aligned}
\vec{q}_c = & -\sigma_s(\vec{E}) \frac{k_B T}{e} \Bigg\{ \left(\frac{\varepsilon_o - \mu}{k_B T} \right) \sum_{n=-\infty}^{\infty} J_n^2(\chi) \\
& - \frac{\Delta_s^*}{2} \left(1 + 3 \sum_{n=-\infty}^{\infty} J_n^2(\chi) \right) \left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*} \right) \\
& - \Delta_z^* \sum_{n=-\infty}^{\infty} J_n^2(\chi) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \Bigg\} \vec{E}_{zn}^* \sin \theta_h \cos \theta_h \\
& - \sigma_s(\vec{E}) \frac{k_B^2 T}{e^2} \Bigg\{ \left(\frac{\varepsilon_o - \mu}{k_B T} \right)^2 \sum_{n=-\infty}^{\infty} J_n^2(\chi)
\end{aligned}
$$

$$- \frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$
$$- 2\Delta_z^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty}^{\infty}J_n^2(\chi)$$
$$+ \frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^*I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$
$$+ \frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$
$$+ \Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^*I_o(\Delta_z^*)}\right)\sum_{n=-\infty}^{\infty}J_n^2(\chi)\Bigg\}\nabla_zT\sin\theta_h\cos\theta_h \quad (196)$$

where

$$\sigma_z = -\frac{e^2\tau n_o w^2\Delta_z b_z^2}{\hbar^2}\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}, \qquad \sigma_s = -\frac{e^2\tau n_o u^2\Delta_s b_s^2}{\hbar^2}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}$$

Eq. (195) and Eq. (196) are the results for weak electric field, $\vec{E}$. The coefficients of $\nabla_z T$ in Eq. (195) and Eq. (196) defines the carrier thermal conductivity. Let κ_{ec} and κ_{ez} be the circumferential and axial components of the carrier thermal conductivity, respectively as:

$$\kappa_{ec} = -\sigma_s(\vec{E})\frac{k_B^2T}{e^2}\sin\theta_h\cos\theta_h$$
$$\times\Bigg\{\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty}^{\infty}J_n^2(\chi)$$
$$- \frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$
$$- 2\Delta_z^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty}^{\infty}J_n^2(\chi)$$
$$+ \frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^*I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$
$$+ \frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$

$$+\Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right)\sum_{n=-\infty}^{\infty} J_n^2(\chi)\Bigg\} \quad (197)$$

and similarly

$$\begin{aligned}
\kappa_{ez} = -\frac{k_B^2 T}{e^2}\Bigg\{\sigma_z(\vec{E})\Bigg[\left(\frac{\varepsilon_o-\mu}{k_B T}\right)^2 \sum_{n=-\infty}^{\infty} J_n^2(\chi) \\
-\frac{\Delta_z^*}{2}\left(\frac{\varepsilon_o-\mu}{k_B T}\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\left(1+3\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right) \\
-2\Delta_s^*\left(\frac{\varepsilon_o-\mu}{k_B T}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\sum_{n=-\infty}^{\infty} J_n^2(\chi) \\
+\frac{\Delta_z^{*2}}{2}\left(1-\frac{3I_o(\Delta_z^*)}{\Delta_z^* I_1(\Delta_z^*)}+\frac{6}{\Delta_z^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right) \\
+\frac{\Delta_s^*\Delta_z^*}{2}\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\left(1+3\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right) \\
+\Delta_s^{*2}\left(1-\frac{I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)}\right)\sum_{n=-\infty}^{\infty} J_n^2(\chi)\Bigg] \\
+\sigma_s(\vec{E})\sin^2\theta_h\Bigg[\left(\frac{\varepsilon_o-\mu}{k_B T}\right)^2 \sum_{n=-\infty}^{\infty} J_n^2(\chi) \\
-\frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o-\mu}{k_B T}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right) \\
-2\Delta_z^*\left(\frac{\varepsilon_o-\mu}{k_B T}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty}^{\infty} J_n^2(\chi) \\
+\frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right) \\
+\frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right) \\
+\Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right)\sum_{n=-\infty}^{\infty} J_n^2(\chi)\Bigg]\Bigg\} \quad (198)
\end{aligned}$$

Lattice Conductivity in FSWCNT

The LBM which is derived from the Boltzmann equation in the kinetic theory of phonons, is used to execute the lattice simulation. A Chapman-Enskog expansion is used to solve the lattice Boltzmann model (LBM). In the diffusive limit, macroscopic heat transport equations are found, and correlations between numerical parameters and bulk material properties are found for 1D, 2D, and 3D cases, respectively. The BTE is then calculated as follows:

$$\nabla_t f(\vec{r},\vec{k},t) + \vec{v}(\vec{k}).\nabla_r f(\vec{r},\vec{k},t) + e\vec{E}\nabla_k f(\vec{r},\vec{k},t) = \left(\frac{\partial f}{\partial t}\right)_{col} \tag{199}$$

The BTE is represented as a phonon transport equation explaining the balance between variation, advection, and scattering of phonons in the absence of external driving force as:

$$\nabla_t N(\vec{r},\vec{k},t) + \vec{v}_g.\nabla_r N(\vec{r},\vec{k},t) = \left(\frac{\partial N}{\partial t}\right)_{col} \tag{200}$$

where $N(\vec{r},\vec{k},t)$ denotes the phonon distribution function, and $N(\vec{r},\vec{k},t)d\vec{r}d\vec{k}$ denotes the phonon probability distribution function with a spatial interval $(\vec{r},\vec{r}+d\vec{r})$ and a wavevector interval $(\vec{k},\vec{k}+d\vec{k})$ at a given time t. Through the relation $\vec{v}_g = \nabla_k \omega$, the wavevector $\vec{k} = 2\pi/\lambda$ is connected to the phonon group velocity $vecvg$ and the phonon frequency ω. The BGK approach gives the collision term in Eq. (200) as:

$$\left(\frac{\partial N}{\partial t}\right)_{col} = -\frac{N(\vec{r},\vec{k},t) - N_o(\vec{k})}{\tau_R} \tag{201}$$

The phonon group velocity $\vec{v}_g$ is constant, and relaxation time τ_R is frequency independent, assuming Debye's linear dispersion relation. As a result of these approximations, Eq. (201) becomes:

$$\frac{\partial N(\vec{r},\vec{k},t)}{\partial t} + \vec{v}_g.\nabla_r N(\vec{r},\vec{k},t) = -\frac{N(\vec{r},\vec{k},t) - N_o(\vec{k})}{\tau_R} \quad (202)$$

where $N_o(\vec{k}) = N_o^{eq}(\vec{k})$ is the Planck's distribution function as

$$N_o^{eq}(\vec{k}) = \frac{1}{\exp(\hbar\omega/k_B T) - 1} \quad (203)$$

The reduced Planck constant is $\hbar$, and Eq. (202) is rewritten in terms of phonon energy as follows:

$$\frac{\partial e(\vec{r},\vec{k},t)}{\partial t} + \vec{v}_g.\nabla_r e(\vec{r},\vec{k},t) = -\frac{e(\vec{r},\vec{k},t) - e_o^{eq}(\vec{k})}{\tau_R} \quad (204)$$

The phonon energy density in equilibrium and non-equilibrium is represented as:

$$e^{eq} = \int \hbar\omega f^{eq}(\vec{r},\vec{k},t)\mathscr{D}(\omega)d\omega \qquad e = \int \hbar\omega f(\vec{r},\vec{k},t)\mathscr{D}(\omega)d\omega \quad (205)$$

with the phonon density of state $\mathscr{D}(\omega)$, which comprises the transverse and longitudinal branches. The LBM for the phonons is found as a discrete variant of the phonon Boltzmann equation in Eq. (204) in the hydrodynamic regime $q\ell \ll 1$ as:

$$\frac{\partial e_i(\vec{x},t)}{\partial t} + \vec{c}_i.\nabla_r e_i(\vec{x},t) = -\frac{e_i(\vec{x},t) - e_i^{eq}(\vec{x},t)}{\tau_R} \quad (206)$$

The discrete velocities are represented by $\vec{c}_i(i = 1,2.....n)$. On the left side of Eq. (204), the spatial and time derivatives of $e_i(\vec{x},t)$ are approximated to first order finite difference, yielding:

$$\frac{e_i(\vec{x}+\vec{c}_i\delta t,t+\delta t)-e_i(\vec{x},t)}{\delta t}+\vec{c}_i.\frac{e_i(\vec{x}+\vec{c}_i\delta t,t+\delta t)-e_i(\vec{x},t)}{\delta \vec{x}}=-\frac{e_i(\vec{x},t)-e_i^{eq}(\vec{x},t)}{\tau_R} \tag{207}$$

Replacing $\delta\vec{x}=\vec{c}_i\delta t$, where $\delta\vec{x}$ is the spatial step and δt is the time step and $\tau=\tau_R/\delta t$ is the dimensionless relaxation time;

$$\frac{e_i(\vec{x}+\vec{c}_i\delta t,t+\delta t)-e_i(\vec{x},t)}{\delta t}+\frac{e_i(\vec{x}+\vec{c}_i\delta t,t+\delta t)-e_i(\vec{x},t)}{\vec{c}_i\delta t}=-\frac{e_i(\vec{x},t)-e_i^{eq}(\vec{x},t)}{\tau_R} \tag{208}$$

$$\frac{e_i(\vec{x}+\vec{c}_i\delta t,t+\delta t)-e_i(\vec{x},t)}{\delta t}=-\frac{e_i(\vec{x},t)-e_i^{eq}(\vec{x},t)}{\tau} \tag{209}$$

The discrete equilibrium phonon energy densities are obtained from Eq. (205) as:

$$e_i^{eq}=\frac{1}{n}\int\hbar\omega\frac{\mathscr{D}(\omega)d\omega}{\exp(\hbar\omega/k_BT)-1} \tag{210}$$

The phonon energy density at equilibrium is reduced to: using Debye's approximation and a laborious simplification as

$$e^{eq}=\frac{9Nk_BT}{V}\left(\frac{T}{\Theta_D}\right)^3\int_0^{\Theta_D/T}\frac{x^3dx}{\exp(x)-1}. \tag{211}$$

Here N/V is the number density of crystal atoms, Θ_D the Debye temperature and $x\equiv\hbar\omega/k_BT$. The FSWCNT local crystal temperature must be determined via an inverse numerical integration from the local phonon energy density based on Eq. (211). This inverse computation is complex and only necessary for very low temperature ($T\ll\Theta_D$) situation where the temperature dependency of heat capacity is dominant for mathematical simplicity and clear interpretation of the results. In this thesis, high temperatures are considered where the heat capacity approaches a constant such that Eq. (198) reduces to $e^{eq} \approx C_v T$. Therefore, Eq. (210) becomes:

$$e_i^{eq} = \frac{1}{n} e^{eq} = \frac{1}{n} C_v T \tag{212}$$

and C_v is given as:

$$C_v = \frac{9Nk_B T}{V} \left(\frac{T}{\Theta_D} \right)^3 \int_0^{\Theta_D/T} \frac{x^3 dx}{\exp(x) - 1} \tag{213}$$

For hydrodynamics studies, the equilibrium distribution functions for phonon LBM and classical LBM are established in a comparable way. In classical LBM, the discrete equilibrium distribution function is created using the local Maxwell-Boltzmann distribution, which determines the likelihood of detecting particles in velocity space. Discrete lattice velocities with different weight factors are used to characterize the mesoscopic kinetics of particles in order to recreate the macroscopic hydrodynamics. Under Debye's assumptions in phonon LBM, i.e. a uniform velocity space, the phonon group velocities have the same value in all directions. The phonon equilibrium distribution function, or Bose-Einstein distribution, defines the probability of locating fictitious particles in energy space while avoiding phonon energy density evolution in frequency space. As a result, discrete equilibrium function in Eq. (210) effectively defines an energy probability distribution in coordinate space, which explains why the same weight factors are used frequently in this

LBM to represent phonon transport in isotropic FSWCNT. Because the photon and the phonon are both bosons, a development of the frequency-dependency phonon distribution function in LBM, as well as a similar unsolved problem in LBM photon (radiative) transmission, remains an outstanding challenge. The

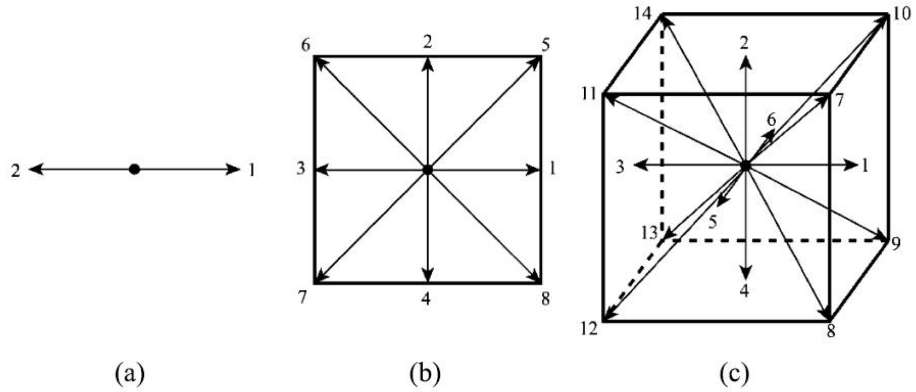

Figure 14: Dimensions of Lattice Structures for LBM at: (a) D1Q2, (b) D2Q8, and (c) D3Q14.

microscopic velocities of the particle c_i along the various directions on the lattice is given as:

$$c_i = \begin{cases} c\left(\cos\left(\frac{(i-1)\pi}{2}\right), \sin\left(\frac{(i-1)\pi}{2}\right)\right), & i=1,2,3,4 \\ \sqrt{2}c\left(\cos\left(\frac{(i-5)\pi}{2}+\frac{\pi}{4}\right), \sin\left(\frac{(i-5)\pi}{2}+\frac{\pi}{4}\right)\right), & i=5,6,7,8 \end{cases} \tag{214}$$

A Cartesian grid is the most common implementation of LBM. This demonstrates that phonons travel a distinct distance in each direction. The phonons have the same group velocity in all directions, and the propagating phonons have the same time step in all directions. A weight factor must be intentionally given to synchronize the transmission of phonons for a single time step. LBM weight factors are built using generalized lattice tensors. The phonons travelling in either direction arrive at the next grid point, where LBM is used to find a solution. To account for the varying distances traveled by each direction, the apparent discrete velocities are calculated using Eq. (214). The discrete velocities are employed to construct appropriate weight factors, w_i, using tensor moment expansions.

In addition to preserving rotational invariance, weight factors are required to maintain constant phonon propagation in each direction. Rotational invariance is the ability to rotate the lattice in 90^o increments without changing the findings. This is an essential feature in LBM because the streaming mechanism is the same in both directions. A rotationally invariant Cartesian lattice must satisfy the conservation equations, which are moments about the lattice speed. Odd moments will vanish due to symmetry, however even moments can be used to determine weight factors and directional lattice speeds. The odd moment equations for the D2Q8 lattice (radiation has no stationary node) with discrete velocities, as shown in Eq. (210), disappear due to symmetry:

$$\sum_{i=1}^{8} c_{i\alpha} w_i = 0, \qquad \sum_{i=1}^{8} c_{i\alpha} c_{i\beta} w_i = 0 \tag{215}$$

and the even moments can be expressed as:

$$\begin{gathered} \sum_{i=1}^{8} w_i = 1 \qquad \sum_{i=1}^{8} c_{i\alpha} c_{i\beta} w_i = \delta_{\alpha\beta} \\ \sum_{i=1}^{8} c_{i\alpha} c_{i\beta} c_{i\gamma} c_{i\eta} w_i = \delta_{\alpha\beta}\delta_{\gamma\eta} + \delta_{\alpha\gamma}\delta_{\beta\eta} + \delta_{\alpha\eta}\delta_{\beta\gamma} \end{gathered} \tag{216}$$

where δ is the Kronecker Delta function.

A Chapman-Enskog expansion is used to solve the phonon LBM in Eq. (196) to get the macroscopic heat transfer equation. As indicated in Figure

14, lattice structures such as D1Q2, D2Q8, and D3Q14 were investigated. In this analysis, the central-point component with vanishing lattice velocity is not taken into account. This discrepancy arises from the fact that in phonon transport, the conservation principles of particle number and mass are no longer guaranteed. However, because phonons are bosons with zero static mass, their energy and velocity can never approach zero, or they will vanish, which is a fundamental principle of classical harmonic oscillators:

$$\mathscr{E} = \frac{\hbar\omega}{2}\left(n + \frac{1}{2}\right) \tag{217}$$

For the Chapman-Enskog expansion, we'll use a conventional D2Q8 lattice, but identical approaches can be used for various lattice configurations. Because the lattice time step is usually small, a Taylor expansion of $e_i(\vec{x} + \vec{c}_i\delta t, t + \delta t)$ is performed in second order around $e_i(\vec{x}, t)$:

$$\begin{aligned} e_i(\vec{x}+\vec{c}_i\delta t, t+\delta t) = e_i(\vec{x},t) + \delta t\left(\frac{\partial}{\partial t} + \vec{c}_{i\alpha}\frac{\partial}{\partial x_\alpha}\right)e_i(\vec{x},t) \\ +\frac{1}{2}(\delta t)^2\left(\frac{\partial}{\partial t} + \vec{c}_{i\alpha}\frac{\partial}{\partial x_\alpha}\right)^2 e_i(\vec{x},t) \end{aligned} \tag{218}$$

Substitution of Eq. (218) into Eq. (209) gives rise to:

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \vec{c}_{i\alpha}\frac{\partial}{\partial x_\alpha}\right)e_i(\vec{x},t) + \frac{1}{2}(\delta t)\left(\frac{\partial}{\partial t} + \vec{c}_{i\alpha}\frac{\partial}{\partial x_\alpha}\right)^2 e_i(\vec{x},t) \\ = -\frac{e_i(\vec{x},t) - e_i^{eq}(\vec{x},t)}{\tau_R} \end{aligned} \tag{219}$$

Expandng $\partial/\partial t$ and introducing two time scales t_1 and t_2 and one spatial scale

$\vec{x}_1$, results in:

$$\frac{\partial}{\partial x_\alpha} = \varepsilon \frac{\partial}{\partial x_{1\alpha}} \qquad \frac{\partial}{\partial t} = \varepsilon \frac{\partial}{\partial t_1} + \varepsilon^2 \frac{\partial}{\partial t_2}. \tag{220}$$

The Knudson number (Kn) is a ratio of phonon mean free path ℓ to the characteristic length of the FSWCNT size L, where ε is a tiny factor frequently chosen as Kn. Within first order, the discrete phonon energy densities were asymptotically expanded: $e_i(\vec{x},t) = e_i^{(0)}(\vec{x},t) + \varepsilon e_i^{(1)}(\vec{x},t) + \ldots$. The following orders of magnitude were obtained by substituting this expression into Eq. (219): $\varepsilon^0, \varepsilon^1, \varepsilon^2$, respectively:

$$e_i^{(0)}(\vec{x},t) = e_i^{eq}(\vec{x},t) \tag{221}$$

$$\frac{\partial e_i^{(0)}(\vec{x},t)}{\partial t_1} + c_{i\alpha} \frac{\partial e_i^{(0)}(\vec{x},t)}{\partial x_{1\alpha}} = -\frac{1}{\tau_R} e_i^{(1)}(\vec{x},t) \tag{222}$$

$$\frac{\partial e_i^{(0)}(\vec{x},t)}{\partial t_2} + \left(1 - \frac{1}{2\tau}\right) \frac{\partial e_i^{(1)}(\vec{x},t)}{\partial t_1} + \left(1 - \frac{1}{2\tau}\right) c_{i\alpha} \frac{\partial e_i^{(1)}(\vec{x},t)}{\partial x_{1\alpha}} = 0 \tag{223}$$

Conservation laws for phonon momentum and energy were ensured during the collision process. The heat flux, $\vec{q}$, was asymptotically expanded within first order as:

$$\vec{q}_\alpha(\vec{x},t) = \vec{q}_\alpha^{(0)}(\vec{x},t) + \varepsilon \vec{q}_\alpha^{(1)}(\vec{x},t) + \varepsilon^2 \vec{q}_\alpha^{(2)}(\vec{x},t) + \ldots \tag{224}$$

Sum over i for Eq. (222) and Eq. (223) respectively, and combining the two equations in terms of space and time t_1, t_2 and $\vec{x}_1$ yields the energy balance equation expressed as:

$$\frac{\partial e}{\partial t'} + \frac{\partial q_\alpha}{\partial x_\alpha} = 0 \tag{225}$$

with the realistic time scale t' related to the time scale t in LBM by $t' = (1 - 1/2\tau)t$. The concerning macroscopic variables: local temperature and heat flux were computed from the discrete phonon energy density through: $e = \sum_i e_i = C_v T$ and $q_\alpha(\vec{x},t) = \sum_i c_{i\alpha} e_i(\vec{x},t)$. Multiplying c_i on both sides of Eq. (222) and Eq. (223), and summing over i respectively, and finally combining the two equations gives rise to the heat flux as:

$$\vec{q}_\alpha = -\frac{3}{4}\tau_R c^2 \frac{\partial e}{\partial x_\alpha} \tag{226}$$

Eq. (226) is the well known classical Fourier law, where $de = C_v dT$ and from Eq. (226) the lattice thermal conductivity is:

$$\kappa_\ell = C_v \tau_R \frac{v_g^2}{3} \tag{227}$$

with the lattice speed and phonon group velocity related as:

$$c = \frac{2}{3} v_g. \tag{228}$$

As a result, the solution of the Chapman-Enskog expansion to the LBM was used to recover the Fourier's law. Table 1 summarizes the three different types of lattice structures and the accompanying lattice speeds.

Table 1: Relations between Lattice Speed and Phonon Group Speed

Lattice type	Discrete lattice velocities	Lattice speed
D1Q2	$(\pm1,0,0)c$	$c=\frac{1}{\sqrt{3}}v_g$
D2Q8	$(\pm1,0,0)c$, $(0,\pm1,0)c$, $(\pm1,\pm1,0)c$	$c=\frac{2}{3}v_g$
D3Q14	$(\pm1,0,0)c$, $(0,\pm1,0)c$, $(0,0,\pm1)c$, $(\pm1,\pm1,\pm1)c$	$c=\sqrt{\frac{7}{15}}v_g$

Thermoelectric figure of merit in FSWCNT

ZT is commonly used to assess the efficiency of a solid-state thermopower generator. The requirement to maximize ZT by attaining a large α, high σ, but low κ is a fundamental part of thermoelectricity research. The dimensionless ZT and Z are defined by the relation:

$$Z=\frac{\alpha^2\sigma}{\kappa_e+\kappa_\ell} \qquad ZT=\frac{\alpha^2\sigma T}{\kappa_e+\kappa_\ell} \tag{229}$$

The ZT describes the feasibility of the practical application of the FSWCNT. A dimensionless figure of merit of $ZT>3$ is practically feasible for thermoelectric devices.

Entropy and energy dissipated in FSWCNT

The carriers traveling in the FSWCNT are mechanically worked by an electric field in the presence of a current. The scalar product, $\vec{J}\cdot\vec{E}$, clearly equals the work done per unit time and volume. The heat generated by this work is dispersed through the exterior connections. As a result, in the external homogeneous conductor mentioned, the quantity of heat evolved per unit time and volume is:

$$\vec{J}\cdot\vec{E} = \sigma E^2 = J^2/\sigma \tag{230}$$

This is Joule's law and the entropy of the FSWCNT increases due to the evolution of heat. The entropy of the volume element dV grows by dQ/T when a certain amount of heat $dQ = \vec{J}\cdot\vec{E}dV$ is evolved. As a result, the rate of change of the FSWCNT's total entropy is given as:

$$\frac{ds}{dt} = \int(\vec{J}\cdot\vec{E})dV \tag{231}$$

Since the entropy increases, its derivative must be positive. Putting $\vec{J} = \sigma\vec{E}$, shows that the conductivity is also positive. The symmetry of the kinetic coefficients gives a relation between the coefficient β and the coefficient α in

$$\vec{J} = \sigma(\vec{E} - \alpha\nabla T) \tag{232}$$

To derive this, the rate of change of the total entropy of the FSWCNT is calculated. The amount of heat evolved per unit time and volume is $-\nabla\cdot\vec{q}$. Hence:

$$\frac{ds}{dt} = -\int\frac{\nabla\cdot\vec{q}}{T}dV \tag{233}$$

Using the equation $div\vec{J} = 0$, yields:

$$\frac{\nabla\cdot\vec{q}}{T} = \frac{1}{T}\{div(\vec{q} - \phi\vec{J}) + div\phi\vec{J}\} = \frac{1}{T}div(\vec{q} - \phi\vec{J}) - \frac{\vec{J}\cdot\vec{E}}{T} \tag{234}$$

The first term is integrated by parts giving;

$$\frac{ds}{dt} = \int \frac{\vec{J}\cdot\vec{E}}{T} dV - \int \frac{(\vec{q}-\phi\vec{J})\cdot\nabla T}{T^2} dV \tag{235}$$

Taking $\vec{J}$ and $\vec{q}$ into account, this quantity contains an amount $\phi\vec{J}$, which arises from the fact that each charged particle carries an energy $e\phi$. The difference $\vec{q}-\phi\vec{J}$, on the other hand, is independent of the potential and may be represented as a linear function of the gradients $\nabla\phi$ and ∇T, analogous to the current density's Eq. (232). This can presently be written as:

$$\vec{J} = \sigma T\frac{\vec{E}}{T} - \sigma\alpha\nabla T^2\frac{\nabla T}{T^2} \tag{236}$$

$$\vec{q} - \phi\vec{J} = \beta T\frac{\vec{E}}{T} - \gamma T^2\frac{\nabla T}{T^2} \tag{237}$$

the coefficients $\sigma\alpha T^2$ and βT must be equal. Thus, $\beta = \sigma\alpha T$ so that $\vec{q}-\phi\vec{J} = \sigma\alpha T\vec{E} - \gamma\nabla T$. Finally, expressing $\vec{E}$ in terms of $\vec{J}$ and by Eq. (233), results in

$$\vec{q} = (\phi + \alpha T)\vec{J} - \kappa\nabla T \tag{238}$$

where $\kappa = \gamma - T\alpha^2\sigma$ is the ordinary thermal conductivity, which gives the heat flux when no electric current is present. it is worth noting that the requirement that ds/dt be positive imposes no new constraints on the thermoelectric coefficients. Substituting Eq. (231) and Eq. (238) in Eq. (235) yields;

$$\frac{ds}{dt} = \int \left(\frac{J^2}{\sigma T} + \frac{\kappa(\nabla T)^2}{T^2}\right) dV > 0 \tag{239}$$

whence it is observed that only the coefficients of thermal and electrical conductivity must be positive. The rate of entropy change in the axial direction is given as:

$$\frac{ds_{zz}}{dt} = \int \left(\frac{J_z^2}{\sigma_{zz} T} + \frac{\kappa_{zz} (\nabla_z T)^2}{T^2} \right) dV > 0 \tag{240}$$

Similarly, along the circumferential direction;

$$\frac{ds_{cz}}{dt} = \int \left(\frac{J_c^2}{\sigma_{cz} T} + \frac{\kappa_{cz} (\nabla_s T)^2}{T^2} \right) dV > 0 \tag{241}$$

Consider the quantity of heat $-div\vec{q}$ evolved per unit time and volume in the FSWCNT. Taking the divergence of Eq. (235) yields;

$$Q = -div\vec{q} = \nabla(\kappa \nabla T) + \vec{J} \cdot \vec{E} + \vec{J} \cdot \nabla(\alpha T) \tag{242}$$

or substituting Eq. (231),

$$Q = \nabla(\kappa \nabla T) + \frac{J^2}{\sigma} - T\vec{J} \cdot \nabla \alpha \tag{243}$$

The first term on RHS is the thermal conduction, whilst the second is Joule's heat and the third which describes the thermoelectric effects, is of particular interest. As a result, the work done on the axial carriers by the external field is given as:

$$Q_{zz} = \nabla_z (\kappa_{zz} \nabla_z T) + \frac{J_z^2}{\sigma_{zz}} - T\vec{J}_{zz} \cdot \nabla_z \alpha_{zz} \tag{244}$$

$$Q_{zz} = \nabla_z \left(-\frac{k_B^2 T}{e^2} \left\{ \sigma_z(\vec{E}) \left[\left(\frac{\varepsilon_o - \mu}{k_B T} \right)^2 \sum_{n=-\infty}^{\infty} J_n^2(\chi) \right. \right. \right.$$

$$
\begin{aligned}
&- \frac{\Delta_z^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)\\
&- 2\Delta_s^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\sum_{n=-\infty}^{\infty}J_n^2(\chi)\\
&+ \frac{\Delta_z^{*2}}{2}\left(1-\frac{3I_o(\Delta_z^*)}{\Delta_z^*I_1(\Delta_z^*)}+\frac{6}{\Delta_z^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)\\
&+ \frac{\Delta_s^*\Delta_z^*}{2}\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)\\
&+ \Delta_s^{*2}\left(1-\frac{I_1(\Delta_s^*)}{\Delta_s^*I_o(\Delta_s^*)}\right)\sum_{n=-\infty}^{\infty}J_n^2(\chi)\Bigg]\\
&+ \sigma_s(\vec{E})\sin^2\theta_h\Bigg[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty}^{\infty}J_n^2(\chi)\\
&- \frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)\\
&- 2\Delta_z^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty}^{\infty}J_n^2(\chi)\\
&+ \frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^*I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)\\
&+ \frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)\\
&+ \Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^*I_o(\Delta_z^*)}\right)\sum_{n=-\infty}^{\infty}J_n^2(\chi)\Bigg]\Bigg\}\nabla T\Bigg)+\frac{j_z^2}{\sigma_{zz}}\\
&+T\vec{J}_{zz}\cdot\nabla_z\left(\frac{\sigma_z(\vec{E})}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_z^*\frac{I_o(\Delta_z^*)}{I_1(\Delta_z)}+2-\Delta_s^*\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right]\right.\\
&\left.+\frac{\sigma_s(\vec{E})\sin^2\theta_h}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z)}{I_o(\Delta_z^*)}\right]\right)
\end{aligned}
\tag{245}
$$

Similarly, the workdone in the circumferential direction is also given as:

$$
Q_{cz}=\nabla_s(\kappa_{cz}\nabla_sT)+\frac{J_c^2}{\sigma_{cz}}-T\vec{J}_{cz}\cdot\nabla_s\alpha_{cz} \tag{246}
$$

$$Q_{cz} = \nabla_s \left(-\sigma_s(\vec{E}) \frac{k_B^2 T}{e^2} \sin\theta_h \cos\theta_h \left\{ \left(\frac{\varepsilon_o - \mu}{k_B T} \right)^2 \sum_{n=-\infty}^{\infty} J_n^2(\chi) \right.\right.$$
$$- \frac{\Delta_s^*}{2} \left(\frac{\varepsilon_o - \mu}{k_B T} \right) \left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*} \right) \left(1 + 3 \sum_{n=-\infty}^{\infty} J_n^2(\chi) \right)$$
$$- 2\Delta_z^* \left(\frac{\varepsilon_o - \mu}{k_B T} \right) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \sum_{n=-\infty}^{\infty} J_n^2(\chi)$$
$$+ \frac{\Delta_s^{*2}}{2} \left(1 - \frac{3 I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)} + \frac{6}{\Delta_s^{*2}} \right) \left(1 + \sum_{n=-\infty}^{\infty} J_n^2(\chi) \right)$$
$$+ \frac{\Delta_z^* \Delta_s^*}{2} \left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*} \right) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \left(1 + 3 \sum_{n=-\infty}^{\infty} J_n^2(\chi) \right)$$
$$\left.\left. + \Delta_z^{*2} \left(1 - \frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)} \right) \sum_{n=-\infty}^{\infty} J_n^2(\chi) \right\} \nabla T \right) + \frac{\vec{J}_c^2}{\sigma_{cz}}$$
$$- T \vec{J}_{cz} \cdot \nabla_s \left(-\frac{k_B}{e} \left[\left(\frac{\varepsilon_o - \mu}{k_B T} \right) - \Delta_s^* \frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} + 2 - \Delta_z^* \frac{I_1(\Delta_z)}{I_o(\Delta_z^*)} \right] \right) \quad (247)$$

Peltier coefficient for FSWCNT

The analytical deductions to establish whether the FSWCNT obeys the Onsager relations is beyond the scope of this study. However, the Onsager relation is invoked to obtain the Peltier coeficient for FSWCNT for possible refridgeration applications. The carrier current densities for the FSWCNT along the z and s directions are respectively, quoted as:

$$\vec{J}_c = -\sigma_s(\vec{E}) \sin\theta_h \cos\theta_h \vec{E}_{zn}^*$$
$$- \sigma_s(\vec{E}) \frac{k_B}{e} \sin\theta_h \cos\theta_h \left[\left(\frac{\varepsilon_o - \mu}{k_B T} \right) - \Delta_s^* \frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} + 2 - \Delta_z^* \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \right] \nabla_z T \quad (248)$$

$$\vec{J}_z = -(\sigma_z(\vec{E}) + \sigma_s(\vec{E}) \sin^2\theta_h) \vec{E}_{zn}^*$$
$$- \sigma_z(\vec{E}) \frac{k_B}{e} \left[\left(\frac{\varepsilon_o - \mu}{k_B T} \right) - \Delta_z^* \frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)} + 2 - \Delta_s^* \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} \right] \nabla_z T$$

$$-\sigma_s(\vec{E})\frac{k_B}{e}\sin^2\theta_h\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right]\nabla_zT \quad (249)$$

In the compact form, Eq. (248) and Eq. (249) takes the form;

$$\vec{J}_c=\sigma_{cz}\vec{E}_{zn}^*-\sigma_{cz}\alpha_{cz}\nabla_zT \quad (250)$$

$$\vec{J}_z=\sigma_{zz}\vec{E}_{zn}^*-\sigma_{zz}\alpha_{zz}\nabla_zT \quad (251)$$

When comparing theory with experiment, representation of $\vec{q}$ in terms of $\vec{E}_{zn}^*$ is inconvenient; instead, $\vec{q}$ must be expressed in terms of $\vec{J}$ and ∇_zT. Hence, making the electric field the subject $\vec{E}_{zn}^*$ from the circumferential component in Eq. (248) yields;

$$\vec{E}_{zn}^*=-\frac{\vec{J}_c}{\sigma_s(\vec{E})\sin\theta_h\cos\theta_h}-\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right]\nabla_zT \quad (252)$$

Substituting $\vec{E}_{zn}^*$ from Eq. (252) into Eq. (196) yields;

$$\vec{q}_c=\sigma_s(\vec{E})\frac{k_BT}{e}\sin\theta_h\cos\theta_h\left\{\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\sum_{n=-\infty}J_n^2(\chi)-\frac{\Delta_s^*}{2}\left(1+3\sum_{n=-\infty}J_n^2(\chi)\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)-\Delta_z^*\sum_{n=-\infty}J_n^2(\chi)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right\}\frac{J_c}{\sigma_s(\vec{E})\sin\theta_h\cos\theta_h}$$
$$+\sigma_s(\vec{E})\frac{k_BT}{e}\sin\theta_h\cos\theta_h\left\{\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\sum_{n=-\infty}J_n^2(\chi)-\frac{\Delta_s^*}{2}\left(1+3\sum_{n=-\infty}J_n^2(\chi)\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\right.$$

$$
- \Delta_z^* \sum_{n=-\infty} J_n^2(\chi) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \Bigg\} \frac{k_B}{e} \left[\left(\frac{\varepsilon_o - \mu}{k_B T} \right) - \Delta_s^* \frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} + 2 - \Delta_z^* \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \right] \nabla_z T
$$

$$
- \sigma_s(\vec{E}) \frac{k_B^2 T}{e^2} \sin\theta_h \cos\theta_h \Bigg\{ \left(\frac{\varepsilon_o - \mu}{k_B T} \right)^2 \sum_{n=-\infty} J_n^2(\chi)
- \frac{\Delta_s^*}{2} \left(\frac{\varepsilon_o - \mu}{k_B T} \right) \left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*} \right) \left(1 + 3 \sum_{n=-\infty} J_n^2(\chi) \right)
- 2\Delta_z^* \left(\frac{\varepsilon_o - \mu}{k_B T} \right) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \sum_{n=-\infty} J_n^2(\chi)
+ \frac{\Delta_s^{*2}}{2} \left(1 - \frac{3 I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)} + \frac{6}{\Delta_s^{*2}} \right) \left(1 + \sum_{n=-\infty} J_n^2(\chi) \right)
+ \frac{\Delta_z^* \Delta_s^*}{2} \left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*} \right) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \left(1 + 3 \sum_{n=-\infty} J_n^2(\chi) \right)
+ \Delta_z^{*2} \left(1 - \frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)} \right) \sum_{n=-\infty} J_n^2(\chi) \Bigg\} \nabla_z T \quad (253)
$$

Simplifying yields:

$$
\vec{q}_c = \frac{k_B T}{e} \Bigg\{ \left(\frac{\varepsilon_o - \mu}{k_B T} \right) \sum_{n=-\infty} J_n^2(\chi)
- \frac{\Delta_s^*}{2} \left(1 + 3 \sum_{n=-\infty} J_n^2(\chi) \right) \left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*} \right)
- \Delta_z^* \sum_{n=-\infty} J_n^2(\chi) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \Bigg\} \vec{J}_c
$$

$$
- \sigma_s(\vec{E}) \frac{k_B^2 T}{e^2} \sin\theta_h \cos\theta_h \Bigg\{ \Bigg\{ \left(\frac{\varepsilon_o - \mu}{k_B T} \right)^2 \sum_{n=-\infty} J_n^2(\chi)
- \frac{\Delta_s^*}{2} \left(\frac{\varepsilon_o - \mu}{k_B T} \right) \left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*} \right) \left(1 + 3 \sum_{n=-\infty} J_n^2(\chi) \right)
- 2\Delta_z^* \left(\frac{\varepsilon_o - \mu}{k_B T} \right) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \sum_{n=-\infty} J_n^2(\chi)
+ \frac{\Delta_s^{*2}}{2} \left(1 - \frac{3 I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)} + \frac{6}{\Delta_s^{*2}} \right) \left(1 + \sum_{n=-\infty} J_n^2(\chi) \right)
+ \frac{\Delta_z^* \Delta_s^*}{2} \left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*} \right) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \left(1 + 3 \sum_{n=-\infty} J_n^2(\chi) \right)
+ \Delta_z^{*2} \left(1 - \frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)} \right) \sum_{n=-\infty} J_n^2(\chi) \Bigg\}
$$

$$
\begin{aligned}
&-\left\{\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\sum_{n=-\infty} J_n^2(\chi)\right.\\
&-\frac{\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)\\
&\left.-\Delta_z^*\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right\}\\
&\times\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right]\Bigg\}\nabla_z T \quad (254)
\end{aligned}
$$

From Eq. (249) $\vec{E}_{zn}^*$ is given as;

$$
\begin{aligned}
\vec{E}_{zn}^* &= -\frac{\vec{J}_z}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\\
&-\frac{\sigma_z(\vec{E})}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_z^*\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}+2-\Delta_s^*\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right]\nabla_z T\\
&-\frac{\sigma_s(\vec{E})}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\sin^2\theta_h\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right]\nabla_z T
\end{aligned}
\quad (255)
$$

Substituting $\vec{E}_{zn}^*$ from Eq. (249) into Eq. (195)

$$
\begin{aligned}
\vec{q}_z = -\frac{k_BT}{e}&\left\{\sigma_z(\vec{E})\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\sum_{n=-\infty} J_n^2(\chi)\right.\right.\\
&-\frac{\Delta_z^*}{2}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)-\Delta_s^*\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\Bigg]\\
&+\sigma_s(\vec{E})\sin^2\theta_h\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\sum_{n=-\infty} J_n^2(\chi)\right.\\
&-\frac{\Delta_s^*}{2}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\\
&\left.\left.-\Delta_z^*\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right]\right\}\frac{\vec{J}_z}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\\
&+\frac{k_B^2T}{e^2}\left\{\sigma_z(\vec{E})\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\sum_{n=-\infty} J_n^2(\chi)\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\Delta_z^*}{2}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)-\Delta_s^*\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\Bigg] \\
&+ \sigma_s(\vec{E})\sin^2\theta_h\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\sum_{n=-\infty} J_n^2(\chi)\right. \\
&- \frac{\Delta_s^*}{2}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)-\Delta_z^*\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\Bigg]\Bigg\} \\
&\times\Bigg\{\frac{\sigma_z(\vec{E})}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_z^*\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}+2-\Delta_s^*\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right] \\
&+\frac{\sigma_s(\vec{E})}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\sin^2\theta_h\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right]\Bigg\}\nabla_z T \\
&- \frac{k_B^2T}{e^2}\Bigg\{\sigma_z(\vec{E})\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty} J_n^2(\chi)\right. \\
&- \frac{\Delta_z^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right) \\
&- 2\Delta_s^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\sum_{n=-\infty} J_n^2(\chi) \\
&+ \frac{\Delta_z^{*2}}{2}\left(1-\frac{3I_o(\Delta_z^*)}{\Delta_z^* I_1(\Delta_z^*)}+\frac{6}{\Delta_z^{*2}}\right)\left(1+\sum_{n=-\infty} J_n^2(\chi)\right) \\
&+ \frac{\Delta_s^*\Delta_z^*}{2}\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right) \\
&+ \Delta_s^{*2}\left(1-\frac{I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)}\right)\sum_{n=-\infty} J_n^2(\chi)\Bigg] \\
&+ \sigma_s(\vec{E})\sin^2\theta_h\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty} J_n^2(\chi)\right. \\
&- \frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right) \\
&- 2\Delta_z^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty} J_n^2(\chi) \\
&+ \frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty} J_n^2(\chi)\right) \\
&+ \frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right) \\
&+ \Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right)\sum_{n=-\infty} J_n^2(\chi)\Bigg]\Bigg\}\nabla_z T \qquad (256)
\end{aligned}
$$

Simplifying further yields:

$$
\begin{aligned}
\vec{q}_z = \frac{k_B T}{e} \Bigg\{ & \frac{\sigma_z(\vec{E})}{(\sigma_z(\vec{E}) + \sigma_s(\vec{E})\sin^2\theta_h)} \Bigg[\left(\frac{\varepsilon_o - \mu}{k_B T} \right) \sum_{n=-\infty} J_n^2(\chi) \\
& - \frac{\Delta_z^*}{2} \left(1 + 3 \sum_{n=-\infty} J_n^2(\chi) \right) \left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)} - \frac{2}{\Delta_z^*} \right) - \Delta_s^* \sum_{n=-\infty} J_n^2(\chi) \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} \Bigg] \\
& + \frac{\sigma_s(\vec{E})\sin^2\theta_h}{(\sigma_z(\vec{E}) + \sigma_s(\vec{E})\sin^2\theta_h)} \Bigg[\left(\frac{\varepsilon_o - \mu}{k_B T} \right) \sum_{n=-\infty} J_n^2(\chi) \\
& - \frac{\Delta_s^*}{2} \left(1 + 3 \sum_{n=-\infty} J_n^2(\chi) \right) \left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*} \right) \\
& - \Delta_z^* \sum_{n=-\infty} J_n^2(\chi) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \Bigg] \Bigg\} \vec{J}_z \\
& - \frac{k_B^2 T}{e^2} \Bigg\{ \sigma_z(\vec{E}) \Bigg[\left(\frac{\varepsilon_o - \mu}{k_B T} \right)^2 \sum_{n=-\infty} J_n^2(\chi) \\
& - \frac{\Delta_z^*}{2} \left(\frac{\varepsilon_o - \mu}{k_B T} \right) \left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)} - \frac{2}{\Delta_z^*} \right) \left(1 + 3 \sum_{n=-\infty} J_n^2(\chi) \right) \\
& - 2\Delta_s^* \left(\frac{\varepsilon_o - \mu}{k_B T} \right) \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} \sum_{n=-\infty} J_n^2(\chi) \\
& + \frac{\Delta_z^{*2}}{2} \left(1 - \frac{3 I_o(\Delta_z^*)}{\Delta_z^* I_1(\Delta_z^*)} + \frac{6}{\Delta_z^{*2}} \right) \left(1 + \sum_{n=-\infty} J_n^2(\chi) \right) \\
& + \frac{\Delta_s^* \Delta_z^*}{2} \left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)} - \frac{2}{\Delta_z^*} \right) \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} \left(1 + 3 \sum_{n=-\infty} J_n^2(\chi) \right) \\
& + \Delta_s^{*2} \left(1 - \frac{I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)} \right) \sum_{n=-\infty} J_n^2(\chi) \Bigg] \\
& + \sigma_s(\vec{E})\sin^2\theta_h \Bigg[\left(\frac{\varepsilon_o - \mu}{k_B T} \right)^2 \sum_{n=-\infty} J_n^2(\chi) \\
& - \frac{\Delta_s^*}{2} \left(\frac{\varepsilon_o - \mu}{k_B T} \right) \left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*} \right) \left(1 + 3 \sum_{n=-\infty} J_n^2(\chi) \right) \\
& - 2\Delta_z^* \left(\frac{\varepsilon_o - \mu}{k_B T} \right) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \sum_{n=-\infty} J_n^2(\chi) \\
& + \frac{\Delta_s^{*2}}{2} \left(1 - \frac{3 I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)} + \frac{6}{\Delta_s^{*2}} \right) \left(1 + \sum_{n=-\infty} J_n^2(\chi) \right) \\
& + \frac{\Delta_z^* \Delta_s^*}{2} \left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*} \right) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \left(1 + 3 \sum_{n=-\infty} J_n^2(\chi) \right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \Delta_z^{*2}\left(1 - \frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right) \sum_{n=-\infty} J_n^2(\chi)\Bigg]\Bigg\}\nabla_z T \\
&+ \frac{k_B^2 T}{e^2}\Bigg\{\sigma_z(\vec{E})\Bigg[\left(\frac{\varepsilon_o-\mu}{k_B T}\right)\sum_{n=-\infty} J_n^2(\chi) \\
&- \frac{\Delta_z^*}{2}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)} - \frac{2}{\Delta_z^*}\right) - \Delta_s^*\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\Bigg] \\
&+ \sigma_s(\vec{E})\sin^2\theta_h\Bigg[\left(\frac{\varepsilon_o-\mu}{k_B T}\right)\sum_{n=-\infty} J_n^2(\chi) \\
&- \frac{\Delta_s^*}{2}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*}\right) \\
&- \Delta_z^*\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\Bigg]\Bigg\} \\
&\times \Bigg\{\frac{\sigma_z(\vec{E})}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\left[\left(\frac{\varepsilon_o-\mu}{k_B T}\right) - \Delta_z^*\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)} + 2 - \Delta_s^*\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right] \\
&+ \frac{\sigma_s(\vec{E})\sin^2\theta_h}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\left[\left(\frac{\varepsilon_o-\mu}{k_B T}\right) - \Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} + 2 - \Delta_z^*\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right]\Bigg\}\nabla_z T
\end{aligned} \tag{257}
$$

$$
\begin{aligned}
\vec{q}_z = \frac{k_B T}{e}&\Bigg\{\frac{\sigma_z(\vec{E})}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\Bigg[\left(\frac{\varepsilon_o-\mu}{k_B T}\right)\sum_{n=-\infty} J_n^2(\chi) \\
&- \frac{\Delta_z^*}{2}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)} - \frac{2}{\Delta_z^*}\right) - \Delta_s^*\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\Bigg] \\
&+ \frac{\sigma_s(\vec{E})\sin^2\theta_h}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\Bigg[\left(\frac{\varepsilon_o-\mu}{k_B T}\right)\sum_{n=-\infty} J_n^2(\chi) \\
&- \frac{\Delta_s^*}{2}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*}\right) \\
&- \Delta_z^*\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\Bigg]\Bigg\}\vec{J}_z \\
&- \frac{k_B^2 T}{e^2}\Bigg\{\sigma_z(\vec{E})\Bigg[\left(\frac{\varepsilon_o-\mu}{k_B T}\right)^2\sum_{n=-\infty} J_n^2(\chi) \\
&- \frac{\Delta_z^*}{2}\left(\frac{\varepsilon_o-\mu}{k_B T}\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)} - \frac{2}{\Delta_z^*}\right)\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)
\end{aligned}
$$

$$
\begin{aligned}
&- 2\Delta_s^* \left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\sum_{n=-\infty} J_n^2(\chi) \\
&+ \frac{\Delta_z^{*2}}{2}\left(1-\frac{3I_o(\Delta_z^*)}{\Delta_z^* I_1(\Delta_z^*)}+\frac{6}{\Delta_z^{*2}}\right)\left(1+\sum_{n=-\infty} J_n^2(\chi)\right) \\
&+ \frac{\Delta_s^*\Delta_z^*}{2}\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right) \\
&+ \Delta_s^{*2}\left(1-\frac{I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)}\right)\sum_{n=-\infty} J_n^2(\chi)\Bigg] \\
&+ \sigma_s(\vec{E})\sin^2\theta_h\Bigg[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty} J_n^2(\chi) \\
&- \frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right) \\
&- 2\Delta_z^* \left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty} J_n^2(\chi) \\
&+ \frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty} J_n^2(\chi)\right) \\
&+ \frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right) \\
&+ \Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right)\sum_{n=-\infty} J_n^2(\chi)\Bigg] \\
&+ (\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)\Bigg\{\frac{\sigma_z(\vec{E})}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\Bigg[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\sum_{n=-\infty} J_n^2(\chi) \\
&- \frac{\Delta_z^*}{2}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right) \\
&- \Delta_s^*\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\Bigg]\Bigg\} \\
&+ \frac{\sigma_s(\vec{E})\sin^2\theta_h}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\Bigg[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\sum_{n=-\infty} J_n^2(\chi) \\
&- \frac{\Delta_s^*}{2}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right) \\
&- \Delta_z^*\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\Bigg] \\
&\times\Bigg\{\frac{\sigma_z(\vec{E})}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\Bigg[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_z^*\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}+2-\Delta_s^*\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\Bigg]
\end{aligned}
$$

$$+ \frac{\sigma_s(\vec{E})\sin^2\theta_h}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right]\Bigg\}\Bigg\}\nabla_z T \tag{258}$$

Eq. (254) and Eq. (258) are in the form of the Onsager relations quoted as:

$$\vec{q}_c = \Pi_{cz}\vec{J}_c - \kappa_{cz}\nabla_z T \tag{259}$$

$$\vec{q}_z = \Pi_{zz}\vec{J}_z - \kappa_{zz}\nabla_z T \tag{260}$$

where κ is the carrier thermal conductivity when the carrier current density $\vec{J}$ is zero and Π is the Peltier coefficient is given as $\Pi = \alpha T$. As usual α is the thermopower. Comparing the equations, the circumferential component of the Peltier coefficient as:

$$\Pi_{cz} = \frac{k_B}{e}\Bigg\{\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\sum_{n=-\infty} J_n^2(\chi) - \frac{\Delta_s^*}{2}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right) - \Delta_z^*\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\Bigg\}T \tag{261}$$

and the axial component is obtained as;

$$\Pi_{zz} = \frac{k_B}{e}\Bigg\{\frac{\sigma_z(\vec{E})}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\Bigg[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\sum_{n=-\infty} J_n^2(\chi) - \frac{\Delta_z^*}{2}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right) - \Delta_s^*\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\Bigg] + \frac{\sigma_s(\vec{E})\sin^2\theta_h}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\Bigg[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\sum_{n=-\infty} J_n^2(\chi) - \frac{\Delta_s^*}{2}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right) - \Delta_z^*\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\Bigg]\Bigg\}T \tag{262}$$

Comparing Eq. (259) and Eq. (260), the thermal conductivity is obtained as:

$$
\begin{aligned}
\kappa_{cz} = \sigma_s(\vec{E})\frac{k_B^2 T}{e^2}\sin\theta_h\cos\theta_h \Bigg[\Bigg\{ & \left(\frac{\varepsilon_o-\mu}{k_B T}\right)^2 \sum_{n=-\infty} J_n^2(\chi) \\
& - \frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o-\mu}{k_B T}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right) \\
& - 2\Delta_z^*\left(\frac{\varepsilon_o-\mu}{k_B T}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty} J_n^2(\chi) \\
& + \frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty} J_n^2(\chi)\right) \\
& + \frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right) \\
& + \Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right)\sum_{n=-\infty} J_n^2(\chi)\Bigg\} \\
& - \Bigg\{\left(\frac{\varepsilon_o-\mu}{k_B T}\right)\sum_{n=-\infty} J_n^2(\chi) \\
& - \frac{\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right) \\
& - \Delta_z^*\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\Bigg\} \\
& \times \left[\left(\frac{\varepsilon_o-\mu}{k_B T}\right)-\Delta_s\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right]\Bigg] \qquad (263)
\end{aligned}
$$

$$
\begin{aligned}
\kappa_{zz} = \frac{k_B^2 T}{e^2}\Bigg\{\sigma_z(\vec{E})\Bigg[& \left(\frac{\varepsilon_o-\mu}{k_B T}\right)^2 \sum_{n=-\infty} J_n^2(\chi) \\
& - \frac{\Delta_z^*}{2}\left(\frac{\varepsilon_o-\mu}{k_B T}\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right) \\
& - 2\Delta_s^*\left(\frac{\varepsilon_o-\mu}{k_B T}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\sum_{n=-\infty} J_n^2(\chi) \\
& + \frac{\Delta_z^{*2}}{2}\left(1-\frac{3I_o(\Delta_z^*)}{\Delta_z^* I_1(\Delta_z^*)}+\frac{6}{\Delta_z^{*2}}\right)\left(1+\sum_{n=-\infty} J_n^2(\chi)\right) \\
& + \frac{\Delta_s^*\Delta_z^*}{2}\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \Delta_s^{*2}\left(1-\frac{I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)}\right)\sum_{n=-\infty} J_n^2(\chi)\Bigg] \\
&+ \sigma_s(\vec{E})\sin^2\theta_h\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty} J_n^2(\chi)\right. \\
&- \frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right) \\
&- 2\Delta_z^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty} J_n^2(\chi) \\
&+ \frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty} J_n^2(\chi)\right) \\
&+ \frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right) \\
&\left.+ \Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right)\sum_{n=-\infty} J_n^2(\chi)\right] \\
&+ (\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)\left\{\frac{\sigma_z(\vec{E})}{\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\sum_{n=-\infty} J_n^2(\chi)\right.\right. \\
&\left.- \frac{\Delta_z^*}{2}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)-\Delta_s^*\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right] \\
&+ \frac{\sigma_s(\vec{E})\sin^2\theta_h}{\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\sum_{n=-\infty} J_n^2(\chi)\right] \\
&\left.- \frac{\Delta_s^*}{2}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)-\Delta_z^*\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right\} \\
&\times\left\{\frac{\sigma_z(\vec{E})}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_z^*\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}+2-\Delta_s^*\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right]\right. \\
&\left.\left.+ \frac{\sigma_s(\vec{E})\sin^2\theta_h}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right]\right\}\right\}\nabla_z T
\end{aligned}
\tag{264}
$$

Lorentz Number Calculation

Transport of heat and charge is commonly estimated using the measured σ using the Wiedemann-Franz law $\kappa = T\sigma L$, where L is the Lorentz number.

The Lorentz number along the circumferential direction L_{cz} is given as:

$$
\begin{aligned}
L_{cz} = \frac{k_B^2 T}{e^2} \Bigg[\Bigg\{ & \left(\frac{\varepsilon_o - \mu}{k_B T}\right)^2 \sum_{n=-\infty} J_n^2(\chi) \\
& - \frac{\Delta_s^*}{2} \left(\frac{\varepsilon_o - \mu}{k_B T}\right) \left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*}\right) \left(1 + 3 \sum_{n=-\infty} J_n^2(\chi)\right) \\
& - 2\Delta_z^* \left(\frac{\varepsilon_o - \mu}{k_B T}\right) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \sum_{n=-\infty} J_n^2(\chi) \\
& + \frac{\Delta_s^{*2}}{2} \left(1 - \frac{3 I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)} + \frac{6}{\Delta_s^{*2}}\right) \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \\
& + \frac{\Delta_z^* \Delta_s^*}{2} \left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*}\right) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \left(1 + 3 \sum_{n=-\infty} J_n^2(\chi)\right) \\
& + \Delta_z^{*2} \left(1 - \frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right) \sum_{n=-\infty} J_n^2(\chi) \Bigg\} \\
- \Bigg\{ & \left(\frac{\varepsilon_o - \mu}{k_B T}\right) \sum_{n=-\infty} J_n^2(\chi) - \frac{\Delta_s^*}{2} \left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*}\right) \left(1 + 3 \sum_{n=-\infty} J_n^2(\chi)\right) \\
& - \Delta_z^* \sum_{n=-\infty} J_n^2(\chi) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \Bigg\} \\
& \times \left[\left(\frac{\varepsilon_o - \mu}{k_B T}\right) - \Delta_s \frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} + 2 - \Delta_z^* \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \right] \Bigg]
\end{aligned}
\tag{265}
$$

and the axial Lorentz number is given as:

$$
\begin{aligned}
L_{zz} = \frac{k_B^2 T}{e^2(\sigma_z(\vec{E}) + \sigma_s(\vec{E}) \sin^2 \theta_h)} \Bigg\{ \sigma_z(\vec{E}) \Bigg[& \left(\frac{\varepsilon_o - \mu}{k_B T}\right)^2 \sum_{n=-\infty} J_n^2(\chi) \\
& - \frac{\Delta_z^*}{2} \left(\frac{\varepsilon_o - \mu}{k_B T}\right) \left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)} - \frac{2}{\Delta_z^*}\right) \left(1 + 3 \sum_{n=-\infty} J_n^2(\chi)\right) \\
& - 2\Delta_s^* \left(\frac{\varepsilon_o - \mu}{k_B T}\right) \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} \sum_{n=-\infty} J_n^2(\chi) \\
& + \frac{\Delta_z^{*2}}{2} \left(1 - \frac{3 I_o(\Delta_z^*)}{\Delta_z^* I_1(\Delta_z^*)} + \frac{6}{\Delta_z^{*2}}\right) \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \\
& + \frac{\Delta_s^* \Delta_z^*}{2} \left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)} - \frac{2}{\Delta_z^*}\right) \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} \left(1 + 3 \sum_{n=-\infty} J_n^2(\chi)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \Delta_s^{*2}\left(1-\frac{I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)}\right)\sum_{n=-\infty} J_n^2(\chi)\Bigg] \\
&+ \sigma_s(\vec{E})\sin^2\theta_h\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty} J_n^2(\chi)\right. \\
&- \frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right) \\
&- 2\Delta_z^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty} J_n^2(\chi) \\
&+ \frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty} J_n^2(\chi)\right) \\
&+ \frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right) \\
&\left.+ \Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right)\sum_{n=-\infty} J_n^2(\chi)\right] \\
&+ (\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)\left\{\frac{\sigma_z(\vec{E})}{\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\sum_{n=-\infty} J_n^2(\chi)\right.\right. \\
&\left.- \frac{\Delta_z^*}{2}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)-\Delta_s^*\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right] \\
&+ \frac{\sigma_s(\vec{E})\sin^2\theta_h}{\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\sum_{n=-\infty} J_n^2(\chi)\right] \\
&\left.- \frac{\Delta_s^*}{2}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)-\Delta_z^*\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right\} \\
&\times\left\{\frac{\sigma_z(\vec{E})}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_z^*\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}+2-\Delta_s^*\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right]\right. \\
&\left.\left.+ \frac{\sigma_s(\vec{E})\sin^2\theta_h}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right]\right\}\right\}\nabla_z T
\end{aligned}
\tag{266}
$$

Chapter Summary

Theories and models were developed and used to study the acoustic metrics in FSWCNT in the hypersound regime, $q\ell \gg 1$. The chapter also elaborated on the calculation of the thermoelectric metrics was used to estimate the ZT of the FSWCNT. In addition to the metrics deduced, the entropy and the energy carried by the elecrons and the Onsager relation for the FSWCNT was also deduced.

CHAPTER FOUR

RESULTS AND DISCUSSION

Introduction

In Chapter Three, the tractable analytical approach developed was used to solve and predict the acoustoelectric and thermoelectric metrics in FSWCNT analytically and computationally in this chapter (i.e., Chapter Four). Metrics such as the acoustoelectric current $\vec{J}_z^{AE}$, Hall-like current $\vec{J}_y^{AME}$, acoustoelectric gain Γ_q, carrier conductivity σ, thermopower α, power factor $\mathscr{P}$, carrier thermal current density κ_e and the lattice conductivity κ_ℓ, were obtained. The calculations were based on solving the BTE for quasi-one-dimensional carrier miniband transport. With optimized values for Δ_s, Δ_z, θ_h, n_o and τ, the dependence of the metrics on T, q, ω_q, $\vec{E}_o$, and $\vec{E}_{s,z}$, were studied numerically. This chapter highlighted some of these results.

Acoustoelectric effect in FSWCNT

The wavelength of the acoustic wave used in this work, $\lambda = 2\pi/q$, was shorter than the mean-free path of FSWCNT carriers in the hypersound regime, $q\ell \gg 1$. In Eq. (87) and Eq. (90), the expressions for current densities in FSWCNT ($\vec{j}_z^{FSWCNT}$) and SWCNT ($\vec{j}_z^{SWCNT}$) were presented, respectively. The obtained acoustoelectric current was found to be substantially dependent on the acoustic wavenumber (q), frequency (ω_q), and temperature (T) in both circumstances. When $\omega_q \gg 12\gamma_o aq$ and $\omega_q \gg 4\gamma_o \sin(a\hbar q/2)/\hbar$ for FSWCNT and SWCNT, respectively, a transparency window was observed, which was a result of the conservation laws of energy and momentum. Only carriers with momenta $\hbar q/2$ interacted with the acoustic phonons, as a result. There would be no attenuation of acoustic waves traveling through the FSWCNT if the acoustic flux had frequencies spanning from GHz to THz, and hence the acoustocurrent would be zero. The acoustocurrent ($\vec{j}$) dependency on q was strongly nonlinear between Eq. (100) and Eq. (103), as observed in Refs.[194, 195] for the condition:

$$G(p_z, \hbar q) = ib^*[\sin(p_z + \hbar q)d - \sin p_z d], \tag{267}$$

where b^* is the resonance integral's derivative with regard to the interatomic distance. Eq. (100) and Eq. (103) were analyzed numerically with the following parameters: $\omega_q = 10^{11}\text{s}^{-1}$, $v_s = 2.5 \times 10^3\text{m/s}$, $\Phi = 10^5\text{Wb/m}^2$ and $\ell = 10^{-4}cm$. By solving the BTE under τ-approximation, the acoustocurrent density was obtained under a quasi-static field $\omega\tau \ll 1$.

In the Ohmic conductivity regime, the current density had a linear dependence on q at $E = 0$. Figures 15 and 16 showed how when q grew, the current density reached a maximum and then dropped off, which resulted in negative conductivity. Figure 15 showed the acoustocurrent density for FSWCNT, which was about four orders of magnitude lower than Figure 16 showed for SWCNT. Owing to degenerate feature (i.e., metallic) of SWCNT, the carrier-phonon interactions were stronger than in FSWCNT. As a result, more intraminiband carriers engage with acoustic phonons in SWCNT than in FSWCNT, which resulted in a large acoustocurrent.

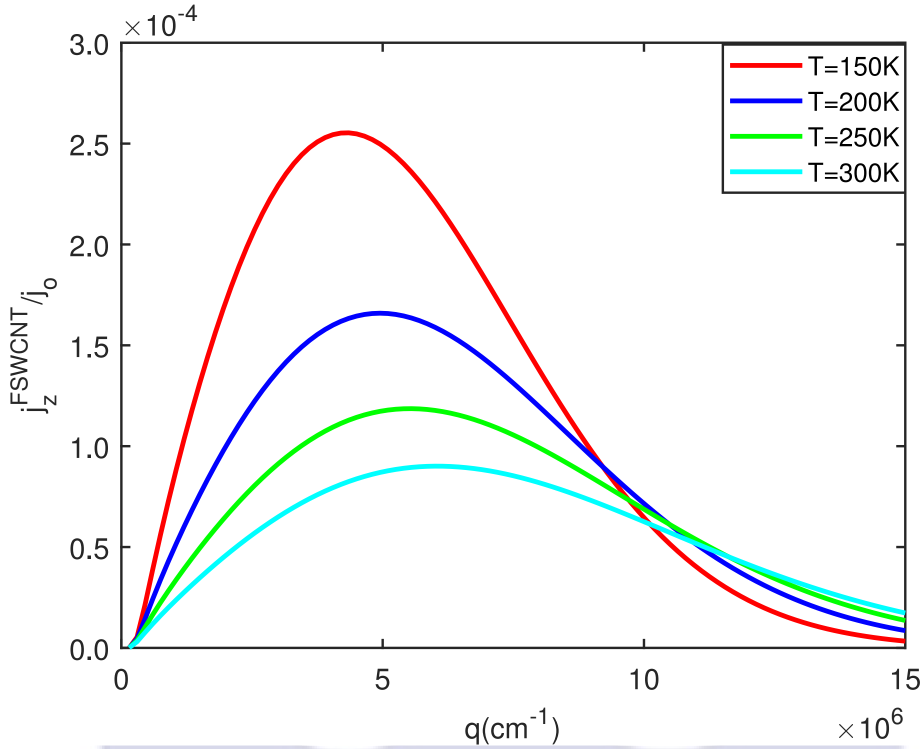

Figure 15: The Dependency of Axial Current Density j_z^{AE} on Wavenumber q for varied Temperature T: FSWCNT.

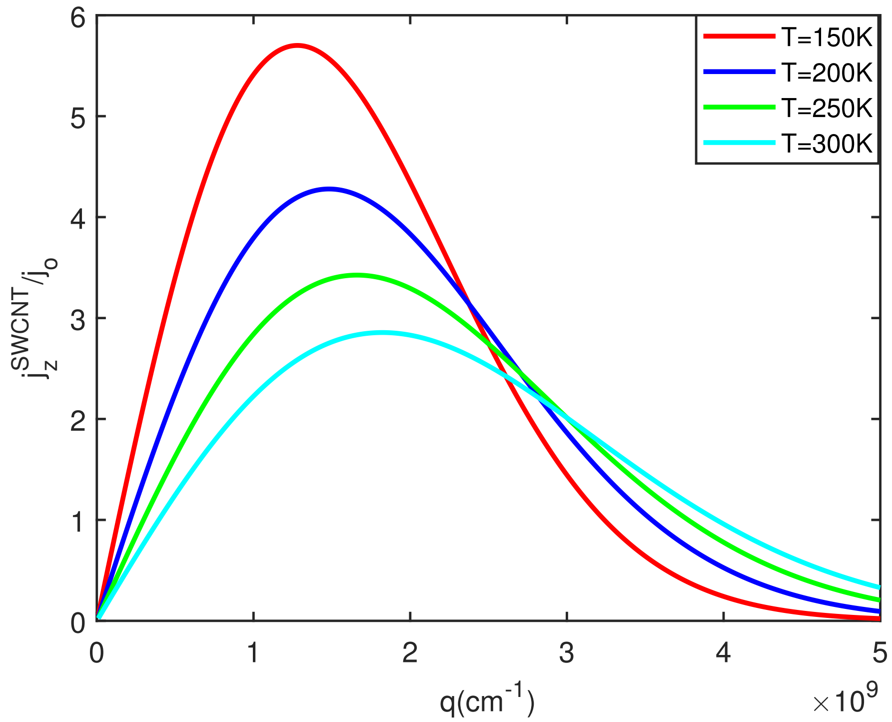

Figure 16: The Dependency of Axial Current Density j_z^{AE} on Wavenumber q for varied Temperatures T: SWCNT.

However, in SWCNT more phonons with high wavenumbers (i.e., $q \approx 10^9 cm^{-1}$) engage in energy-momentum transfer than in FSWCNT (i.e., $q \approx 10^6 cm^{-1}$), which could cause the FSWCNT to vapourize. The commencement of Bloch-like oscillations of the intraminiband carrier current, which was a result of Bragg's reflection of the carriers at the band boundaries, was related to the nonlinear rise and decline of carrier behaviour. The phonons exchanged their energies and momenta to the carriers that moved within the Brillouin zone because the carriers were under a lattice potential field. They acquired specific momenta and were reflected off the margins of the Brillouin zone when they reached the top of the BZ. Owing to considerable non-parabolicity of the FSWCNT's dispersion, increasing the temperature reduced the acoustocurrent and shifted the peak to a high q, as shown in Figures 15 and 16.

The dependency of the acoustoelectric current on temperature for various wavenumbers were shown in Figures 17 and 18. As the temperature increased, the acoustocurrent increased at first, peaks and then declined as the temperature increased, which resulted in a negative conductivity. For both FSWCNT and SWCNT, the peak current shifted towards high temperatures as q was increased. This implied that large current was obtained at low q values and low temperatures at room temperature for FSWCNT in Figure 17 and below $100K$ for SWCNT in Figure 18. This was an intriguing observation, as it showed that the FSWCNT could be used as an acoustoelectric device such as acoustic diodes, acoustic transistors and transducers at $300K$ against $100K$ for the SWCNT.

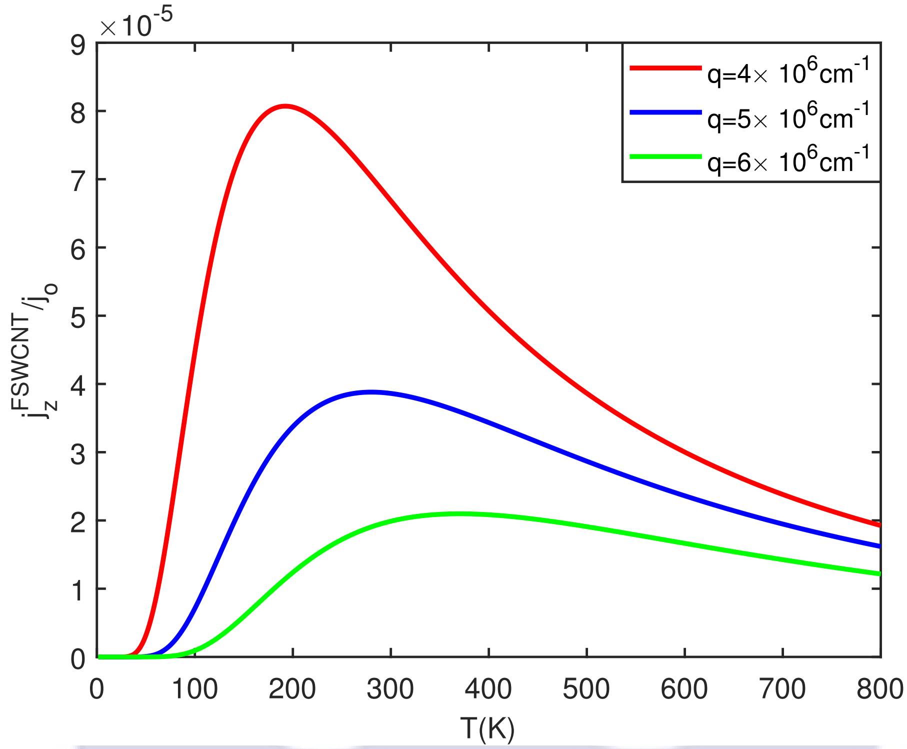

Figure 17: The Dependency of Current Density j_z^{AE} on Temperature T for varied Wavenumber q: FSWCNT.

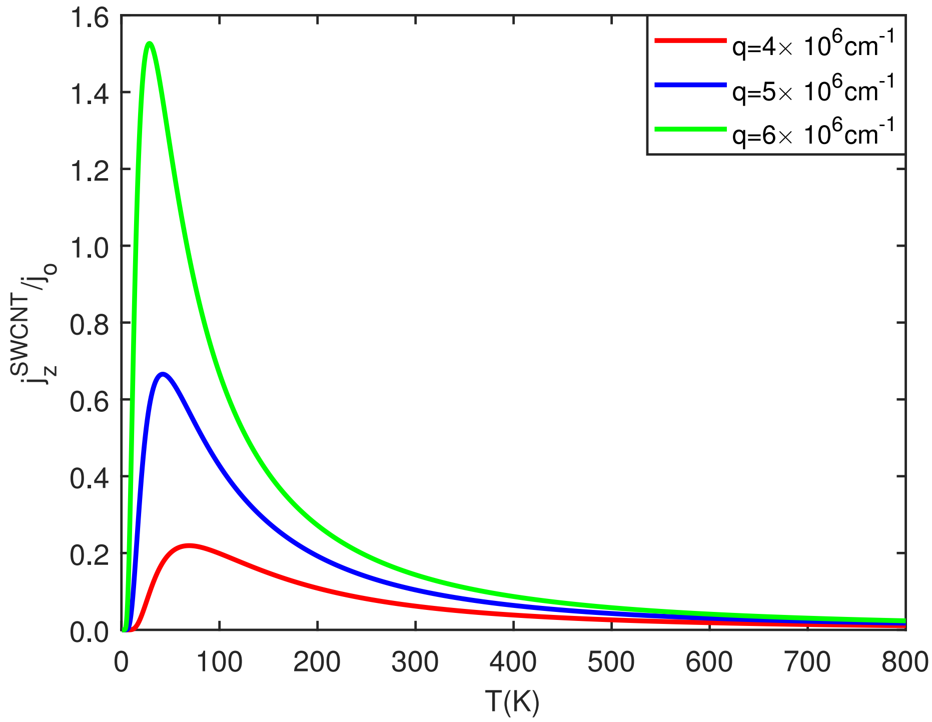

Figure 18: The Dependency of Current Density j_z^{AE} on Temperature T for varied Wavenumber q: SWCNT.

Increased value of q decreased the current density in FSWCNT, but increased the current density in SWCNT. The SWCNT was degenerate, which meant there were more intraminiband carriers which interacted strongly through deformation potential at any given time, and resulted in a large current. As illustrated in Figures 17 and 18, the SWCNT had a greater acoustocurrent $\approx$ in orders of five magnitude than FSWCNT.

Furthermore, at high temperatures, heating of the non-degenerate carrier gas supplied extra energy to the carriers. Once the carriers reached the NDV regime, the energy associated with thermal excitation was lost by the creation of propagating high-field electric domains [92], making the transition to the population inversion, $W_{dc} > 0$, inaccessible. The onset of NDV induced the establishment of field and charge domains due to the rise of space-charge density variations that prohibited the unwavering monitoring of the Esaki-Tsu curve [92].

Physics of carrier trajectory in the wave dragging regime

In the absence of an external field, the dynamics of intraminiband carriers propagating along the FSWCNT were investigated. A deformation potential was generated by a longitudinal coherent acoustic wave which propagated down the FSWCNT, which resulted in periodic change of the FSWCNT's conduction band edge. The strain wave generated was assumed to propagate down the major axis ($z-$axis) in this analysis. In the semiclassical regime, the potential energy obtained as a result of the strain (S) on the lattice was:

$$V_s = \Lambda S. \tag{268}$$

The strain, $S(z,t)$, caused by a coherent acoustic wave which propagated along the $z-$axis of FSWCNT was calculated as followed:

$$S(z,t) = -S_o \sin(qz + \omega_q t). \tag{269}$$

The frequency of this acoustic wave was $\omega_q = v_s q$, for a linear dispersion. As a result, the maximum strain was calculated as:

$$S_o = qD, \tag{270}$$

where S_o was the mechanical displacement amplitude and D was the displacement of the FSWCNT lattice determined from the acoustic wave. The potential energy generated by the acoustic wave was found by putting Eq. (269) and Eq. (270) into Eq. (268) as;

$$V_s(z,t) = -\mathscr{U} \sin(qz + \omega_q t), \tag{271}$$

where the amplitude of the acoustic wave was $\mathscr{U} = \Lambda S_o$. The semiclassical motion equations, which are identical to Hamilton's equations, were as follows:

$$v_z = \frac{\partial \mathscr{H}}{\partial p_z} = \frac{3\sqrt{3}b}{2\hbar} \sin(3ap_z) \qquad \frac{dp_z}{dt} = \frac{\partial \mathscr{H}}{\partial z} = q\mathscr{U} \cos(q(z+z_o) - \omega_q t) \tag{272}$$

with the semiclassical Hamiltonian given as: $\mathscr{H}(z, p_z) = \varepsilon(p_z) + V(z,t)$. The drift velocity was solved numerically by making use of Eq. (272) , and taking $v_z = 0$, and $p_z = 0$ when $t = 0$, to determine the carrier trajectories in the absence of scattering. The drift velocity (v_d) of the intraminiband carriers dependence on the acoustic wavenumber (q) was presented in Figure 19.

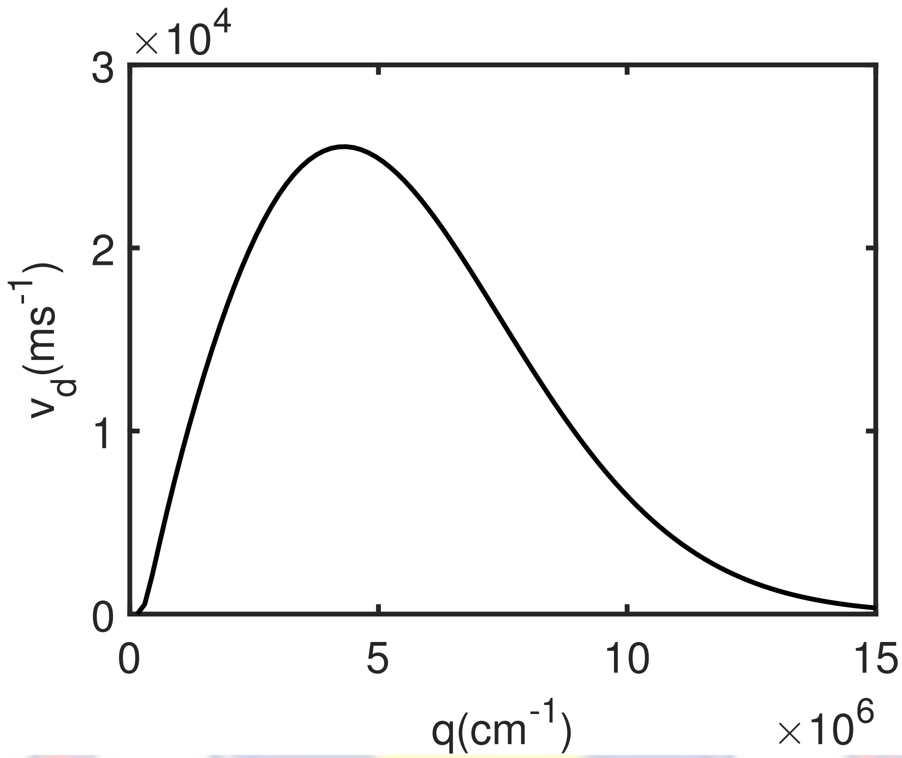

Figure 19: Carrier Dynamics for $q = 5 \times 10^5 cm^{-1}$, with Initial Values of $z = 0$, and $p_z = 0$ where High Frequency Oscillations were Driven by the Acoustic Wave.

The carrier trajectory was made up of regular, nearly sinusoidal oscillations superimposed on a linear gradient v_z, implied that the acoustic wave dragged the carrier through the FSWCNT. The carrier motion was examined in the rest frame of the acoustic wave, where the carrier's location, $z'(t) = z(t) - v_s t$, confirmed this image. Further changes of the trajectory resulted in the following:

$$z(t) = v_s t + \frac{\lambda}{4}[1 - \cos(\omega_R t)] \tag{273}$$

where ω_R was the frequency for motion to and fro across the crystal potential wells. Figure 20 showed the region $\omega_R \ll 1/\tau$, which indicated that the drift velocity curve was linear, as was the case with a typical Ohmic current-voltage relationship. The carriers were scattered in this domain before being permitted

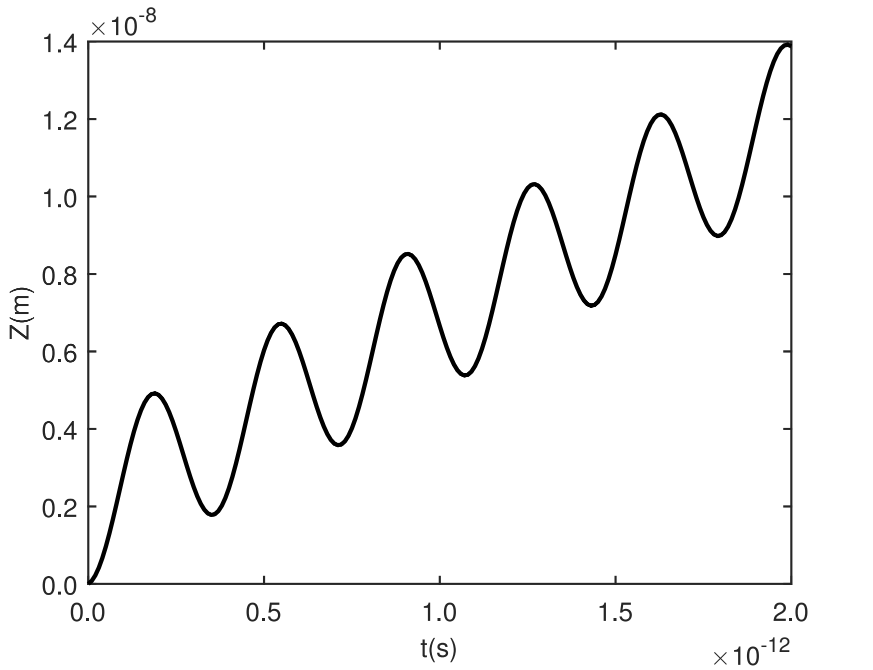

Figure 20: Carrier Dynamics in Real Space for $q = 4 \times 10^5 cm^{-1}$, with Initial Values of $z = 0$, and $p_z = 0$, where the Carrier was Dragged in the FSWCNT with the Velocity of the Acoustic Wave.

to travel far down the dispersion curve, and the FSWCNT behaved like a pure conductor. In other words, the carriers could only access the lower, parabolic part of the dispersion curve before scattering. As a result, no Bloch oscillation occurred, and Ohmic behaviour was observed. When $\omega_R = 1/\tau$ and the intraminiband carriers' drift velocity was at its greatest, the carriers were allowed to cross roughly 0.8 of the Brillouin zone length before being scattered.

Figure 21 also showed the region $\omega_R > 1/\tau$, which indicated that carriers were suppressed as the field increased, also known as the negative differential conductivity zone. More carriers were permitted to reach the Brillouin zone boundary, and Bloch oscillated before scattering in this region. Bloch oscillations caused the carriers to be confined, which suppressed transport. Furthermore, when the field rose, the carrier's odds of executing a

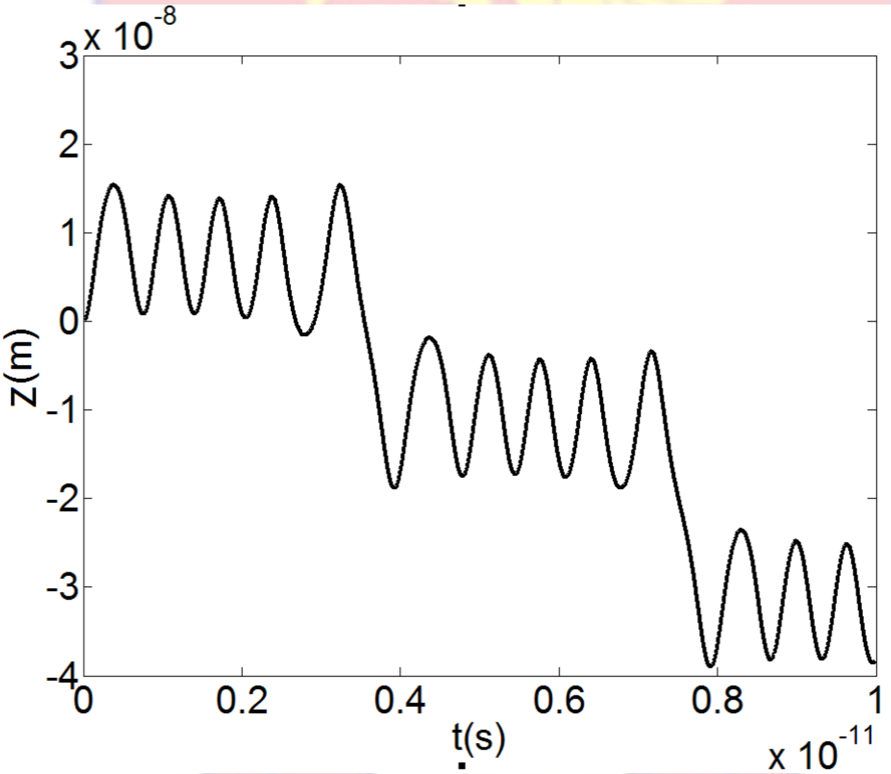

Figure 21: Carrier Dynamics for $q = 5 \times 10^5 cm^{-1}$, with Initial Values of $z = 0$, and $p_z = 0$ where High Frequency Bloch Oscillations were Driven by Acoustic Waves

single to numerous Bloch oscillations increased, and the localizing effect of these oscillations became stronger and stronger, and resulted in negative differential velocity (NDV). Intraminiband conduction carriers experienced collective high frequency oscillations with frequencies ranging from GHz or sub-THz to THz when NDV was present.

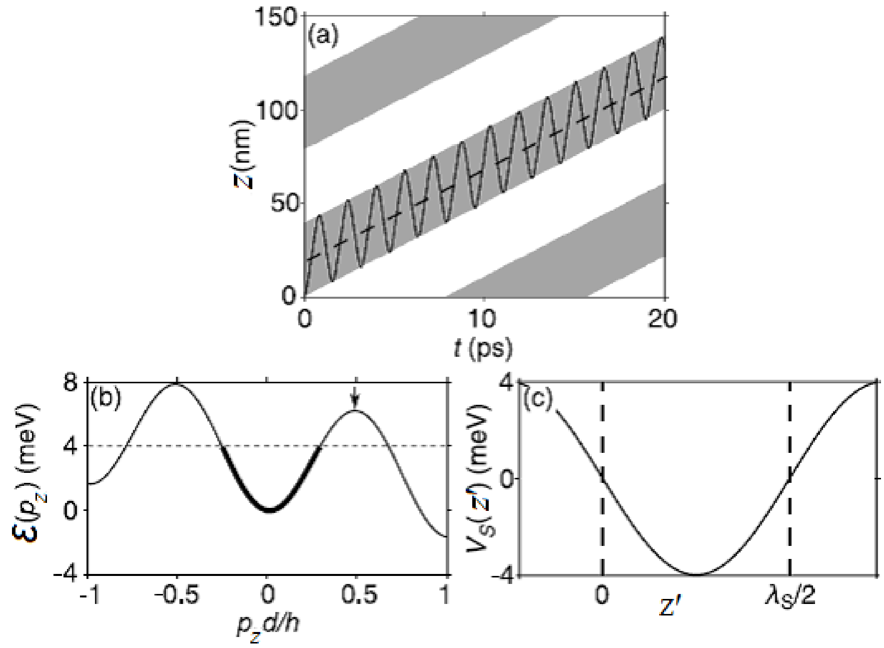

Figure 22: (a) Bold Curve: Carrier Trajectory, $z(t)$, Calculated for $\mathscr{U} = 4meV$. $V_s(z,t) > 0$ (white regions) and $V_s(z,t) < 0$ (gray regions). Broken Line has Gradient v_s. (b) is $\varepsilon'(p_z)$ where the Dotted Line Shows when $\mathscr{U} = 4meV$. At the Arrowed Peak, $\varepsilon'(p_z) = \mathscr{U}_c$. (c) shows $V_s(z')$, where Dotted Lines Mark Turning Points of Orbit in (a) [91].

Figure 22(a) showed $z(t)$ trajectory obtained numerically from Eq. (272), when $\omega_q = 5 \times 10^{11} rad/s$ and taking $\mathscr{U} = 4meV$ to the left of the peak drift velocity in the $v_d(q)$ curve. The trajectory was made up of regular, sinusoidal oscillations placed on a gradient v_s of a linear background. The amplitude of the oscillations was half the acoustic wave's wavelength, λ. The dashed line in Figure 22(a) depicted the following relationship:

$$\langle z(t) \rangle = v_s t + \frac{\lambda}{4}. \tag{274}$$

The carrier was clearly carried across the lattice by the acoustic wave in this illustration. The motion of the carrier within the rest frame of the acoustic wave was researched to better understand the trajectory in the acoustic wave dragging regime. The Hamiltonian defined in the rest frame of the FSWCNT was used to do this coordinate translation. This gave the total energy of the carrier, but it was not a constant of motion due to the explicit time dependency of the acoustic wave potential (see Eq. (271)).

The Lagrangian corresponding to the Hamiltonian was defined to aid the transformation from the laboratory (x,y,z) frame to the rest frame (x,y,z') of the acoustic wave. The Lagrangian was an explicit function of generalized coordinates, such as position coordinates and generalized velocities in this example. The Lagrangian, $\mathscr{L}$, was given as followed when the Legendre transform was used as:

$$\mathscr{L}(z,\dot{z},t) = \dot{z}p_z[\dot{z}] - \mathscr{H}(z,p_z[\dot{z}],t) = \dot{z}p_z[\dot{z}] - \varepsilon(p_z) - V_s(qz-\omega_q t) \qquad (275)$$

where $p_z[\dot{z}]$ was the dependency of the canonical momentum on $\dot{z} = \partial\mathscr{H}/\partial p_z$. Making the coordinate transformation of $z(t)$ into the moving frame of the acoustic wave $z'(t)$ with the following substitution:

$$z'(t) = z(t) - v_s t \qquad \Longrightarrow \dot{z}' = \dot{z}(t) - v_s, \qquad (276)$$

thus, the transformed Lagrangian, $\mathscr{L}'$ was;

$$\mathscr{L}'(z'(t),\dot{z}(t)) = (\dot{z}'+v_s)p_z[\dot{z}'+v_s] - \varepsilon(p_z[\dot{z}'+v_s]) - V_s(z'). \qquad (277)$$

The carrier experienced a static potential energy (time independent) in the acoustic wave frame, and therefore the acoustic wave potential in Eq. (271) became:

$$V_s(z') = -\mathscr{U}\sin(qz'), \tag{278}$$

where $\omega_q = v_s q$. The generalized momentum in the moving frame, p_z', taking into consideration that $\dot{z}'(t) = \dot{z} - v_s$ was deduced as:

$$p_z' = \frac{\partial \mathscr{L}}{\partial \dot{z}'} = p_z[\dot{z}' + v_s] = p_z \tag{279}$$

which showed that the canonical momentum was the same in both reference frames. This implied, the Hamiltonian in the moving frame was: $\mathscr{H}'(z', p_z') = \mathscr{H}'(z', p_z)$. As a result of the Legendre transformation, the Hamiltonian in the moving frame was:

$$\mathscr{H}'(z', p_z) = \dot{z}' p_z - \mathscr{L}' = \varepsilon(p_z) - v_s p_z + V_s(z'). \tag{280}$$

$\mathscr{H}'$ was a motion constant that was independent of time; nevertheless, it should be noted that it was not the total energy of the system, $\mathscr{H}$. In the moving frame, the system's modified dispersion relation was:

$$\varepsilon'(p_z) = \varepsilon(p_z) - v_s p_z, \tag{281}$$

where $\mathscr{H}'$ was equal to the sum of the kinetic and potential energy:

$$\mathscr{H}'(p_z) = \varepsilon'(p_z) - V_s(z'). \tag{282}$$

Using Hamilton's equations, the related equations of motion in the frame of the acoustic wave were as followed:

$$v_z' = v_z - v_s = \frac{dz'}{dt} = \frac{\partial \mathscr{H}'}{\partial p_z} = \frac{3\sqrt{3}\Delta b}{2\hbar} \sin\left(\frac{3\sqrt{3}p_z b}{2\hbar}\right) - v_z \tag{283}$$

$$\frac{dp_z}{dt} = \frac{\partial \mathscr{H}'}{\partial z'} = q\mathscr{U}\cos(qz'). \tag{284}$$

Initially, carrier trajectories were simulated starting at rest, so that $z(t=0) = p_z(t=0) = z'(t=0) = 0$. Thus, at $t = 0$, $\mathscr{H}' = 0$. Since $\mathscr{H}'$ was a constant of motion, it was therefore equal to 0 for all t. This implied that;

$$\varepsilon'(p_z) = -V_s(z'), \tag{285}$$

and therefore $\varepsilon'(p_z)$ could only take values between $\pm\mathscr{U}$, determined by amplitude $V_s(z')$. Figure 22(b) was a plot of the effective dispersion curve, $\varepsilon'(p_z)$ versus p_z, for $\mathscr{H} = 4meV$ corresponding to the trajectory in Figure 22(a). The horizontal dotted line and the lower axis in the figure mark $\pm 4meV$ i.e., the maximum and minimum values that $\varepsilon'(p_z)$ could possibly obtained. The dotted line revealed that the carrier could only access the lower parabolic region of the $\varepsilon'(p_z)$ curve, around $p_z = 0$, shown as a thick curve in Figure 22(b).

Since the minimum value of $\varepsilon'(p_z)$ that the carrier could attain was ≈ 0, its maximum potential energy was also ≈ 0. The carrier was therefore confined within the first Brillouin zone in the rest frame of the acoustic wave and oscillated back and forth across first Brillouin zone between the turning points at $z' = 0$ and $\lambda_s/2$ (see vertical dashed lines in Figure 22(c)), where $\lambda_s = 2\pi/q$ was the acoustic wavelength. Since the carrier remained within the parabolic region of $\varepsilon'(p_z)$, where its effective mass was constant, $z'(t)$ was a nearly harmonic function of t. Consequently, in Figure 22(a), the carrier trajectory could be approximated as:

$$z(t) = v_s t + \frac{\lambda_s}{4}[1 - \cos(\omega_R t)]. \tag{286}$$

In Figure 22(a), the gray patches depicted the region where $V_s \leq 0$, indicating that the carrier was trapped within the zone. Increasing $\mathscr{U}/q$ above $4meV$ had no noticeable influence on the carrier orbits at first. The carriers were of the type, $z(t) = v_s t + f(t)$, and continued to be pulled through the FSWCNT. The carrier could visit the nonparabolic portions of $\varepsilon'(p_z)$ as the value of $\mathscr{U}/q$ indicated by the top dotted line in Figure 22(b) grew, and the trajectory became less periodic.

The carrier was no longer trapped within the acoustic wave when $\mathscr{U}/q_c$ reached the critical value, $\mathscr{U}_c/q_c$, which was equal to the local maximum of $\varepsilon'(p_z)$, as shown by the arrow in Figure 22(b). The carriers could now reach the first minizone's boundary, and their trajectories abruptly change from closed to open orbits in p_z, which could traverse many minizones. Analytically, the value of $\mathscr{U}_c/q_c$ could be approximated by looking at Eq. (281). From Eq. (282), the local maximum of $\varepsilon'(p_z)$ occurred when;

$$\frac{d\varepsilon(p_z)}{dp_z} = v_s \tag{287}$$

$$\sin\left(\frac{3\sqrt{3}p_z b}{2\hbar}\right) = \frac{2\hbar v_s}{3\sqrt{3}\Delta b} \tag{288}$$

For small-angles, keep in mind that the Brillouin zone extended for $2\hbar\pi/3\sqrt{3}b$, and that the local maximum arrowed in Figure 22(b) along the major axis occurred when:

$$p_z = \frac{2\hbar\pi}{3\sqrt{3}b} - \frac{4\hbar^2 v_s}{27\Delta b^2} \tag{289}$$

Physics of carrier trajectory in the Bloch oscillation regime

Figure 23(a) showed $z(t)$ calculated for $\mathscr{U} = 15meV > \mathscr{U}_c$. High frequency oscillations were interrupted by jumps (arrowed) in the negative z direction in the trajectory. Bloch oscillations produced by the acoustic wave started the high-frequency variations in $z(t)$. When V_s was highest at the centers of the white and gray stripes in Figure 23(a), the acoustic force was temporarily too weak (zero) to induce Bloch oscillations, and the jumps in $z(t)$ occurred.

The variation of $\varepsilon'(p_z)$ and $V_s'(z)$ in the rest frame of the acoustic wave, shown respectively in Figures 23(b) and 23(c), was regarded to fully explained the form of the trajectory in Figure 23(a). The carrier was initially at $z' = 0$, where the strong gradient of $V_s(z')$ (see Figure 23(c) causes p_z to rapidly climb up to the edge of the first minizone (labeled 0 in Figure 23(b)), reversed v_z and also v_z'. The carrier continued to experience a high positive force after crossing the minizone boundary, which increased p_z through minizones 1-9 in Figure 23(b), resulted in Bloch oscillations within Bracket 1 in Figure 23(a).

As p_z increased, the average value of $\varepsilon'(p_z)$ dropped, and therefore $V_s(z')$ increased to keep $\mathscr{H}'$ in Eq. (280)= 0 as the carriers advanced up the

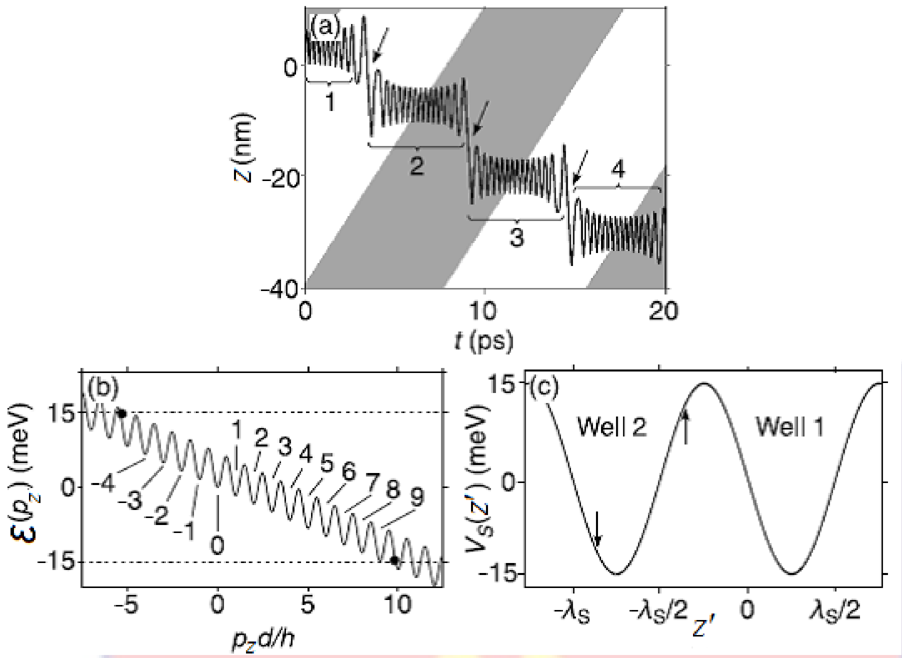

Figure 23: a) Bold Curve: Carrier Trajectory, $z(t)$, Calculated for $\mathscr{U} = 15meV$. $V_s(z,t) > 0$ (white regions) and $V_s(z,t) < 0$ (gray regions). Bloch Oscillation Bursts, within Numbered Brackets, are Separated by Sudden Jumps (arrowed). (b) $\varepsilon'(p_z)$ where Dotted Lines Mark $\pm\mathscr{U}$ when $\mathscr{U} = 15meV$. The Left and Right Circles Mark where $\varepsilon'(p_z) = \mathscr{U}$ and $-\mathscr{U}$ respectively. Numbers Label Different Minizones. (c) Shows Adjacent Energy Wells (1 and 2) in $V_s(z')$ [91].

left-hand side in Figure 23(c). As the carriers ascend the well wall, $dV_s(z')/dz'$ falls, lowering the frequency and increasing the amplitude of the Bloch oscillations, as illustrated by the $z(t)$ curve within Bracket 1 in Figure 23(a). When the carrier reached the top of Well 1 (see Figure 23(c)), $V_s(z') = \mathscr{U}$, $\varepsilon'(p_z)$ achieved its lowest possible value of $-\mathscr{U}$ (lower dotted horizontal line in Figure 23(b)), and p_z could no longer grow. Instead, because the acoustic force was immediately zero, p_z was briefly pinned at the intersection of $\varepsilon'(p_z)$ and the lower dotted line (right hand filled circle in Figure 23(c)).

The carrier jumped backwards along the segment of the $z(t)$ curve represented by the left hand arrow in Figure 23 due to the enormous negative velocity at this intersection, $d\varepsilon'/dp_z \approx -5.6 \times 10^4 ms^{-1}$ (a). The carrier was transferred to the point indicated by the right hand arrow in Well 2 (Figure 23(c)) by this jump. The acoustic wave exerted a large negative force on the carrier at this position, causing p_z to decrease and caused another burst of Bloch oscillations until $\varepsilon'(p_z)$ reached its maximum value (upper dotted line in Figure 23(b)) and $V_s(z')$ reached its minimum value of $-\mathscr{U}$ in Well 2. Then, the carrier again jumped backwards, along the $z(t)$ trajectory marked by the central arrow in Figure 23(a), with velocity, $d\varepsilon'/dp_z \approx -6.8 \times 10^4 ms^{-1}$, approximately equal to $d\varepsilon'/dp_z$ at the intersection (left hand filled circle in Figure 23(b)) between $\varepsilon'(p_z)$ and the upper dotted line.

This jump transferred the carrier to the position marked by the left hand arrow in Well 2 (see Figure 23(c)), where a large positive force caused p_z rapidly to increase, triggering the Bloch oscillation burst within Bracket 3 in Figure 23(a). The carrier then jumped backward again, this time along the $z(t)$ trajectory marked by the central arrow in Figure 23(a), with a velocity of $d\varepsilon'/dp_z \approx -6.8 \times 10^4 ms^{-1}$, which was roughly equal to $\varepsilon'(p_z)$ at the intersection (left-hand filled circle in Figure 23(b)) between $\varepsilon'(p_z)$. This jump moved the carrier to the left hand arrow point in Well 2 (see Figure 23(c)), where a strong positive force caused p_z to rapidly grow, causing the Bloch oscillation burst within Bracket 3 in Figure 23(a). Afterwards, the cycle repeated itself, with each Bloch oscillation burst causing the carrier to jump backward. Within each burst, the number of Bloch oscillations, N_{BO}, equals the number of unique minizones that the carrier can traverse. When the feasible energy range of the applied acoustic wave, $2\mathscr{U}$, was divided by the slope of the effective dispersion curve, $2v_s\hbar\pi/b$, the result was:

$$N_{BO} \approx 2\mathscr{U}\frac{b}{v_s\hbar} \tag{290}$$

FSWCNT as a low voltage current amplifier acoustic device

The acoustocurrent density obtained in Eq. (106) showed a strong dependence on the acoustic wavenumber, frequency, temperature and the dimensionless electric field, $(1 - v_d/v_s)$. Eq. (106) could be solved explicitly under two scenarios: (i) in the absence of an electric field; when $\omega_q \gg 12\gamma_o aq$, $j_z^{AE} = 0$, there was no absorption of acoustic waves and thus no acoustoelectric current present. The FSWCNT under such a condition could be used as a current filter, and this phenomena had been observed in Eq. (100) and Eq. (103) (see Figure 16−Figure 18). (ii) In the presence of a weak field, $\omega\tau \ll 1$, which yielded Eq. (106) and exhibited a strong nonlinear dependence of j_z^{AE} on $(1 - v_d/v_s)$. The acoustocurrent density, j_z^{AE}/j_o, in this case rose to a maximum and fell off in a manner similar to that observed in NDC in Figures 16-18, when $1 \gg v_d/v_s$ [196-202]. Conversely, when $1 \ll v_d/v_s$, j_z^{AE}/j_o decreased to a minimum value and started to rise (see Figure 24) [196, 203, 204]. In the presence of weak scattering ($\omega\tau \ll 1$), the carrier gained energy in the vicinity of the electric field and moved upward until it reached the point where Bloch waves reached the boundary of the Brillouin zone [205-210]. The nonlinear dependency of j_z^{AE}/j_o on $1 - v_d/v_s$ for varied q values was shown in Figure 24. It could be observed that as the q increased, the peak of j_z^{AE}/j_o increased to a maximum before it fell to its minimum value. The reason being that, there were more acoustic phonons to trade their energies and momenta to the intraminiband carriers to generate a high acoustoelectric current [211].

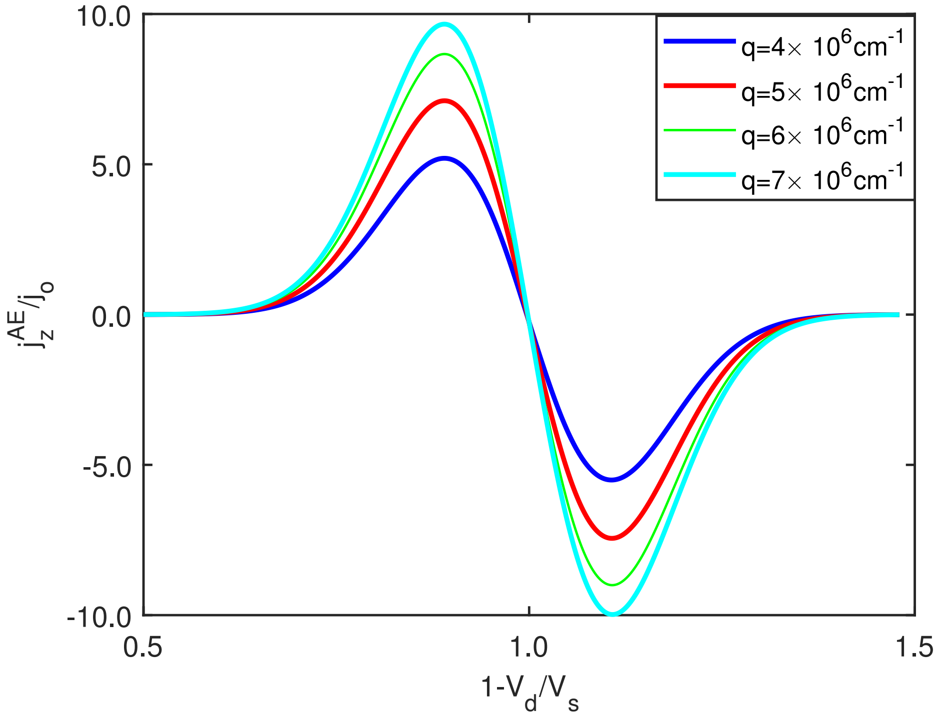

Figure 24: The Dependence of Normalized Axial Current Density j_z^{AE}/j_o on $1-v_d/v_s$ for varied Wavenumber q at $T = 300K$.

Numerically, when $q = 4\times10^6 cm^{-1}$, $(j_z^{AE}/j_o)_{max} = 5.193$, $q = 5\times10^6 cm^{-1}$, $(j_z^{AE}/j_o)_{max} = 7.114$, $q = 6\times10^6 cm^{-1}$, $j_z^{AE}/j_o = 8.668$ and $q = 7\times10^6 cm^{-1}$, $(j_z^{AE}/j_o)_{max} = 9.662$. The corresponding minimum currents were :$q = 4\times10^6 cm^{-1}$, $(j_z^{AE}/j_o)_{min} = -5.504$, $q = 5\times10^6 cm^{-1}$, $(j_z^{AE}/j_o)_{min} = -7.439$, $q = 6\times10^6 cm^{-1}$, $(j_z^{AE}/j_o)_{min} = -9.0$ and $q = 7\times10^6 cm^{-1}$, $(j_z^{AE}/j_o)_{min} = -9.998$. The negative j_z^{AE}/j_o observed when $1 \gg v_d/v_s$ was due to the intraminiband carriers which reversed direction and moved opposite to the field which was attributed to strong hypersound flux with an increasing electric field.

The dependency of j_z^{AE}/j_o on $1-v_d/v_s$ for varying temperature, was shown in Figure 25. It could be inferred that there was a decrease in the peak values of j_z^{AE}/j_o as the temperature increased. Numerically, for $T = 150K$,

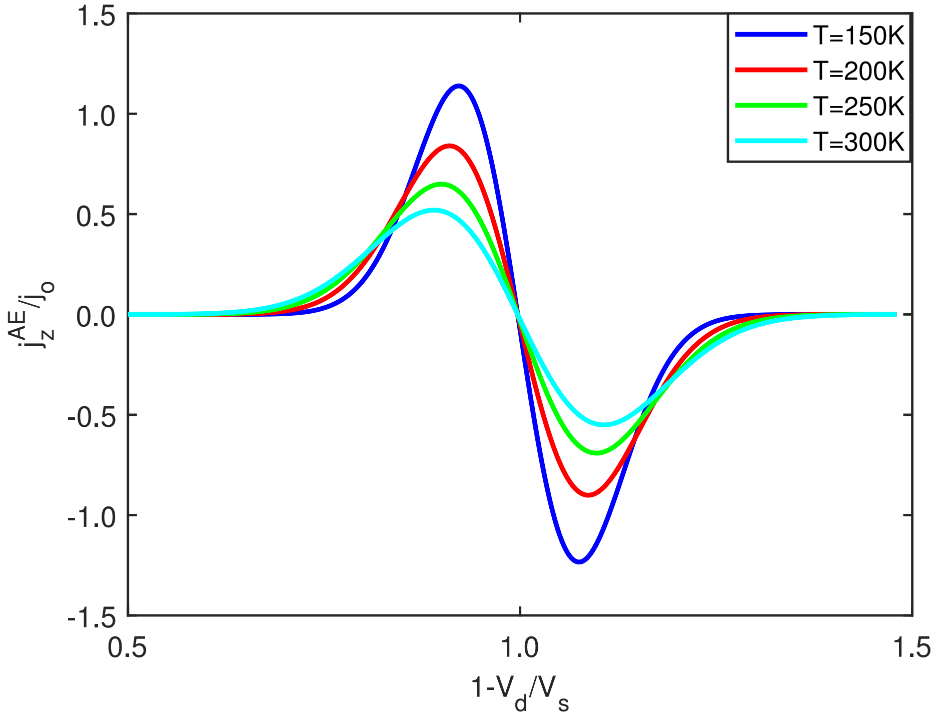

Figure 25: The Dependence of Normalized Axial Current Density j_z^{AE}/j_o on $1 - v_d/v_s$ for varied Temperature T.

$(j_z^{AE}/j_o)_{max} = 1.139$, $T = 200K$, $(j_z^{AE}/j_o)_{max} = 0.8383$, $T = 250K$, $(j_z^{AE}/j_o)_{max} = 0.6464$ and $T = 300K$, $(j_z^{AE}/j_o)_{max} = 0.5193$. The corresponding minimum current densities occurred at: for $T = 150K$, $(j_z^{AE}/j_o)_{min} = -1.234$, $T = 200K$, $(j_z^{AE}/j_o)_{min} = -0.9008$, $T = 250K$, $(j_z^{AE}/j_o)_{min} = -0.6911$ and $T = 300K$, $(j_z^{AE}/j_o)_{min} = -0.5504$. This was because increasing temperature increased the scattering processes in the FSWCNT. The majority of carriers in this case acquired a high drift velocity, and attaind a high kinetic energy. These energetic carriers which were the majority carriers underwent interminiband transition which allowed only a few to perform intraminiband transition. Thus, these few intraminiband carriers interacted with the co-propagating acoustic phonons leading to a decrease in the current, j_z^{AE}/j_o. The I-V charateristic curve for the varying temperatures intersected at different values which indicated that at these points of intersections they had the same q values.

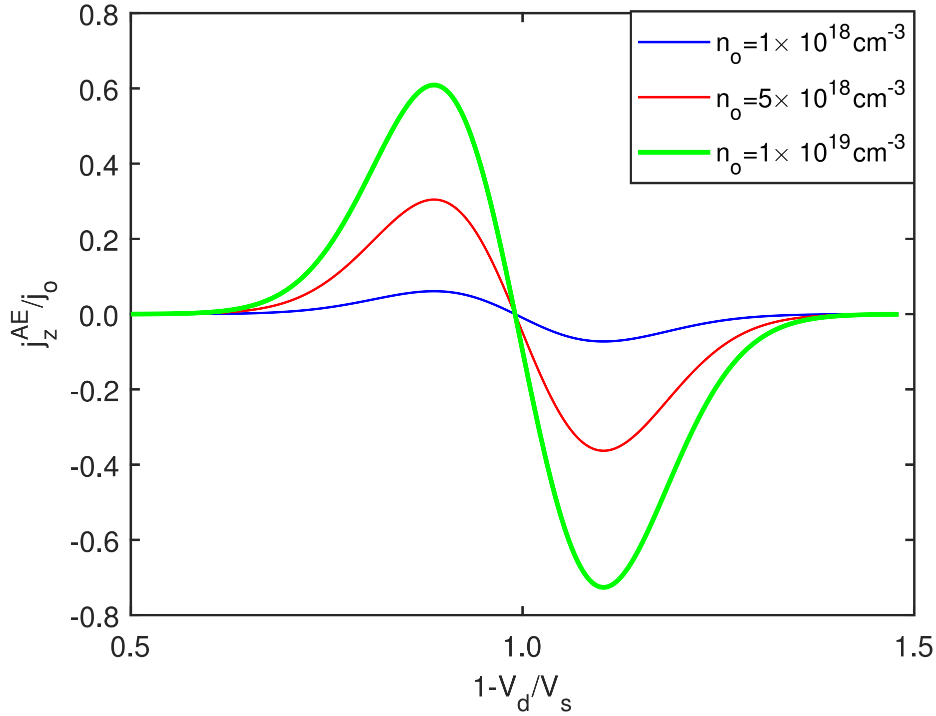

Figure 26: The Dependence of Normalized Axial Current Density j_z^{AE}/j_o on $1-v_d/v_s$ for varied Carrier Concentration n_o at $T=300K$.

The acoustocurent as shown in Figure 26 was seen to be highly sensitive to the carrier concentration, n_o, and worked better for moderate n_o within $10^{16}-10^{19}cm^{-3}$ without introducing strong carrier-carrier interactions. Higher n_o increased the reverse current without screening out the deformation potential field to lower the current density. Again, increasing the carrier concentration increased the AE current, because more intraminiband carriers interacted with the acoustic phonon to generate the current densities. This meant the carrier concentration could be used to tune the FSWCNT to obtain a high AE current at room temperature and could hold potential for current amplifying acoustic material for ultrasound current source density imaging (UCSDI) and AE hydrophone devices [212].

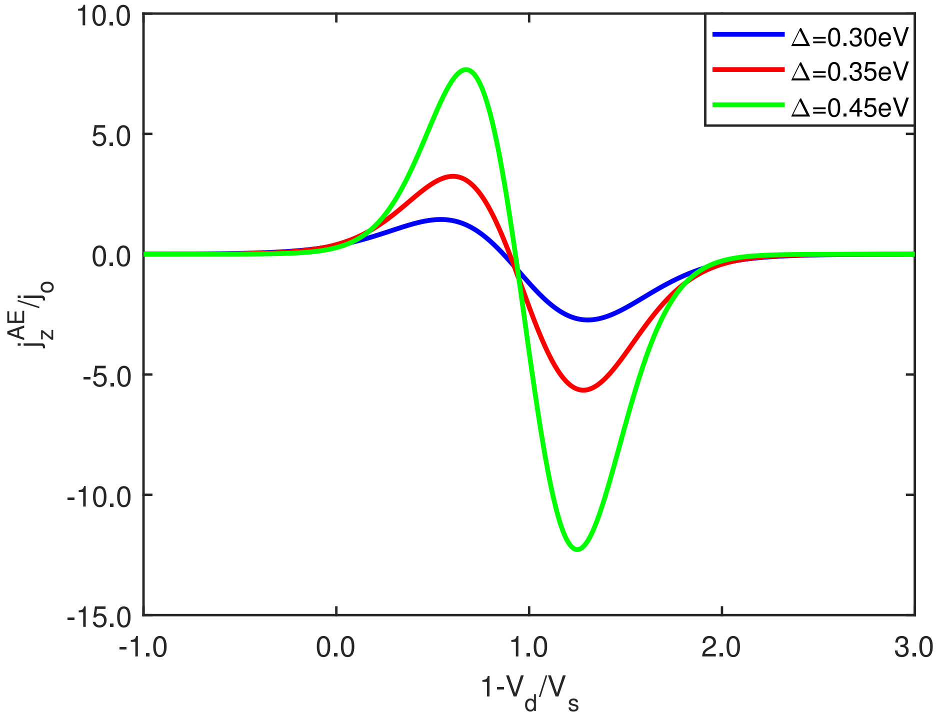

Figure 27: The Dependence of Axial Current Density j_z^{AE} on $1-v_d/v_s$ for varied Carrier-phonon parameter Δ at $T=300K$

However, a strong nonlinear dependency of j_z^{AE}/j_o on $1-v_d/v_s$ for different values of carrier-phonon interactions, Δ, was observed in Figure 27. Increasing Δ, increased the reverse current significantly to about 2.5 times the forward current. This implied there were more intraminiband carriers which interacted strongly with the co-propagating acoustic phonons to generate a high current in the reverse direction. The strong non-parabolicity of the band relation in FSWCNT was assumed to be a contributory factor to the bahaviour observed.

In momentum space, the distinct behaviour of carrier transport was accompanied by a sharp decrease in $v_d(E_z)$ dependency. According to the findings, the critical behaviour of the drift velocity was directly related to the appearance of Bloch oscillations. Similarly, carrier intraminiband movement under the influence of a weak probing field had a dramatic effect on the transition between diffusive motion types. For a set of values of E_z, unique profiles of absorption and gain were formed (see Figures 24- 27). Eq. (93) was used to create these profiles. Following that, using Eq. (108), a distinguishing Lorentzian-like absorption profile that was centred at $0 < 1 - v_d/v_s < 1$ was generated (see Figures 24- 27). Because the system was driven forcefully enough to create non-linear effects, the sub-harmonic resonance of the absorption was invoked. The rise in E_o appears to alter carrier transport in some way, resulting in a gain in the high-frequency range. Gain could be attributed to nonlinear oscillations (see Figures 24- 27), which represented the carrier's localized motion. As a result, a large enough E_o caused the maximum crystal momentum to rise towards the Brillouin zone boundary. At $p_z = \pi$, the Brillouin zone border coincided with the saddle points defined by dispersion relation. When the carrier trajectory approached the saddle points separatrix, which was significant in the gain realization, these non-linearities become more prominent. Furthermore, the gain in Figures 24-27 occurs at frequencies higher than the resonance frequency. In other words, decreasing the temperature of the carrier streamed as it approached the critical value of E_{cr}, which defines the maximum value of v_d, would have had a direct effect on the gain (see Figures 24- 27). In this situation, the crossover frequency was determined by $1 - v_d/v_s$. As a result, the gain in this situation was magnified, which was due to extremely nonlinear oscillations that approached the separatrix. Carriers encountered Bragg reflections when $E_o/E_{cr} > 2$, which contributed to low-frequency gain instability.

Physics of carrier dynamics under external electric field

The mobility of the carriers was greatly influenced in the presence of a dc electric field. Because the electric field served to transfer energy to the carrier system, the carrier distribution when the electric field was present was moved to higher energies than when the electric field was absent. The carriers inhabit low-energy states in the absence of the electric field, while higher-energy states were empty. As a result, the carriers were more likely to absorb a phonon than to emit one, because the former process required them to move from highly occupied low-energy states to practically empty higher-energy ones, whereas the latter process required them to do the opposite. The electric field caused the carrier distribution to shift in such a way that higher-energy states became occupied and lower-energy levels became unoccupied.

As a result, the chances of phonon absorption falls while the probability of phonon emission rises. When the likelihood of emission surpassed that of absorption at a critical field E_{cr}, a beam of phonons passing the FSWCNT was amplified rather than absorbed. The carrier-phonon coupling could be seen as phonon SASER process, where the dc electric field acted as a pump, and inverted the population of carriers so that phonon emission could occur within the same miniband. To put it another way, the electric field sends energy to the phonon system through the carrier system. When the component of v_d coming from the electric field along $\hat{k}$ surpasses the velocity of sound, the carrier population inversion occurs, i.e. $\hat{k} \cdot \hat{v}_d > v_s$.

The gain in acoustic wave via carrier-phonon coupling and photon Cerenkov emission have certain similarities. In the latter, when the carriers' velocity surpasses the photon velocity in the FSWCNT, Cerenkov radiation of photons occurs. The emission of phonons happens in this analysis when the carrier velocity surpasses the phonon velocity in the FSWCNT. As a result, phonon gained could be viewed as phonon Cerenkov emission by carriers. In addition, when the velocity of drift makes an angle ϑ with the propagation path, the phonons are emitted into a cone with an angular half-width ϑ_{cr}, where $\cos\vartheta_{cr} = v_s/v_d$. Phonon emission becomes maximum at the critical angle ϑ_{cr}, which is at the edge of the Mach cone. Maximum phonon absorption, on the other hand, occurs at the opposite end of the Mach cone. When $\hat{k} \cdot \hat{v}_d = v_s$, the carriers comes into resonance with the acoustic wave, which causes this. As a result of this resonant behaviour, absorption increases as the drift field increases, reaching a maximum and then turning negative at the critical drift field.

Induced Hall-like current by acoustic phonons in FSWCNT

From Eq. (106) and Eq. (118), the axial acoustoelectric and Hall-like current against $(1 - v_d/v_s)$, with varied wavenumber q, and the acoustoelectric and Hall-like current against q for varied temperature T are displayed in Figures 28 and 29 and Figures 30 and 31, respectively.

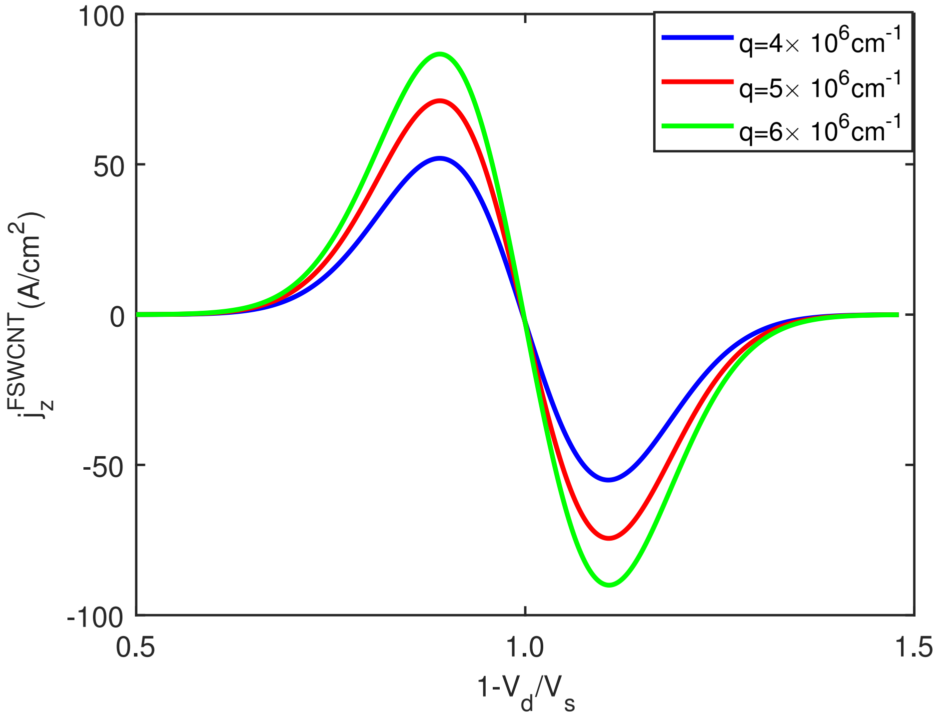

Figure 28: Dependence of Axial Current Density j_z^{AE} on $(1 - v_d/v_s)$ for varied Wavenumber q at $T = 300K$

When the electric field was negative, the current rises to a maximum, then falls off as in NDC (see Figures 28 and 29). On the contrary, when the electric field was positive the carrier current decreased, reached a minimum and increased again. This could be attributed to the Bragg's reflection at the band edge. Furthermore, the ratio of the height of the absorption peak to that of the amplification peak for successive q differred by one. Increasing the q tends to increase the axial acoustoelectric current density, $\vec{j}_z^{AE}$, as shown in Figure 28 and 29 but the Hall-like current density $\vec{j}_y^{AME}$ decreased in the y-direction due to the increased scattering along that direction (see Figure 29).

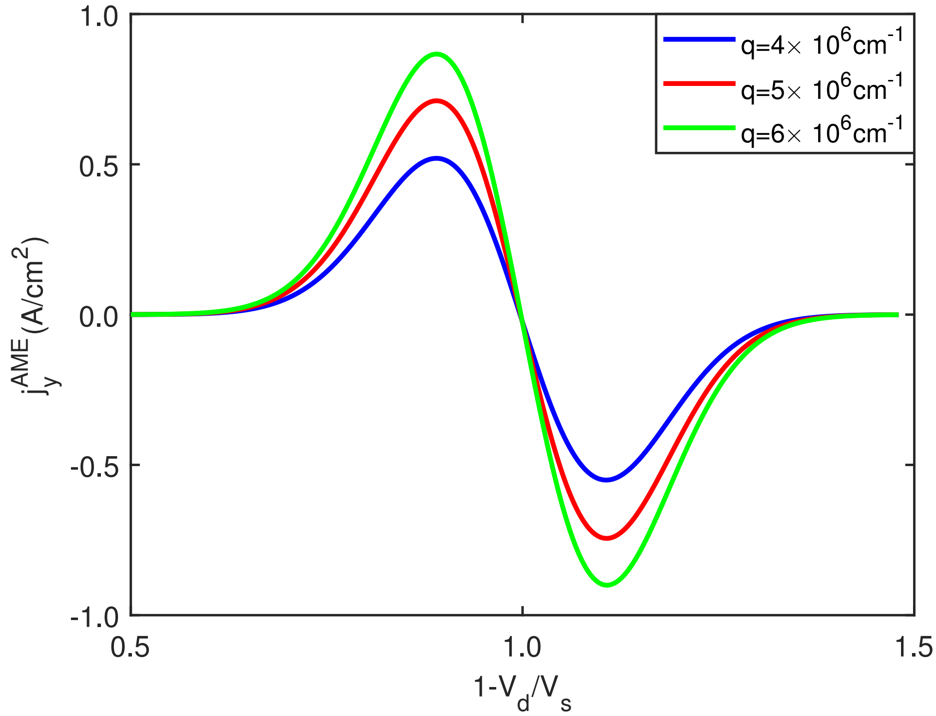

Figure 29: Dependence of Axial Current j_y^{AME} on $(1 - v_d/v_s)$ for varied Wavenumber q at $T = 300K$.

Though Bloch oscillations controlled the transport process, increasing the temperature in the presence of the external electric field increased the conductivity as shown in Figures 30 and 31, with the higher magnitude observed at room temperature, $T = 300K$. This was because increasing the temperature ($T = 300K$), decreased the band gap of the semiconductor FSWCNT, creating new conducting pathways where the majority of the carriers in this case underwent intraminiband transition. Consequently, a high intraminiband current interacted strongly with the phonons to create a high acoustoelectric and Hall-like current densities. In Figure 30, the $\vec{j}_z^{AE}$ was observed to be about two orders of magnitude greater the Hall-like $\vec{j}_z^{AME}$ (see Figure 31) as a result of the scattering processes.

Employing Eq. (106) and Eq. (118) a Lorentzian-like absorption profile (see

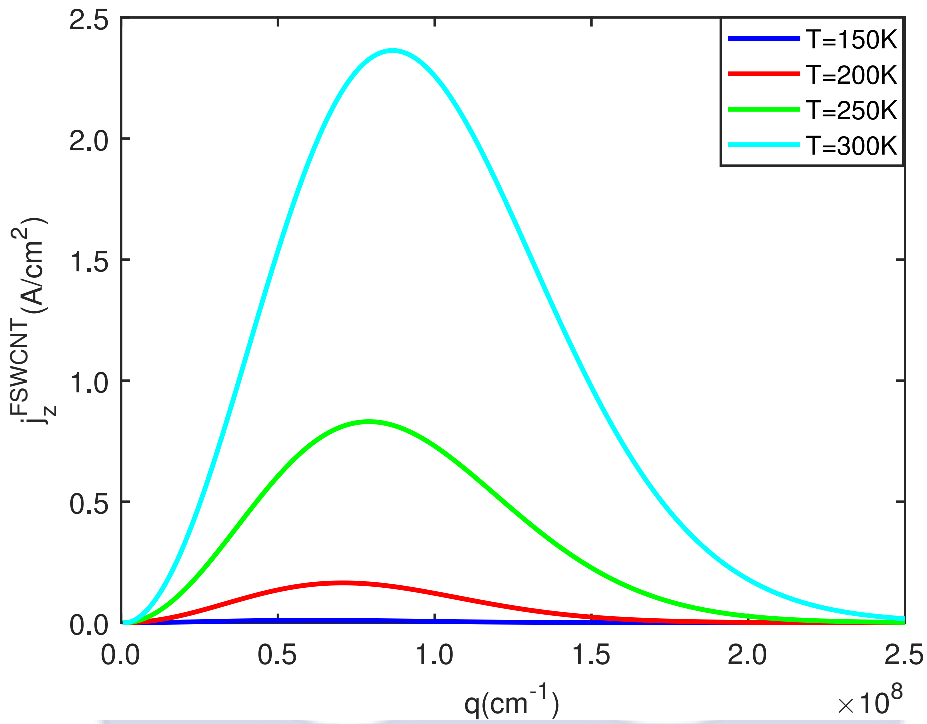

Figure 30: Dependency of Axial Current Density j_z^{AE} on Wavenumber q for varied Temperature T at $(1 - v_d/v_s) = 0.5$.

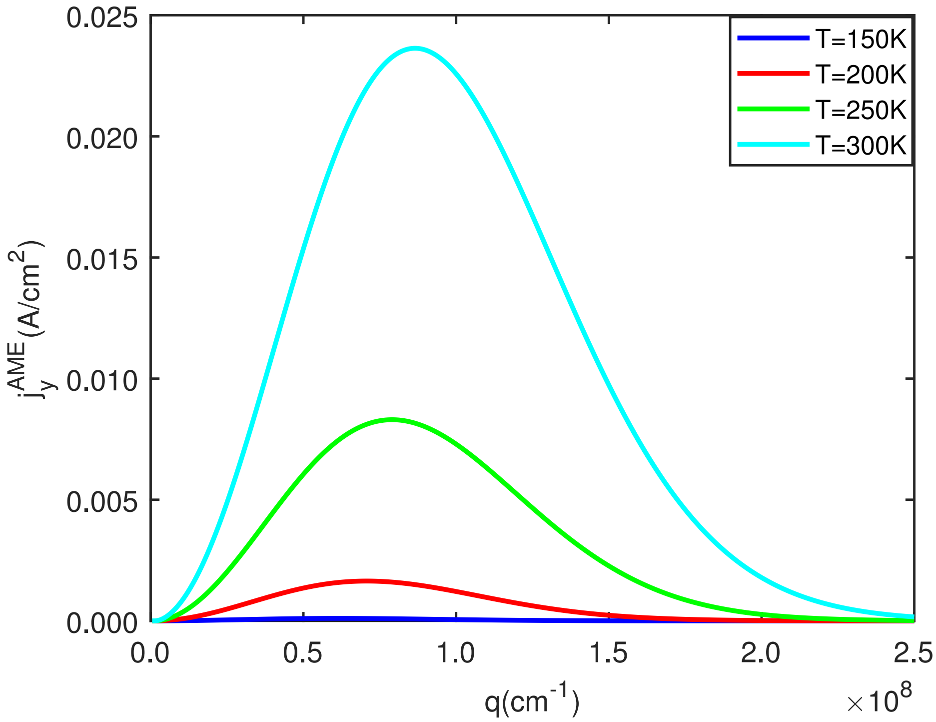

Figure 31: The Dependency of Hall-like Current Density j_y^{AME} on Wavenumber q for varied Temperature T at $(1 - v_d/v_s) = 0.5$.

Figures 30 and 31) was obtained. Thus, because the system was driven forcefully enough by the $ac-$field to create non-linear effects, the sub-harmonic resonance of the absorption was invoked. Amplification of the probe field was not achievable in this scenario, as evidenced by acoustoelectric and Hall-like current densities (see Figures 30 and 31) that stay positive as the probe field frequency increased.

Similarly, the dependency of $\vec{j}_z^{AE}$ and $\vec{j}_y^{AME}$ on q when the electric field is switched off was displayed in Figures 32 and 33, respectively. In the absence of the external electric field, there existed an inherent $dc-$field generated by the phonons which transferred their energy and momentum to the carrier so that the carrier moved with a drfit velocity v_d through the semiconductor. The peak currents $\vec{j}_z^{AE}$ and $\vec{j}_y^{AME}$ in Figures 32 and 33 respectively, decreased with increase in temperature, in contrast to what was observed in the presence of the external electric field. The carriers in this case had only a single path for conduction. Increasing the temperature increased the kinetic energy of the carriers and the majority of these carriers undergo interminiband transition. Thus, only a handful undergo intraminiband transition and this resulted in low conductivity as observed. Owing to the conservation laws of energy and momentum, only carriers with momentum $p_z > \hbar q/2$ interacted with the acoustic phonons. Passing a sound flux with high frequency through the sample, $\Gamma(q)$ as well as $\vec{j}_z^{AE}$ and $\vec{j}_y^{AME}$ went to zero. Moreover, the nonlinear behaviour was attributed to the magnetic field directing a few carriers in the Hall direction leading to weak interaction with acoustic phonons.

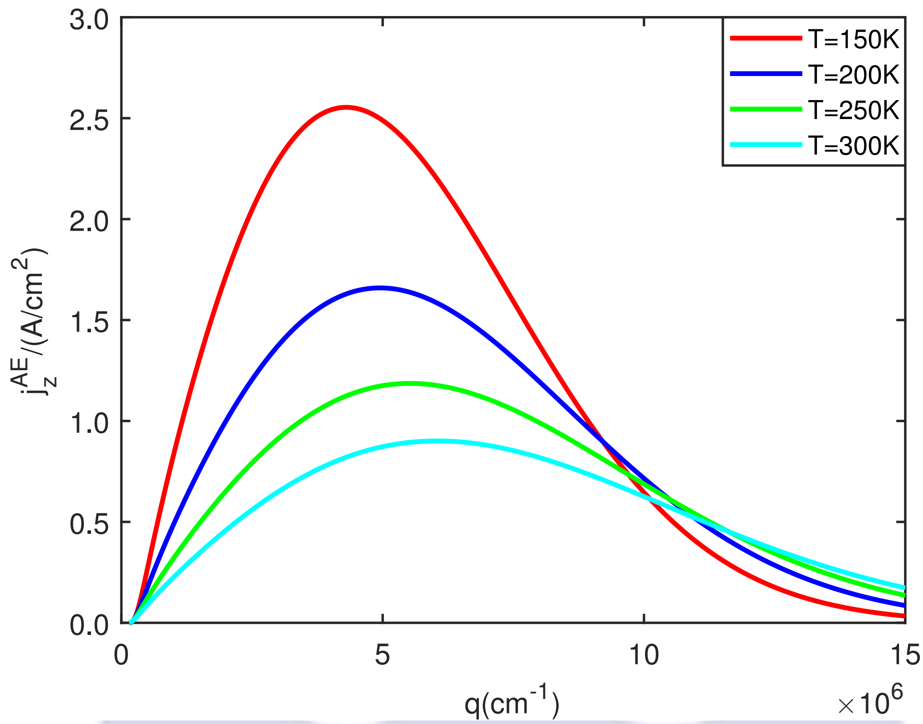

Figure 32: The Dependency of Axial Current Density j_z^{AE} on Wavenumber q for varied Temperature T.

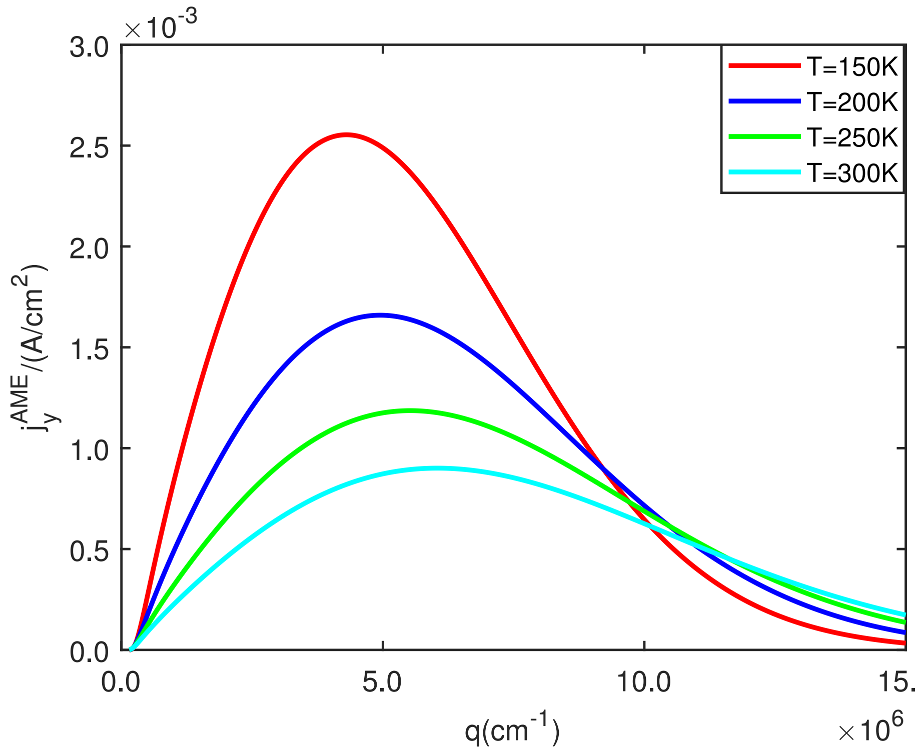

Figure 33: Dependency of Hall-like Current Density j_y^{AME} on Wavenumber q for varied Temperature T at $\Omega\tau = 0.01$.

In Figure 34 and 35, the dependence of $\vec{j}_z^{AE}$ and $\vec{j}_y^{AME}$ on T in the absence of the electric field was shown. The $\vec{j}_z^{AE}$ and $\vec{j}_y^{AME}$ rise linearly to a maximum and begun to drop and shifted towards higher temperatures. Both $\vec{j}_z^{AE}$ and $\vec{j}_y^{AME}$ exhibited linear dependence on T at $E = 0$ (in the region of ohmic conductivity). Thus, the heating of the non-degenerate carrier gas provided additional energy to the carriers at high temperatures. The energy associated to thermal excitation was dissipated by the development of propagating high-field electric domains [92], making the transition to the population inversion, $W_{dc} > 0$, unavailable once the carriers reached the NDC region. Thus, increasing q resulted in a strong carrier-phonon interaction so that more phonons trade their energy and momentum to the carrier to increase the $\vec{j}_z^{AE}$ and $\vec{j}_y^{AME}$.

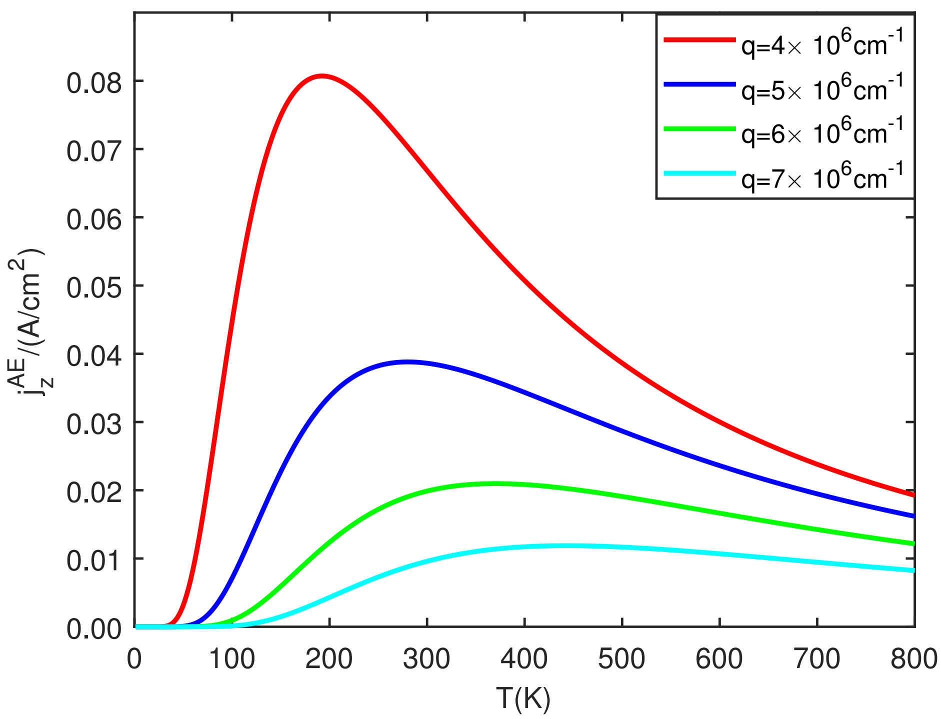

Figure 34: Dependency of Axial Current j_z^{AE} on Temperature T for varied Wavenumbers q.

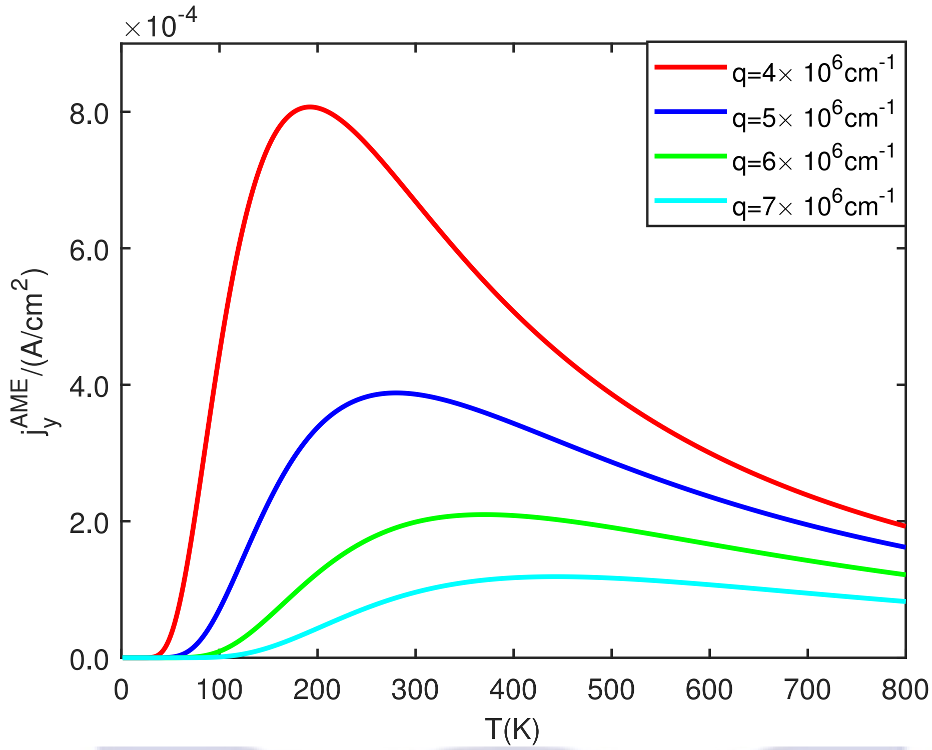

Figure 35: Dependency of Hall-like Current j_y^{AME} on Temperature T for varied Wavenumber q at $\Omega\tau = 0.01$

However, this was not the case because at high temperatures, the energy of the carrier-phonon interactions increased due to increase in their kinetic energies and collisions with other excitations. This resulted in a handful of carrier undergoing intraminiband transition leading to a decrease in both $\vec{j}_z^{AE}$ and $\vec{j}_y^{AME}$. There was a threshold temperature for which the $\vec{j}_z^{AE}$ and $\vec{j}_y^{AME}$ turned on: $T = 35K$ for $q = 4 \times 10^6 cm^{-1}$, T=56K for $q = 5 \times 10^6 cm^{-1}$, $T = 84K$ for $q = 6 \times 10^6 cm^{-1}$ and $T = 112K$ for $q = 7 \times 10^6 cm^{-1}$. The $\vec{j}_z^{AE} \gg \vec{j}_y^{AME}$ as observed in Figures 34 and 35 respectively, since the scattering was high along the Hall direction (y-axis).

The dependence of $\vec{j}_z^{AE}$ and $\vec{j}_y^{AME}$ on temperature for different values of carrier-phonon interaction parameter, Δ, were also presented in Figures 36 and 37. As temperature rises, the net $\vec{j}_z^{AE}$ and $\vec{j}_y^{AME}$ decreased but shifted more toward high temperatures. This suggested that higher current was obtained at lower q values and at low temperatures in both $\vec{j}_z^{AE}$ and $\vec{j}_y^{AME}$. Therefore, q can be used to tune the current to high temperatures for room temperature applications. The shift towards right (i.e., higher temperatures) in the current was attributable to the non-parabolicity of the band structure which was very strong in FSWCNT. This accounted for the curve intersections of the plots for different values of Δ

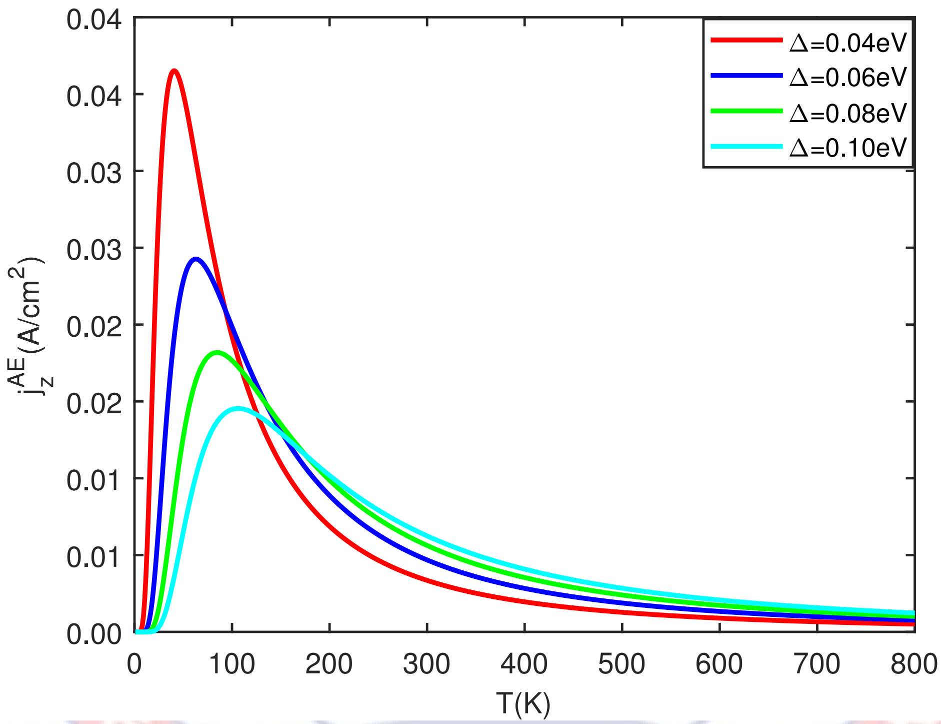

Figure 36: The Dependency of Axial current $\vec{j}_z^{AE}$ on Temperature T for varied Carrier-phonon parameter Δ.

in $\vec{j}_z^{AE}$ and $\vec{j}_y^{AME}$ (Figures 36 and 37). At these temperatures, different values of Δ had the same q values and so the same $\vec{j}_z^{AE}$ and $\vec{j}_y^{AME}$. From Figure 36, as Δ increased to $\Delta = 0.10eV$, the peak of $\vec{j}_y^{AME}$ fell drastically due to increased scattering in the Hall direction.

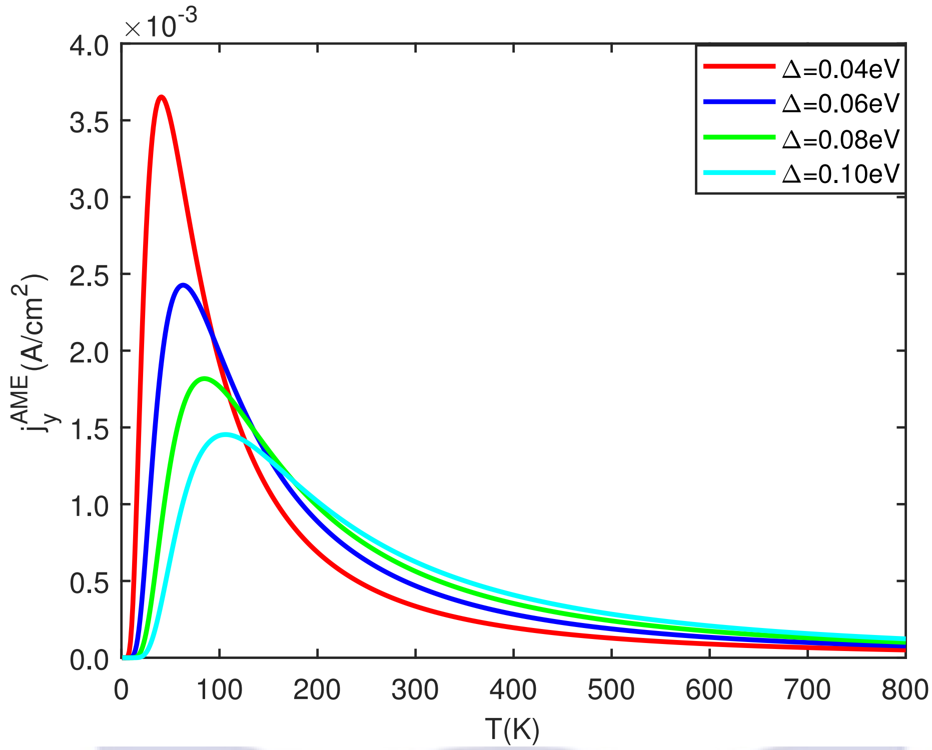

Figure 37: The Dependency of Axial Current j_z^{AE} on Temperature T for varied Carrier-phonon Interaction Δ at $\Omega\tau = 0.01$.

Furthermore, the Hall-like field's dependency $\vec{E}_{SAME}$ on q in the absence of an electric field is calculated in Eq. (122) and the corresponding drift velocity given in Eq. (123) and displayed in Figures 38 and 39. Following Ref. [190], the Hall-like field calculated had a linear dependency on the magnetic field H but highly nonlinear with respect to q. The magnetic field's orientation produced a Hall-like field in the Hall direction that opposed carrier mobility, which resulted in an absorption graph. As the temperature increased, the peak of the Hall-like field decreased but shifted toward high q values (see Figure 38). For varying magnetic fields, the Hall-like field strength increased to its peak and then fell. The graph obtained increased with increasing magnetic field at $T = 300K$ as shown Figure 39.

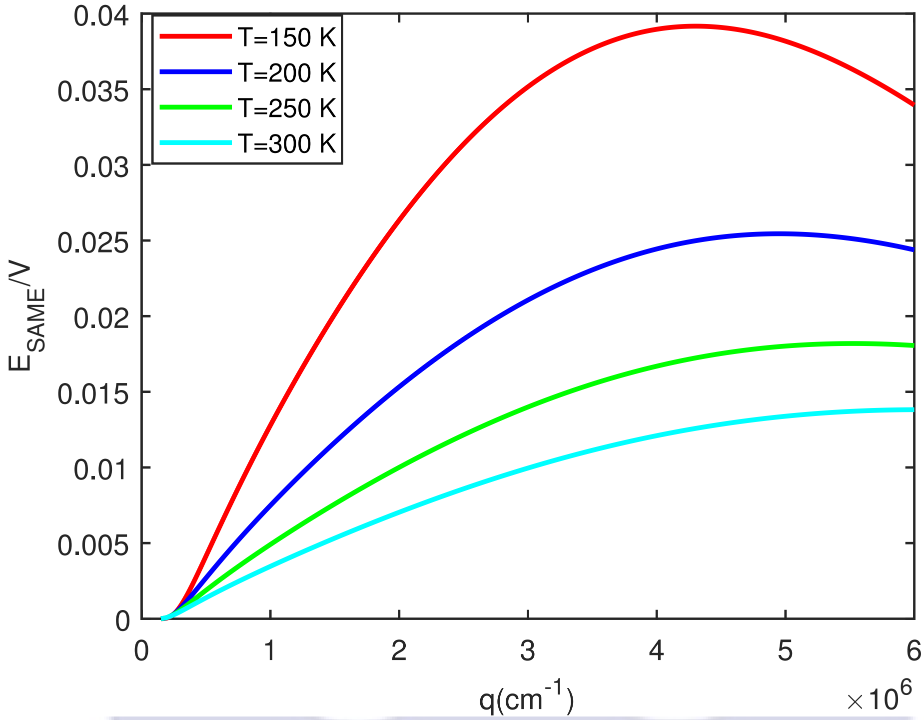

Figure 38: Dependence of Hall-like field E_{SAME} on Wavenumber q for varied Temperature T.

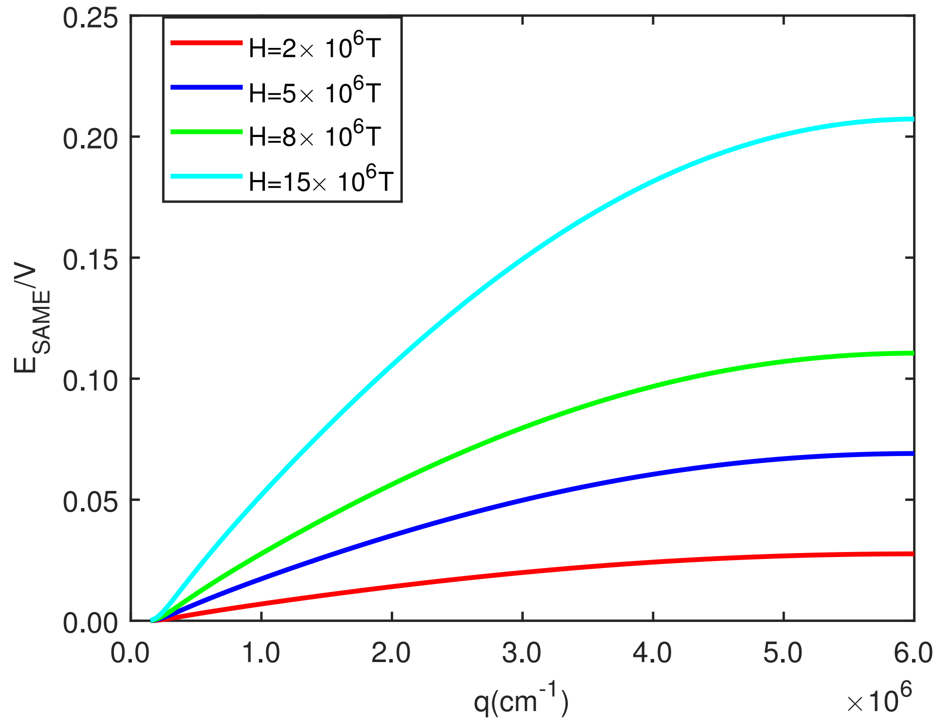

Figure 39: Dependence of Hall-like field E_{SAME} on Wavenumber (q) for varied Magnetic Field H at $T = 300K$.

Moreover, the Hall-like field's dependency E_{SAME} on $(1 - v_d/v_s)$ when the external electric field was switched on as calculated in Eq. (122) were presented in Figures 40 and 41. It was observed that the orientation of the H-field produced a Hall-like field in the Hall direction which opposed the carrier motion and thus, produced an absorption graph when $1 \ll v_d/v_s$. However, H-field orientation reverses in the Hall direction and move along with the carrier motion with little opposition and thus produced a gain graph when $1 \gg v_d/v_s$. As the temperature increased, the peak of the Hall-like field decreased in both directions but shifted toward high $1 - v_d/v_s$ values (see Figure 40).This was very interesting because, at $T = 300\ K$ increasing the magnitude of the H-field produced a high Hall-like field which drove the carriers for a higher Hall-like current to be obtained (see Figure 41). In the presence of a magnetic field, the threshold electric field for amplification was increased due to the dc magnetoresistance. Furthermore, the magnetic field decreases the dielectric relaxation frequency and increased the diffusion frequency. The presence of a transverse magnetic field greatly reduced the carrier drift velocity necessary to form the acoustoelectric domains [213-216].

The carriers in this regime scatter long before they reach the Brillouin zone's boundary. Furthermore, increasing the electric field above the point where the drift velocity achieved its maximum value permitted the carriers to go further along the dispersion curve before scattering, finally nearing the Brillouin zone's edge. This caused carrier transport to be suppressed, as evidenced by the decrease in drift velocity as E_o was increased. On the contrary, with large enough electric fields, the average miniband energy

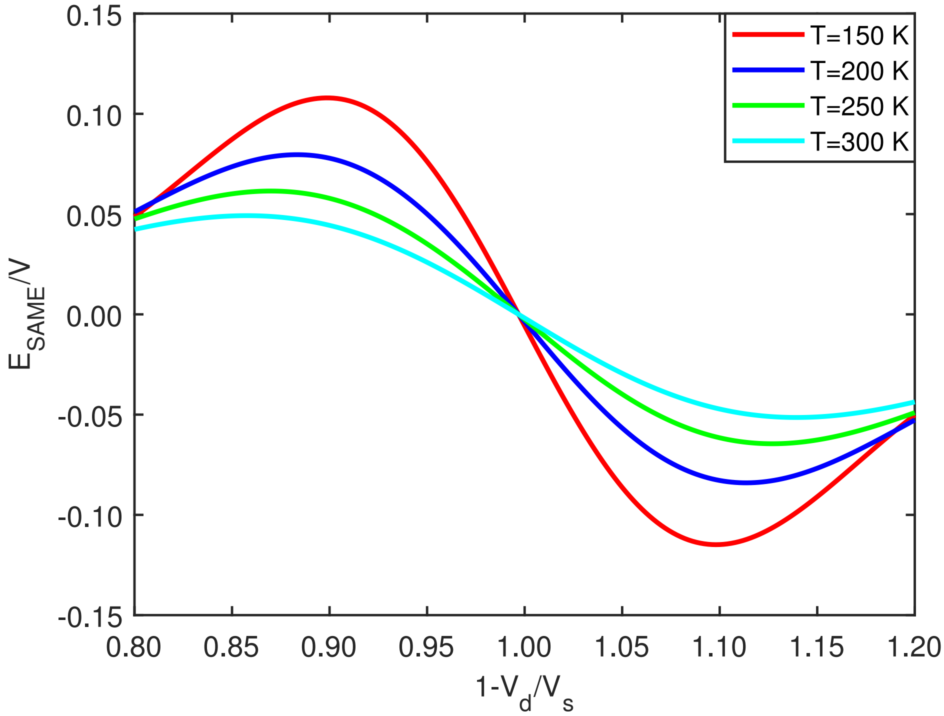

Figure 40: Dependence of Hall-like field E_{SAME} on $1 - v_d/v_s$ for varied Temperature T.

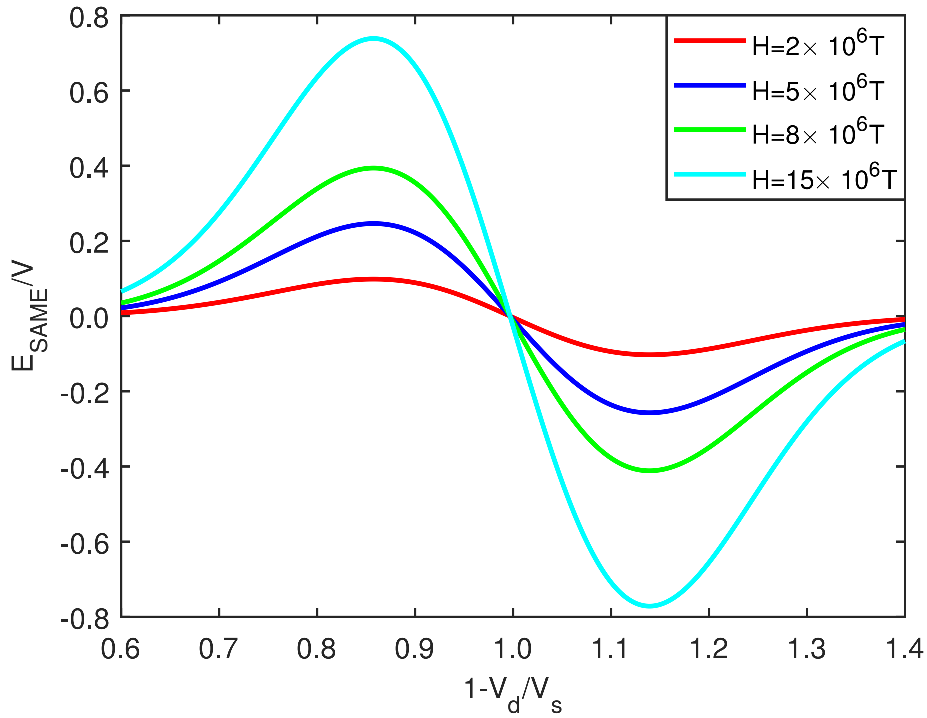

Figure 41: Dependence of Hall-like field E_{SAME} on $1 - v_d/v_s$ for varied Magnetic Field H at $T = 300K$.

continued to rise, $E_o > E_{cr}(\omega_B\tau > 1)$. After being Bragg reflected, additional carriers could perform effective Bloch oscillations in the area of negative differential velocity, $\partial v_d/\partial E_o < 0$. Both v_d and W_{dc} disappearred asymptotically in the limit of a strong electric field, $\omega_B\tau \gg 1$, implying that the carriers were homogeneously distributed within the Brillouin zone.

FSWCNT as a SASER device

A novel idea of monochromatic acoustic phonon amplification in the THz frequency zone was examined in this formulation. Coherent phonons propagated in the forward and backward directions along the FSWCNT due to impulsive phonon stimulation by a picosecond laser pulse, resulting in a stationary acoustic wave. Phonons are created when an acoustic wave interacts with an electrically driven intraminiband transition carrier current within a carrier miniband. The intravalley or intraminiband character of the carrier transport allowed for much higher currents than interminiband carrier and thus, a much stronger phonon amplification by more than 200% has been reported [211].

The pertubation theory of carrier transition is employed, where the FSWCNT carriers are anticipated to migrate away from the lattice ions. carrier-carrier and phonon-phonon interactions as well as phonon losses are ignored in this formulation but the carrier-phonon interactions are considered to be weak and hence treated as a perturbation. As can be seen from Eq. (151), $\Gamma(\omega)/\Gamma_o$ becomes negative whenever $v_d > v_s$, corresponding to amplification of the acoustic phonons. This occurs because the laser field gives the carriers a drift velocity, $v_d = eE_o\Delta_2 a^2/\Omega$. It is worth noting that whenever $\omega_q \gg 6\Delta_1 aq/(1 - v_d/v_s)$, there appears a transparency window i.e $\Gamma(\omega)/\Gamma_o = 0$. This is a consequence of the conservation laws of energy and momentum. Hence, the phonon amplification is obtained for a particular band of phonon wavevectors. This behaviour of the FSWCNT suggests that it can be used as a phonon filter. Again, $\Gamma_q(\omega)/\Gamma_o$ can be quite large when ω_q is reasonably close to $6\Delta_1 aq/(1 - v_d/v_s)$. At resonance, $\omega_q(1 - v_d/v_s) = 6\Delta_1 aq$ and the absorption coefficient becomes so large that the initial criteria are broken down, therefore such a situation is not considered in this thesis. To provide a physical interpretation to Eq. (151), a numerical approach was adopted to model the metrics using the following parameters: $\omega_q = 10^{11}\text{s}^{-1}$, $v_s = 2.5 \times 10^3\text{m/s}$, $\Phi = 10^5\text{Wb/m}^2$ $\ell = 10^{-4}cm$ and $q = 10^6 cm^{-1}$.

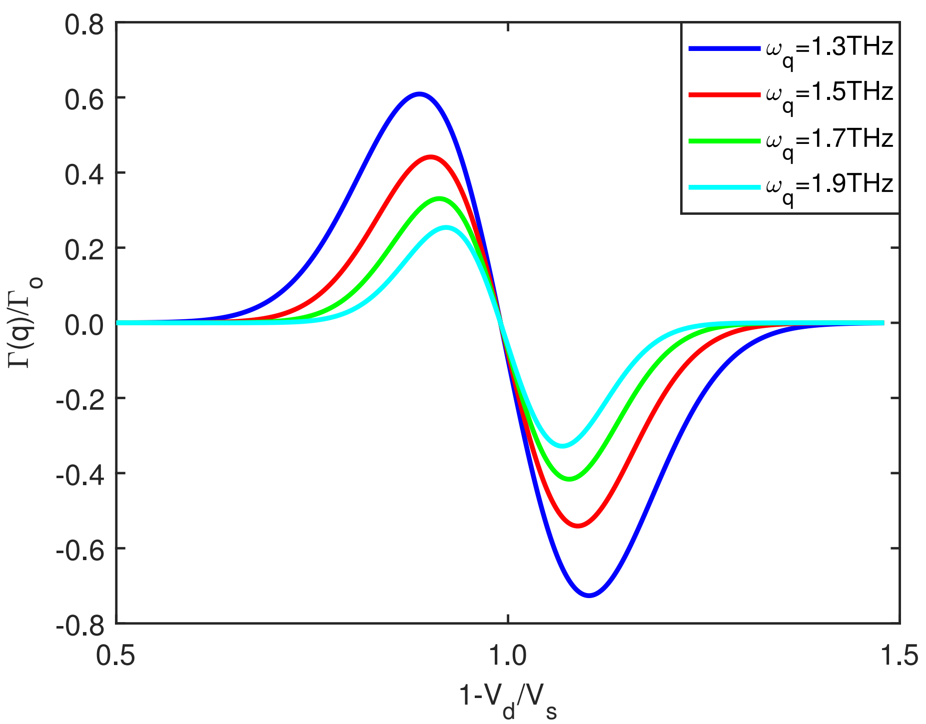

Figure 42: Dependence of $\Gamma_q(\omega)/\Gamma_o$ on $1 - v_d/v_s$ for varied Frequencies ω_q at $T = 300K$.

In Figure 42, a plot of the acoustoelectric gain $\Gamma(\omega)/\Gamma_o$, dependency on $1 - v_d/v_s$, for varied values of acoustic frequencies ω_q, at $T = 300K$ was displayed. It was observed that there is a threshold field of $(1 - v_d/v_s) = 0.61$, where $\Gamma(\omega)/\Gamma_o$ rises to be positive indicating an absorption of the acoustic waves to a maximum value before it falls slowly and approaches zero when $v_d/v_s \approx 1$, and then $\Gamma(\omega)/\Gamma_o$ becomes more negative indicating that amplification surpasses absorption at this point. This is due to a shift in the external electric field's sign. This behaviour was observed for varying values of ω_q. Per numerical analysis for $\omega_q = 1.3THz$, $\Gamma_{amp}/\Gamma_{abs} \approx 1.193$, for $\omega_q = 1.5THz$, $\Gamma_{amp}/\Gamma_{abs} \approx 1.225$, for $\omega_q = 1.7THz$, $\Gamma_{amp}/\Gamma_{abs} \approx 1.259$ and for for $\omega_q = 1.9THz$, $\Gamma_{amp}/\Gamma_{abs} \approx 1.294$. It was inferred that when the carriers kinetic energy was equal or less than that of the sound wave, then the drift velocity (v_s) of the carriers was lower than the velocity of the sound wave in the FSWCNT. This lead to the condition $1 \gg v_d/v_s$ and thus, the acoustic waves transferred their energy and momentum to the carriers, leading to the absorption of acoustic phonons. Under external bias as the acoustic frequency (ω_q) was increased, the phonons acquired high kinetic energy and thus, only a handful undergo intraminiband transition to interact with the carriers. This led to a decrease in magnitude of the gain when the frequencies increased (see Figure 42).

Increasing the external bias further increased v_d slightly above v_s in the FSWCNT, leading to the condition $1 \ll v_d/v_s$ and thus, the carriers transferred their energy and momenta to the acoustic waves, yielding an amplification of acoustic phonons. As the acoustic frequency (ω_q) increases the phonons acquire high kinetic energy and thus, only a handful undergo intraminiband transition to interact with the carriers. This leads to a decrease in magnitude of the gain when the frequencies increases (see Figure 42) [211]. Increasing the electric field further increases the carrier drift velocity slightly above the phase velocity of sound in the FSWCNT.

In otherwords, under the external bias the ratio of carriers with kinetic energy higher than the phonon energy were enhanced, resulting in an enhanced stimulated emission of acoustic phonons and thus, a net gain in the acoustic wave. Amplification happened when carriers transferred more energies and momenta to acoustic waves than their Ohmic losses allowed. Changes in carrier transport characteristics can also be seen as a result of energy transfer. For example, under the correct conditions, the deformation potential coupling can intensify the increase of indigenous acoustic flux, resulting in substantial electrical nonlinearities and the production of acoustoelectric domains with accompanying current oscillations [213].

Increasing temperature decreases the absorption as well as the amplification coefficients for different temperature values as observed in Figure 43. This is because increasing temperature increases the scattering process in the FSWCNT. The majority of carriers in this case acquire a higher velocity, shorter collision time, and attains a higher kinetic energy. This energetic carriers which are the majority carriers undergo interminiband transition allowing only a handful to undergo intraminiband transition. Thus, it is the intraminiband carrier current generated that interacts with the acoustic phonons. Only a few intraminiband carriers can interact with the

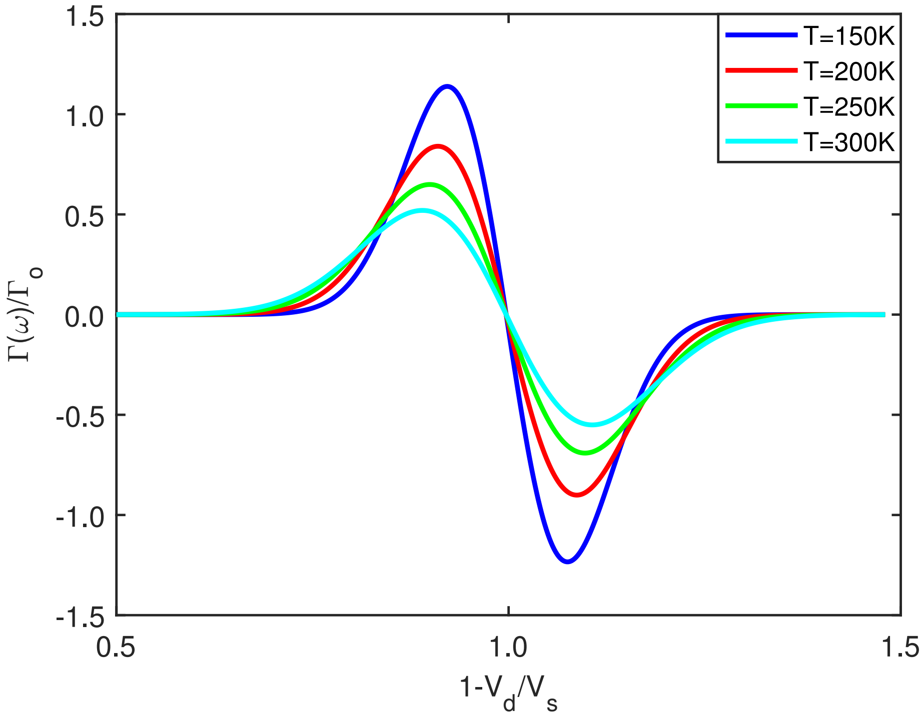

Figure 43: Dependence of $\Gamma_q(\omega)/\Gamma_o$ on $1 - v_d/v_s$ for varied Temperature T.

co-propagating phonons leading to a reduction in stimulated emission of phonons. Evaluating the ratios yields the same ratio for all temperatures i.e. $\Gamma_{amp}/\Gamma_{abs} \approx 1.4$. The graph of the different values of T converges at $1 - v_d/v_s = 0.84$ for $\Gamma(\omega)/\Gamma_o = +0.45cm^{-1}$ and $1 - v_d/v_s = 1.166$ for $\Gamma(\omega)/\Gamma_o = -0.426cm^{-1}$.

The dependence of $\Gamma(\omega)/\Gamma_o$ on $1 - v_d/v_s$ for fixed values of Δ is presented in Figure 44. The amplification surpasses absorption for various values of Δ. A plot of $\Gamma(\omega)/\Gamma_o$ against $1 - v_d/v_s$ for different values of Δ increases gain Γ_{amp} strongly to about 2-folds in comparison to the absorption, Γ_{abs}. At low temperatures, about 3-4 folds of amplification in comparison to the absorption is observed. The computed ratios are as follows: for $\Delta = 0.30eV$, $\Gamma_{amp}/\Gamma_{abs} \approx 2.0$, for $\Delta = 0.35eV$, $\Gamma_{amp}/\Gamma_{abs} \approx 1.75$ and for $\Delta = 0.40eV$, $\Gamma_{amp}/\Gamma_{abs} \approx 1.6$. Δ is responsible for the carrier-phonon interactions in the FSWCNT and so increasing the Δ means strong coupling between the carriers and phonons thus, yielding strong amplification.

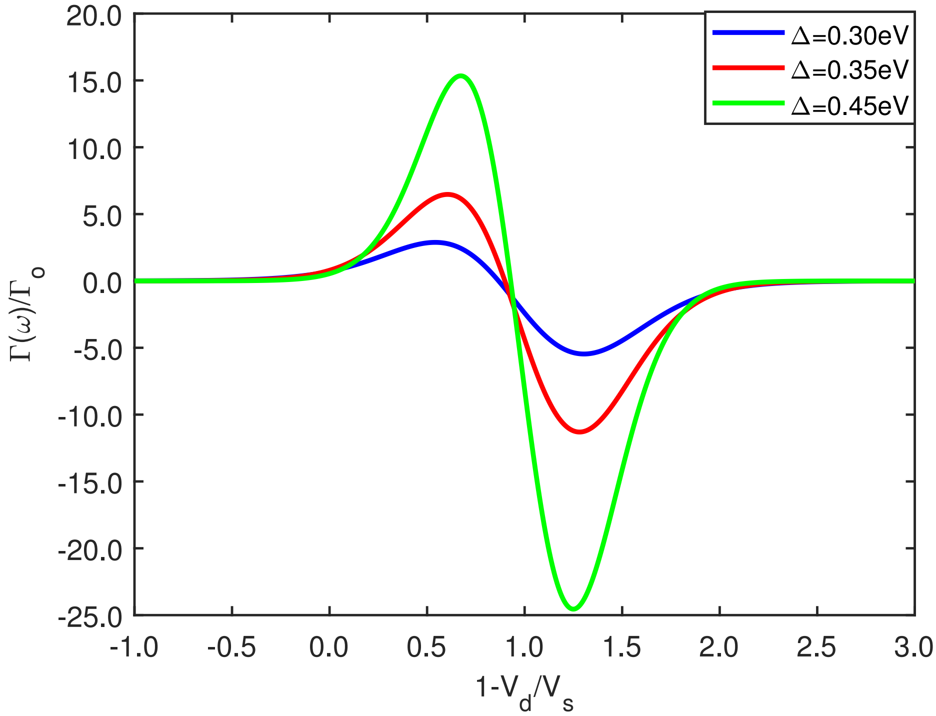

Figure 44: Dependence of $\Gamma_q(\omega)/\Gamma_o$ on $1 - v_d/v_s$ for varied Δ at $T = 300K$.

However, it was realised that the acoustoelectric amplification is high at low temperatures $(T = 100K)$ for the same energies of the carrier-phonon interactions as shown in Figure 45. This is obvious since at low temperature, the scattering of carriers decreases and so more carriers undergo intraminiband transition to interact with the co-propagating acoustic phonons to generate a high acoustoelectric current. The amplification obtained in this case is twice that obtained at $T = 300K$. Thus, if contemporary methods can be employed for the FSWCNT to function at low temperatures, then a high amplification can be obtained.

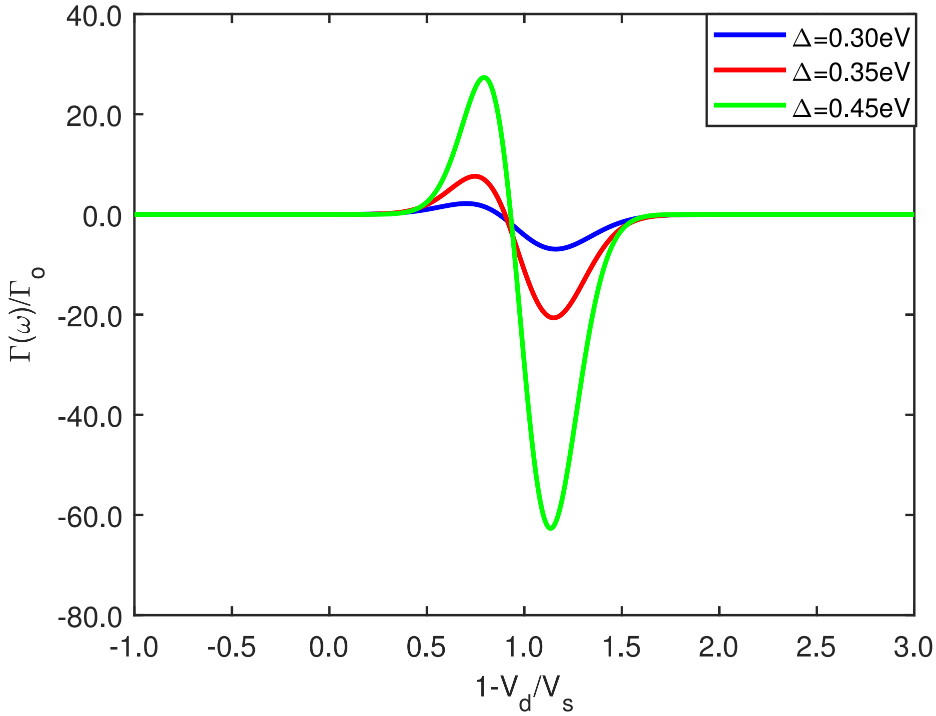

Figure 45: Dependence of $\Gamma_q(\omega)/\Gamma_o$ on $1 - v_d/v_s$ for varied Carrier Concentration n_o at $T = 100K$.

Moreover, the Cerenkov emission requires the carrier velocity surpass sound velocity, if the propagation of the sound was along the z-axis. An electric momentum displacement caused a population inversion of acoustic phonons. However, when the kinetic energy of the carriers was equal to or less than the kinetic energy of the acoustic wave, the net amplification of the acoustic wave was reduced. As a result, the single carrier dynamics were substantially influenced by the sound wave amplitude. Furthermore, different types of carrier oscillations were discovered to be associated with different dynamical regimes, with THz frequencies significantly beyond the GHz frequency of sound waves. The foregoing findings emphasized the importance of events related with carrier interactions with high-frequency phonons. Such results provide up new avenues for studying carrier dynamics in FSWCNTs, in particular.

In Figure 46, $\Gamma(\omega)/\Gamma_o$ is observed to be highly sensitive to the surface carrier concentration, n_o. Although $\Gamma(\omega)/\Gamma_o$ is also very sensitive to v_d as well, $\Gamma(\omega)/\Gamma_o$ for FSWCNT works very well for low and moderate n_o within $10^{15} - 10^{19} cm^{-3}$. Higher n_o increases the amplification without screening out the piezoelectric field to lower the amplification as in heterstructure interface like AlGaN/GaN [213]. In other words, carrier-carrier interaction is neglible in this study. For $n_o = 10^{18} cm^{-3}$, $\Gamma_{amp}/\Gamma_{abs} \approx 1.10$, for $n_o = 5 \times 10^{18} cm^{-3}$, $\Gamma_{amp}/\Gamma_{abs} \approx 1.11$, for $n_o = 1 \times 10^{19} cm^{-3}$, and $\Gamma_{amp}/\Gamma_{abs} \approx 1.20$. Under

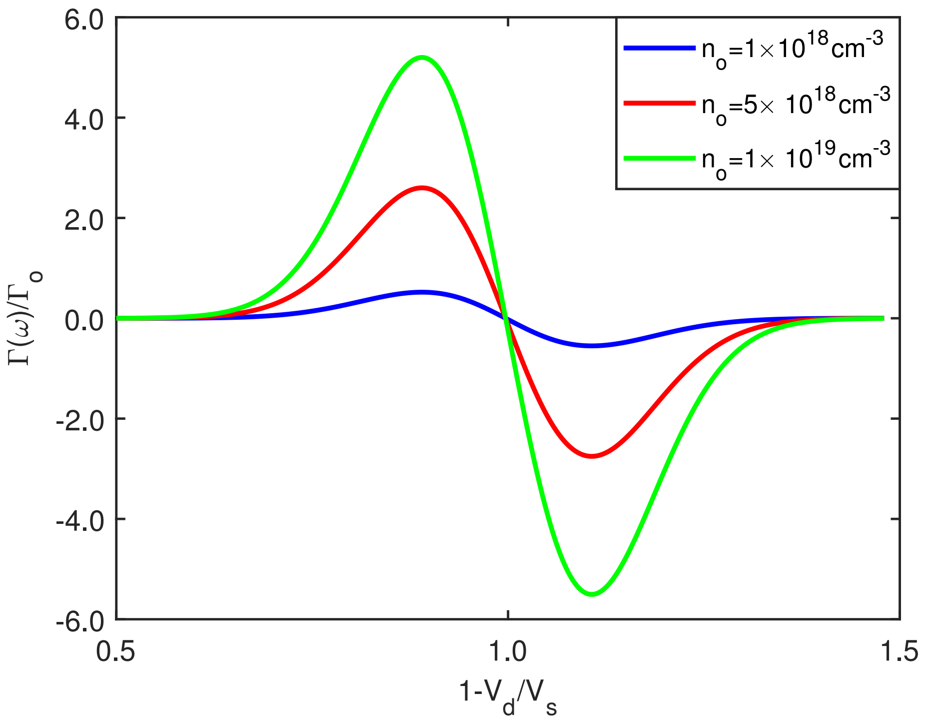

Figure 46: Dependence of $\Gamma_q(\omega)/\Gamma_o$ on $1 - v_d/v_s$ for varied Carrier Concentration n_o at $T = 300K$.

external bias, the intraminiband carriers interacted with the coherent phonon excitations mostly by means of deformation potential coupling. Piezoelectric coupling [217, 218] was feeble as a result of the lattice symmetry and screening at high carrier densities. Carriers with kinetic energies higher than the FSWCNT phonon energy enhanced the sound amplitude by stimulated intraminiband discharge of acoustic phonons, which exploited the population inversion between carrier states in the same miniband. This Cerenkov instrument acted mostly on the acoustic wave moving in the forward direction, i.e., in the same direction as the carriers. In other words, the electric predisposition moved the Fermi distribution of carriers to large k-vectors, presenting an asymmetric carrier distribution in k-space and enabled phonon amplification.

Electrical resistivity in FSWCNT

Numerical results for the study of the axial resistivity ρ_{zz} in Eq. (176) dependence on T was shown in Figures 47-50 for varied values of the overlapping integral for jumps $\Delta_{s,z}$, carrier concentration n_o and $dc-$field E_o. Figure 47 showed that the E_o increased ρ_{zz} from E_o to $3E_o$ but decreased beyond $3E_o$. This was because increasing the $dc-$field from E_o to $3E_o$ energized the carrier gas but weakly.

Thus, only a few of the carriers undergo intraminiband transport, resulting in low intraminiband current and a low conductivity and thus, a high electrical resistivity, ρ_{zz}. Furthermore, an increase to $4E_o$ energises the carriers moderately and the number of carriers which undergoes intraminiband transport increases, resulting in a high conductivity and thus, a low resistivity.

Figure 48 showed that, varying Δ_s and keeping Δ_z fixed decreased ρ_{zz} slowly due to the strong carrier scattering along Δ_z. A profound decrease in ρ_{zz}

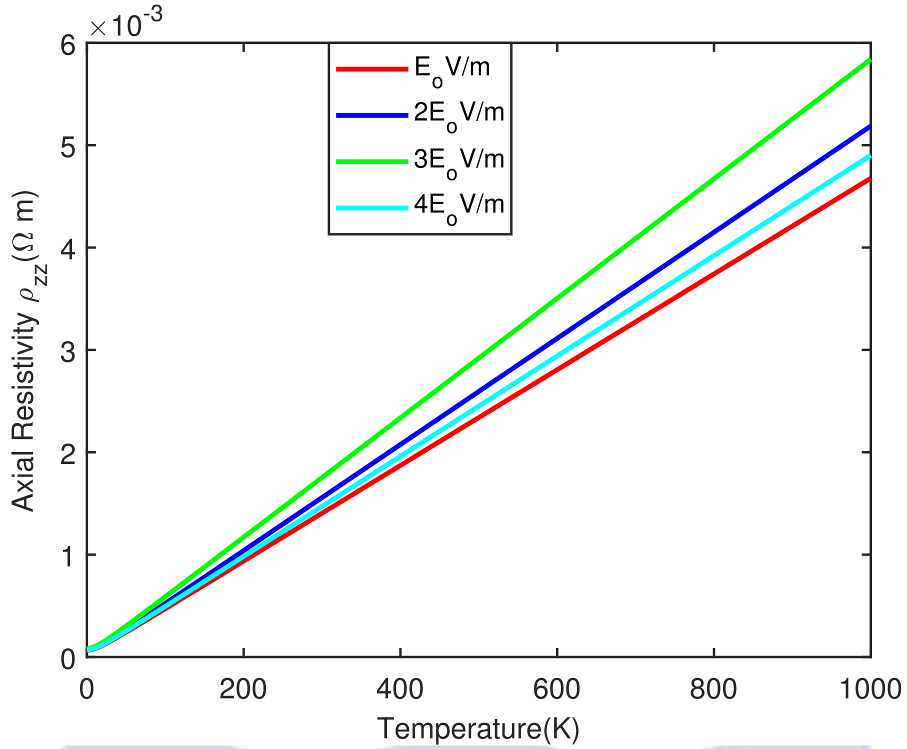

Figure 47: Dependency of Axial Resistivity (ρ_{zz}) on Temperature for varied Values of E_o with $\Delta_s = 0.015eV$, $\Delta_z = 0.024eV$, $n_o = 10^{19}cm^{-3}$, and $E_z = 10^2V/m$.

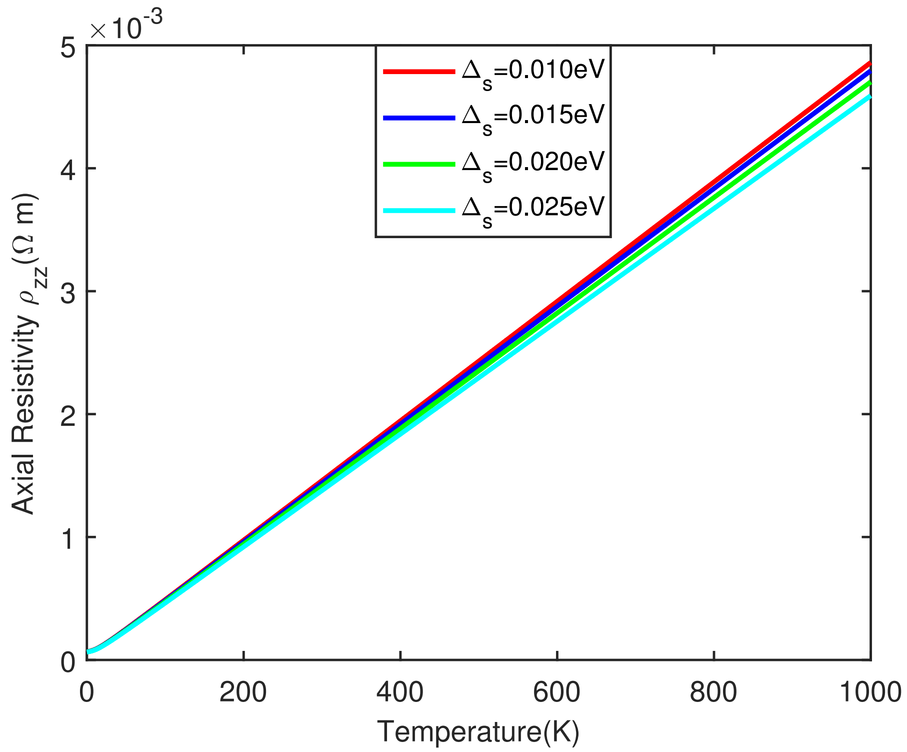

Figure 48: Dependency of Axial Resistivity (ρ_{zz}) on Temperature for varied Values of Δ_s with $\Delta_z = 0.024eV$, $n_o = 10^{19}cm^{-3}$, $E_o = 10^4V/m$, and $E_z = 10^2V/m$.

was observed for varying Δ_z and keeping Δ_s fixed (see Figure 49). This was because varying Δ_z reduced the carrier scattering and so the differetial conductivity increased and thus, decreasing ρ_{zz}. Moreover, the dependency of ρ_{zz} on temperature for varied carrier n_o was displayed in Figure 50. The ρ_{zz} decreased for varying values of n_o. This was due to the drop in carrier scatterered along the axial direction where carrier-carrier interaction was minimal. However, FSWCNT exhibited anomalous resistance $R(T)$ behaviour. This behaviour, could be explained with phenomena such as weak localization contact and tube-tube interactions within a bundle [225-227] in an array of FSWCNT.

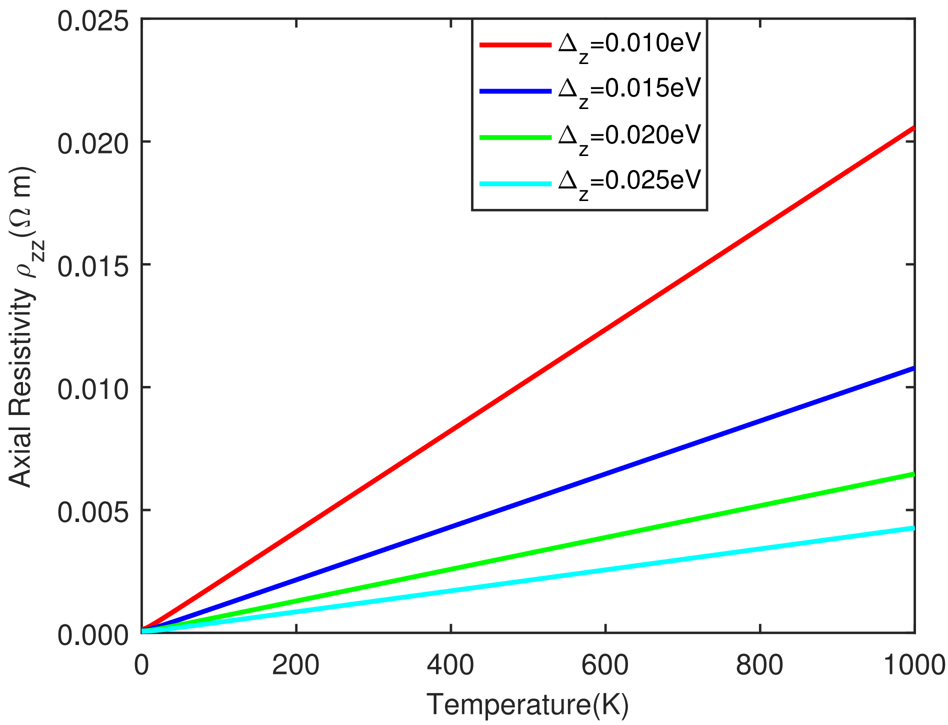

Figure 49: Dependency of Axial Resistivity (ρ_{zz}) on Temperature for varied Values of Δ_z with $\Delta_s = 0.024eV$, $n_o = 10^{19}cm^{-3}$, $E_o = 10^4V/m$, and $E_z = 10^2V/m$.

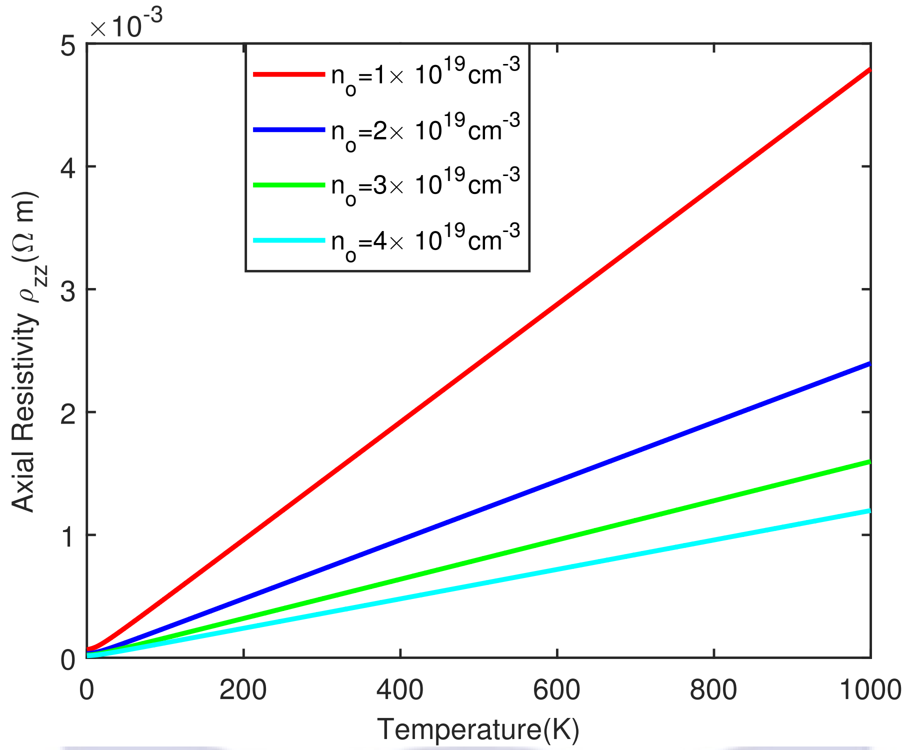

Figure 50: Dependency of Axial Resistivity (ρ_{zz}) on Temperature for varied Values of n_o with $\Delta_s = 0.015eV$, $\Delta_z = 0.024eV$, $E_o = 10^4 V/m$, and $E_z = 10^2 V/m$.

Giant thermoelectric power in FSWCNT

A calculation of the axial current density in the presence of both applied field, $\vec{E}$, and temperature gradient, ∇T, which yielded the differential electrical conductivity, σ, and the thermoelectric power, α was obtained. The thermoelectric power, α, was highly anisotropic and depended on the carrier concentration, n_o, external field $\vec{E}$, temperature, T, and carrier-phonon parameter, Δ_s and Δ_z. For an further understanding of the complex equations obtained in Eq. (177) and Eq. (179), a numerical analysis was performed but of most interest to the study was the axial component in Eq. (179).

Figures 51-54 illustrate the axial thermopower, α_{zz}, dependency on temperature, T, with the identical values of Δ_s, Δ_z, n_o, and E_o. Figures 51-54

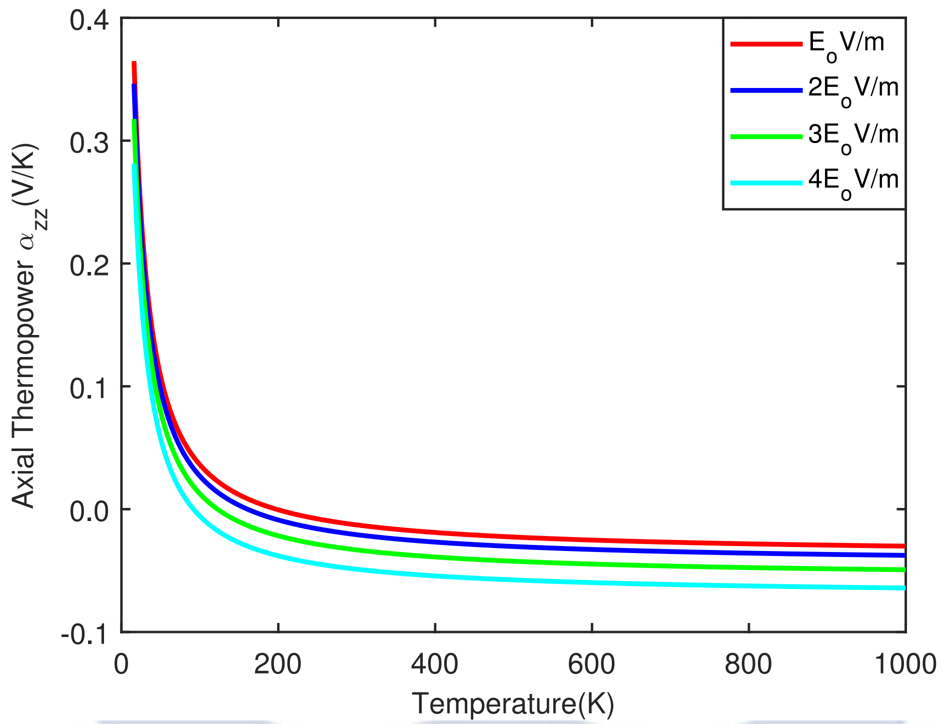

Figure 51: Dependency of Axial Thermopower (α_{zz}) on Temperature for varied values of: E_o with $\Delta_s = 0.015eV$, $\Delta_z = 0.024eV$, $n_o = 10^{19}cm^{-3}$, and $E_z = 10^3V/m$.

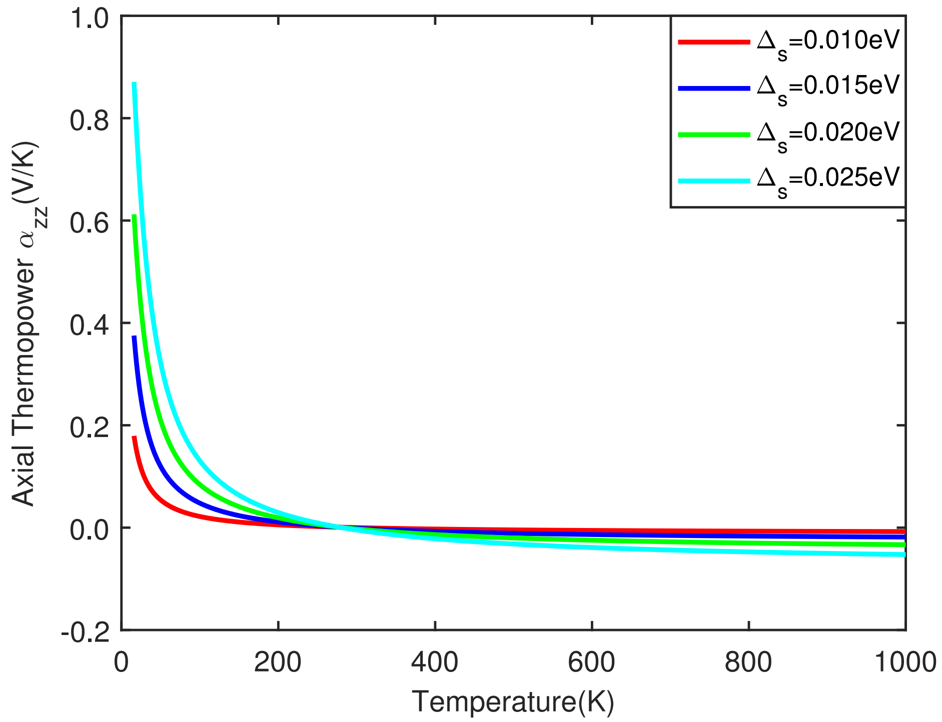

Figure 52: Dependency of Axial Thermopower (α_{zz}) on Temperature for varied Values of: Δ_s with $\Delta_z = 0.024eV$, $n_o = 10^{19}cm^{-3}$, $E_o = 10^4V/m$, and $E_z = 10^3V/m$.

showed that α_{zz} dropped with rising temperature and gradually led to a low fixed value at high temperatures (i.e, hyperbolic). This behaviour had been observed in semiconducting nanotubes with $\alpha_{zz} \approx T^{-1}$ [219]. The axial thermopower α_{zz} in Figure 51 was high at low temperatures, but it rapidly declined as the temperature increased, switching from a p-type to a n-type behaviour. The FSWCNT functioned completely as a p-type semiconductor in Figure 52, but when the value of Δ_s grew, it switched to a n-type behaviour at higher temperatures. For all values of Δ_s, the transition from p-type to n-type occurred at the same temperature. Figure 53 showed similar behaviour when Δ_z was changed; however, the transition occurred at low and different temperatures for the various Δ_z values. This pattern indicated that as Δ_z increased above a certain point, α_{zz} became exclusively p-type. Owing to charge transfer from the SWCNT to the fluorine doping, α_{zz} drifts steadily from a n-type to a p-type as the doping concentration increased (see Figure 54). The n_o, Δ_s, and Δ_z tunable behaviour of α_{zz} has the potential for room temperature thermoelectric application.

When comparing the results of this study to the experimentally measured α_{zz} in Ref. [220], it could be seen that the theoretical curves and the experimental curves coincide pretty well. Surprisingly, turning points occurred in two situations: (i) when Δ_z was significantly greater than Δ_s, and (ii) when Δ_s was significantly greater than Δ_z. At 298K, the dominant carrier shifted from hole to carrier, indicating that there was a temperature threshold. TEP exhibited this behaviour for weakly interacting systems at low lattice temperatures under linear response circumstances. Boundary scattering

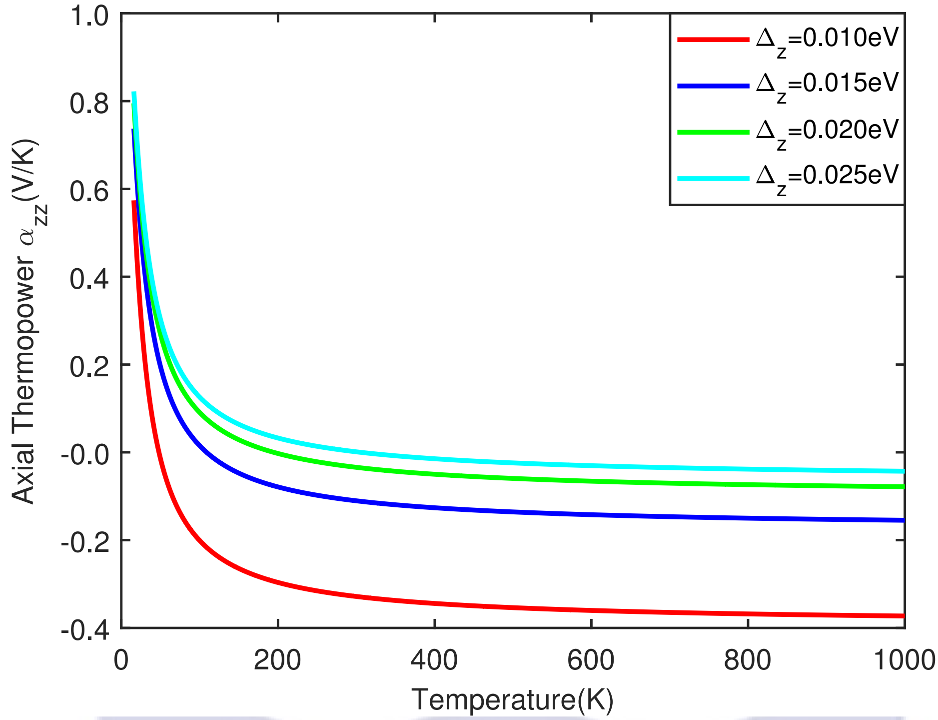

Figure 53: Dependency of Axial Thermopower (α_{zz}) on Temperature for varied values of: Δ_z, with $\Delta_s = 0.024eV$, $n_o = 10^{19}cm^{-3}$, $E_o = 10^4V/m$, and $E_z = 10^3V/m$.

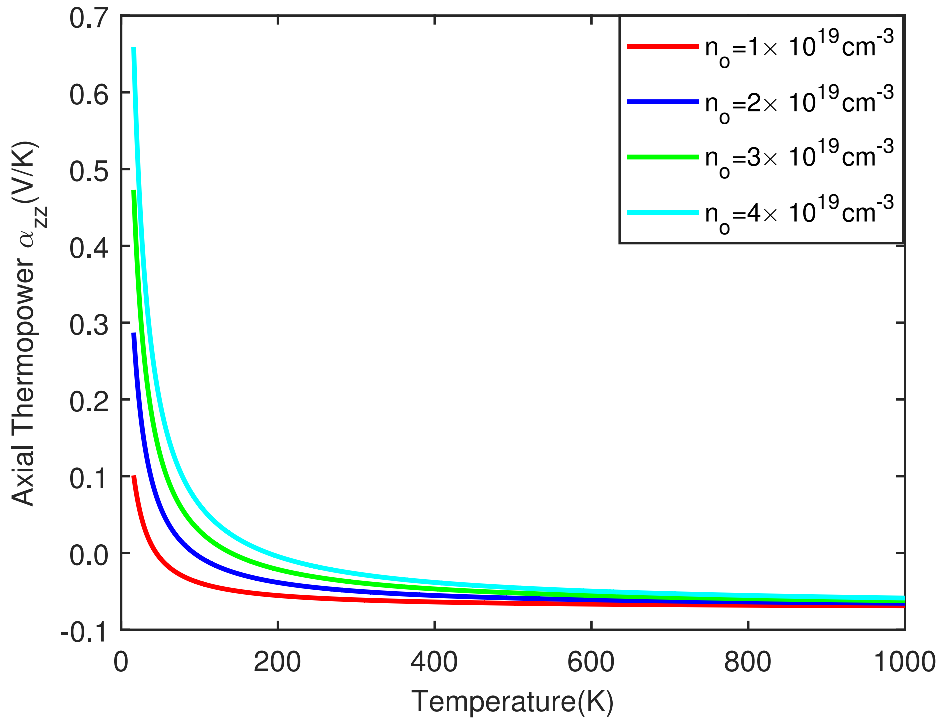

Figure 54: Dependency of Axial Thermopower (α_{zz}) on Temperature for varied Values of: n_o with $\Delta_s = 0.015eV$, $\Delta_z = 0.024eV$, $E_o = 10^4V/m$, and $E_z = 10^3V/m$.

determined the phonon relaxation time here, but at high lattice temperatures, phonon-phonon scatterings were prominent in defining the phonon relaxation time. The drag TEP was maximum if the phonons were scattered mainly by carriers. However, it was obvious that the large carrier concentration of FSWCNT was solely responsible for the phonon scattering and thus, ensured that the TEP was dependent on the phonon drag mechanism [221, 222].

Figures 55 and 56 demonstrated α_{zz}-temperature dependency in the presence and absence of the $ac-$ and $dc-$fields. When the ac source E_z intensity was between 0 and $2.7 \times 10^2 V/m$, α_{zz} did not vary. Figure 55 depicted this, with α_{zz} (ac off) and α_{zz} (ac on) overlapping. In this situation, the ac component modulated the dc component. The carriers with no energy have their momenta and kinetic energies updated by the ac. As a result, the

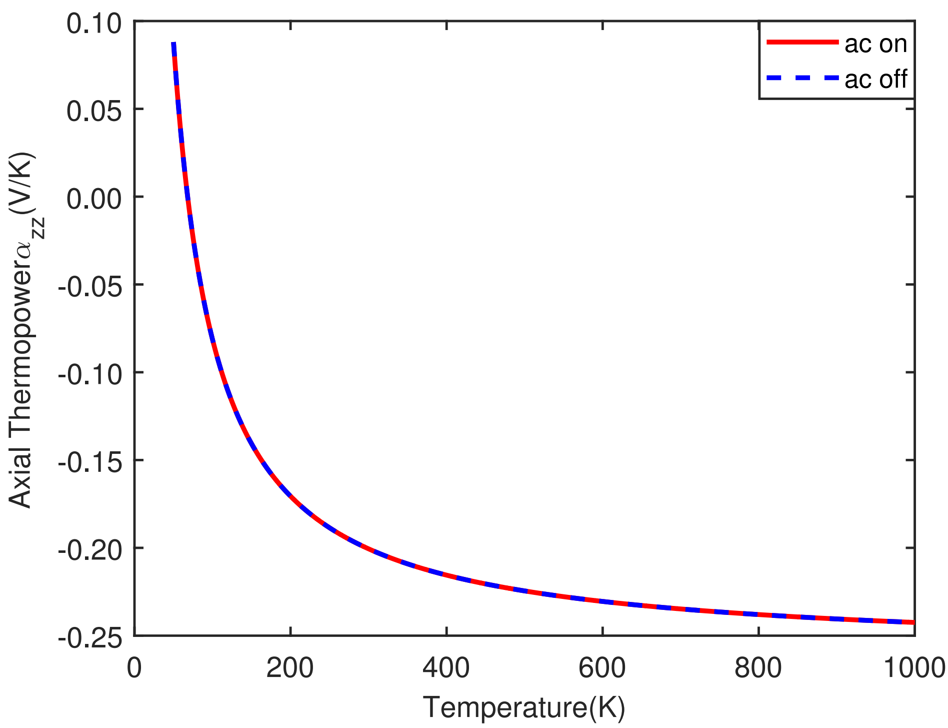

Figure 55: Dependency of Axial Thermopower (α_{zz}) on Temperature for $n_o = 10^{19} cm^{-3}$, $E_o = 10^4 V/m$, $E_z = 10^2 V/m$, $\Delta_s = 0.024eV$ and $\Delta_z = 0.015eV$ for $ac-$on and $ac-$off.

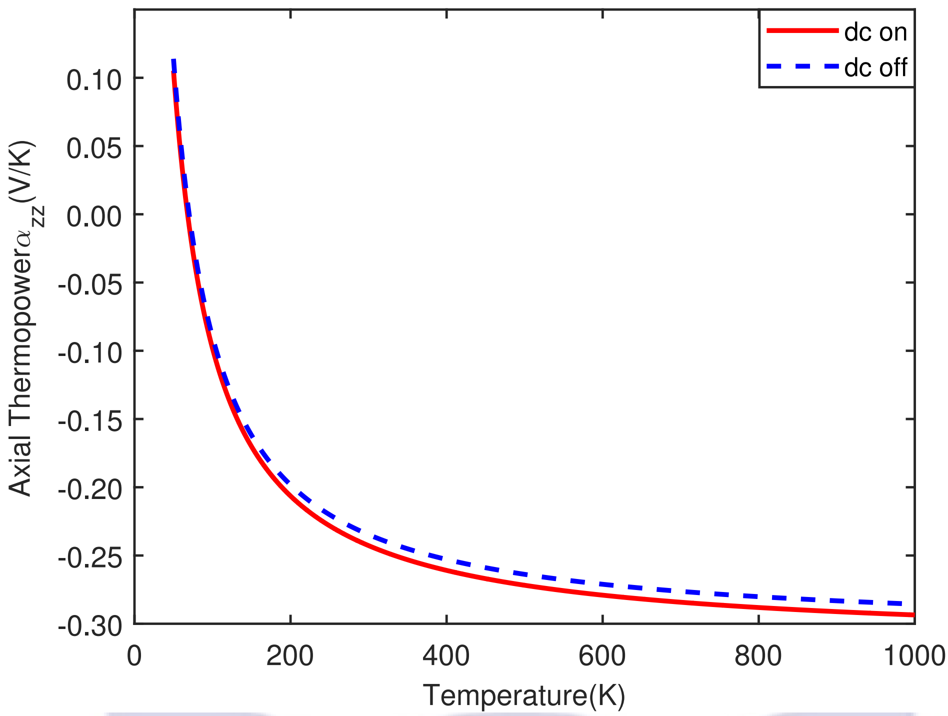

Figure 56: Dependency of Axial Thermopower (α_{zz}) on Temperature for $n_o = 10^{19}cm^{-3}$, $E_o = 10^4 V/m$, $E_z = 10^2 V/m$, $\Delta_s = 0.024eV$ and $\Delta_z = 0.015eV$ for $dc-$on and $dc-$off.

bulk of carriers with the requisite momenta and energies transition into the intraminiband, generating a significant intraminiband current to augment α_{zz}. Figure 56, on the other hand, showed a drop in α_{zz} when E_z was turned off, because just a few carriers now had the momentum and energy to generate the intraminiband current.

The dependency of axial thermopower for FSWCNT, α_{zz}, and that for SL, α_{zz}^{SL}, on temperature was displayed in Figure 57. The α_{zz} was observed to be far greater than α_{zz}^{SL} (i.e $\alpha_{zz} \gg \alpha_{zz}^{SL}$). This high value could be attributable to a variety of FSWCNT mechanisms, including double periodicity, carrier concentration, and strong phonon drag mechanism.

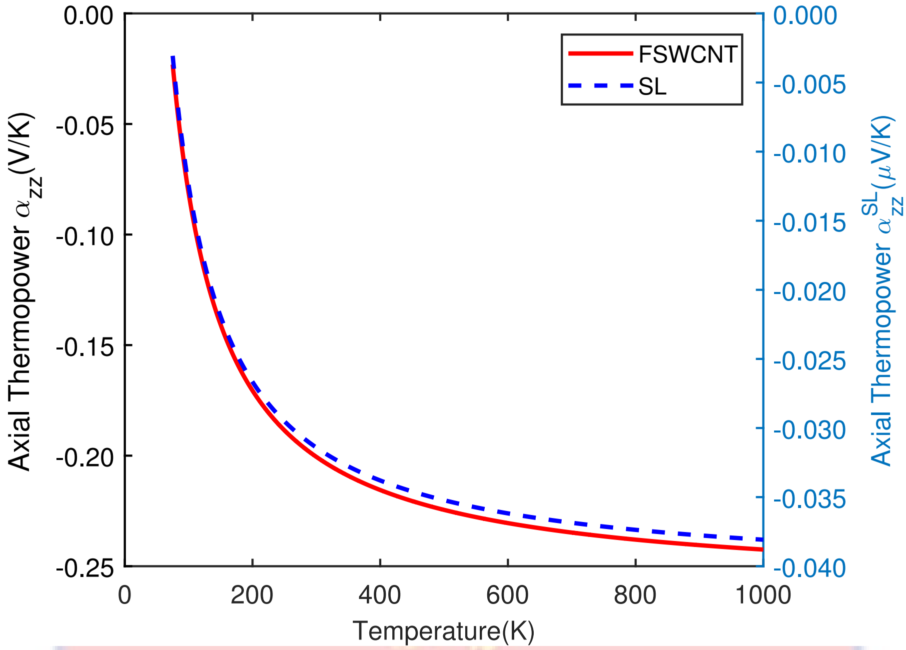

Figure 57: Dependency of Axial Thermopower (α_{zz}) on Temperature for: (left) FSWCNT ($\Delta_s = 0.030eV$ and $\Delta_z = 0.015eV$) and (right) SL ($\Delta = 7meV$ and $d = 12.5nm$) at $n_o = 10^{19}cm^{-3}$, $E_o = 10^4V/m$, and $E_z = 10^2V/m$.

Tunable electrical power factor in FSWCNT

The electrical power factor of the nondegenerate FSWCNT was calculated using a tractable mathematical technique [223, 224]. The electrical power factor $\mathscr{P}$ was extremely nonlinear in terms of carrier concentration n_o, external field E, temperature T, and the real overlapping integrals for jumps (Δ_s and Δ_z). For an understanding of Eq. (180) and Eq. (181), a numerical analyses of the axial component, $\mathscr{P}_{zz}$, was performed which was of more interest to the study. The dependency of $\mathscr{P}_{zz}$ on temperature for same values of Δ_s, Δ_z, n_o, and E_o used in Figures 51- 57 was presented in Figures 58-61.

When Δ_s was less than Δ_z, $\mathscr{P}_{zz}$ increased to a resonance maximum and decreased gradually. The majority of the carriers that contributed to $\mathscr{P}_{zz}$ were electrons, as shown by the trend in Figures 58-61. Owing to the intraminiband current created by states with energy $\varepsilon(\vec{p}) > \mu$, when $\Delta_s < \Delta_z$, the intraminiband current grew for different values of static field E_o, as $\mathscr{P}_{zz}$ increased (see Figure 58). Increased carrier-phonon interactions along the circumerential path, on the other hand, increased scattering likelihood, which resulted in a reduction in the number of carriers that performed the intraminiband transition. As a result, the generated intraminiband current decreased, affecting the intensity of $\mathscr{P}_{zz}$ (see Figure 59). Figure 60 showed similar results; however, owing to the increased carrier-phonon interactions along the axial direction Δ_z, the maximum value of $\mathscr{P}_{zz}$ was higher than in Figure 59 and was found at low values of Δ_z ($\Delta_s > \Delta_z$). This lowered the scattering probability, which resulted in majority of carriers performing intraminiband transition. Thus, the intraminiband current generated increased, increasing the intensity of $\mathscr{P}_{zz}$. A profound increase in $\mathscr{P}_{zz}$ was observed as carrier concentration was changed in Figure 61. The magnitude of $\mathscr{P}_{zz}$ was unaffected by this, instead it caused the $\mathscr{P}_{zz}$ peak to shift to a higher temperature. The $\mathscr{P}_{zz}$'s tunability suggested that it could be used throughout a higher temperature range. By filtering the piezoelectric field, increasing n_o kept the peak of $\mathscr{P}_{zz}$ constant. Figure 62 is an expansion of Figure 61 that demonstrated how the axial and circumferential carrier-phonon parameters, in addition to n_o, contributed to the tunability of $\mathscr{P}_{zz}$.

Figures 63 and 64 demonstrated $\mathscr{P}_{zz}$ dependency on temperature in the presence and absence of $ac-$ and $dc-$fields. $\mathscr{P}_{zz}$ did not vary when the intensity of the ac source (E_z) was between 0 and $2.7 \times 10^2 V/m$ ($0 < E_z < 2.7 \times 10^2$). This was seen in Figure 63, where $\mathscr{P}_{zz}$ (ac off) and $\mathscr{P}_{zz}$

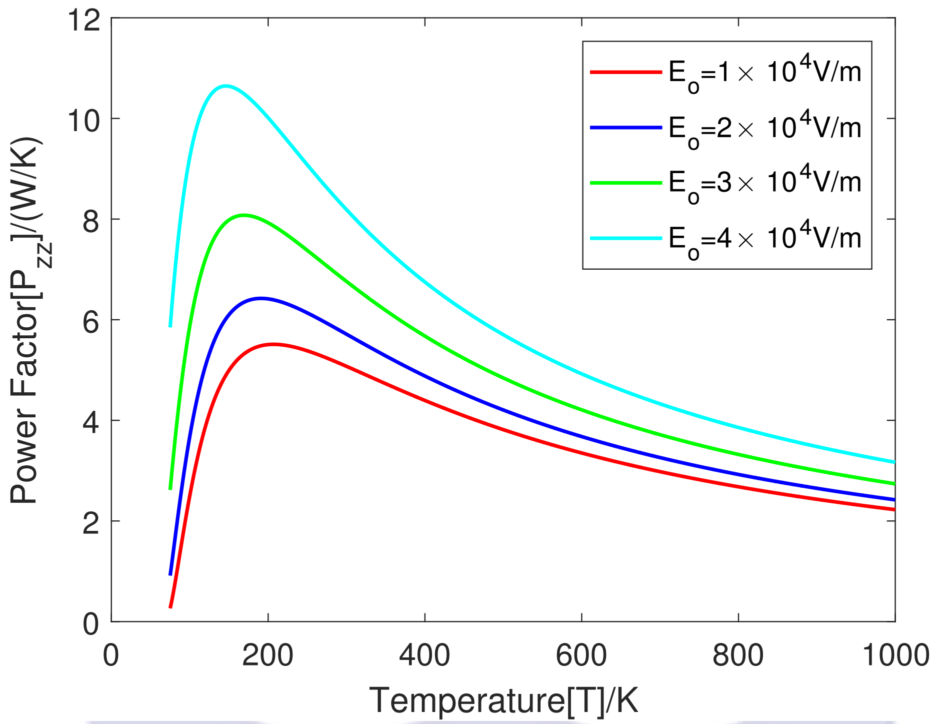

Figure 58: Dependency of Axial Power Factor ($\mathscr{P}_{zz}$) on Temperature for varied Values of: E_o with $\Delta_s = 0.015eV$, $\Delta_z = 0.024eV$, $n_o = 10^{19}cm^{-3}$, and $E_z = 10^2V/m$.

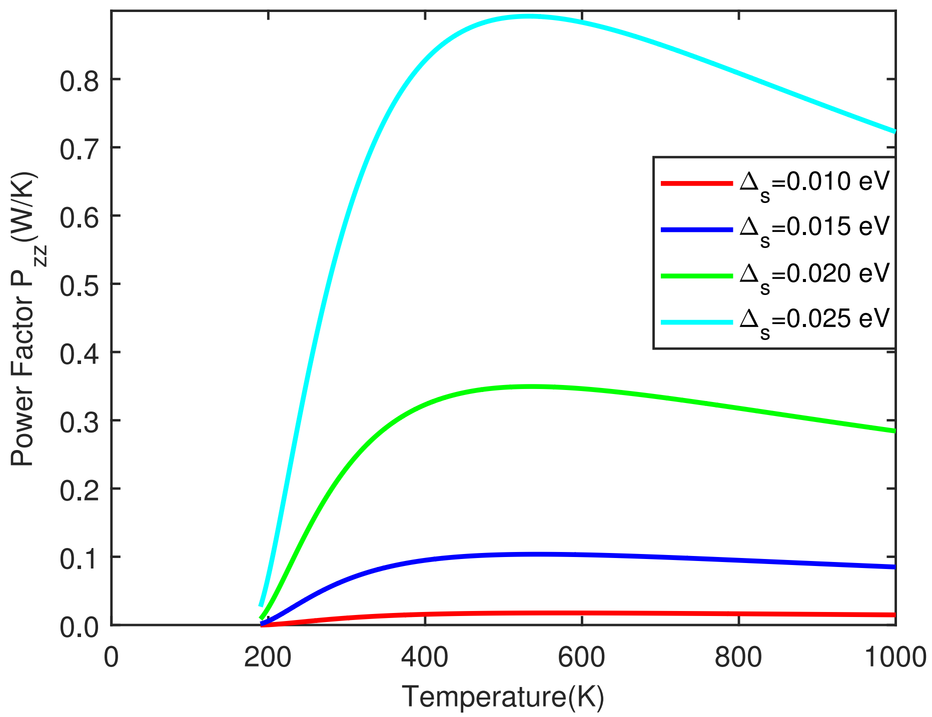

Figure 59: Dependency of Axial Power Factor ($\mathscr{P}_{zz}$) on Temperature for varied Values of: Δ_s with $\Delta_z = 0.024eV$, $n_o = 10^{19}cm^{-3}$, $E_o = 10^4V/m$, and $E_z = 10^2V/m$.

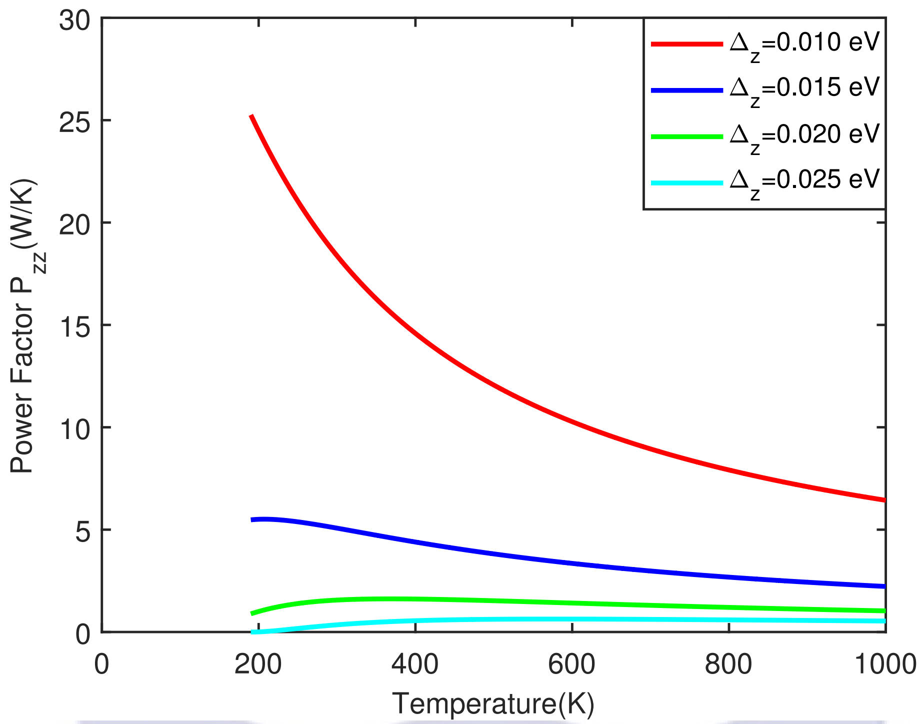

Figure 60: Dependency of Axial Power Factor ($\mathscr{P}_{zz}$) on Temperature for varied Values of: E_o with $\Delta_s = 0.015eV$, $\Delta_z = 0.024eV$, $n_o = 10^{19}cm^{-3}$, and $E_z = 10^2V/m$.

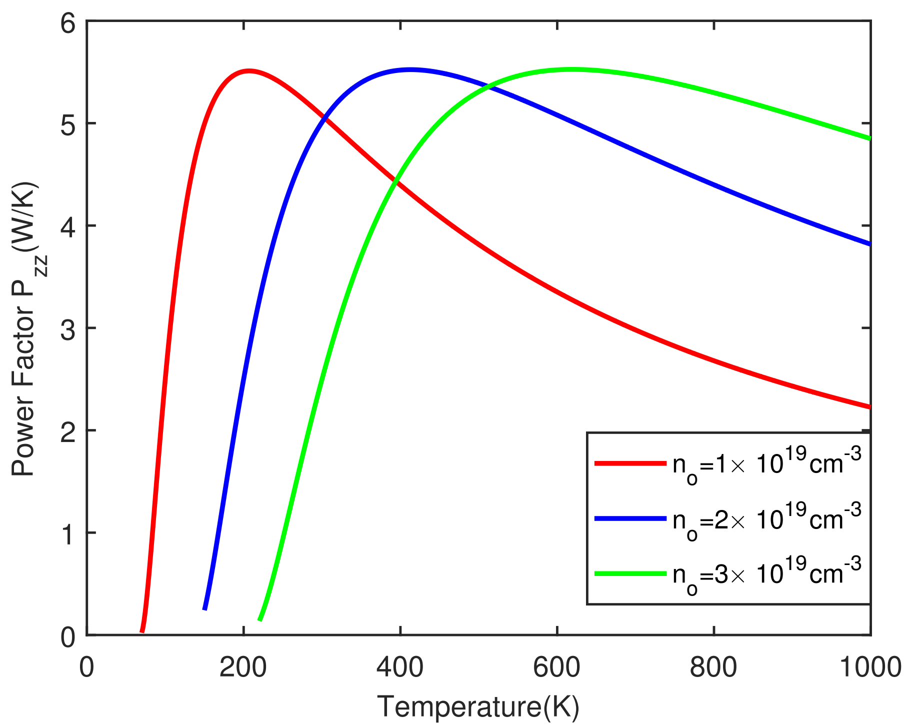

Figure 61: Dependency of Axial Power Factor ($\mathscr{P}_{zz}$) on Temperature for varied Values of: Δ_s with $\Delta_z = 0.024eV$, $n_o = 10^{19}cm^{-3}$, $E_o = 10^4V/m$, and $E_z = 10^2V/m$.

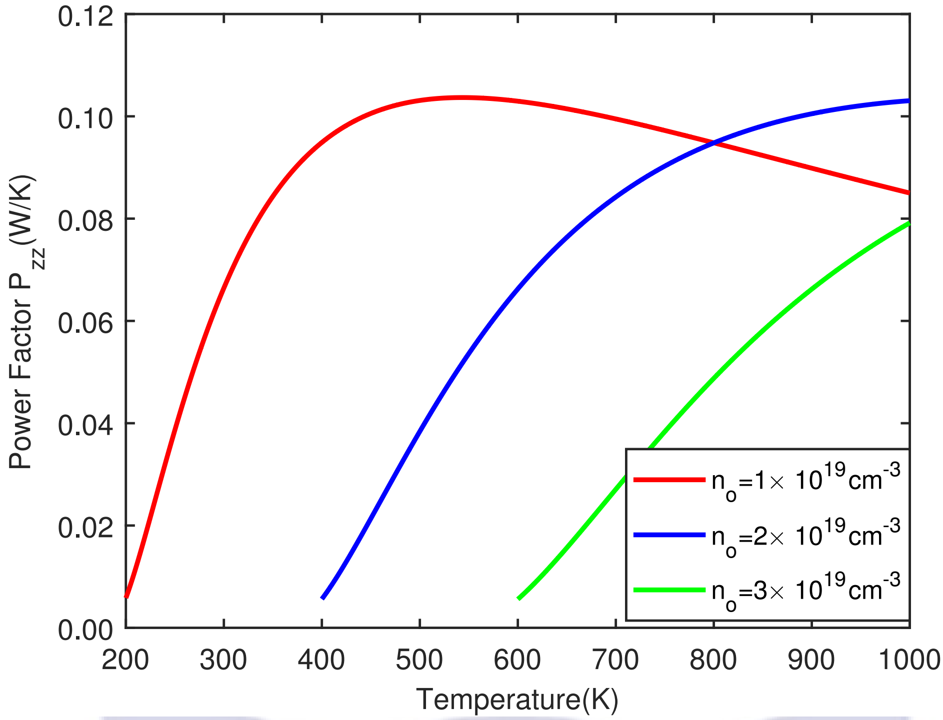

Figure 62: Dependency of Axial Power Factor ($\mathscr{P}_{zz}$) on Temperature for Different Values of n_o at: $\Delta_s = 0.015eV$, $\Delta_z = 0.024eV$, $E_o = 10^4 V/m$, and $E_z = 10^2 V/m$.

(*ac* on) intersected. In this case, the *ac* modulated the *dc* component. The *ac* updated the momenta and kinetic energies of carriers that had less energies. To enhance $\mathscr{P}_{zz}$, the majority of carriers with requisite momenta and energies performed intraminiband transition and generated a large intraminiband current. When E_z was turned off, however, Figure 64 showed a drop in $\mathscr{P}_{zz}$ because just a few carriers had the appropriate energies to generate the intraminiband current.

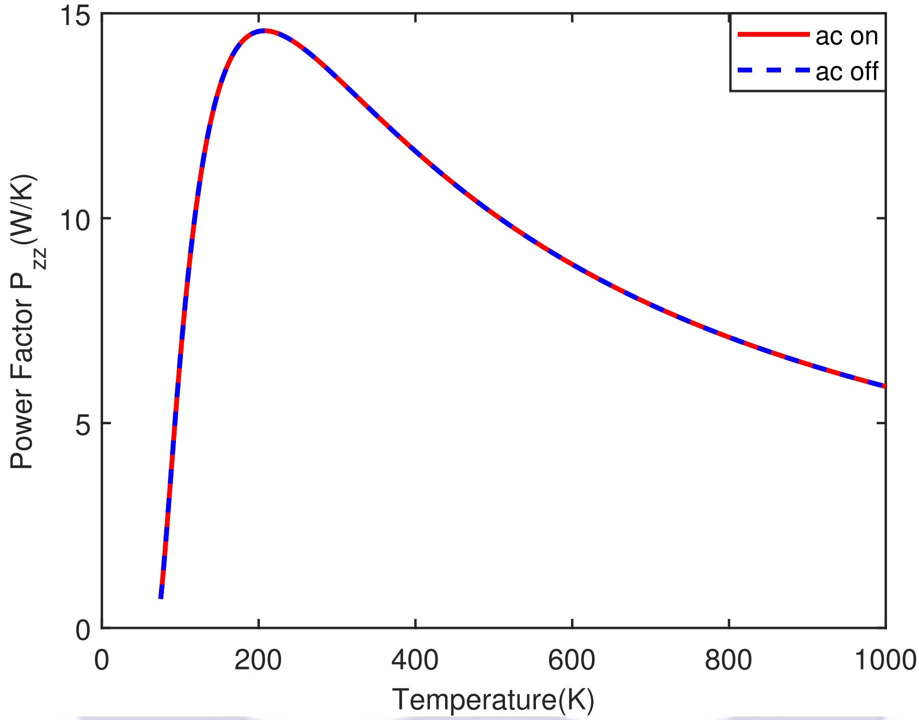

Figure 63: Dependency of Axial Power Factor ($\mathscr{P}_{zz}$) on Temperature for $n_o = 10^{19} cm^{-3}$, $E_o = 10^4 V/m$, $E_z = 10^2 V/m$, $\Delta_s = 0.024eV$ and $\Delta_z = 0.015eV$ for $ac-$on and $ac-$off.

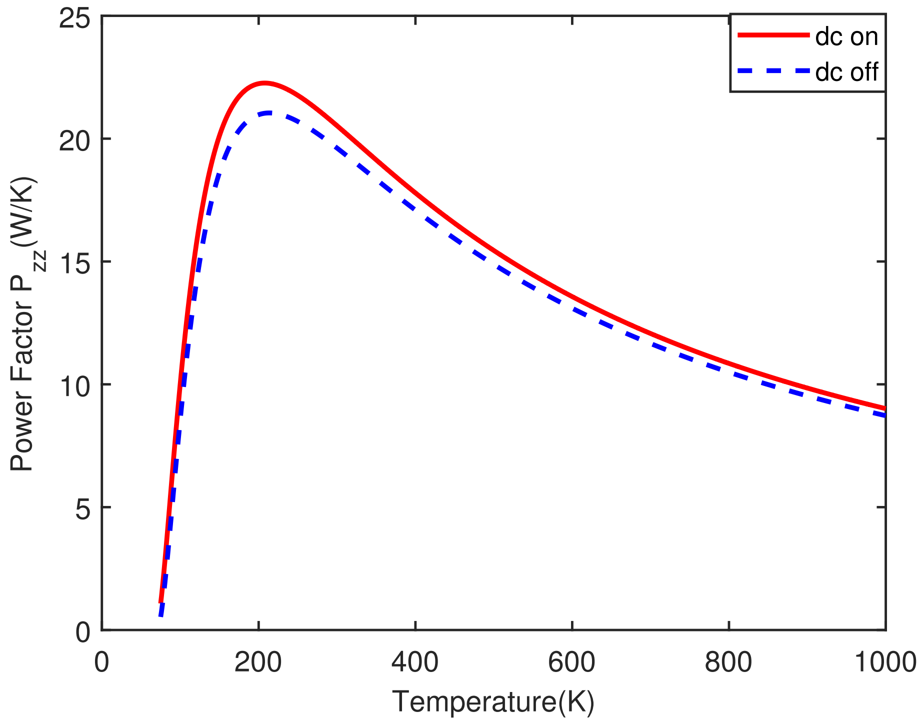

Figure 64: Dependency of Axial Power Factor ($\mathscr{P}_{zz}$) on Temperature for $n_o = 10^{19} cm^{-3}$, $E_o = 10^4 V/m$, $E_z = 10^2 V/m$, $\Delta_s = 0.024eV$ and $\Delta_z = 0.015eV$ for $dc-$on and $dc-$off.

Practically, a comparison was made between results obtained in this work and that of SL. The energy band for the lowest miniband carriers for the SL was given as:

$$\varepsilon_{SL}(\vec{p}_z) = \frac{\Delta}{2}(1 - \cos\frac{\vec{p}_z d}{\hbar}), \tag{291}$$

where the carrier miniband velocity $v(\vec{p}_z) = (\Delta d/\hbar)\sin(\vec{p}_z d/\hbar)$. Following the approach for FSWCNT, the axial current density for the SL was obtained as:

$$\vec{J}_z^{SL} = -\frac{\sigma(\vec{E})}{e}\left\{\nabla_z\left(\frac{\mu}{e} - \varphi\right) + \left[(\Delta - \mu) + 3k_B T - \Delta\frac{I_o}{I_1}\right]\frac{\nabla_z T}{T}\right\}. \tag{292}$$

α_{zz}^{SL} was the thermopower given as:

$$\alpha_{zz}^{SL} = \frac{k}{e}\left\{\left(\frac{\Delta - \mu}{k_B T}\right) + 3 - \Delta^*\frac{I_o(\Delta^*)}{I_1(\Delta^*)}\right\}\Theta, \tag{293}$$

where $\Theta = [1 + (e\vec{E}d\tau)^2]/[1 + (2e\vec{E}d\tau)^2]$. For weak electric field, $e\vec{E}d\tau \ll 1$,

$$\alpha_{zz}^{SL} = \frac{k_B}{e}\left\{\left(\frac{\Delta - \mu}{k_B T}\right) + 3 - \Delta^*\frac{I_o(\Delta^*)}{I_1(\Delta^*)}\right\}, \tag{294}$$

and when $\Delta \ll k_B T$, α_{zz}^{SL} yields:

$$\alpha_{zz}^{SL} = \frac{k}{e}\left\{-\frac{\mu}{k_B T} + 1\right\}. \tag{295}$$

Thus, the electrical power factor was obtained as:

$$\mathscr{P}_{zz}^{SL} = (\alpha_{zz}^{SL})^2\Theta. \tag{296}$$

Here, $\mu = k_B T ln(\pi\hbar^2 nd/m^* k_B T)$. The dependence of the $\mathscr{P}_{zz}$ for FSWCNT and $\mathscr{P}_{zz}^{SL}$ on temperature was displayed in Figure 65. The $\mathscr{P}_{zz}$, was observed to be far greater than $\mathscr{P}_{zz}^{SL}$ (i.e., $\mathscr{P}_{zz} \gg \mathscr{P}_{zz}^{SL}$). This high value could be attributable to a number of properties of FSWCNT, including its high non-parabolicity, carrier concentration, and phonon drag mechanism which scatters carriers purely by phonons. Moreover, more carriers performed intraminiband transition to generate high intraminiband current in FSWCNT than in SL to contribute to the $\mathscr{P}_{zz}$ and thus, making $\mathscr{P}_{zz}$ far greater than $\mathscr{P}_{zz}^{SL}$ as shown below.

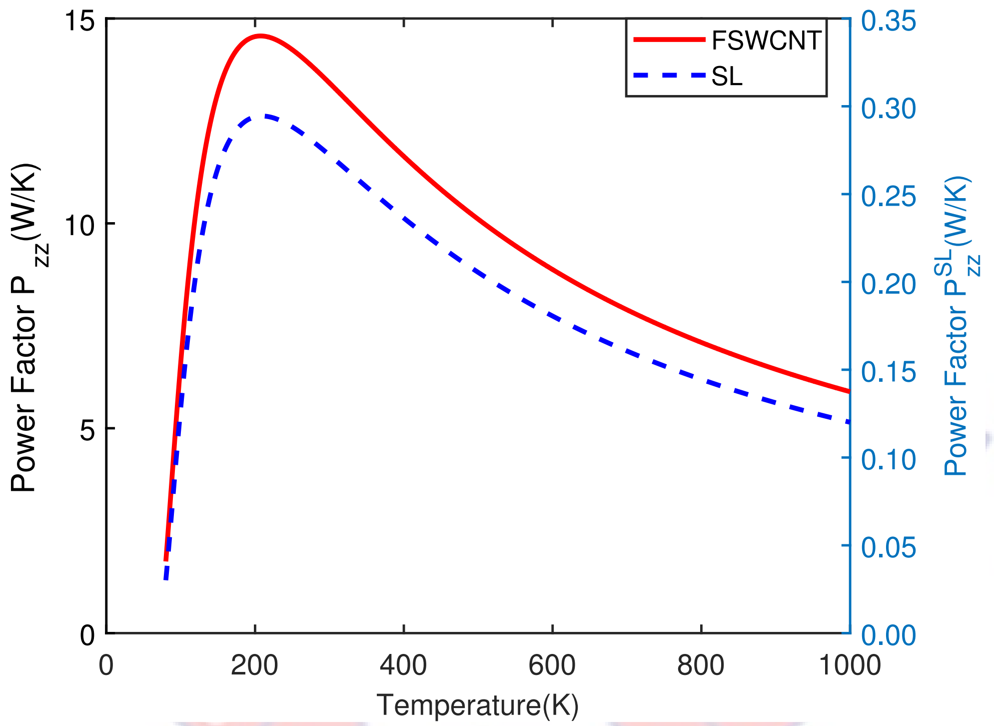

Figure 65: Dependency of $\mathscr{P}_{zz}$, on Temperature for FSWCNT and SL.

Thermal conductivity in FSWCNT

This section elucidated the computational analysis of the effective lattice conductivity of the nondegenerate FSWCNT when BTE and LBM was utilized. The differential carrier thermal conductivity, κ_e, and lattice conductivty, κ_ℓ, obtained were summed to yield the effective thermal conductivity, κ_{zz}. κ_{zz} obtained was observed to be highly anisotropic and depended on the carrier concentration, temperature, and carrier-phonon interaction parameters (Δ_s and Δ_z). In the simulation of the lattice component using the phonon LBM, the spatial derivative responsible for the introduction of the advection-diffusion term was included. The advection-diffusion process occurred when both advection and diffusion occur at the same time. The advection referred to the movement of heat, mass, and momentum by bulk motion of fluid particles; consequently, heat flow was modeled as a fluid moving through the FSWCNT lattice. The phenomenon of advection-diffusion with and without a source or reaction term such as an electric field is very common in nature and in industrial and engineering applications, usually called transport problem. The simulation result for the lattice conductivty dependence on temperature using the phonon LBM was presented in Figure 66.

The simulation process took into account both thermally stimulated transverse and longitudinal phonon modes that interacted via normal, N, and Umklapp, U-processes. The relaxation time model approximates the collision terms, with τ_U and τ_N reflecting the time(s) required for U and $N-$processes to relax the phonon distribution to their corresponding equilibrium distribution i.e. Planck's distribution. During the simulation isoflux, isothermal, and adiabatic boundary conditions were taken into account. The boundary condition was set to a periodic temperature gradient to handle the scenario where heat is transported in regions with large aspect ratio. κ_ℓ increased and peaked around $100K$ and then decreased with increasing temperature. The peak

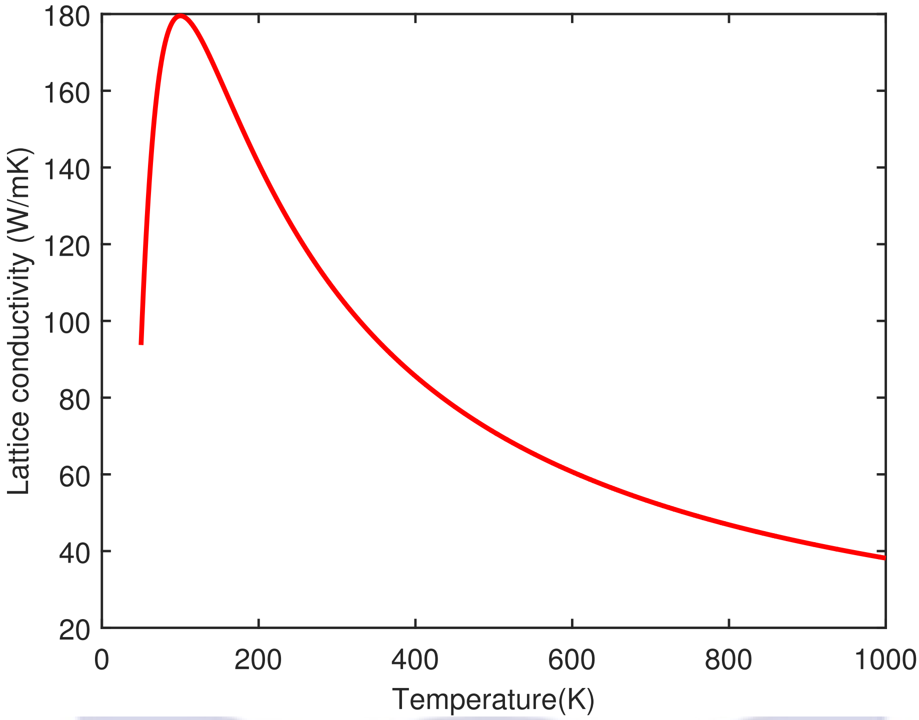

Figure 66: Lattice Thermal Conductivity Simulation Using Phonon Lattice Boltzmann Model for FSWCNT.

value of κ_ℓ ($180W/mK$) was comparable to that of SWCNT bundles, reported to be $150W/mK$ by Shi et al. [230]. The κ_ℓ then decreased to $107.2W/mK$ at $T = 300K$ which indicates the onset of a three-phonon scattering as the temperature increased. A D1Q2 simulation executed for 17,000 time step was performed. The data was used to obtain the temperature dependency of κ_ℓ, which was then shown.

The thermal conductivity κ_{zz}, dependency on temperature for different doping concentrations, *no*, was shown in Figures 67 and 68. In the high temperature zone, boundary and phonon-phonon scattering dominated the thermal conductivity when n_o was in the range $10^{18}cm^{-3}$ (see Figure 67). As devices like FSWCNT with bigger mass and radius disparities between the foreign F and host atoms C, the scattering of phonons by impurities implied a significant loss in thermal conductivity for low doping concentrations. This explained the greater decreased in FSWCNT's thermal conductivity shown in Figure 67. At decreasing temperatures, the phonon-impurity scattering rate became weaker. At low doping concentrations, the thermal conductivities of the FSWCNTs are poor, showing that phonon-boundary scattering and Umklampp processes prevail over phonon-impurity scattering at high temperatures.

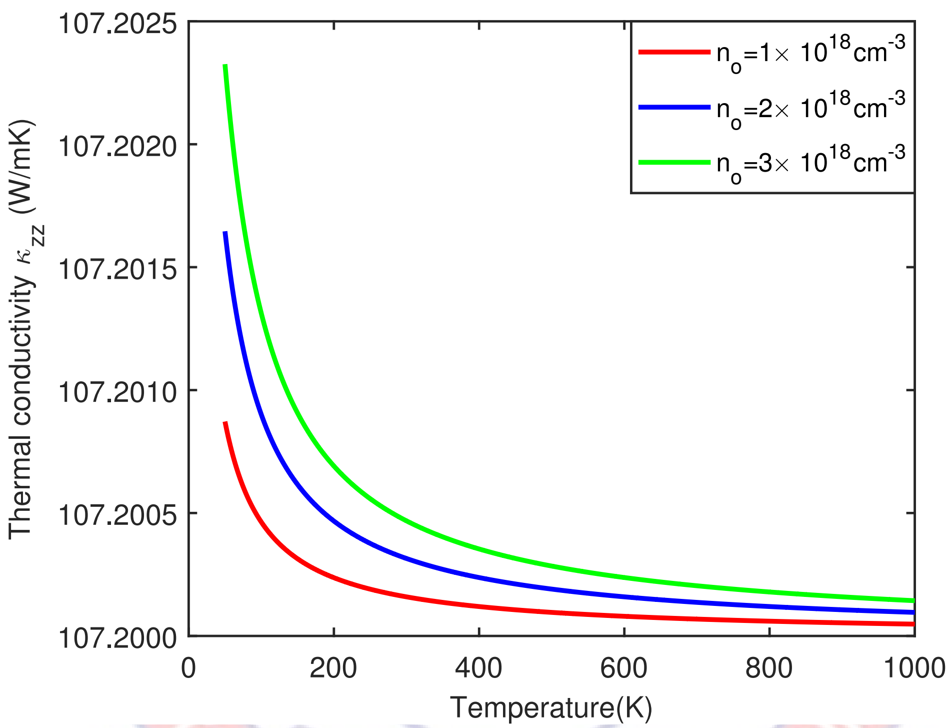

Figure 67: The Dependency of κ_{zz} on Temperature with $\Delta_s = 0.020eV$ and $\Delta_z = 0.013eV$ for $n_o = 1 \times 10^{18} cm^{-3}$, $n_o = 2 \times 10^{18} cm^{-3}$ and $n_o = 3 \times 10^{18} cm^{-3}$.

As doping concentration was increased to the range $10^{19} cm^{-3}$, the thermal conductivity begun to show linear behaviour at low temperature regions where phonon-imperfection and phonon-boundary scattering dominated (see Figure 68). Regardless of the level of doping concentration, phonon-boundary scattering contributed significantly to the lowering of the

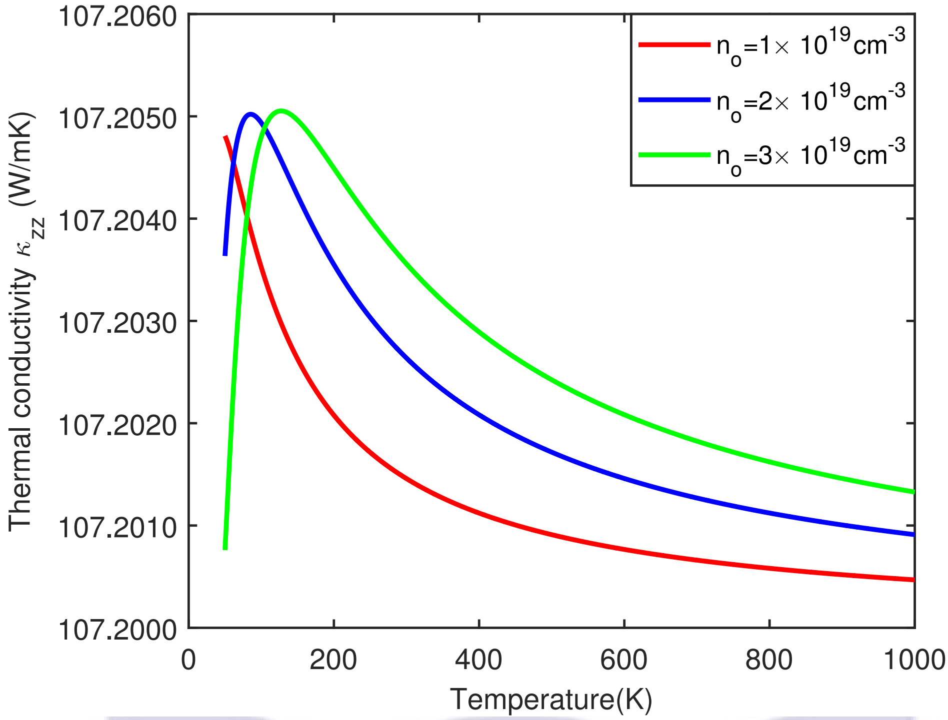

Figure 68: The Dependency of κ_{zz} on Temperature with $\Delta_s = 0.020eV$ and $\Delta_z = 0.013eV$ for (a) $n_o = 1 \times 10^{18} cm^{-3}$, $n_o = 2 \times 10^{18} cm^{-3}$ and $n_o = 3 \times 10^{18} cm^{-3}$.

thermal conductivity. The level of impurities in the FSWCNT determined the peak conductivities, which were marginal at low temperatures and shifted to high temperatures as the doping concentration increased. The peak values of thermal conductivity, however, remained the same at doping levels of $10^{19} cm^{-3}$. It did not change as the doping concentration increased, but it was somewhat higher at room temperature for $10^{19} cm^{-3}$ than $10^{18} cm^{-3}$. This minor increase in the former was due to a rise in doping levels as the FSWCNT transitioned from non-degenerate to degenerate character (see Figure 68).

Figures 69-72 showed the dependency of the axial thermal conductivity on temperature for varied values of Δ_z while Δ_s was fixed. Since FSWCNT was a non-degenerate semiconductor, it was noticed that the lattice contribution (κ_ℓ) dominated the thermal conductivity. The thermal energy was propagated in the presence of ∇T via wave packets made up of several normal modes. The Debye approximation for the phonon LBM included resistive phonon scattering processes, which were referred to as U-processes. For $\Delta_z = 0.013, 0.014$ and $0.015eV$, a strong dependency of the axial thermal conductivity on low temperature U-processes ($\exp(\Theta/T)$), where boundary scattering determined the phonon relaxation time and high temperature $U-$processes (T^{-1}), was observed (see Figure 69).

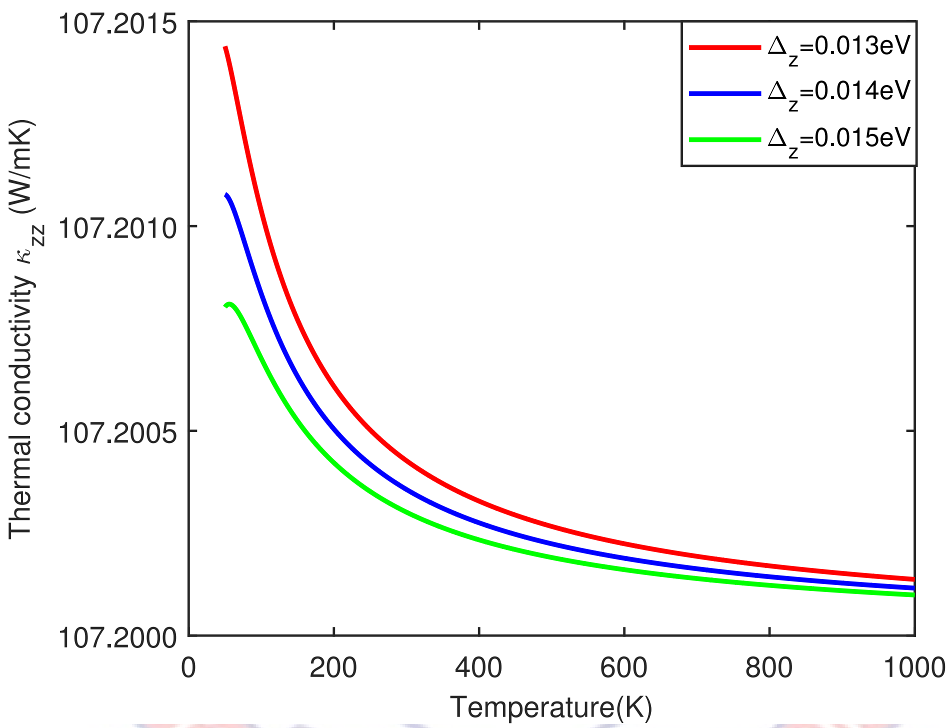

Figure 69: The Dependency of κ_{zz} on Temperature for varied Δ_z and $\Delta_s = 0.015eV$.

The U-process was found to be extremely strong, which indicated high phonon-phonon interactions. Because the U-processes tend to restore non-equilibrium phonon distribution to equilibrium in the FSWCNT and also give rise to thermal resistance [231], the total phonon momenta were not conserved during this interaction (phonon-phonon). The N-process, on the

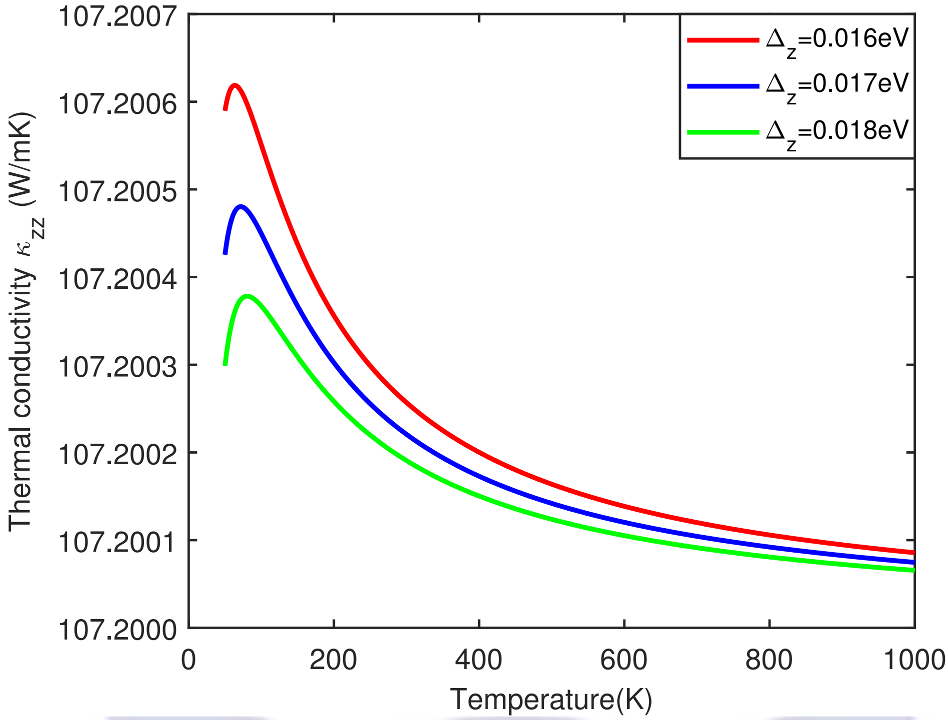

Figure 70: The Dependency of κ_{zz} on Temperature for varied Δ_z and $\Delta_s = 0.015eV$.

other hand, was non-resistive and conserved total crystal momentum but did not contributed to thermal resistance while having a significant impact on lattice contributions (κ_ℓ). These N-processes had a negative influence on energy transfer between phonon modes, which prohibitted substantial departures from the equilibrium distribution. The dependency of κ_{zz} varied with T^2 for $\Delta_z = 0.016$, 0.017, and $0.018eV$, as shown in Figure 70. As Δ_z was increased from Δ_z = 0.019-0.024eV, a linear conductivity was revealed, as seen in Figures 71 and 72. This was caused by phonon-imperfections and phonon-boundary scattering, which occurred as a result of doping processes and increased doping concentration, $\kappa \approx T^3$.

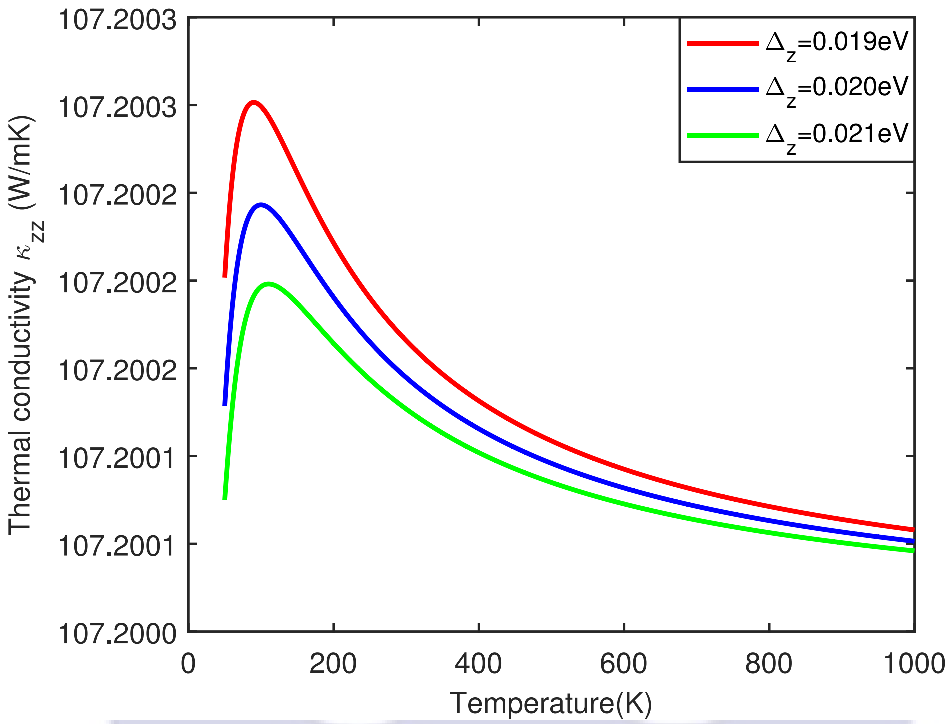

Figure 71: The Dependency of κ_{zz} on Temperature for varied Δ_z and $\Delta_s = 0.015eV$.

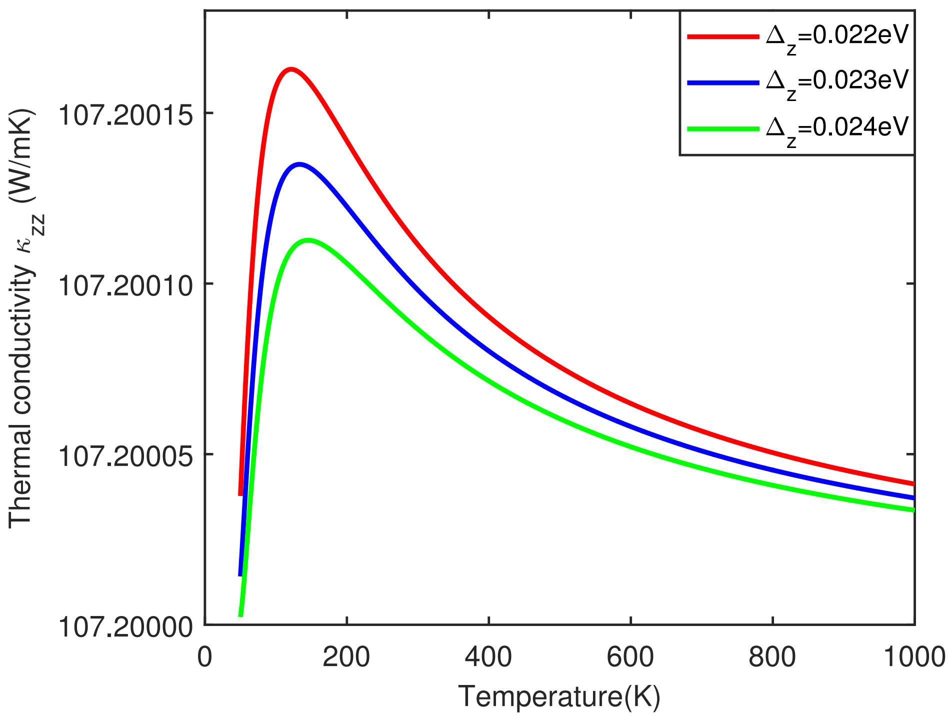

Figure 72: The Dependency of κ_{zz} on Temperature for varied Δ_z and $\Delta_s = 0.015eV$.

Figure 73 showed the dependency of κ_{zz} on temperature when the $ac-$field was switched on and off. The $ac-$field modulated the $dc-$field, which increased κ_{zz} by increasing κ_{ℓ} and thus, reduced κ_{e}. The reason being that the $ac-$field heated the carrier gas and set them to vibrate with large amplitude. This amplitude of vibration scattered the majority of the intraminiband carriers that carried the thermal energy $(\varepsilon - \mu)$ and thus, reduced the carrier contribution for only the lattice component to dominate the transport. Thus, the excess energy carried by the thermally excited carriers through the FSWCNT was nearly negligible as compared to that of the phonons [177, 180]. Since the $ac-$field was weak, majority of low frequency

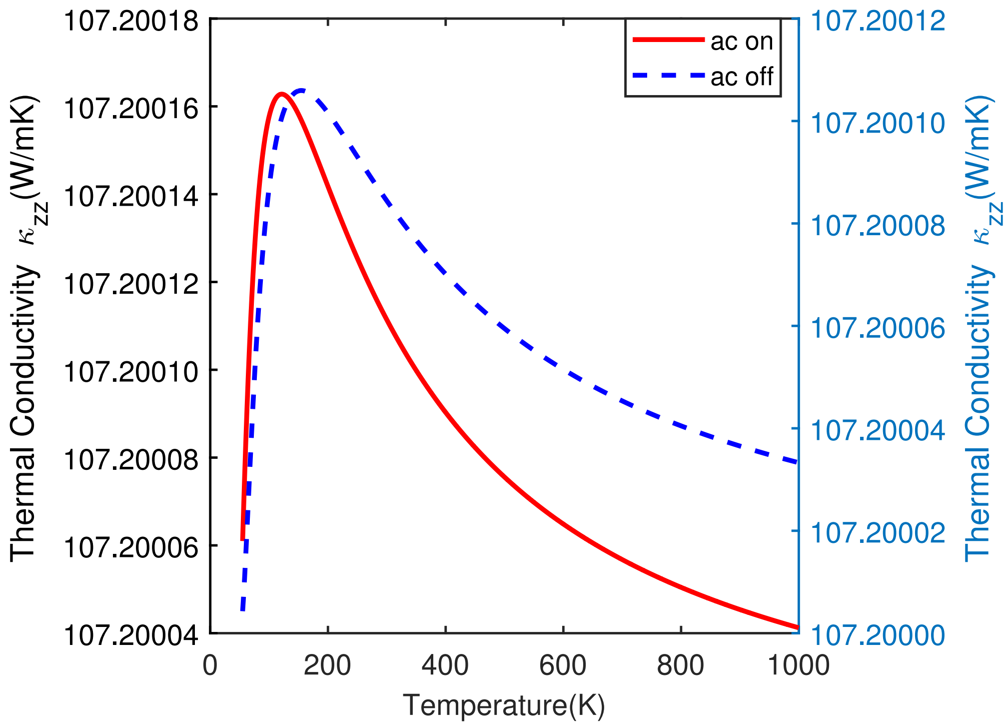

Figure 73: The Dependency of Axial Thermal Conductivity κ_{zz} on Temperature T for $n_o = 10^{19} cm^{-3}$, $E_o = 0.5 \times 10^5 V/m$, $E_z = 10^2 V/m$ $\Delta_s = 0.015 eV$ and $\Delta_z = 0.022 eV$.

phonons were generated which resembled acoustic waves because, their frequencies were small at longer wavelenghts but with high group velocity.

Thus, these branch of phonons generated termed the “acoustic branch”. This was because the acoustic waves generated were linked with the motion of center of mass of FSWCNT’s primitive cell. However, these acoustic phonons carried the majority of the thermal energy because of their high group velocity. The optical phonons were thus, not effective in the transport of thermal energy due to their small group velocity, but they also affected heat conduction by interacting with the acoustic phonons that were the main heat transporters.

Figure of merit in FSWCNT

To predict the practicability of FSWCNT as a thermoelement at 300K, it was worthy to study ZT numerically and evaluate it. For an understanding of Eq. (229), analysis of the equation was performed numerically. The computed results for ZT dependency of temperature were presented in Figures 74-79 for varying values of Δ_z and fixed values of Δ_s which depended on the carrier-phonon interactions (Δ_s and Δ_z).

When Δ_s was varied, the ZT climbed linearly and saturated as the temperature rises over 100K for $\Delta_z = 0.015eV$. The ZT was saturated because carriers in FSWCNT reached their maximum velocity (saturation velocity) in the presence of high fields. The average velocity of the FSWCNT carriers with time was proportional to the strength of ∇T. The higher the FSWCNT carriers’ mobility, the greater the drift velocity and, as a result, the higher the intraminiband current values for a given field intensity. Due to multiple factors that eventually limit the movement of the carriers in the FSWCNT, a limit was achieved about 100K where further increase in high field values did not allowed the FSWCNT carriers to travel any faster, having hit saturation velocity. The FSWCNT carrier velocity did not rise when the applied field ∇T increased beyond this temperature ($T = 100K$), because the carriers lost energy through greater levels of interaction with the lattice, by generating phonons, and photons, because the carrier energy was substantial enough to do so. Thus, for given Δ_z, scattering of intraminiband carriers by phonons increased at high fields (i.e., high ∇T), which resulted in high-frequency carrier dynamics that were highly dependent on the size of ∇T. The carrier velocity was drifted through the FSWCNT and allowed to perform drifting periodic orbits in THz frequencies, which resulted in a saturated resonant increase of ZT. Figure 74 showed that the ZT value was very high i.e., $ZT > 6$. As the value of Δ_z increased from $0.015eV$ to $0.025eV$, this value of ZT progressively saturated (Figures 74 and 75).

When Δ_z was fixed and low, the carrier scattering along the axial direction decreased with increasing values of Δ_s. Because fewer intraminiband carriers are scattered by lattice vibrations, a greater number of intraminiband carriers contribute to carrier conductivity (σ) and, as a result power factor, $\alpha^2\sigma$. As $\alpha^2\sigma T \gg \kappa$, the ZT continued to rise, reaching $ZT > 6$. When Δ_z was fixed, however, carrier scattering along the axial path begun to increase for increasing values of Δ_s. This meant that the number of intraminiband carriers scattered by lattice vibrations was greater, and thus a smaller number of intraminiband carriers contributed to the carrier conductivity σ, and as a result, to the power factor, $\alpha^2\sigma$. Thus, $\alpha^2\sigma T \ll \kappa$ and the ZT continued to decrease, $ZT < 6$ (Figures 76-79).

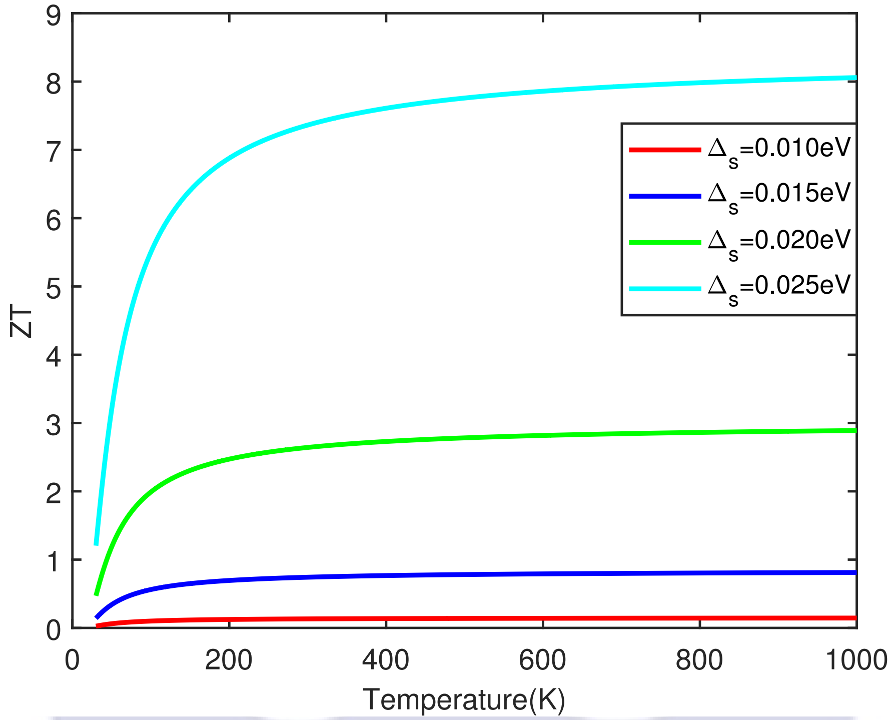

Figure 74: The Dependency of ZT on Temperature for $n_o = 10^{19}cm^{-3}$ and $E_z = 10^2V/m$ for varied Δ_s at $\Delta_z = 0.015eV$.

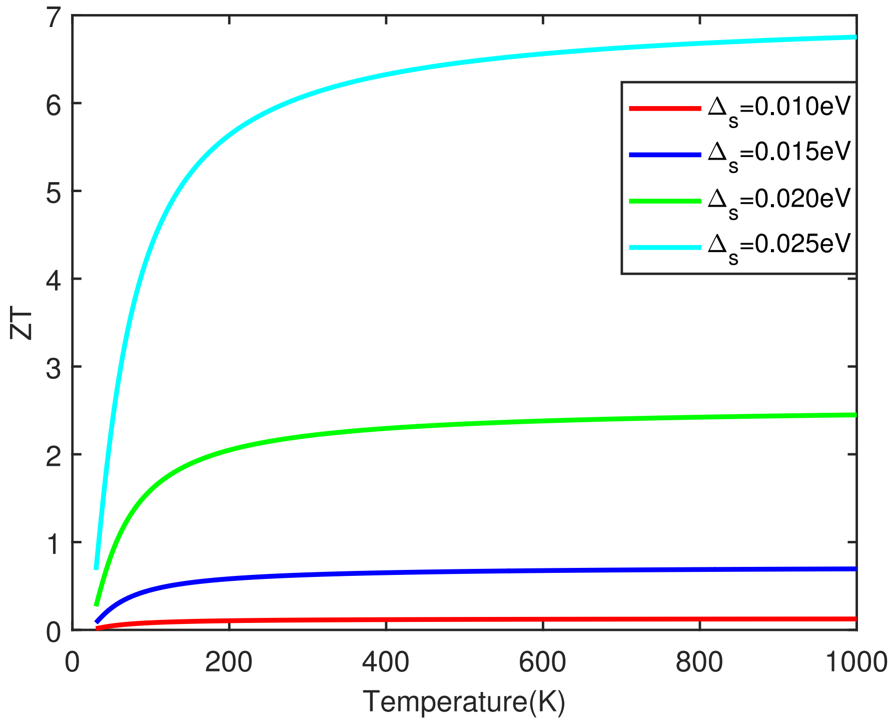

Figure 75: The Dependency of ZT on Temperature for $n_o = 10^{19}cm^{-3}$ and $E_z = 10^2V/m$ for varied Δ_s at $\Delta_z = 0.016eV$.

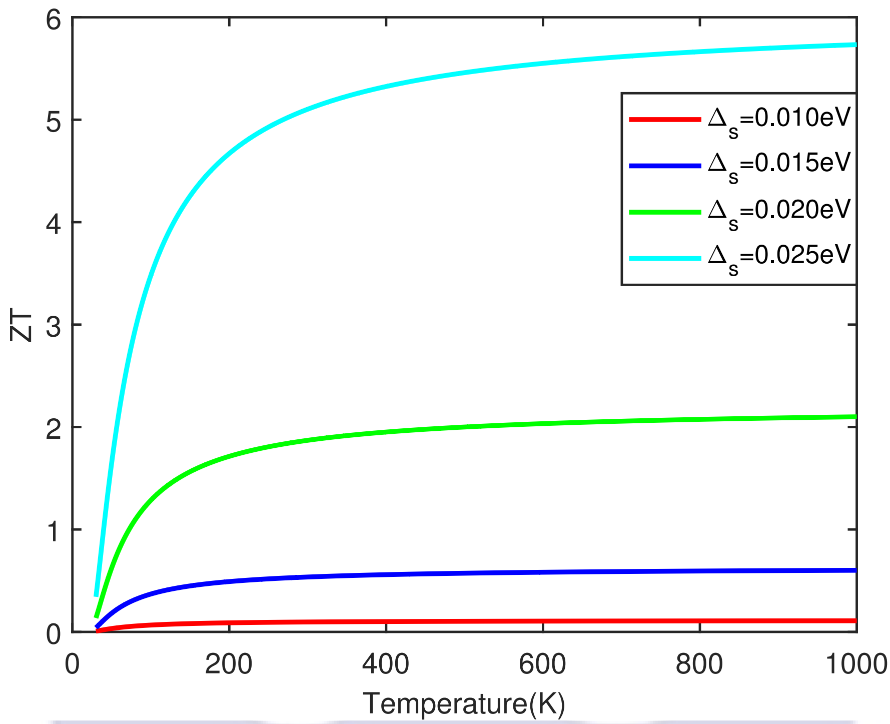

Figure 76: The Dependency of ZT on Temperature for $n_o = 10^{19} cm^{-3}$ and $E_z = 10^2 V/m$ for varied Δ_s at $\Delta_z = 0.017eV$.

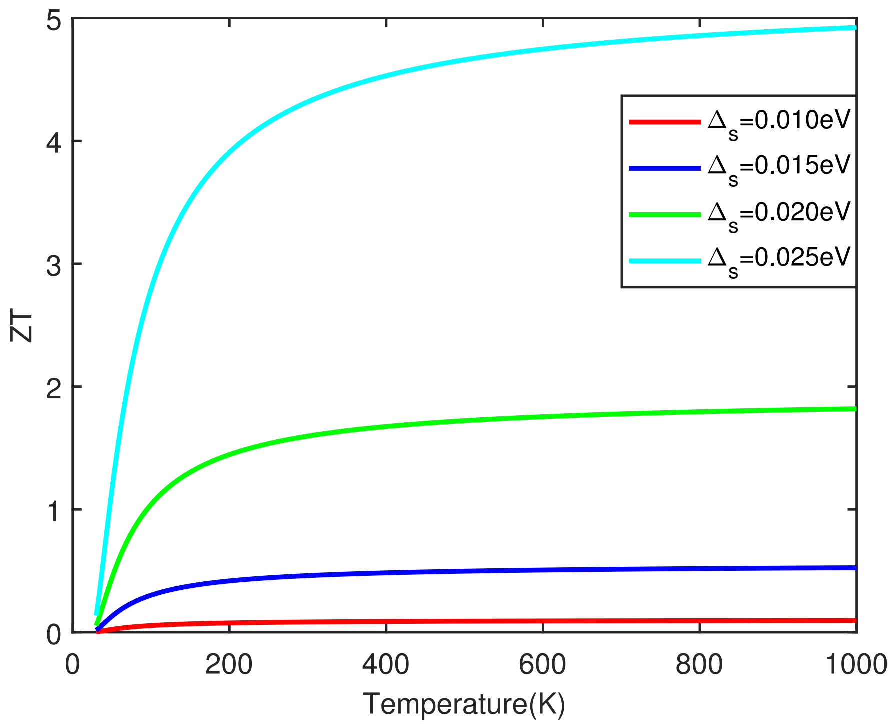

Figure 77: The Dependency of ZT on Temperature for $n_o = 10^{19} cm^{-3}$ and $E_z = 10^2 V/m$ for varied Δ_s at $\Delta_z = 0.018eV$.

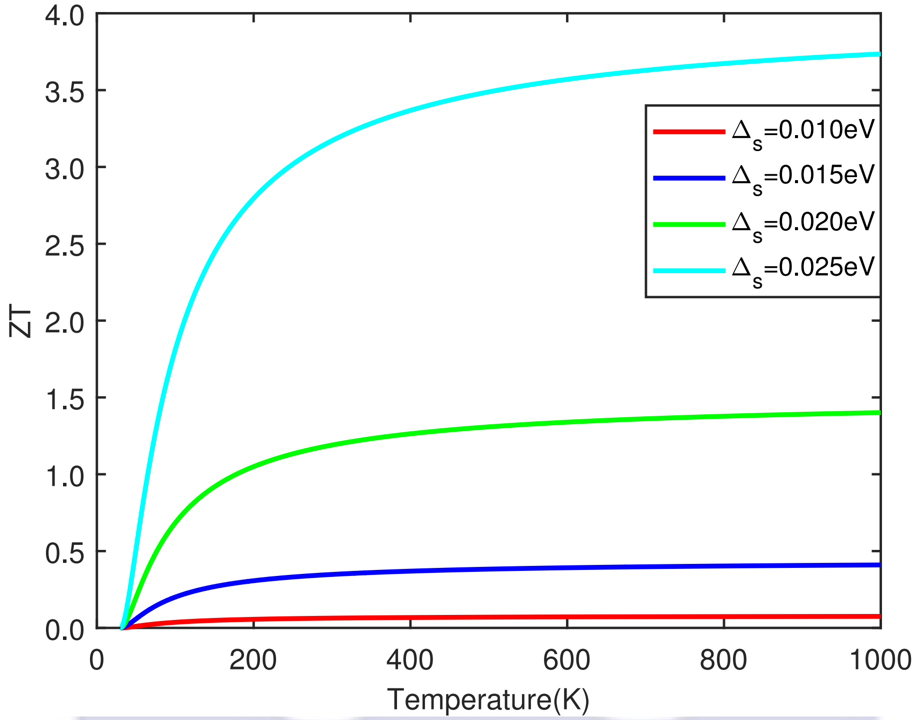

Figure 78: The Dependency of ZT on Temperature for $n_o = 10^{19} cm^{-3}$ and $E_z = 10^2 V/m$ for varied Δ_s at $\Delta_z = 0.020eV$.

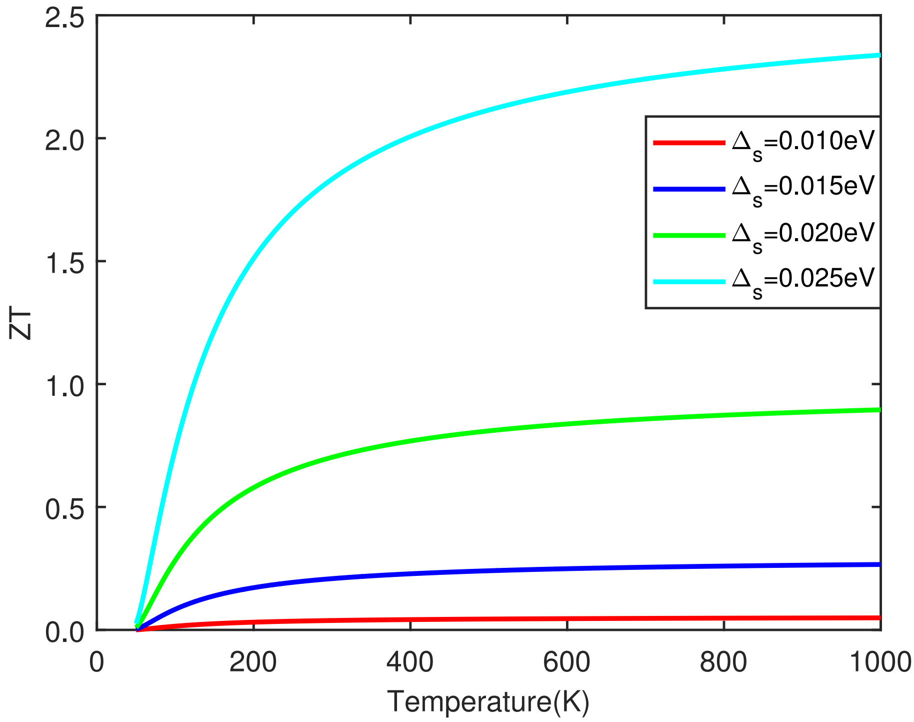

Figure 79: The Dependency of ZT on Temperature for $n_o = 10^{19} cm^{-3}$ and $E_z = 10^2 V/m$ for varied Δ_s at $\Delta_z = 0.024eV$.

The dependency of ZT for varied Δ_z values when Δ_s was fixed was shown in Figures 80-85. Figures 74-79 showed a comparable behaviour to that of the ZT. ZT in FSWCNT saturated beyond 100 K because carriers reached their maximum velocity in the presence of high fields. This saturation was based on the earlier assumption of the constant relaxation time made when the semiclassical BTE was invoked. The higher the carriers' mobility, the greater the drift velocity and, thus, the higher the intraminiband current for a given field intensity. Owing to a number factors that eventually limit the movement of carriers in FSWCNT, a limit was achieved about 100 K where further increase in high field values did not allowed carriers to travel any faster, due to a number of mechanisms in the FSWCNT having reached saturation velocity. Figures 80- 85 showed a relatively low ZT that gradually increased as the value of Δ_s grew from $0.015eV$ to $0.024eV$ (see Figures 80- 85). When Δ_s was fixed and low, the carrier scattering along the axial direction increased with increasing values of Δ_z. Because more intraminiband carriers are scattered by lattice vibrations, a lower number of intraminiband carriers contribute to carrier conductivity (σ) and, as a result power factor, $\alpha^2\sigma$. As $\alpha^2\sigma T \ll \kappa$, the ZT continued to fall, reaching $ZT < 6$. When Δ_s was fixed, however, carrier scattering along the axial path begun to decrease for increasing values of Δ_z. This meant that the number of intraminiband carriers scattered by lattice vibrations was lower, and thus a large number of intraminiband carriers contributed to the carrier conductivity σ, and as a result, to the power factor, $\alpha^2\sigma$. Thus, as $\alpha^2\sigma T \gg \kappa$, the ZT continued to rise i.e. $ZT > 20$ (Figures 80-85).

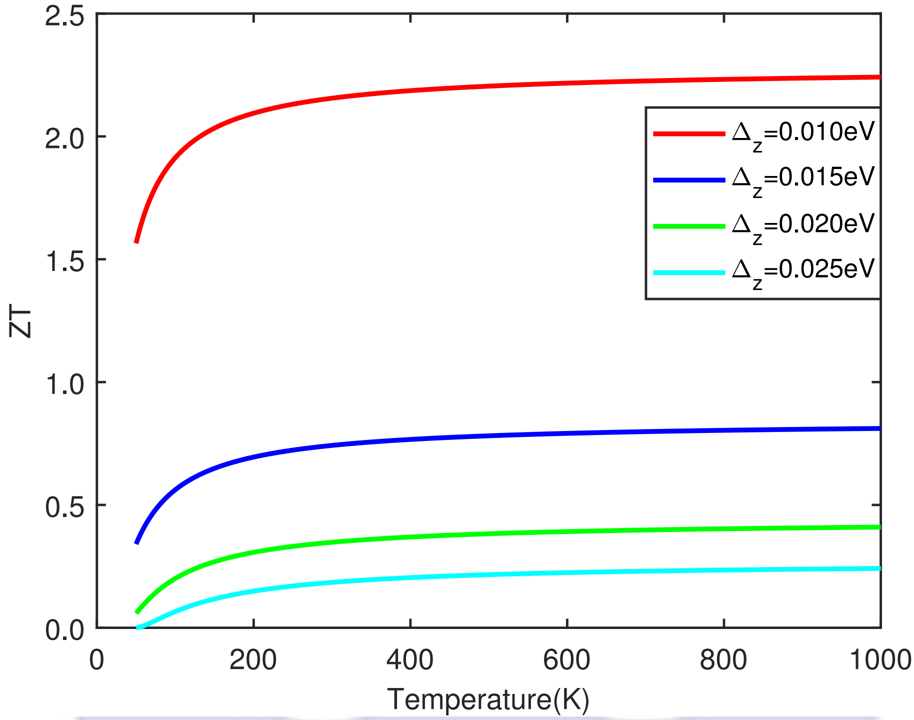

Figure 80: The Dependency of ZT on Temperature for varied Δ_z at $\Delta_s = 0.015eV$ and $n_o = 10^{19}cm^{-3}$.

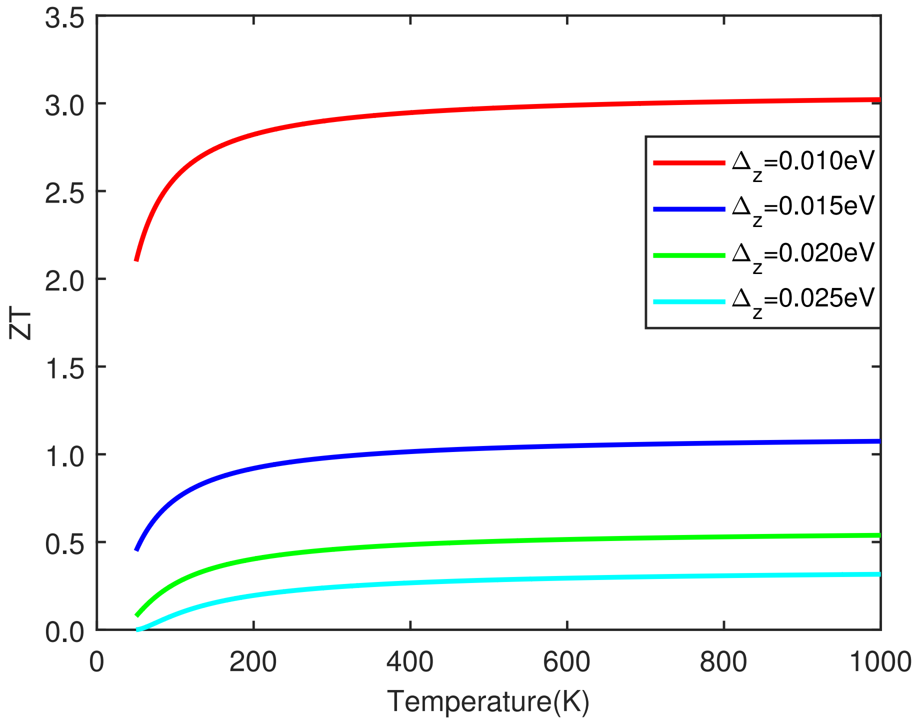

Figure 81: The Dependency of ZT on Temperature for varied Δ_z at $\Delta_s = 0.016eV$ and $n_o = 10^{19}cm^{-3}$.

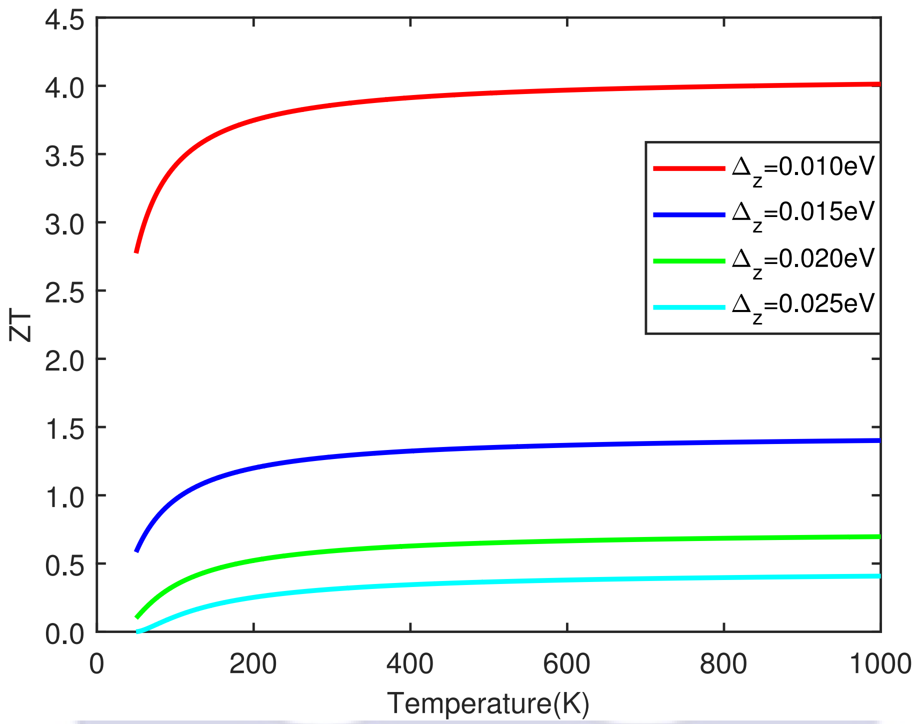

Figure 82: The Dependency of ZT on Temperature for varied Δ_z at $\Delta_s = 0.017eV$ and $n_o = 10^{19}cm^{-3}$.

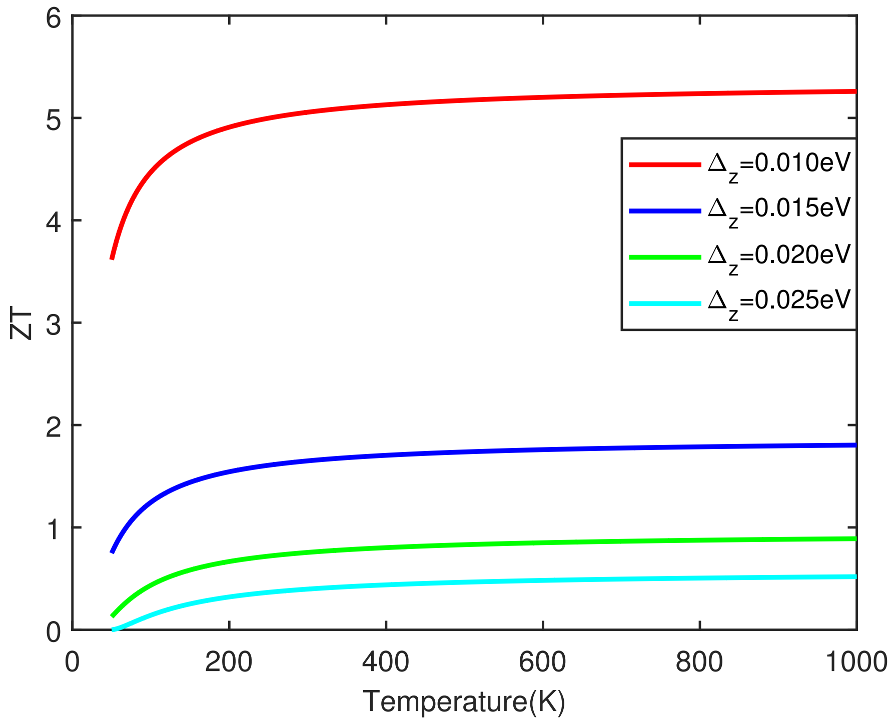

Figure 83: The Dependency of ZT on Temperature for varied Δ_z at $\Delta_s = 0.018eV$ and $n_o = 10^{19}cm^{-3}$.

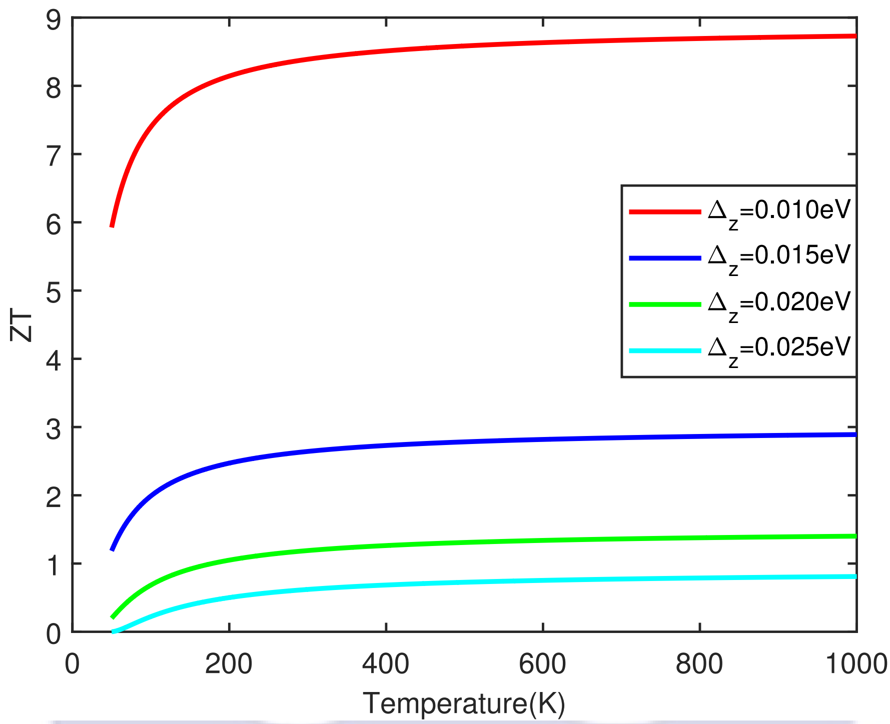

Figure 84: Dependency of ZT on Temperature T for varied Δ_z at $\Delta_s = 0.020eV$ and $n_o = 10^{19}cm^{-3}$.

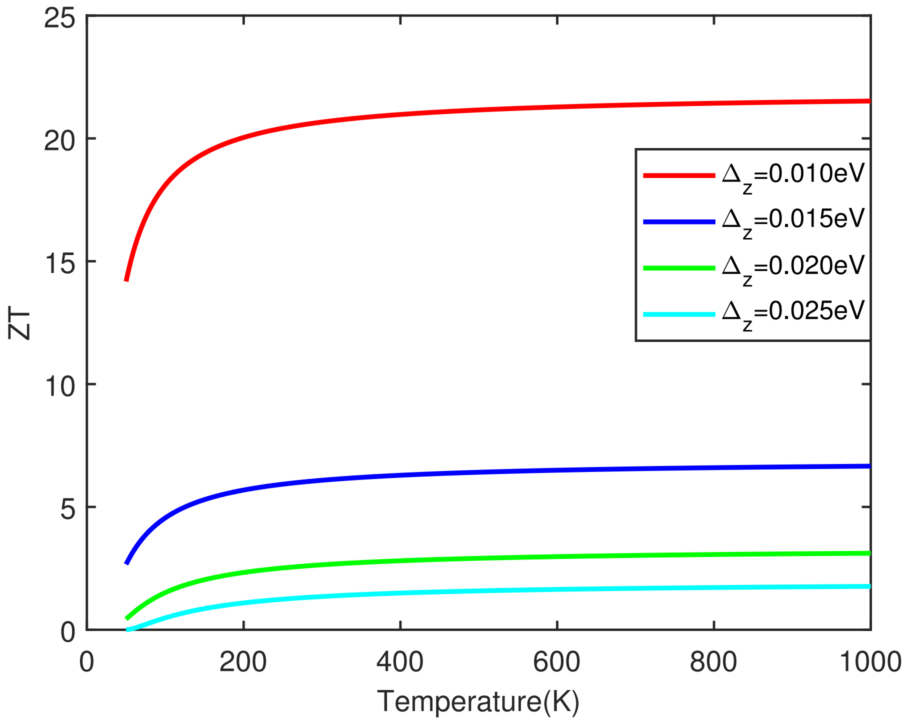

Figure 85: The Dependency of ZT on Temperature for varied Δ_z at $\Delta_s = 0.024eV$ and $n_o = 10^{19}cm^{-3}$.

The dependency of ZT on temperature for varied values of n_o for fixed values of, Δ_s and Δ_z were displayed in Figures 86 and 87. Fixing $\Delta_s = 0.024eV$ and $\Delta_z = 0.015eV$ and varying the carrier concentrations at $n_o = 1 \times 10^{19} cm^{-3}$, $1.5 \times 10^{19} cm^{-3}$ and $2 \times 10^{19} cm^{-3}$ was displayed in Figure 86. A significant increase in ZT ($ZT > 10.5$) in comparison to what had been reported in superlattice nanowires [232] at $T = 300K$ was observed. This was beacuse increasing the carrier concentration did not screen out the piezoelectric field to lower the ZT. In other words, the carrier-carrier interaction was minimal and non-significant for the FSWCNT. In Figure 87, the carrier-phonon interations were fixed at $\Delta_s = 0.015eV$ and $\Delta_z = 0.024eV$ and varying the carrier concentrations at $n_o = 1 \times 10^{19} cm^{-3}$, $1.5 \times 10^{19} cm^{-3}$ and $2 \times 10^{19} cm^{-3}$, it was observed that the ZT was remarkably low but kept rising gradually.

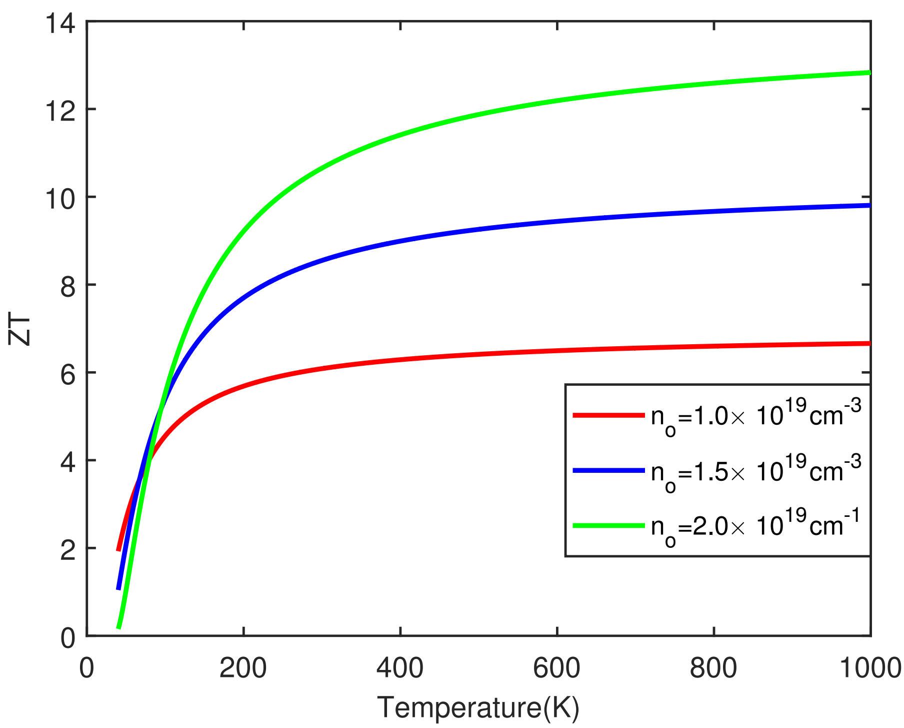

Figure 86: The Dependency of ZT on Temperature for varied Carrier Concentration n_o at $\Delta_s = 0.024eV$ and $\Delta_z = 0.015eV$.

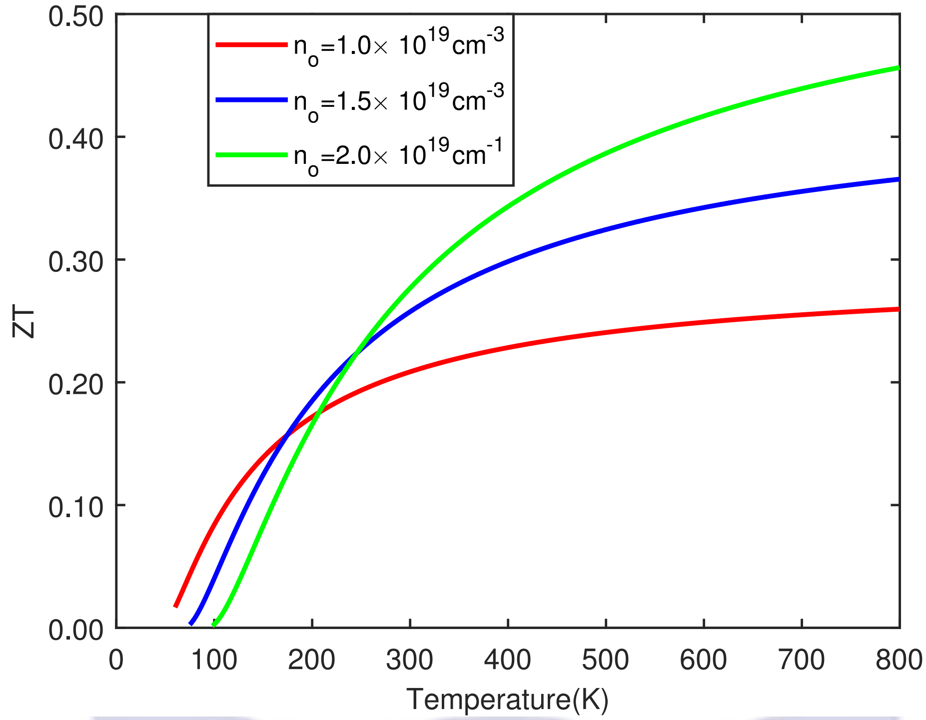

Figure 87: The Dependency of ZT on Temperature for varied Carrier Concentration n_o at $\Delta_s = 0.015eV$ and $\Delta_z = 0.024eV$.

Chapter Summary

In summary, the calculations of the acoustic and thermoelectric metrics carried out in Chapter Three using the tractable analytical approach were analysed with numerical and computational schemes. Carrier miniband transport introduced by Δ_s and Δ_z was observed to influence the transport processes strongly. Metrics such as the the acoustoelectric current $\vec{J}_z^{AE}$, the Hall-like current density, $\vec{J}_y^{AME}$, amplification coefficient, $\Gamma_q(\omega)$, electrical conductivity, σ, thermoelectric power, α, electrical power factor, $\mathscr{P}$, carrier current density, $\vec{J}$, carrier thermal current density, and the lattice thermal conductivity, κ_ℓ were analysed and discussed with their dependence on temperature, carrier-phonon interaction and carrier concentration presented pictorially.

CHAPTER FIVE

SUMMARY, CONCLUSION AND REOMENDATION(S)

Overview

The optimization of thermo-physical properties of a nondegenerate FSWCNT was carried out using the BTE where τ is assumed to be constant. This chapter focussed on the conclusions made with regards to the acoustoelectric and the thermoelectric metrics obtained. The recommendations and pertinent suggestions made in relation to the study to enhance the thermo-physical metrics was addressed as well.

Summary

The thesis was composed of five main chapters. In Chapter One, a background to the study of thermoelectrics and an introduction to the FSWCNT band structure using the Huckel matrix approach was discussed. The general and specific objective(s) as well as the scope of the work and the organisation of the thesis were also discussed. Chapter Two reviewed literature on acoustoelectricity, thermoelectricity, carrier, thermal and optical properties and the principles underlying the theory used in the study. Moreover, discussions on phonon LBM for studying the lattice thermal conductivity was also elaborated. In Chapter Three, a tractable mathematical model was developed and used to obtain the acoustoelectric and thermoelectric metrics. The chapter elucidated the development of the phonon LBM from the

Boltzmann's kinetic equation used to deduce the lattice thermal conductivity. The results, analysis and discussion on acoustoelectric and thermoelectric metrics from the models were presented graphically in Chapter Four. In this chapter (Chapter Five), the conclusions and pertinent suggestions drawn from the study and recommendations made to enhance the thermo-physical properties of FSWCNT are presented.

Conclusion(s)

In the first part of Chapter Three, the acoustodynamics was investigated in a nondegenerate FSWCNT under the condition $ql \gg 1$. For $E(t) = 0$, a phenomenon of negative conductivity was observed due to the strong non-parabolicity of FSWCNT and SWCNT band structures. Comparatively, the acoustoelectric current density was observed to be far less in FSWCNT than in SWCNT ($j_z^{FSWCNT}/j_o \ll j_z^{SWCNT}/j_o$). This was due to carrier-phonon coupling in SWCNT being more stronger than that of FSWCNT since SWCNT was metallic and FSWCNT a semicondutor. Thus, there were more intraminiband carriers interacting with the acoustic phonons to generate a high acoustoelectric current in SWCNT than in FSWCNT.

In the presence of an external electric field, $E(t) \neq 0$, a strong nonlinear dependence of j_z^{AE}/j_o on $1 - v_d/v_s$, for Δ, q and T was observed. The acoustoelectric current obtained was highly nonlinear and depended on the attenuation of acoustic phonons by electric field driven carriers experiencing intraminiband transition. The acoustoelectric current obtained holds potential for low voltage current amplifying acoustic device in the THz frequency regime and is vital to devices such as phonon filters, phonon spectrometer, acoustic diodes, acoustic transistors, and micro-nanocarrier gadgets. Furthermore, the carrier concentration can be used to tune the acoustoelectric current for room temperature applications.

In the presence of weak magnetic field, a strong nonlinear dependence of the Hall-like current, j_y^{AME}, on Ω, q, Δ, induced by acoustic phonons was also observed. Qualitatively, the Hall-like current existed even when the relaxation time does not depend on the carrier energy but has a strong spatial dispersion, and gives different results compared to that obtained in bulk semiconductors. In the case of a constant relaxation, the effect is only present for non-degenerate carrier gas but absent for degenerate carrier gas. For $\omega_q = 10^{12}Hz$ and $H = 10^6 Wb/m^2$, the Hall-like current obtained has a value of about $0.09Acm^{-2}$ at $T = 300K$. Moreover, analytical investigation of gain in coherent acoustic phonons in a nondegenerate FSWCNT utilising the BTE shows a highly nonlinear, stimulated emission of phonons by electrically driven carriers experiencing intraminiband transition. This study has a potential for the generation of intense sources of reasonable acoustic phonons in the sub-THz regime which is vital for SASER generation.

Secondly, α_{zz} and $\mathscr{P}_{zz}$ of the nondegenerate FSWCNT was carried out invoking a tractable analytical approach by benefiting from the BTE. FSWCNT parameters, Δ_s, Δ_z, θ_h, E_o and E_z were found to influence α_{zz} strongly. Numerically, as Δ_z increased beyond $0.017eV$, the FSWCNT switched from a p-type to an n-type semiconductor. It was worthy that when E_z was above $2.3 \times 10^2 V/m$, the α_{zz} values decreased. Interestingly, varying the carrier concentration, n_o, could be used to tune the FSWCNT to ascertain which carriers contributed to the thermoelectric power. In otherwords, it could be used as a thermoelectric diagnostic tool. The ZT was also found to be influenced by the FSWCNT parameters Δ_s, Δ_z, and n_o strongly. Optimizing Δ_s, Δ_z, T and n_o can yield a ZT greater 6 ($ZT > 6$). Interestingly, varying the n_o could be used to tune the FSWCNT to operate at higher temperatures. Thus, looking at the high ZT obtained for the weak electric field, this study proposes non-degenerate semiconductor FSWCNT with non-parabolic double periodic band as a good material for a thermoelement.

Recommendation(s)

It is possible to make an extension of this approach by applying a strong magnetic field to the acoustoelectric metrics and weak magnetic field to the thermoelectric metrics in the studies. Further work can also be done on the following; acoustoelectric effect in FSWCNT, amplification of acoustic phonons in the presence of external electric and magnetic fields, effect on laser on acoustoelectric effect in FSWCNT in the quantum regime.

REFERENCES

[1] Vedernikov, M. V., and E. K. Iordanishvili. ”AF Ioffe and origin of modern semiconductor thermoelectric energy conversion.” In Seventeenth International Conference on Thermoelectrics. Proceedings ICT98 (Cat. No. 98TH8365), pp. 37-42. IEEE, 1998.

[2] Boukai, Akram I., Yuri Bunimovich, Jamil Tahir-Kheli, Jen-Kan Yu, William A. Goddard Iii, and James R. Heath. ”Silicon nanowires as efficient thermoelectric materials.” nature 451, no. 7175 (2008): 168-171..

[3] Balandin, Alexander, and Kang L. Wang. ”Significant decrease of the lattice thermal conductivity due to phonon confinement in a free-standing semiconductor quantum well.” Physical Review B 58, no. 3 (1998): 1544.

[4] Balandin, Alexander, and Kang L. Wang. ”Effect of phonon confinement on the thermoelectric figure of merit of quantum wells.” Journal of Applied Physics 84, no. 11 (1998): 6149-6153.

[5] Hicks, L. D., and M. S. Dresselhaus. ”Use of quantum-well superlattices to obtain a high figure of merit from nonconventional thermoelectric materials.” MRS Online Proceedings Library (OPL) 326 (1993).

[6] Whitlow, L. W., and T. Hirano. ”Superlattice applications to thermoelectricity.” Journal of Applied Physics 78, no. 9 (1995): 5460-5466.

[7] Dresselhaus, M. S. Nanostructures and energy conversion, Proceedings of Rosenhow Symposium on Future Trends of Heat Transfer, Cambridge, MA. 2003.

[8] Böttner, Harald, Gang Chen, and Rama Venkatasubramanian. ”Aspects of thin-film superlattice thermoelectric materials, devices, and applications.” MRS bulletin 31, no. 3 (2006): 211-217.

[9] Koga, T., S. B. Cronin, M. S. Dresselhaus, J. L. Liu, and K. L. Wang. ”Experimental proof-of-principle investigation of enhanced Z 3D T in (001) oriented Si/Ge superlattices.” Applied Physics Letters 77, no. 10 (2000): 1490-1492.

[10] Khabashesku, Valery N., W. Edward Billups, and John L. Margrave. ”Fluorination of single-wall carbon nanotubes and subsequent derivatization reactions.” Accounts of chemical research 35, no. 12 (2002): 1087-1095.

[11] Bettinger, Holger F. ”Experimental and Computational Investigations of the Properties of Fluorinated SingleWalled Carbon Nanotubes.” ChemPhysChem 4, no. 12 (2003): 1283-1289.

[12] Nakajima, T., S. Kasamatsu, and Y. Matsuo. ”Synthesis and characterization of fluorinated carbon nanotube.” European journal of solid state and inorganic chemistry 33, no. 9 (1996): 831-840.

[13] Mickelson, E. T., C. B. Huffman, A. G. Rinzler, R. E. Smalley, R. H. Hauge, and J. L. Margrave. ”Fluorination of single-wall carbon nanotubes.” Chemical physics letters 296, no. 1-2 (1998): 188-194.

[14] Van Lier, Gregory, Christopher P. Ewels, Filippo Zuliani, Allesandro De Vita, and Jean-Christophe Charlier. ”Theoretical analysis of fluorine addition to single-walled carbon nanotubes: functionalization routes and addition patterns.” The Journal of Physical Chemistry B 109, no. 13 (2005): 6153-6158.

[15] Bettinger, Holger F., Konstantin N. Kudin, and Gustavo E. Scuseria. ”Thermochemistry of fluorinated single wall carbon nanotubes.” Journal of the American Chemical Society 123, no. 51 (2001): 12849-12856.

[16] Boul, P. J., J. Liu, E. T. Mickelson, C. B. Huffman, L. M. Ericson, I. W. Chiang, K. A. Smith et al. ”Reversible sidewall functionalization of buckytubes.” Chemical Physics Letters 310, no. 3-4 (1999): 367-372.

[17] Kelly, K. F., I. W. Chiang, E. T. Mickelson, R. H. Hauge, J. L. Margrave, X. Wang, G. E. Scuseria, C. Radloff, and N. J. Halas. ”Insight into the mechanism of sidewall functionalization of single-walled nanotubes: an STM study.” Chemical physics letters 313, no. 3-4 (1999): 445-450.

[18] Saini, Rajesh K., Ivana W. Chiang, Haiqing Peng, R. E. Smalley, W. E. Billups, Robert H. Hauge, and John L. Margrave. ”Covalent sidewall functionalization of single wall carbon nanotubes.” Journal of the American Chemical Society 125, no. 12 (2003): 3617-3621.

[19] Stevens, Joel L., Aaron Y. Huang, Haiqing Peng, Ivana W. Chiang, Valery N. Khabashesku, and John L. Margrave. ”Sidewall amino-functionalization of single-walled carbon nanotubes through

fluorination and subsequent reactions with terminal diamines.” Nano Letters 3, no. 3 (2003): 331-336.

[20] Valentini, L., D. Puglia, Ilaria Armentano, and J. M. Kenny. ”Sidewall functionalization of single-walled carbon nanotubes through CF4 plasma treatment and subsequent reaction with aliphatic amines.” Chemical Physics Letters 403, no. 4-6 (2005): 385-389.

[21] Dettlaff-Weglikowska, Urszula, Viera Skakalova, Jannik Meyer, Jiri Cech, Bernd G. Mueller, and Siegmar Roth. ”Effect of fluorination on electrical properties of single walled carbon nanotubes and C60 peapods in networks.” Current Applied Physics 7, no. 1 (2007): 42-46.

[22] Zhou, Chongfu, Satish Kumar, Condell D. Doyle, and James M. Tour. ”Functionalized single wall carbon nanotubes treated with pyrrole for electrochemical supercapacitor membranes.” Chemistry of materials 17, no. 8 (2005): 1997-2002.

[23] Lee, Ji Yeong, Kay Hyeok An, Jeong Ku Heo, and Young Hee Lee. ”Fabrication of supercapacitor electrodes using fluorinated single-walled carbon nanotubes.” The Journal of Physical Chemistry B 107, no. 34 (2003): 8812-8815.

[24] Seifert, Gotthard, Thomas Köhler, and Thomas Frauenheim. ”Molecular wires, solenoids, and capacitors by sidewall functionalization of carbon nanotubes.” Applied Physics Letters 77, no. 9 (2000): 1313-1315.

[25] Hamwi, Andre, Philippe Gendraud, Helene Gaucher, Sylvie Bonnamy, and Francois Beguin. ”Electrochemical properties of carbon nanotube fluorides in a lithium cell system.” Molecular Crystals and Liquid Crystals Science and Technology. Section A. Molecular Crystals and Liquid Crystals 310, no. 1 (1998): 185-190.

[26] Hamwi, A., H. Alvergnat, S. Bonnamy, and F. Beguin. ”Fluorination of carbon nanotubes.” Carbon 35, no. 6 (1997): 723-728.

[27] Peng, Haiqing, Zhenning Gu, Jiping Yang, J. L. Zimmerman, P. A. Willis, M. J. Bronikowski, R. E. Smalley, R. H. Hauge, and J. L. Margrave. ”Fluorotubes as cathodes in lithium electrochemical cells.” Nano Letters 1, no. 11 (2001): 625-629.

[28] Root, Michael J. ”Comparison of fluorofullerenes with carbon monofluorides and fluorinated carbon single wall nanotubes: thermodynamics and electrochemistry.” Nano Letters 2, no. 5 (2002): 541-543.

[29] Hayashi, T., M. Terrones, C. Scheu, Y. A. Kim, M. Rühle, T. Nakajima, and M. Endo. ”NanoTeflons: structure and EELS characterization of fluorinated carbon nanotubes and nanofibers.” Nano Letters 2, no. 5 (2002): 491-496.

[30] Gupta, Vinay, Tsuyoshi Nakajima, Yoshimi Ohzawa, and Boris Žemva. ”A study on the formation mechanism of graphite fluorides by Raman spectroscopy.” Journal of Fluorine Chemistry 120, no. 2 (2003): 143-150.

[31] Hayashi T., Reply to ”A Comment on NanoTeflons: Structure and EELS Characterization of Fluorinated Carbon Nanotubes and Nanofibers”, Nano Letters 4 (2004) 1001-02

[32] Khabashesku, V. N., J. L. Margrave, and E. V. Barrera. ”Functionalized carbon nanotubes and nanodiamonds for engineering and biomedical applications.” Diamond and Related Materials 14, no. 3-7 (2005): 859-866.

[33] Bauschlicher Jr, Charles W. ”Hydrogen and fluorine binding to the sidewalls of a (10, 0) carbon nanotube.” Chemical physics letters 322, no. 3-4 (2000): 237-241.

[34] Bahr, Jeffrey L., and James M. Tour. ”Covalent chemistry of single-wall carbon nanotubes.” Journal of Materials Chemistry 12, no. 7 (2002): 1952-1958.

[35] Li, W. Z., J. G. Wen, and Z. F. Ren. ”Straight carbon nanotube Y junctions.” Applied Physics Letters 79, no. 12 (2001): 1879-1881.

[36] Holzinger, Michael, Otto Vostrowsky, Andreas Hirsch, Frank Hennrich, Manfred Kappes, Robert Weiss, and Frank Jellen. ”Sidewall functionalization of carbon nanotubes.” Angewandte chemie international edition 40, no. 21 (2001): 4002-4005.

[37] Molodtsov, S. L., S. I. Fedoseenko, D. V. Vyalikh, I. E. Iossifov, R. Follath, S. A. Gorovikov, M. M. Brzhezinskaya et al. ”High-resolution Russian–German beamline at BESSY.” Applied Physics A 94, no. 3 (2009): 501-505.

[38] Jaffe, Richard L. "Quantum chemistry study of fullerene and carbon nanotube fluorination." The Journal of Physical Chemistry B 107, no. 38 (2003): 10378-10388.

[39] Ewels, Chris P., Gregory Van Lier, Jean-Christophe Charlier, Malcolm I. Heggie, and Patrick R. Briddon. "Pattern formation on carbon nanotube surfaces." Physical review letters 96, no. 21 (2006): 216103.

[40] Sadykov, N. R., E. Yu Kocherga, and P. N. D'yachkov. "Nonlinear current in modified nanotubes with exposure to alternating and constant electric fields." Russian Journal of Inorganic Chemistry 58, no. 8 (2013): 951-955.

[41] Sadykov, N. R., N. A. Scorkin, and E. A. Akhljustina. "Effect of electromagnetic radiation on an array of weakly interacting carbon nanotubes in the presence of nanosecond pulses." Semiconductors 47, no. 9 (2013): 1246-1251.

[42] Sadykov, N. R., and N. A. Scorkin. "Generation of electromagnetic radiation based on nanotubes under a constant electric field and an electromagnetic wave field." Semiconductors 46, no. 2 (2012): 159-164.

[43] Kittel, Charles, Paul McEuen, and Paul McEuen. Introduction to solid state physics. Vol. 8. New York: Wiley, 1996.

[44] Winters, Michael. Electron transport studies in epitaxial graphene on SiC. Chalmers Tekniska Hogskola (Sweden), 2013.

[45] Ziman, John M. Principles of the Theory of Solids. Cambridge university press, 1972.

[46] Srednicki, Mark. Quantum field theory. Cambridge University Press, 2007.

[47] Sarma, S. Das, Shaffique Adam, E. H. Hwang, and Enrico Rossi. ”Electronic transport in two-dimensional graphene.” Reviews of modern physics 83, no. 2 (2011): 407.

[48] Mahan, Gerald D., and K. R. Subbaswamy. Local density theory of polarizability. Springer Science Business Media, 2013.

[49] Griffiths, David J., and Darrell F. Schroeter. Introduction to quantum mechanics. Cambridge University Press, 2018.

[50] Martel, R., V. Derycke, Jörg Appenzeller, S. Wind, and Phaedon Avouris. ”Carbon nanotube field-effect transistors and logic circuits.” In Proceedings 2002 Design Automation Conference (IEEE Cat. No. 02CH37324), pp. 94-98. IEEE, 2002.

[51] Martel, Richard, T. Schmidt, H. R. Shea, T. Hertel, and Ph Avouris. ”Single-and multi-wall carbon nanotube field-effect transistors.” Applied physics letters 73, no. 17 (1998): 2447-2449.

[52] Ktitorov, S. A., G. S. Simin, and Sindalov. VY. ”Bragg reflections and high-frequency conductivity of an electronic solid-state plasma.” Soviet Physics Solid State, USSR 13, no. 8 (1972): 1872.

[53] Maksimenko, Anton S., and Gregory Ya Slepyan. "Negative differential conductivity in carbon nanotubes." Physical Review Letters 84, no. 2 (2000): 362.

[54] Yu, Alan. "A Study of Carbon Nanotubes and Their Applications in Transistors." School of Electrical and Computer Engineering (2004): 1-32.

[55] Mukhopadhyay, Sumit, Jens T. Birkholzer, Jean-Philippe Nicot, and Seyyed A. Hosseini. "A model comparison initiative for a CO2 injection field test: an introduction to Sim-SEQ." Environmental Earth Sciences 67, no. 2 (2012): 601-611.

[56] Slepyan, G. Ya, S. A. Maksimenko, V. P. Kalosha, J. Herrmann, E. E. B. Campbell, and I. V. Hertel. "Highly efficient high-order harmonic generation by metallic carbon nanotubes." Physical Review A 60, no. 2 (1999): R777.

[57] Dresselhaus, Mildred S., and Phaedon Avouris. "Introduction to carbon materials research." Carbon nanotubes (2001): 1-9.

[58] Mensah, S. Y., F. K. A. Allotey, N. G. Mensah, and V. W. Elloh. "Laser induced amplification of hypersound in nondegenerate semiconductor superlattices." Superlattices and microstructures 33, no. 1-2 (2003): 41-52.

[59] Maksimendo, A. S., and G. Ya Slepyan. Nonlinear Properties of Carbon Nanotubes in a Strong Electric Field. Belarus State Univ. Minsk Inst. of Nuclear Problems, 2000.

[60] Apostolakis, Apostolos. ”High-frequency acoustoelectronic phenomena in miniband superlattices.” PhD diss., Loughborough University, 2017.

[61] Seidu, S.S. Negative Differential Conductivity and Terahertz Gnereation in Carbon Nanotubes, PhD Thesis, University of Cape coast, Department of Physics, Ghana, 2011.

[62] Bloch, Felix. ”Quantum mechanics of electrons in crystal lattices.” Z. Phys 52 (1928): 555-600.

[63] Khvesyuk, V. I., A. A. Barinov, B. Liu, and W. Qiao. ”A review to the specific problems of nano thermal physics.” In Journal of Physics: Conference Series, vol. 1683, no. 2, p. 022073. IOP Publishing, 2020.

[64] Wacker, Andreas. ”Semiconductor superlattices: a model system for nonlinear transport.” Physics Reports 357, no. 1 (2002): 1-111.

[65] Kroemer, Herbert. ”On the derivation of $.dk/dt = F$, the k-space form of Newton’s law for Bloch waves.” American Journal of Physics 54, no. 2 (1986): 177-178.

[66] Brown, E. ”Bloch electrons in a uniform magnetic field.” Physical Review 133, no. 4A (1964): A1038.

[67] Ando, T., and S. Mori. ”Effective-mass theory of semiconductor heterojunctions and superlattices.” Surface Science 113, no. 1-3 (1982): 124-130.

[68] Bouchard, A. M., and Marshall Luban. ”Bloch oscillations and other dynamical phenomena of electrons in semiconductor superlattices.” Physical Review B 52, no. 7 (1995): 5105.

[69] Lyssenko, V. G., G. Valušis, F. Löser, T. Hasche, K. Leo, M. M. Dignam, and K. Köhler. ”Direct measurement of the spatial displacement of Bloch-oscillating electrons in semiconductor superlattices.” Physical review letters 79, no. 2 (1997): 301.

[70] Zener, Clarence. ”A theory of the electrical breakdown of solid dielectrics.” Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character 145, no. 855 (1934): 523-529.

[71] Leo, Karl, Peter Haring Bolivar, Frank Brüggemann, Ralf Schwedler, and Klaus Köhler. ”Observation of Bloch oscillations in a semiconductor superlattice.” Solid State Communications 84, no. 10 (1992): 943-946.

[72] Waschke, Christian, Hartmut G. Roskos, Ralf Schwedler, Karl Leo, Heinrich Kurz, and Klaus Köhler. ”Coherent submillimeter-wave emission from Bloch oscillations in a semiconductor superlattice.” Physical review letters 70, no. 21 (1993): 3319.

[73] Sudzius, M., V. G. Lyssenko, G. Valusis, F. Löser, T. Hasche, M. M. Dignam, K. Köhler, and K. Leo. ”Direct measurement of the spatial displacement of Bloch-oscillating electrons in

semiconductor superlattices." Physica E: Low-dimensional Systems and Nanostructures 2, no. 1-4 (1998): 437-440.

[74] Peik, Ekkehard, Maxime Ben Dahan, Isabelle Bouchoule, Yvan Castin, and Christophe Salomon. "Bloch Oscillations of Atoms in an optical potential." In Atomic Physics 15: Proceedings Of The Fifteenth International Conference On Atomic Physics, p. 358. World Scientific, 1997.

[75] Gustavsson, Mattias, Elmar Haller, M. J. Mark, Johann G. Danzl, Gabriel Rojas-Kopeinig, and H-C. Nägerl. "Control of interaction-induced dephasing of Bloch oscillations." Physical review letters 100, no. 8 (2008): 080404.

[76] Ridley, Brian K. Quantum processes in semiconductors. Oxford university press, 2013.

[77] Bernardini, Fabio, Vincenzo Fiorentini, and David Vanderbilt. "Spontaneous polarization and piezoelectric constants of III-V nitrides." Physical Review B 56, no. 16 (1997): R10024.

[78] Bester, Gabriel, Xifan Wu, David Vanderbilt, and Alex Zunger. "Importance of second-order piezoelectric effects in zinc-blende semiconductors." Physical review letters 96, no. 18 (2006): 187602.

[79] Sun, Yongke, Scott E. Thompson, and Toshikazu Nishida. Strain effect in semiconductors: theory and device applications. Springer Science & Business Media, 2009.

[80] Glavin, B. A., V. A. Kochelap, T. L. Linnik, K. W. Kim, and M. A. Stroscio. ”Generation of high-frequency coherent acoustic phonons in superlattices under hopping transport. I. Linear theory of phonon instability.” Physical review B 65, no. 8 (2002): 085303.

[81] Huynh, Agnès, Bernard Perrin, and A. Lemaître. ”Semiconductor superlattices: A tool for terahertz acoustics.” Ultrasonics 56 (2015): 66-79.

[82] Spector, Harold N. ”Amplification of acoustic waves through interaction with conduction electrons.” Physical Review 127, no. 4 (1962): 1084.

[83] Colvard, C., T. A. Gant, M. V. Klein, R. Merlin, R. Fischer, H. Morkoc, and A. C. Gossard. ”Folded acoustic and quantized optic phonons in (GaAl) As superlattices.” Physical Review B 31, no. 4 (1985): 2080.

[84] Kent, A. J., R. N. Kini, N. M. Stanton, M. Henini, B. A. Glavin, V. A. Kochelap, and T. L. Linnik. ”Acoustic phonon emission from a weakly coupled superlattice under vertical electron transport: observation of phonon resonance.” Physical review letters 96, no. 21 (2006): 215504.

[85] Schomburg, Ekkehard, N. V. Demarina, and Karl Friedrich Renk. ”Amplification of a terahertz field in a semiconductor superlattice via phase-locked k-space bunches of Bloch oscillating electrons.” Physical Review B 67, no. 15 (2003): 155302.

[86] Czerniuk, Thomas, Christian Brüggemann, Jan Tepper, Sebastian Brodbeck, Christian Schneider, Martin Kamp, Sven Höfling

et al. ”Lasing from active optomechanical resonators.” Nature communications 5, no. 1 (2014): 1-6.

[87] Tzianaki, Eirini, Makis Bakarezos, George D. Tsibidis, Yannis Orphanos, Panagiotis A. Loukakos, Constantine Kosmidis, Panos Patsalas, Michael Tatarakis, and Nektarios A. Papadogiannis. ”High acoustic strains in Si through ultrafast laser excitation of Ti thin-film transducers.” Optics express 23, no. 13 (2015): 17191-17204.

[88] Temnov, Vasily V., Christoph Klieber, Keith A. Nelson, Tim Thomay, Vanessa Knittel, Alfred Leitenstorfer, Denys Makarov, Manfred Albrecht, and Rudolf Bratschitsch. ”Femtosecond nonlinear ultrasonics in gold probed with ultrashort surface plasmons.” Nature communications 4, no. 1 (2013): 1-6.

[89] Plombon, J. J., Ebrahim Andideh, Valery M. Dubin, and Jose Maiz. ”Influence of phonon, geometry, impurity, and grain size on copper line resistivity.” Applied physics letters 89, no. 11 (2006): 113124.

[90] Lundstrom, M. ”Fundamentals of carrier transport (Cambridge Univ Pr, 2000).” Cited on: 45.

[91] Greenaway, M. T., Alexander G. Balanov, D. Fowler, A. J. Kent, and T. M. Fromhold. ”Using acoustic waves to induce high-frequency current oscillations in superlattices.” Physical Review B 81, no. 23 (2010): 235313.

[92] Esaki, Leo, and Ray Tsu. ”Superlattice and negative differential conductivity in semiconductors.” IBM Journal of Research and Development 14, no. 1 (1970): 61-65.

[93] Bass, F. G., V. V. Zorchenko, and V. I. Shashora. ”Theory of galvanomagnetic and high-frequency phenomena in super-lattice semiconductors.” Soviet Physics Semiconductors-USSR 15, no. 3 (1981): 263-266.

[94] Pippard, Alfred Brian. The dynamics of conduction electrons. Gordon and Breach, 1965.

[95] Budd, Herbert F. ”The generalized path variable method.” Journal of the Physical Society of Japan 18, no. 1 (1963): 142-142.

[96] MacCallum, Crawford. ”Kinetic integral solutions of the Boltzmann equation.” Physical Review 132, no. 2 (1963): 930.

[97] Ignatov, A. A., and Yu A. Romanov. ”Nonlinear electromagnetic properties of semiconductors with a superlattice.” physica status solidi (b) 73, no. 1 (1976): 327-333.

[98] Winnerl, S., E. Schomburg, J. Grenzer, H-J. Regl, A. A. Ignatov, A. D. Semenov, K. F. Renk et al. ”Quasistatic and dynamic interaction of high-frequency fields with miniband electrons in semiconductor superlattices.” Physical Review B 56, no. 16 (1997): 10303.

[99] Hyart, Timo, Jussi Mattas, and Kirill N. Alekseev. ”Model of the influence of an external magnetic field on the gain of terahertz radiation

from semiconductor superlattices." Physical review letters 103, no. 11 (2009): 117401.

[100] Brozak, G., M. Helm, F. DeRosa, C. H. Perry, M. Koza, R. Bhat, and S. J. Allen Jr. "Thermal saturation of band transport in a superlattice." Physical review letters 64, no. 26 (1990): 3163.

[101] Grahn, H. T., K. Von Klitzing, K. Ploog, and G. H. Döhler. "Electrical transport in narrow-miniband semiconductor superlattices." Physical Review B 43, no. 14 (1991): 12094.

[102] Sibille, A., J. F. Palmier, M. Hadjazi, H. Wang, G. Etemadi, E. Dutisseuil, and F. Mollot. "Limits of semiclassical transport in narrow miniband GaAs/AlAs superlattices." Superlattices and microstructures 13, no. 2 (1993): 247.

[103] Poyser, C. L., A. V. Akimov, A. G. Balanov, R. P. Campion, and A. J. Kent. "A weakly coupled semiconductor superlattice as a harmonic hypersonic-electrical transducer." New Journal of Physics 17, no. 8 (2015): 083064.

[104] Wan, J., C. Martijn De Sterke, and M. M. Dignam. "Dynamic localization and quasi-Bloch oscillations in general periodic ac-dc electric fields." Physical Review B 70, no. 12 (2004): 125311.

[105] Tolpygo, KR, and Z. I. Uritskii. "On the Theory of Polaron Mobility." Soviet Physics JETP-USSR 3, no. 5 (1956): 725-731.

[106] Weinreich, Gabriel, T. M. Sanders Jr, and Harry G. White. ”Acoustoelectric effect in n-type germanium.” Physical Review 114, no. 1 (1959): 33.

[107] Pomerantz, M. ”Amplification of microwave phonons in germanium.” Physical Review Letters 13, no. 9 (1964): 308.

[108] Hutson, A. R., J. H. McFee, and D. L. White. ”Ultrasonic amplification in CdS.” Physical Review Letters 7, no. 6 (1961): 237.

[109] Eckstein, S.G., 1963. Resonant amplification of sound by conduction electrons. Physical Review, 131(3), p.1087.

[110] Gokhale, Vikrant J., Yonghyun Shim, and Mina Rais-Zadeh. ”Observation of the acoustoelectric effect in gallium nitride micromechanical bulk acoustic filters.” In 2010 IEEE International Frequency Control Symposium, pp. 524-529. IEEE, 2010.

[111] Ariza-Flores, Augusto, and Isaac Rodríguez-Vargas. ”Electron subband structure and mobility trends in pn delta-doped quantum wells in Si.” Progress In Electromagnetics Research Letters 1 (2008): 159-165.

[112] Vasilopoulos, P., M. Charbonneau, and C. M. Van Vliet. ”Linear and nonlinear electrical conduction in quasi-two-dimensional quantum wells.” Physical Review B 35, no. 3 (1987): 1334.

[113] Bau, Nguyen Quang, and Hoang Dinh Trien. ”The nonlinear absorption coefficient of strong electromagnetic waves caused by electrons confined in quantum wires.” J. Korean Phys. Soc 56, no. 1 (2010): 120.

[114] Busse, L. J., and J. G. Miller. ”Response characteristics of a finite aperture, phase insensitive ultrasonic receiver based upon the acoustoelectric effect.” The Journal of the Acoustical Society of America 70, no. 5 (1981): 1370-1376.

[115] Mason, Warren P. ”Physical acoustics and the properties of solids.” The Journal of the Acoustical Society of America 28, no. 6 (1956): 1197-1206.

[116] Bau, Nguyen Quang, Le Dinh, and Tran Cong Phong. ”Absorption coefficient of weak electromagnetic waves caused by confined electrons in quantum wires.” Journal-Korean Physical Society 51, no. 4 (2007): 1325.

[117] Komirenko, S. M., K. W. Kim, A. A. Demidenko, V. A. Kochelap, and M. A. Stroscio. ”Generation and amplification of sub-THz coherent acoustic phonons under the drift of two-dimensional electrons.” Physical Review B 62, no. 11 (2000): 7459.

[118] Zhao, X. F., J. Zhang, S. M. Chen, and W. Xu. ”Cerenkov acoustic-phonon emission generated electrically from a polar semiconductor.” Journal of Applied Physics 105, no. 10 (2009): 104514.

[119] Dompreh, Kwadwo A., Natalia G. Mensah, and Samuel Y. Mensah. ”Amplification of Hypersound in Graphene with degenerate energy dispersion.” arXiv preprint arXiv:1503.07360 (2015).

[120] Komirenko, S. M., K. W. Kim, A. A. Demidenko, V. A. Kochelap, and M. A. Stroscio. ”Amplification of transverse acoustic phonons in

quantum well heterostructures with piezoelectric interaction." Journal of Applied Physics 90, no. 8 (2001): 3934-3941.

[121] Mensah, S. Y., F. K. A. Allotey, and S. K. Adjepong. "Acoustoelectric effect in a semiconductor superlattice." Journal of Physics: Condensed Matter 6, no. 34 (1994): 6783.

[122] Dompreh, K. A., S. Y. Mensah, S. S. Abukari, F. Sam, and N. G. Mensah. "Amplification of Acoustic Waves in Graphene Nanoribbon in the Presence of External Electric and Magnetic Field." arXiv preprint arXiv:1410.8064 (2014).

[123] Shmelev, G. M., Mensah, S. Y., Tsurkan, G. I. (1988). Hypersound amplification by a superlattice in a nonquantised electric field. Journal of Physics C: Solid State Physics, 21(33), L1073.

[124] Bau, N. Q., L. T. Hung, and N. D. Nam. "The nonlinear absorption coefficient of a strong electromagnetic wave by confined electrons in quantum wells under the influences of confined phonons." Journal of Electromagnetic Waves and Applications 24, no. 13 (2010): 1751-1761.

[125] Glavin, B. A., V. A. Kochelap, and T. L. Linnik. "Generation of high-frequency coherent acoustic phonons in a weakly coupled superlattice." Applied physics letters 74, no. 23 (1999): 3525-3527.

[126] Rodrigues, C., A. L. A. Fonseca, and O. A. C. Nunes. "Laser-Dressed Sound Absorption Coefficient in Superlattice Structures." physica status solidi (b) 179, no. 1 (1993): K17-K20.

[127] Busse, L. J., and J. G. Miller. ”Response characteristics of a finite aperture, phase insensitive ultrasonic receiver based upon the acoustoelectric effect.” The Journal of the Acoustical Society of America 70, no. 5 (1981): 1370-1376.

[128] Parmenter, Robert H. ”The acousto-electric effect.” Physical Review 89, no. 5 (1953): 990.

[129] Mintmire, John W., Brett I. Dunlap, and Carter T. White. ”Are fullerene tubules metallic?.” Physical review letters 68, no. 5 (1992): 631.

[130] Saito, Riichiro, Mitsutaka Fujita, G. Dresselhaus, and U. MS Dresselhaus. ”Electronic structure of chiral graphene tubules.” Applied physics letters 60, no. 18 (1992): 2204-2206.

[131] Santos, P. V., T. Schumann, M. H. Oliveira Jr, J. M. J. Lopes, and H. Riechert. ”Acousto-electric transport in epitaxial monolayer graphene on SiC.” Applied physics letters 102, no. 22 (2013): 221907.

[132] Wang, Feng, Gordana Dukovic, Louis E. Brus, and Tony F. Heinz. ”The optical resonances in carbon nanotubes arise from excitons.” Science 308, no. 5723 (2005): 838-841.

[133] Hertel, Tobias, and Gunnar Moos. ”Electron-phonon interaction in single-wall carbon nanotubes: A time-domain study.” Physical review letters 84, no. 21 (2000): 5002.

[134] LeRoy, Brian J., S. G. Lemay, J. Kong, and C. Dekker. ”Electrical generation and absorption of phonons in carbon nanotubes.” Nature 432, no. 7015 (2004): 371-374.

[135] Galperin Yu M., and Kagan V. D.(1970), Zh. Eksp. Teor. Fiz. (Soy.Phys.JETP) 59 1657.

[136] Abdelraheem, S. K., D. P. Blyth, and N. Balkan. ”Amplification of ultrasonic waves in bulk GaN and GaAlN/GaN heterostructures.” physica status solidi (a) 185, no. 2 (2001): 247-256.

[137] Sorbel R.(1976), Phys. Stat. Solidi B 77 141.

[138] Mensah, S. Y., N. G. Mensah, V. W. Elloh, G. K. Banini, Frederick Sam, and F. K. A. Allotey. ”Propagation of ultrasonic waves in bulk gallium nitride (GaN) semiconductor in the presence of high-frequency electric field.” Physica E: Low-dimensional Systems and Nanostructures 28, no. 4 (2005): 500-506.

[139] Dompreh, K. A., N. G. Mensah, and S. Y. Mensah. ”Acoustoelectric effect in graphene with degenerate energy dispersion.” Physica E: Low-Dimensional Systems and Nanostructures 85 (2017): 160-163.

[140] Mensah, S. Y., F. K. A. Allotey, and S. K. Adjepong. ”Acoustoelectric effect in a semiconductor superlattice.” Journal of Physics: Condensed Matter 6, no. 34 (1994): 6783.

[141] Fleming, W. J., and J. E. Rowe. ”Acoustoelectric effects in indium antimonide.” Journal of Applied Physics 42, no. 5 (1971): 2041-2047.

[142] Rotter, M., Achim Wixforth, W. Ruile, D. Bernklau, and H. Riechert. ”Giant acoustoelectric effect in GaAs/LiNbO3 hybrids.” Applied physics letters 73, no. 15 (1998): 2128-2130.

[143] Talyanskii, V. I., J. M. Shilton, M. Pepper, C. G. Smith, C. J. B. Ford, E. H. Linfield, D. A. Ritchie, and G. A. C. Jones. ”Single-electron transport in a one-dimensional channel by high-frequency surface acoustic waves.” Physical Review B 56, no. 23 (1997): 15180.

[144] Rocke, C., S. Zimmermann, Achim Wixforth, Jörg P. Kotthaus, G. Böhm, and G. Weimann. ”Acoustically driven storage of light in a quantum well.” Physical Review Letters 78, no. 21 (1997): 4099.

[145] Buitelaar, M. R., V. Kashcheyevs, P. J. Leek, V. I. Talyanskii, C. G. Smith, D. Anderson, G. A. C. Jones, J. Wei, and D. H. Cobden. ”Adiabatic charge pumping in carbon nanotube quantum dots.” Physical review letters 101, no. 12 (2008): 126803.

[146] Mensah, S. Y., F. K. A. Allotey, N. G. Mensah, H. Akrobotu, and G. Nkrumah. ”The influence of external electric field on acoustoelectric effect in a superlattice.” Superlattices and Microstructures 37, no. 2 (2005): 87-97.

[147] Lippens, P. E., M. Lannoo, and J. F. Pouliquen. ”Calculation of the transverse acoustoelectric voltage in a piezoelectricextrinsic semiconductor structure.” Journal of applied physics 66, no. 3 (1989): 1209-1216.

[148] Cunningham, J., M. Pepper, V. I. Talyanskii, and D. A. Ritchie. ”Acoustoelectric current in submicron-separated quantum wires.” Applied Physics Letters 86, no. 15 (2005): 152105.

[149] Bau, N. Q., and N. V. Hieu. ”Theory of the Acoustomagnetoelectric Effect in a Superlattice.” PIERS Proceedings, Xi’an, China, March 22 26 (2010).

[150] Wang, Wei-Shan, Tsung-Tsong Wu, Tai-Hsu Chou, and Yung-Yu Chen. ”A ZnO nanorod-based SAW oscillator system for ultraviolet detection.” Nanotechnology 20, no. 13 (2009): 135503.

[151] Ebbecke, Jens, Christoph J. Strobl, and Achim Wixforth. ”Acoustoelectric current transport through single-walled carbon nanotubes.” Physical Review B 70, no. 23 (2004): 233401.

[152] Reulet, B., A. Yu Kasumov, M. Kociak, R. Deblock, I. I. Khodos, Yu B. Gorbatov, V. T. Volkov, Catherine Journet, and H. Bouchiat. ”Acoustoelectric effects in carbon nanotubes.” Physical Review Letters 85, no. 13 (2000): 2829.

[153] Afonin, V. V., and Yu M. Gal’Perin. ”Acoustoelectric effect and phonon-drag electron thermoelectric power under weak localization conditions.” Semiconductors (Woodbury, NY) 27, no. 1 (1993): 61-65.

[154] Hoskin, M.J.,Morko, J., and Hunsinger, B.J.,Appl. Phys. Lett,Vol.41,332.(1982).

[155] Yamada, Toshiyuki. "Acoustomagnetoelectric effect in bismuth." Journal of the Physical Society of Japan 20, no. 8 (1965): 1424-1437.

[156] Gurevich, V. L., V. B. Pevzner, and G. J. Iafrate. "Giant quantum oscillations of acoustoelectric effect in nanostructures." Physical review letters 77, no. 18 (1996): 3881.

[157] Bau, Nguyen Quang, Nguyen Van Hieu, and Nguyen Vu Nhan. "The quantum acoustomagnetoelectric field in a quantum well with a parabolic potential." Superlattices and Microstructures 52, no. 5 (2012): 921-930.

[158] Margulis, A. D., and Vl A. Margulis. "The quantum acoustomagnetoelectric effect due to Rayleigh sound waves." Journal of Physics: Condensed Matter 6, no. 31 (1994): 6139.

[159] Kogami, Mineichi, and Shoji Tanaka. "Acoustomagnetoelectric and acoustoelectric effects in n-InSb at low temperatures." Journal of the Physical Society of Japan 30, no. 3 (1971): 775-784.

[160] Epshtein, E. M., and Gulyaev, Y.V., Sov. Phys. Solids State, Vol. 9, 28, 1967.

[161] Goldsmid, H. Thermoelectric refrigeration. Springer, 2013.

[162] Pei, Yanzhong, Xiaoya Shi, Aaron LaLonde, Heng Wang, Lidong Chen, and G. Jeffrey Snyder. "Convergence of electronic bands for high performance bulk thermoelectrics." Nature 473, no. 7345 (2011): 66-69.

[163] Liu, Wei, Xiaojian Tan, Kang Yin, Huijun Liu, Xinfeng Tang, Jing Shi, Qingjie Zhang, and Ctirad Uher. ”Convergence of conduction bands as a means of enhancing thermoelectric performance of n-type Mg 2 Si 1− x Sn x solid solutions.” Physical review letters 108, no. 16 (2012): 166601.

[164] Sheng, Caiyu, Dengdong Fan, and Huijun Liu. ”High thermoelectric performance of two-dimensional (PbTe) 2 layer.” arXiv preprint arXiv:1905.13442 (2019).

[165] Snyder, G. Jeffrey, and Eric S. Toberer. ”Complex thermoelectric materials.” Materials for sustainable energy: a collection of peer-reviewed research and review articles from Nature Publishing Group (2011): 101-110.

[166] Zhao, Li-Dong, Shih-Han Lo, Yongsheng Zhang, Hui Sun, Gangjian Tan, Ctirad Uher, Christopher Wolverton, Vinayak P. Dravid, and Mercouri G. Kanatzidis. ”Ultralow thermal conductivity and high thermoelectric figure of merit in SnSe crystals.” Nature 508, no. 7496 (2014): 373-377.

[167] Tritt, Terry M., ed. Thermal conductivity: theory, properties, and applications. Springer Science & Business Media, 2005.

[168] Mensah, S. Y., and G. K. Kangah. ”The thermoelectric effect in a semiconductor superlattice in a non-quantized electric field.” Journal of Physics: Condensed Matter 4, no. 3 (1992): 919.

[169] Hicks, Lyndon D., and Mildred S. Dresselhaus. ”Effect of quantum-well structures on the thermoelectric figure of merit.” Physical Review B 47, no. 19 (1993): 12727.

[170] Mensah, S. Y., F. K. A. Allotey, N. G. Mensah, and G. Nkrumah. ”Giant electrical power factor in single-walled chiral carbon nanotube.” Superlattices and microstructures 33, no. 3 (2003): 173-180.

[171] Wang, Lei, and Baowen Li. ”Thermal logic gates: computation with phonons.” Physical review letters 99, no. 17 (2007): 177208.

[172] Slepyan, Gregory Ya, Sergey A. Maksimenko, Akhlesh Lakhtakia, Oleg M. Yevtushenko, and Anton V. Gusakov. ”Electronic and electromagnetic properties of nanotubes.” Physical Review B 57, no. 16 (1998): 9485.

[173] Mensah, Samuel Yeboah, F. K. A. Allotey, N. G. Mensah, and G. Nkrumah. ”Differential thermopower of a CNT chiral carbon nanotube.” Journal of Physics: Condensed Matter 13, no. 24 (2001): 5653.

[174] Sakyi-Arthur, D., S. Y. Mensah, N. G. Mensah, K. W. Adu, K. A. Dompreh, and R. Edziah. ”Tunable power factor in fluorine-doped single-walled carbon nanotubes.” Journal of Applied Physics 128, no. 24 (2020): 244301.

[175] Sakyi-Arthur, D., Mensah, S.Y., Adu, K.W., Dompreh, K.A., Edziah, R. and Mensah, N.G. Unusual differential thermoelectric power fluorine doped SWCNT. Physical Review Letters (2020) (in progress)

[176] Klemens, P. G. ”Anharmonic decay of optical phonons.” Physical Review 148, no. 2 (1966): 845.

[177] Lin, Yu-Ming, and M. S. Dresselhaus. ”Thermoelectric properties of superlattice nanowires.” Physical review B 68, no. 7 (2003): 075304.

[178] Narumanchi, Sreekant VJ, Jayathi Y. Murthy, and Cristina H. Amon. ”Submicron heat transport model in silicon accounting for phonon dispersion and polarization.” J. Heat Transfer 126, no. 6 (2004): 946-955.

[179] Sobhan, Choondal B., and George P. Peterson. Microscale and nanoscale heat transfer: fundamentals and engineering applications. CRC Press, 2008.

[180] Chen, Gang. Nanoscale energy transport and conversion: a parallel treatment of electrons, molecules, phonons, and photons. Oxford university press, 2005.

[181] Kittel, Charles, Paul McEuen, and Paul McEuen. Introduction to solid state physics. Vol. 8. New York: Wiley, 1996.

[182] Mittal, Arpit, and Sandip Mazumder. ”Monte Carlo study of phonon heat conduction in silicon thin films including contributions of optical phonons.” (2010): 052402.

[183] Majumdar, A. S. M. E. ”Microscale heat conduction in dielectric thin films.” (1993): 7-16.

[184] Tien, C. L., & Chen, G. Challenges in microscale conductive and radiative heat transfer.Journal of Heat Transfer, Vol. 116, 1994 pp. 799 – 807.

[185] Flik, M. I., B. I. Choi, and K. E. Goodson. ”Heat transfer regimes in microstructures.” (1992): 666-674.

[186] Huang, Mei-Jiau, Wen-Yen Chong, and Tai-Ming Chang. ”The lattice thermal conductivity of a semiconductor nanowire.” Journal of applied physics 99, no. 11 (2006): 114318.

[187] Blatt, Frank J., Peter A. Schroeder, Carl L. Foiles, and Denis Greig. ”The Thermoelectric Power of Transition Metals.” In Thermoelectric Power of Metals, pp. 135-190. Springer, Boston, MA, 1976.

[188] Miyamoto, Yoshiyuki, Steven G. Louie, and Marvin L. Cohen. ”Chiral conductivities of nanotubes.” Physical review letters 76, no. 12 (1996): 2121.

[189] Mensah, S. Y., F. K. A. Allotey, and S. K. Adjepong. ”Acoustomagnetoelectric effect in a superlattice.” Journal of Physics: Condensed Matter 8, no. 9 (1996): 1235.

[190] Yamada, Toshiyuki. ”Acoustomagnetoelectric effect in bismuth.” Journal of the Physical Society of Japan 20, no. 8 (1965): 1424-1437.

[191] Mickelson, E. T., C. B. Huffman, A. G. Rinzler, R. E. Smalley, R. H. Hauge, and J. L. Margrave. ”Fluorination of single-wall carbon nanotubes.” Chemical physics letters 296, no. 1-2 (1998): 188-194.

[192] Mensah, S. Y., F. K. A. Allotey, N. G. Mensah, and V. W. Elloh. ”Laser induced amplification of hypersound in nondegenerate semiconductor superlattices.” Superlattices and microstructures 33, no. 1-2 (2003): 41-52.

[193] Mensah, S. Y., F. K. A. Allotey, and S. K. Adjepong. ”The effect of a high-frequency electric field on hypersound amplification in a superlattice.” Journal of Physics: Condensed Matter 6, no. 19 (1994): 3479.

[194] Towe, E. Proceeding of the International Semiconductor Device Research Symposium (ISDRS’95) Held in Charlottesville, Virginia on 5-8 December 1995. Volume 2. Virginia Univ Charlottesville School of Engineering and Applied Science, 1996.

[195] Kryuchkov, S.V. & Mikheev, N.P. Characteristics of the Acoustoelectric Effect in Narrow-Gap Semiconductors. Soviet Physics Solids State, USSR, 16, 104 (1982).

[196] Mensah, S. Y., F. K. A. Allotey, and S. K. Adjepong. ”Acoustoelectric effect in a semiconductor superlattice.” Journal of Physics: Condensed Matter 6, no. 34 (1994): 6783.

[197] Plank, Natalie OV, Gavin A. Forrest, Rebecca Cheung, and Andrew J. Alexander. ”Electronic properties of n-type carbon nanotubes prepared by CF4 plasma fluorination and amino functionalization.” The Journal of Physical Chemistry B 109, no. 47 (2005): 22096-22101.

[198] Zhang, Lei, Jianzhong Yang, Christopher L. Edwards, Lawrence B. Alemany, Valery N. Khabashesku, and Andrew R. Barron. ”Diels–Alder addition to fluorinated single walled carbon nanotubes.” Chemical communications 26 (2005): 3265-3267.

[199] Miyagawa, Hiroaki, and Lawrence T. Drzal. ”Thermo-physical and impact properties of epoxy nanocomposites reinforced by single-wall carbon nanotubes.” Polymer 45, no. 15 (2004): 5163-5170.

[200] Adu, Kofi, Daniel Sakyi-Arthur, Samuel Y. Mensah, Natalie G. Mensah, Kwadwo A. Dompreh, and Raymond Edziah. ”Giant Thermoelectric Figure of Merit in Fluorine Doped Single Walled Carbon Nanotube.” (2021).

[201] Esaki, Leo, and Ray Tsu. ”Superlattice and negative differential conductivity in semiconductors.” IBM Journal of Research and Development 14, no. 1 (1970): 61-65.

[202] Sibille, A., J. F. Palmier, H. Wang, and F. Mollot. ”Observation of Esaki-Tsu negative differential velocity in GaAs/AlAs superlattices.” Physical review letters 64, no. 1 (1990): 52.

[203] Mensah, S. Y., F. KA Allotey, and N. G. Mensah. ”Nonlinear acoustoelectric effect in semiconductor superlattice.” (1999).

[204] Gokhale, Vikrant J., Yonghyun Shim, and Mina Rais-Zadeh. ”Observation of the acoustoelectric effect in gallium nitride micromechanical bulk acoustic filters.” In 2010 IEEE International Frequency Control Symposium, pp. 524-529. IEEE, 2010.

[205] Esaki, L., and L. L. Chang. ”New transport phenomenon in a semiconductor” Superlattice”.” Physical Review Letters 33, no. 8 (1974): 495.

[206] Beltram, Fabio, Federico Capasso, Deborah L. Sivco, Albert L. Hutchinson, Sung-Nee G. Chu, and Alfred Y. Cho. ”Scattering-controlled transmission resonances and negative differential conductance by field-induced localization in superlattices.” Physical review letters 64, no. 26 (1990): 3167.

[207] Romanov, Yu A., and Yu Yu Romanova. ”Self-oscillations in semiconductor superlattices.” Journal of Experimental and Theoretical Physics 91, no. 5 (2000): 1033-1045.

[208] Ignatov, Anatoly A., K. F. Renk, and E. P. Dodin. ”Esaki-Tsu superlattice oscillator: Josephson-like dynamics of carriers.” Physical review letters 70, no. 13 (1993): 1996.

[209] Schomburg, E., T. Blomeier, K. Hofbeck, J. Grenzer, S. Brandl, I. Lingott, A. A. Ignatov et al. ”Current oscillation in superlattices with different miniband widths.” Physical Review B 58, no. 7 (1998): 4035.

[210] Savvidis, Pavlos G., Borys Kolasa, G. Lee, and S. J. Allen. ”Resonant crossover of terahertz loss to the gain of a Bloch oscillating InAs/AlSb superlattice.” Physical review letters 92, no. 19 (2004): 196802.

[211] Shinokita, Keisuke, Klaus Reimann, Michael Woerner, Thomas Elsaesser, Rudolf Hey, and Christos Flytzanis. ”Strong amplification

of coherent acoustic phonons by intraminiband currents in a semiconductor superlattice." Physical review letters 116, no. 7 (2016): 075504.

[212] Wang, Zhaohui. Biomedical applications of acoustoelectric effect. The University of Arizona, 2011.

[213] Abdelraheem, S. K., D. P. Blyth, and N. Balkan. "Amplification of ultrasonic waves in bulk GaN and GaAlN/GaN heterostructures." physica status solidi (a) 185, no. 2 (2001): 247-256.

[214] Hutson, A. R., and Donald L. White. "Elastic wave propagation in piezoelectric semiconductors." Journal of Applied Physics 33, no. 1 (1962): 40-47.

[215] Abe, Yutaka, and Nobuo Mikoshiba. "Ultrasonic amplification in a transverse magnetic field." Japanese Journal of Applied Physics 7, no. 8 (1968): 881.

[216] Kikuchi, Makoto, Hisao Hayakawa, and Yutaka Abe. "Acoustoelectric current oscillation in InSb and its dependence on the transverse magnetic field." Japanese Journal of Applied Physics 5, no. 12 (1966): 1259.

[217] Dumke, W. P., and R. R. Haering. "Ultrasonic amplification in semimetals." Physical Review 126, no. 6 (1962): 1974.

[218] Weinreich, Gabriel, and Harry G. White. "Observation of the acoustoelectric effect." Physical Review 106, no. 5 (1957): 1104.

[219] Sumanasekera, Gamini U., C. K. W. Adu, S. Fang, and P. C. Eklund. ”Effects of gas adsorption and collisions on electrical transport in single-walled carbon nanotubes.” Physical Review Letters 85, no. 5 (2000): 1096.

[220] Hone, J., I. Ellwood, M. Muno, Ari Mizel, Marvin L. Cohen, A. Zettl, Andrew G. Rinzler, and R. E. Smalley. ”Thermoelectric power of single-walled carbon nanotubes.” Physical Review Letters 80, no. 5 (1998): 1042.

[221] Belashchenko, Kirill D., and D. V. Livanov. ”Phonon effects in the thermoelectric power of impure metals.” Journal of Physics: Condensed Matter 10, no. 34 (1998): 7553.

[222] Pitaevskii, L. P., E. M. Lifshitz, and J. B. Sykes. Course of theoretical physics: physical kinetics. Vol. 10. Elsevier, 2017.

[223] Lakhtakia, Akhlesh, Gregory Ya Slepyan, Sergey A. Maksimenko, Anton V. Gusakov, and Oleg M. Yevtushenko. ”Effective medium theory of the microwave and the infrared properties of composites with carbon nanotube inclusions.” Carbon 36, no. 12 (1998): 1833-1839.

[224] Vashaee, Daryoosh, and Ali Shakouri. ”Improved thermoelectric power factor in metal-based superlattices.” physical review letters 92, no. 10 (2004): 106103.

[225] Fischer, J. E., H. Dai, A. Thess, R. Lee, N. M. Hanjani, D. L. Dehaas, and R. E. Smalley. ”Metallic resistivity in crystalline ropes of single-wall carbon nanotubes.” Physical review B 55, no. 8 (1997): R4921.

[226] Delaney, Paul, Hyoung Joon Choi, Jisoon Ihm, Steven G. Louie, and Marvin L. Cohen. ”Broken symmetry and pseudogaps in ropes of carbon nanotubes.” Nature 391, no. 6666 (1998): 466-468.

[227] Kaiser, A. B., G. Düsberg, and S. Roth. ”Heterogeneous model for conduction in carbon nanotubes.” Physical Review B 57, no. 3 (1998): 1418.

[228] Delaney, Paul, Hyoung Joon Choi, Jisoon Ihm, Steven G. Louie, and Marvin L. Cohen. ”Broken symmetry and pseudogaps in ropes of carbon nanotubes.” Nature 391, no. 6666 (1998): 466-468.

[229] Bradley, Keith, Seung-Hoon Jhi, Philip G. Collins, J. Hone, Marvin L. Cohen, S. G. Louie, and A. Zettl. ”Is the intrinsic thermoelectric power of carbon nanotubes positive?.” Physical Review Letters 85, no. 20 (2000): 4361.

[230] Grigorian, L., G. U. Sumanasekera, A. L. Loper, S. L. Fang, J. L. Allen, and P. C. Eklund. ”Giant thermopower in carbon nanotubes: A one-dimensional Kondo system.” Physical Review B 60, no. 16 (1999): R11309.

[231] Shi, Li, Deyu Li, Choongho Yu, Wanyoung Jang, Dohyung Kim, Zhen Yao, Philip Kim, and Arunava Majumdar. ”Measuring thermal and thermoelectric properties of one-dimensional nanostructures using a microfabricated device.” J. Heat Transfer 125, no. 5 (2003): 881-888.

[232] Lin, Yu-Ming, and M. S. Dresselhaus. ”Thermoelectric properties of superlattice nanowires.” Physical review B 68, no. 7 (2003): 075304.

APPENDIX A

ACOUSTODYNAMICS IN FSWCNT

The energy dispersion for the semiconductor FSWCNT is give as:

$$\varepsilon(\vec{p}_z) = \varepsilon_o + \Xi_n \gamma_o \cos^{2N-1}(a\vec{p}_z) \tag{A.1}$$

For $N = 2$, the dispersion relation yields:

$$\varepsilon(\vec{p}_z) = \varepsilon_o + 8\gamma_0 \cos^3(a\vec{p}_z) \tag{A.2}$$

A.2 is expanded as:

$$\varepsilon(\vec{p}) = \varepsilon_o + \Delta_1 \cos(3a\vec{p}_z) + \Delta_2 \cos(a\vec{p}_z) \tag{A.3}$$

where $\Delta_1 = 2\gamma_o$ and $\Delta_2 = 6\gamma_o$. The acoustoelectric current density is defined as:

$$\vec{j}^{AE} = -e \sum_{n,n'} \int U^{ac}_{n,n'} \Psi_i(\vec{p}_z) d^2 p_z. \tag{A.4}$$

Here $\Psi_i(\vec{p})$ is the solution to the BTE in the absence of a magnetic field. $U^{ac}_{n,n'}$ is the electron-phonon interaction and is given as:

$$U^{ac}_{n,n'} = \frac{2\pi\vec{\Phi}}{\omega_q v_s} \sum_{n,n'} \{ |G_{\vec{p}_z - \hbar\vec{q}, \vec{p}_z}|^2 [f(\varepsilon_n(\vec{p}_z - \hbar\vec{q})) - f(\varepsilon_n(\vec{p}_z))] \delta(\varepsilon_n(\vec{p}_z - \hbar\vec{q}) - \varepsilon_n(\vec{p}_z) + \hbar\omega_q)$$
$$+ |G_{\vec{p}_z + \hbar\vec{q}, \vec{p}_z}|^2 [f(\varepsilon_{n'}(\vec{p}_z + \hbar\vec{q})) - f(\varepsilon_{n'}(\vec{p}_z))] \delta(\varepsilon_{n'}(\vec{p}_z + \hbar\vec{q}) - \varepsilon_{n'}(\vec{p}_z) - \hbar\omega_q) \} \tag{A.5}$$

Employing the principle of detailed balance and denoting $\vec{p}_z^{\,'} = \vec{p}_z \pm \hbar\vec{q}$, yields the condition:

The condition $\vec{p}_z^{\,'} = \vec{p}_z \pm \hbar\vec{q}$ is obtained by applying the principle of detailed balance and indicating $vecp'z = vecpzpmhbarvecq$.

$$|G_{p',p}|^2 = |G_{p,p'}|^2 \tag{A.6}$$

and the electron-phonon interaction's matrix element is provided as

$$|G_{p',p}| = \frac{4\pi e \mathscr{K}}{\sqrt{2\rho\omega_q \varepsilon}} \tag{A.7}$$

where $\mathscr{K}$ represents the piezoelectric modulus, ε represents the lattice dielectric constant, and ρ represents the FSWCNT density. After that, the AE current density adopts the following shape:

$$\vec{j}^{AE}_z = -\frac{2e}{(2\pi\hbar)^2} \frac{2\pi\vec{\Phi}}{\omega_q v_s} \sum_{n,n'} |G_{p'_z,p_z}|^2 [f(\varepsilon_{n'}(\vec{p}_z)) - f(\varepsilon_{n'}(\vec{p}_z + \hbar q))]$$

$$\times [\Psi_i(p_z + \hbar q) - \Psi_i(p_z)]\delta(\varepsilon_{n'}(\vec{p}_z + \hbar q) - \varepsilon_{n'}(\vec{p}_z) - \hbar\omega_q)d^2\vec{p}_z \quad \text{(A.8)}$$

where, $\Psi_i(p_z) = l_i(\vec{p}_z)$ is the electron mean free path written as:

$$l_z = \tau v_z \quad \text{(A.9)}$$

$$v_z = \frac{\partial \varepsilon(\vec{p})}{\partial \vec{p}} \quad \text{(A.10)}$$

Substituting A.9 and A.10 into A.8 yields:

$$\vec{j}_z^{AE} = -\frac{16e^3\pi\vec{\Phi}\mathscr{K}^2\tau}{2\varepsilon^2\omega_q^2 v_s\rho\hbar^2}\sum_{n,n'}[f(\varepsilon_{n'}(\vec{p}_z)) - f(\varepsilon_{n'}(\vec{p}_z + \hbar\vec{q}))]$$
$$\times [v_z(p_z + \hbar\vec{q}) - v_z(\vec{p})]\delta(\varepsilon_{n'}(\vec{p}_z + \hbar\vec{q}) - \varepsilon_{n'}(\vec{p}_z) - \hbar\omega_q)dp_z \quad \text{(A.11)}$$

The electron distribution function is given by the shifted fermi-dirac distribution as

$$f_o(p) = \frac{1}{1 + \exp[(\varepsilon(\vec{p}) - \mu)/k_B T]} \quad \text{(A.12)}$$

When A.12 is substituted for A.11, the result is an equation with the term $\mathscr{F}_{1/2}$, which represents the Fermi-Dirac integral of order $1/2$ and is expressed as:

$$\vec{j}_z^{AE} = -\frac{16e^3\pi\vec{\Phi}\mathscr{K}^2\tau}{2\varepsilon^2\omega_q^2 v_s\rho\hbar^2}\sum_{n,n'}[\mathscr{F}_{1/2}(\varepsilon_{n'}(\vec{p}_z)) - \mathscr{F}_{1/2}(\varepsilon_{n'}(\vec{p}_z + \hbar\vec{q}))]$$
$$\times [v_z(p_z + \hbar\vec{q}) - v_z(p_z)]\delta(\varepsilon_{n'}(\vec{p}_z + \hbar q) - \varepsilon_{n'}(\vec{p}_z) - \hbar\omega_q)d\vec{p} \quad \text{(A.13)}$$

The Fermi-Dirac integral $\mathscr{F}_{1/2}$, is given as:

$$\mathscr{F}_{1/2} = \frac{1}{\Gamma(1/2)}\int_0^\infty \frac{\eta_f^{1/2}d\eta}{1 + \exp(\eta - \eta_f)} \quad \text{(A.14)}$$

$(\mu - \varepsilon_c)/k_B T \equiv \eta_f$ and $\Gamma(1/2)$ is the Gamma function of the order $1/2$. For non-degenerate electron gas, A.12 becomes:

$$f_o(\vec{p}) = A^\dagger \exp\left(-\frac{\varepsilon(\vec{p}) - \mu}{k_B T}\right) \quad \text{(A.15)}$$

Solving the Boltzmann equation in the $\tau-$ approximation yields the electron distribution function in the presence of a constant electric field, $\vec{E}(t)$:

$$f(\vec{p}_z) = \int_0^\infty \frac{dt}{\tau}\exp(-t/\tau)f_o(\vec{p}_z - \vec{p}') \quad \text{(A.16)}$$

where $A^\dagger$ is the normalization constant that may be calculated using the normalization condition $\int f(\vec{p})d\vec{p} = n_o$ as follows:

$$A^\dagger = \frac{3n_o a^2}{2I_o(\Delta_1^*)I_o(\Delta_2^*)}\exp\left(\frac{\varepsilon_o - \mu}{k_B T}\right) \quad \text{(A.17)}$$

The dispersion relation yields the following electron velocity:

$$v_z(\vec{p}_z) = -[3a\Delta_1 \sin(3a\vec{p}_z) + a\Delta_2 \sin(a\vec{p}_z)] \tag{A.18}$$

Substituting A.14-A.17 into A.11, the current density is obtained as

$$\begin{aligned}
\vec{j}_z^{AE} = &\frac{4A^\dagger \pi \vec{\Phi} e^3 \mathscr{K}^2 \tau \Theta(1-\alpha^2)}{\hbar^3 \omega_q^2 \varepsilon^2 v_s \rho a q \sqrt{1-\alpha^2}} \int_0^\infty \exp\left(\frac{dt'}{\tau}\right) \\
&\left\{ \sinh\left[\Delta_1^* \cos(3ea\vec{E}t') \sin A \sin\left(\frac{3}{2}a\hbar q\right) + \Delta_2^* \cos(ea\vec{E}t') \sin B \sin\left(\frac{a}{2}\hbar q\right)\right] \right. \\
&\times \sinh\left[\Delta_1^* \cos(3ea\vec{E}t') \cos A \cos\left(\frac{3}{2}a\hbar q\right) + \Delta_2^* \cos(ea\vec{E}t') \cos B \cos\left(\frac{a}{2}\hbar q\right)\right] \\
&- 4\left(\Delta_2^* \sin(ea\vec{E}t') \cos B \sin\left(\frac{a}{2}\hbar q\right) + \Delta_1^* \cos A \sin(3ea\vec{E}t') \sin\left(\frac{3}{2}a\hbar q\right)\right. \\
&\left. + \Delta_1^* \Delta_2^* \sin(ea\vec{E}t') \sin(3ea\vec{E}t') \cos A \cos B \sin\left(\frac{a}{2}\hbar q\right) \sin\left(\frac{3}{2}a\hbar q\right)\right) \\
&\times \cosh\left[\Delta_1^* \cos(3ea\vec{E}t') \cos A \cos\left(\frac{3}{2}a\hbar q\right) + \Delta_2^* \cos(ea\vec{E}t') \cos B \cos\left(\frac{a}{2}\hbar q\right)\right] \\
&\left. \times \cosh\left[\Delta_1^* \cos(3ea\vec{E}t') \sin A \sin\left(\frac{3}{2}a\hbar q\right) + \Delta_2^* \cos(ea\vec{E}t') \sin B \sin\left(\frac{a}{2}\hbar q\right)\right]\right\}
\end{aligned} \tag{D.19}$$

where $\Delta_1^* = \Delta_1/k_B T$, and $\Delta_2^* = \Delta_2/k_B T$. Switching off the external electric field ($\vec{E} = 0$), A.19 reduces to:

$$\begin{aligned}
\vec{j}_z^{FSWCNT} = &\frac{4A^\dagger \pi \vec{\Phi} e^3 \mathscr{K}^2 \tau \Theta(1-\alpha^2)}{\hbar^3 \omega_q^2 \varepsilon^2 v_s \rho a q \sqrt{1-\alpha^2}} \\
&\times \left[\sinh\left\{\Delta_1^* \sin\left(\frac{3}{2}a\hbar q\right) \sin A + \Delta_2^* \sin\left(\frac{a}{2}\hbar q\right) \sin B\right\}\right. \\
&\left. \times \sinh\left\{\Delta_1^* \cos\left(\frac{3}{2}a\hbar q\right) \cos A + \Delta_2^* \cos\left(\frac{a}{2}\hbar q\right) \cos B\right\}\right]
\end{aligned} \tag{A.20}$$

Simplifying A.19 gives;

$$\begin{aligned}
\vec{j}_z^{FSWCNT} = \vec{j}_o &\left[\sinh\left\{\Delta_1^* \sin\left(\frac{3}{2}a\hbar q\right) \sin A + \Delta_2^* \sin\left(\frac{a}{2}\hbar q\right) \sin B\right\}\right. \\
&\left. \times \sinh\left\{\Delta_1^* \cos\left(\frac{3}{2}a\hbar q\right) \cos A + \Delta_2^* \cos\left(\frac{a}{2}\hbar q\right) \cos B\right\}\right]
\end{aligned} \tag{A.21}$$

where

$$\vec{j}_o = \frac{6A^\dagger \pi \vec{\Phi} e^3 \mathscr{K}^2 a \tau \Theta(1-\alpha^2)}{\hbar^3 \omega_q^2 \varepsilon v_s \rho q \sqrt{1-\alpha^2}}, \qquad \alpha = \frac{\omega_q}{12\gamma_o a q}$$

and

$$A = \frac{3}{4}\sin^{-1}\left(\frac{\omega_q}{12\gamma_o aq}\right), \quad B = \frac{1}{4}\sin^{-1}\left(\frac{\omega_q}{12\gamma_o aq}\right)$$

and Θ is a Heaviside step function.
The same approach is used for the FSWCNT to compare the results with those of undoped SWCNT. Using the $\vec{p}_z$ orbital's tight-binding energy dispersion, which is given as:

$$\varepsilon(\vec{p}_z) = \pm\gamma_o\left(1 - 2\cos\left(\frac{\vec{p}_z\sqrt{3}b}{2\hbar}\right)\right) \tag{A.24}$$

From A.22, the acoustocurrent density in undoped SWCNT determined by deformation potential is given by;

$$\vec{j}_z^{SWCNT} = \vec{j}_o^{SWCNT}\sinh\left\{\frac{\hbar\omega_q}{k_BT}\right\}\sinh\left\{\Delta^*\sqrt{1-\alpha^2}\cos\left(\frac{a\hbar q}{2}\right)\right\} \tag{A.25}$$

where

$$\vec{j}_o^{SWCNT} = \frac{\Lambda^2 q^2 e\tau n_o a\Theta(1-\alpha_o^2)}{\pi\hbar^2\omega_q^2 v_s\rho\sin(a\hbar q/2)I_o(\Delta^*)\sqrt{1-\alpha_o^2}} \tag{A.26}$$

and

$$\alpha_o = \frac{\hbar\omega_q}{4\gamma_o\sin(a\hbar q/2)}, \quad \Delta = 4\gamma_o, \quad \Delta^* = \Delta/k_BT$$

APPENDIX B

LOW VOLTAGE-CURRENT ACOUSTIC DEVICE

$$\varepsilon(\vec{p}_z) = \varepsilon_o + \Delta_1 \cos(3a\vec{p}_z) + \Delta_2 \cos(a\vec{p}_z) \tag{B.1}$$

where p_o, is the momentum within the first BZ i.e $-\pi/a \leq p_o \leq \pi/a$, with $\Delta_1 = 2\Delta$, and $\Delta_2 = 6\Delta$.
Following the procedure in Appendix A, in the presence of an external electric field, yields

$$\begin{aligned}
j_z^{AE} &= \frac{4A^\dagger \pi \vec{\Phi} e^3 \mathscr{K}^2 \Theta(1-\alpha^2)}{\hbar^3 \omega_q^2 \varepsilon^2 \rho a q \sqrt{1-\alpha^2}} \int_0^\infty \exp\left(-\frac{dt'}{\tau}\right) \\
&\times \left\{ \sinh\left[\Delta_1^* \cos(3ea\vec{E}t') \sin A \sin\left(\frac{3}{2}a\hbar q\right) + \Delta_2^* \cos(ea\vec{E}t') \sin B \sin\left(\frac{a}{2}\hbar q\right)\right]\right. \\
&\times \sinh\left[\Delta_1^* \cos(3ea\vec{E}t') \cos A \cos\left(\frac{3}{2}a\hbar q\right) + \Delta_2^* \cos(ea\vec{E}t') \cos B \cos\left(\frac{a}{2}\hbar q\right)\right] \\
&- 4\left(\Delta_2^* \sin(ea\vec{E}t') \cos B \sin\left(\frac{a}{2}\hbar q\right) + \Delta_1^* \cos A \sin(3ea\vec{E}t') \sin\left(\frac{3}{2}a\hbar q\right)\right. \\
&\left. + \Delta_1^* \Delta_2^* \sin(p'a) \sin(3ea\vec{E}t') \cos A \cos B \sin\left(\frac{a}{2}\hbar q\right) \sin\left(\frac{3}{2}a\hbar q\right)\right) \\
&\times \cosh\left[\Delta_1^* \cos(3ea\vec{E}t') \cos A \cos\left(\frac{3}{2}a\hbar q\right) + \Delta_2^* \cos(ea\vec{E}t') \cos B \cos\left(\frac{a}{2}\hbar q\right)\right] \\
&\left.\times \cosh\left[\Delta_1^* \cos(3ea\vec{E}t') \sin A \sin\left(\frac{3}{2}a\hbar q\right) + \Delta_2^* \cos(ea\vec{E}t') \sin B \sin\left(\frac{a}{2}\hbar q\right)\right]\right\}
\end{aligned} \tag{B.1}$$

Simplifying B.1 yields:

$$\begin{aligned}
\vec{j}_z^{AE} &= \vec{j}_z^{AE}(0) \left\{ 1 - 4\left(\Delta_2^* \sin\left(\chi\left(1 - \frac{v_d}{v_s}\right)\right) \cos B \sin\left(\frac{a}{2}\hbar q\right)\right.\right. \\
&\left. + \Delta_1^* \cos A \sin\left(3\chi\left(1 - \frac{v_d}{v_s}\right)\right) \sin\left(\frac{3}{2}a\hbar q\right)\right) \\
&\times \coth\left[\Delta_1^* \cos\left(3\chi\left(1 - \frac{v_d}{v_s}\right)\right) \cos A \cos\left(\frac{3}{2}a\hbar q\right)\right. \\
&\left. + \Delta_2^* \cos\left(\chi\left(1 - \frac{v_d}{v_s}\right)\right) \cos B \cos\left(\frac{a}{2}\hbar q\right)\right] \\
&\times \coth\left[\Delta_1^* \cos\left(3\chi\left(1 - \frac{v_d}{v_s}\right)\right) \sin A \sin\left(\frac{3}{2}a\hbar q\right)\right. \\
&\left.\left. + \Delta_2^* \cos\left(\chi\left(1 - \frac{v_d}{v_s}\right)\right) \sin B \sin\left(\frac{a}{2}\hbar q\right)\right]\right\}
\end{aligned} \tag{B.2}$$

where $\vec{j}_z^{AE}(0)$ denotes the acoustoelectric current density in the absence of an external electric field and is defined as:

$$\vec{j}_z^{AE}(0) = \vec{j}_o \left[\sinh\left\{ \Delta_1^* \sin\left(\frac{3}{2}a\hbar q\right) \sin A + \Delta_2^* \sin\left(\frac{a}{2}\hbar q\right) \sin B \right\} \right.$$
$$\left. \times \sinh\left\{ \Delta_1^* \cos\left(\frac{3}{2}a\hbar q\right) \cos A + \Delta_2^* \cos\left(\frac{a}{2}\hbar q\right) \cos B \right\} \right] \quad \text{(B.22)}$$

and $\alpha = \omega_q(1 - v_d/v_s)/12\gamma_o aq$ with

$$\vec{j}_o = \frac{4A^\dagger \pi \vec{\Phi} e^3 \mathscr{K}^2 \tau \Theta(1-\alpha^2)}{\hbar^3 \omega_q^2 \varepsilon^2 \rho a q \sqrt{1-\alpha^2}} \qquad \chi = \hbar\omega_q(1 - v_d/v_s)a/v_s$$

APPENDIX C

HALL-LIKE CURRENT INDUCED BY ACOUSTIC PHONONS

The current density for such an orientation is given as:

$$\vec{j} = \frac{2e}{(2\pi\hbar)^2}\sum_{n,n'}\int U^{ac}_{n,n'}\Psi_i(\vec{p},\vec{H})d^2\vec{p}_z \tag{C.2}$$

$$\vec{v}\frac{\partial\Psi_i}{\partial\vec{p}} + \vec{W}_p\{\Psi\} = \vec{v}_i \tag{C.3}$$

Ψ_i is the root of the kinetic equation given as:

$$\frac{e}{c}\left(\vec{v}\times\vec{H}\right)\frac{\partial\Psi_i}{\partial\vec{p}} + \vec{W}_p\{\Psi\} = \vec{v}_i \tag{C.5}$$

where $\vec{v}_i$ is the electron velocity and $\vec{W}_p\{...\} = (\partial f/\partial\varepsilon...)^{-1}\vec{W}_p(\partial f/\partial\varepsilon)$. The operator $\vec{W}_{\vec{p}}$ is a Hermitian operator and it is the collision operator describing the relaxation of the nonequilibrium distribution of the electron. Assuming τ to be constant, the collision operator has the form $\vec{W}_{\vec{p}} = 1/\tau$. Assuming a solution to C.5 as:

$$\Psi_i = \Psi_i^0 + \Psi_i^1 + \Psi_i^2 + ... \tag{C.6}$$

Substituting C.6 into C.5 and solving by the method of iteration to obtain the zero approximation in the absence of the magnetic field ($\vec{H} = 0$) as:

$$\Psi_i^0 = v_i\tau \tag{C.7}$$

Similarly, the first approximation yields:

$$\Psi_i^1 = -\frac{\tau^2 e}{mc}(\vec{v}\times\vec{H})_i \tag{C.8}$$

and $i = x,y,z$. Substituting C.7 and C.8 into C.3 and using the principle of detailed balance, i.e $|G_{p',p}|^2 = |G_{p,p'}|^2$, the net current density is obtaied as:

$$\begin{aligned}\vec{j}_i = \frac{2e}{(2\pi\hbar)^2}\frac{2\pi\vec{\Phi}}{\omega_q v_s}\sum_{n,n'}\int |G_{\vec{p}_z+\hbar q,\vec{p}_z}|^2[f(\varepsilon_{n'}(\vec{p}_z)) - f(\varepsilon_{n'}(\vec{p}_z+\hbar\vec{q}))]\\ \times[\Psi_i(\vec{p}_z+\hbar\vec{q}) - \Psi_i(\vec{p}_z)]\delta(\varepsilon_{n'}(\vec{p}_z+\hbar\vec{q}) - \varepsilon_{n'}(\vec{p}_z) - \hbar\omega_q)d^2\vec{p}_z\\ -\frac{2e}{(2\pi\hbar)^2}\frac{2\pi\vec{\Phi}}{\omega_q v_s}\frac{e\tau^2}{mc}\sum_{n,n'}\int |G_{\vec{p}_z+\hbar\vec{q},\vec{p}_z}|^2[f(\varepsilon_{n'}(\vec{p}_z)) - f(\varepsilon_{n'}(\vec{p}_z+\hbar\vec{q}))]\\ \times[\Psi_i(\vec{p}_z+\hbar\vec{q}) - \Psi_i(\vec{p}_z)]\delta(\varepsilon_{n'}(\vec{p}_z+\hbar\vec{q}) - \varepsilon_{n'}(\vec{p}_z) - \hbar\omega_q)d^2\vec{p}_z\end{aligned} \tag{C.9}$$

Substituting A.7, A.9 and A.10 into C.9 yields;

$$\vec{j} = -\frac{2\mathscr{K}^2\pi\vec{\Phi}e^3\tau\Theta(1-\alpha^2)}{\hbar^3\omega_q^2\varepsilon\rho a\sqrt{1-\alpha^2}}\sum_{n,n'}\int[f(\varepsilon_{n'}(\vec{p}_z)) - f(\varepsilon_{n'}(\vec{p}_z+\hbar\vec{q}))]$$

$$\times [v_z(\vec{p}_z+\hbar\vec{q})-v_z(\vec{p}_z)]\delta(\varepsilon_{n'}(\vec{p}_z+\hbar\vec{q})-\varepsilon_{n'}(\vec{p}_z)-\hbar\omega_q)d\vec{p}_z$$
$$-\frac{2\mathscr{K}^2\pi\vec{\Phi}e^4\tau^2\Theta(1-\alpha^2)}{\hbar^3\omega_q^2\varepsilon\rho a\sqrt{1-\alpha^2}mc}\sum_{n,n'}\int[f(\varepsilon_{n'}(\vec{p}_z))-f(\varepsilon_{n'}(\vec{p}_z+\hbar\vec{q}))]$$
$$\times[(\vec{v}(\vec{p}_z+\hbar\vec{q})\times\vec{H})-(\vec{v}(\vec{p}_z)\times\vec{H})]\delta(\varepsilon_{n'}(\vec{p}_z+\hbar q)-\varepsilon_{n'}(\vec{p}_z)-\hbar\omega_q)d^2\vec{p}_z \tag{C.13}$$

Similarly, the AME current density in C.13 is obtained after some cumbersome calculations as:

$$\vec{j}_y^{AME}=-\frac{2A^{\dagger}\mathscr{K}^2\pi\vec{\Phi}e^3\tau^2\Theta(1-\alpha^2)\vec{\Omega}}{\hbar^3\omega_q^2\varepsilon\rho a\sqrt{1-\alpha^2}}\int_0^{\infty}\exp\left(-\frac{dt'}{\tau}\right)$$
$$\times\left\{\sinh\left[\Delta_1^*\cos(3eaEt')\sin A\sin\left(\frac{3}{2}a\hbar q\right)+\Delta_2^*\cos(eaEt')\sin B\sin\left(\frac{a}{2}\hbar q\right)\right]\right.$$
$$\times\sinh\left[\Delta_1^*\cos(3eaEt')\cos A\cos\left(\frac{3}{2}a\hbar q\right)+\Delta_2^*\cos(eaEt')\cos B\cos\left(\frac{a}{2}\hbar q\right)\right]$$
$$-4\left(\Delta_2^*\sin(eaEt')\cos B\sin\left(\frac{a}{2}\hbar q\right)+\Delta_1^*\cos A\sin(3eaEt')\sin\left(\frac{3}{2}a\hbar q\right)\right.$$
$$\left.+\Delta_1^*\Delta_2^*\sin(eaEt')\sin(3eaEt')\cos A\cos B\sin\left(\frac{a}{2}\hbar q\right)\sin\left(\frac{3}{2}a\hbar q\right)\right)$$
$$\times\cosh\left[\Delta_1^*\cos(3eaEt')\cos A\cos\left(\frac{3}{2}a\hbar q\right)+\Delta_2^*\cos(eaEt')\cos B\cos\left(\frac{a}{2}\hbar q\right)\right]$$
$$\left.\times\cosh\left[\Delta_1^*\cos(3eaEt')\sin A\sin\left(\frac{3}{2}a\hbar q\right)+\Delta_2^*\cos(eaEt')\sin B\sin\left(\frac{a}{2}\hbar q\right)\right]\right\} \tag{C.17}$$

where $\vec{\Omega}=\mu\vec{H}/\hbar c$, $\Delta_1^*=\Delta_1/k_BT$, $\Delta_2^*=\Delta_2/k_BT$. Simplifying further yields:

$$\vec{j}_y^{AME}=\vec{j}_y(0)\left\{1-4\left(\Delta_2^*\sin\left(\chi\left(1-\frac{v_d}{v_s}\right)\right)\cos B\sin\left(\frac{a}{2}\hbar q\right)\right.\right.$$
$$\left.+\Delta_1^*\cos A\sin\left(3\chi\left(1-\frac{v_d}{v_s}\right)\right)\sin\left(\frac{3}{2}a\hbar q\right)\right)$$
$$\times\coth\left[\Delta_1^*\cos\left(3\chi\left(1-\frac{v_d}{v_s}\right)\right)\cos A\cos\left(\frac{3}{2}a\hbar q\right)\right.$$
$$\left.+\Delta_2^*\cos\left(\chi\left(1-\frac{v_d}{v_s}\right)\right)\cos B\cos\left(\frac{a}{2}\hbar q\right)\right]$$
$$\times\coth\left[\Delta_1^*\cos\left(3\chi\left(1-\frac{v_d}{v_s}\right)\right)\sin A\sin\left(\frac{3}{2}a\hbar q\right)\right.$$
$$\left.\left.+\Delta_2^*\cos\left(\chi\left(1-\frac{v_d}{v_s}\right)\right)\sin B\sin\left(\frac{a}{2}\hbar q\right)\right]\right\} \tag{C.18}$$

Switching off the external electric field from C.18 yields:

$$\vec{j}_y(0)=\vec{j}_{oy}\left[\sinh\left\{\Delta_1^*\sin\left(\frac{3}{2}a\hbar q\right)\sin A+\Delta_2^*\sin\left(\frac{a}{2}\hbar q\right)\sin B\right\}\right.$$

$$\times \sinh\left\{\Delta_1^* \cos\left(\frac{3}{2}a\hbar q\right)\cos A + \Delta_2^* \cos\left(\frac{a}{2}\hbar q\right)\cos B\right\}\Bigg] \quad \text{(C.19)}$$

where

$$\vec{j}_{oy} = -\frac{4e^3 A^\dagger \mathscr{K}^2 \pi \vec{\Phi}\tau^2 \Theta(1-\alpha^2)}{\hbar^4 \omega_q^2 v_s \varepsilon \rho a q \sqrt{1-\alpha^2}} \frac{\mu \vec{H}}{c} \quad \text{(C.20)}$$

and

$$A = \frac{3}{4}\sin^{-1}\left(\frac{\omega_q}{12\Delta a q}\right), B = \frac{1}{4}\sin^{-1}\left(\frac{\omega_q}{12\Delta a q}\right)$$

The following relation exists between the attenuation coefficient Γ_{abs} and E_{SAME}:

$$\Gamma_{abs}\vec{\Phi} = \frac{n_o e \vec{E}_{SAME}}{\Omega} \quad \text{(C.21)}$$

Thus, the Hall-like (surface acoustomagnetoelectric) field $\vec{E}_{SAME}$ yields:

$$\begin{aligned} \vec{E}_{SAME} = &\frac{2A^\dagger \pi \vec{\Phi}^2 K^2 \Theta(1-\alpha^2)}{3\hbar^2 \omega_q^2 \rho v_s \Delta \varepsilon a q \sqrt{1-\alpha^2} n_o e}\left(\frac{\mu H}{\hbar c}\right) \\ &\times \Bigg[\sinh\left\{\Delta_1^* \sin\left(\frac{3}{2}a\hbar q\right)\sin A + \Delta_2^* \sin\left(\frac{a}{2}\hbar q\right)\sin B\right\} \\ &\times \cosh\left\{\Delta_1^* \cos\left(\frac{3}{2}a\hbar q\right)\cos A + \Delta_2^* \cos\left(\frac{a}{2}\hbar q\right)\cos B\right\}\Bigg] \end{aligned} \quad \text{(C.22)}$$

The drift velocity of the electrons yields:

$$\begin{aligned} \vec{v}_d = \mu \vec{E}_{SAME} = &\frac{2A^\dagger \pi \vec{\Phi}^2 K^2 \Theta(1-\alpha^2)}{3\hbar^2 \omega_q^2 \rho v_s \Delta \varepsilon a q \sqrt{1-\alpha^2} n_o e}\left(\frac{\mu^2 H}{\hbar c}\right) \\ &\times \Bigg[\sinh\left\{\Delta_1^* \sin\left(\frac{3}{2}a\hbar q\right)\sin A + \Delta_2^* \sin\left(\frac{a}{2}\hbar q\right)\sin B\right\} \\ &\times \cosh\left\{\Delta_1^* \cos\left(\frac{3}{2}a\hbar q\right)\cos A + \Delta_2^* \cos\left(\frac{a}{2}\hbar q\right)\cos B\right\}\Bigg] \end{aligned} \quad \text{(C.23)}$$

APPENDIX D

HIGH-FREQUENCY AMPLIFICATION OF ACOUSTIC PHONONS

$$\varepsilon(\vec{p}_z) = \varepsilon_o + \Delta_1 \cos(3a\vec{p}) + \Delta_2 \cos(a\vec{p}) \tag{D.1}$$

Employing the Hamiltonian of the electron-phonon system for FSWCNT in the second quantization formalism as

$$H = \sum_{p,\nu} \varepsilon^{(\nu)}(\vec{p}) \left(\vec{p} - \frac{e}{c}\vec{A}(t)\right) a_p^{(\nu+)} a_\nu^{(\nu)} + \sum_k \omega_k b_k^+ b_k \dots$$
$$+ \frac{1}{\sqrt{N}} \sum_{p,k} \sum_{\nu\nu' n} c_k m_{\nu\nu'}(k_z) a_p^{(\nu+)} a_{p-k+ng}^{\nu'} (b_k^+ + b_{-k}) \tag{D.2}$$

$a_p^{(+)}$ and a_p are the creation and annihilation operators of an electron with quasi-momentum p in the νth miniband respectively, and b_k^+ and b_k are the phonon creation and annihilation operators, respectively. N is the number of FSWCNT periods, and $m_{\nu\nu'}$ is given as:

$$m_{\nu\nu'}(k_z) = \int \varphi_{\nu'}^*(z) \varphi_\nu(z) e^{ik_z} dz \tag{D.3}$$

where $\varphi_\nu(z)$ is the wavefunction of the νth state in one of the one-dimensional mini band from which the FSWCNT potential is formed. Proceeding to calculate the attenuation (or amplification) coefficient, equation of motion for the phonons from the Heisenberg formalism is given as:

$$i\frac{\partial}{\partial t}\langle b_q \rangle_t = \langle [b_q, H] \rangle_t = \omega_q \langle b_q \rangle_t + \frac{1}{\sqrt{N}} G_{-q} \sum_p m_{\nu\nu'}(-q) \langle a_p^{(\nu+)} a_{p+qng}^{(\nu)} \rangle_t \tag{D.4}$$

Again for $\langle a_p^{(\nu+)} a_{p+qng}^{(\nu)} \rangle_t$ yields:

$$i\frac{\partial}{\partial t}\langle a_p^{(\nu+)} a_{p+qng}^{(\nu)} \rangle_t = (\varepsilon_{p+q}^{\nu'} - \varepsilon_p^{\nu}) \langle a_p^{(\nu+)} a_{p+qng}^{(\nu')} \rangle_t$$
$$+ \frac{1}{N} \sum_p \sum_{\nu'\nu n} G_{k'k} \Big[M_{\nu'\nu''}(k_z) \langle a_p^{(\nu+)} a_{p+q-k+(n+n')g}^{(\nu')} (b_k + b_{-k}^+)_t \rangle_t$$
$$- M_{\nu''\nu}(q_z) \langle a_{p+k-n'g}^{(\nu)} a_{p+q+ng}^{(\nu')} (b_k + b_{-k}^+)_t \rangle_t \Big] \tag{D.5}$$

Solving D.5 within the initial condition $\langle a_p^{(\nu)} a_{p+qng}^{(\nu+)} \rangle_{t=\infty} = 0$ and substituting into D.4, we obtain

$$i\frac{\partial}{\partial t}\langle b_q \rangle_t = \omega_q \langle b_q \rangle_t - I \sum_p \sum_{\nu'\nu n} G_{-q} M_{\nu\nu'}(-q)$$
$$\times \int_{-\infty}^{t} e^{\int_{t'}^{t} (\varepsilon_{p+q} - \varepsilon_p) dt''} dt' \sum_{\nu'\nu''} G_k [M_{\nu'\nu''}(k_z) \langle a_p^{(\nu+)} a_{p+q-k+(n'+n)g}^{(\nu'')}$$

$$\times (b_k + b_{-k}^+)_t\rangle\tau' - M_{\nu''\nu}(k_z)\langle a_{p+k-ng}^{(\nu+)} a_p + q + (n'+n)g^{(\nu'')}(b_k + b_{-k}^+)\rangle_{t'}] \tag{D.6}$$

For weak electron-phonon interaction and considered as a perturbation, RHS of D.6 is decoupled using

$$\langle a_p^{(\nu+)} a_{p'}^{(\nu')} b_k\rangle = \delta_{kk'}\delta_{\nu\nu'}\langle b_k\rangle_k n_p^\nu \tag{D.7}$$

where $n_p^\nu = \langle a_p^{(\nu+)} a_{p'}^{(\nu')}\rangle_t$ is the electron distribution function. We further obtain

$$\begin{aligned}\frac{\partial}{\partial t}\langle b_q\rangle_t + i\omega_q\langle b_q\rangle_t = &\sum_p \sum_{\nu'\nu n} G_{-q}G_{q+nq}M_{\nu'\nu}(-q_z)M_{\nu'\nu}(-q_z+ng_z)[n_p^{\nu'} - n_{p+q}^{\nu'}]\\ &\times \int_{-\infty}^{t} dt'(\langle b_{q+ng}\rangle_{t'} + \langle b_{-q-ng}^+\rangle_{t'})\exp\Big[i(\varepsilon_p^\nu - \varepsilon_{p+q}^{\nu'})(t-t')\\ &- \frac{3eE_oa\Delta_1}{\Omega}\{\sin(\vec{p}_s+\vec{q})a - \sin\vec{p}_s a\}(\sin\Omega t - \sin\Omega t')\\ &- \frac{eE_oa\Delta_2}{\Omega}\{\sin(\vec{p}_z+\vec{q})a - \sin\vec{p}_z a\}(\sin\Omega t - \sin\Omega t')\Big]\end{aligned} \tag{D.8}$$

Employing planar conditions

$$\begin{aligned}\frac{\partial}{\partial t}\langle b_q\rangle_t + i\omega_q\langle b_q\rangle_t = &\sum_p \sum_{\nu'\nu n} G_{-q}G_{q+nq}M_{\nu'\nu}(-q_z)M_{\nu'\nu}(-q_z+ng_z)[n_p^{\nu'} - n_{p+q}^{\nu'}]\\ &\times \int_{-\infty}^{t} dt'(\langle b_{q+ng}\rangle_{t'} + \langle b_{-q-ng}^+\rangle_{t'})\exp\Big[i(\varepsilon_p^\nu - \varepsilon_{p+q}^{\nu'})(t-t')\\ &- \frac{eE_oa\Delta_2}{\Omega}\{\sin(p_z+q)a - \sin p_z a\}(\sin\Omega t - \sin\Omega t')\Big]\end{aligned} \tag{D.9}$$

Eliminating $\langle b_{-q-ng}^+\rangle_t$ and taking the Fourier transform of the component

$$B_q(\omega) = \int_{-\infty}^{\infty}\langle b_q\rangle_t \exp(i\omega t)dt \tag{D.10}$$

yields

$$\begin{aligned}(\omega-\omega_q)B_q(\omega) = \sum_{k=-\infty}\sum_{\nu'\nu} &G_{-q}G_{q+ng}M_{\nu\nu'}(-q_z)M_{\nu'\nu}(q_z-ng_z)\\ &\times \frac{2\omega_{q+ng}}{\omega+\omega_{q+ng}-\Omega k}B_q(\omega-k)M_k(q,\omega)\end{aligned} \tag{D.11}$$

where

$$M_k(q,\omega) = \sum_{\ell=-\infty} J_\ell(\xi)J_{\ell+k}(\xi)\Pi(q,\omega+\ell\Omega) \tag{D.12}$$

$$\Pi_{\nu\nu'}(q,\omega+\ell\Omega) = \sum_p \frac{n_p^\nu - n_{p+q}^{\nu'}}{\varepsilon_{p+q}^{\nu'} - \varepsilon_p^\nu - \omega_q} \tag{D.13}$$

and

$$\xi = \frac{eE_o a\Delta_2}{\Omega^2}[\sin(\vec{p}_z+\vec{q})a - \sin(\vec{p}_z a)] \tag{D.14}$$

Subsequently, we shall confine ourselves to the case when the electron gas is non-degenerate and only the lowest miniband is filled, i.e. $n_p^\nu = n_p$. We shall also limit ourselves to $k = 0$, since when $k \neq 0$, the summation give terms of higher perturbation. Then for $n = 0$ we obtain from D.11

$$\omega^2 - \omega_q^2 - 2\omega_q^2 G_q^2 M_o(q,\omega) = 0 \tag{D.15}$$

This simplifies to

$$\omega - \omega_q - G_q^2 \sum_{\ell=-\infty}^{\infty} J_\ell^2(\xi)\Pi_q^0(\omega_q - \Omega\ell) = 0 \tag{D.16}$$

Then the phonon amplification in the presence of the electromagnetic wave will be given by

$$\Gamma(\omega) = -Im\omega = \sum_{\ell=-\infty}^{\infty} J_\ell^2(\xi)\Gamma_q^0(\omega_q + \ell\Omega) \tag{D.17}$$

where

$$\Gamma_q^0(\omega) = \pi G_q^2 \sum_p U_{\nu,\nu'}^{ac} \tag{D.18}$$

$U_{\nu,\nu'}^{ac}$ is the electron-phonon interaction expressed as:

$$U_{\nu,\nu'}^{ac} = \sum_{\nu,\nu'} \{|G_{p-\hbar q,p}|^2[n(\vec{p}-\hbar q) - n(p)]\delta(\varepsilon_\nu(\vec{p}-\hbar\vec{q}) - \varepsilon_\nu(p) + \hbar\omega_q - \xi)$$
$$+|G_{\vec{p}+\hbar\vec{q},\vec{p}}|^2[n(\vec{p}+\hbar\vec{q}) - n(\vec{p})]\delta(\varepsilon_{\nu'}(\vec{p}+\hbar\vec{q}) - \varepsilon_{\nu'}(\vec{p}) - \hbar\omega_q + \xi)\} \tag{D.19}$$

i.e. the imaginary part of D.17 is the polarization vector. For large values, the Bessel function $J_\ell(\xi)$ is small except when the order is equal to the argument. From D.16 we obtain

$$\xi = \frac{eE_o a^2\Delta_2 q}{\Omega^2} \tag{D.20}$$

Suming over $|\ell|$ by employing the approximation as in ref. [6]

$$\sum_{\ell=-\infty} J_\ell^2(\xi)\delta(E - \ell\Omega) \approx \frac{1}{2}[\delta(E-\xi) + \delta(E+\xi)] \tag{D.21}$$

where $E = \varepsilon(\vec{p}+\hbar\vec{q}) - \varepsilon(\vec{p}) - \hbar\omega_q$. $G(\vec{p}\pm\hbar\vec{q},\vec{p})$ is the matrix element of the electron-phonon interaction. Letting $\vec{p}' = \vec{p}\pm\hbar\vec{q}$ scattering into/out of a state p'/p is the same hence

$$|G_{p',p}|^2 = |G_{p,p'}|^2 \tag{D.22}$$

The attenuation or amplification coefficient becomes;

$$\Gamma_q(\omega) = \frac{\pi}{2}\sum_p |G_{p',p}|^2[n(p) - n(p+\hbar q)]$$

$$\times \left\{ \delta(\varepsilon_v(p+\hbar q) - \varepsilon_v(p) - \hbar\omega_q - \xi) + \delta(\varepsilon_v(p+\hbar q) - \varepsilon_v(p) + \hbar\omega_q + \xi) \right\} \tag{D.23}$$

The first $\delta-$function in D.23 corresponds to the emission and the second to the absorption of phonons. The number of photons absorbed or emitted is the same order of magnitude as the ratio of the classical oscillatory energy of the electron to that of the phonon

$$\ell = \frac{2e^2E_o^2/(e\tau\Omega^2/\mu)}{\Omega} \tag{D.24}$$

Multiple photon absorption or emission processes ($\ell \gg 1$) are valid for laser fields where

$$E_o \gg \left(\frac{\tau\Omega^3}{2e\mu}\right)^{1/2} \tag{D.25}$$

For low electron temperature, and for $kT \ll \xi$ the emission term is neglible compared to the absorption term. This is justified provided $\Delta_2 \gg kT$.
D.24 becomes

$$\Gamma_q(\omega) = \frac{\pi|G_{p',p}|^2}{2}\sum_n [n(\vec{p}) - n(\vec{p}+\hbar\vec{q})]$$
$$\times\, \delta(\varepsilon_v(\vec{p}+\hbar\vec{q}) - \varepsilon_v(\vec{p}) - \hbar\omega_q + \xi)d^2\vec{p} \tag{D.26}$$

Changing the summation to an integral due to the continuity of the carrier states as a function of the momentum p, as a result we obtain

$$\Gamma_q(\omega) = \frac{2\pi\Phi|G_{p',p}|^2}{\omega_q v_s}\int [n(\vec{p}) - n(\vec{p}+\hbar\vec{q})]$$
$$\times\, \delta(\varepsilon_v(\vec{p}+\hbar\vec{q}) - \varepsilon_v(\vec{p}) - \hbar\omega_q + \xi)d^2\vec{p} \tag{D.27}$$

Employing cylindrical coordinates in relation to the tubular geometry of FSWCNT yields:

$$\begin{aligned}
\Gamma(\omega) = \Gamma(0)\Bigg\{ 1 - 4\Bigg(&\Delta_2^*\sin\left(\chi\left(1-\frac{v_d}{v_s}\right)\right)\cos B\sin\left(\frac{a}{2}\hbar q\right) \\
&+ \Delta_1^*\cos A\sin\left(3\chi\left(1-\frac{v_d}{v_s}\right)\right)\sin\left(\frac{3}{2}a\hbar q\right)\Bigg) \\
&\times\tanh\left[\Delta_1^*\cos\left(3\chi\left(1-\frac{v_d}{v_s}\right)\right)\cos A\cos\left(\frac{3}{2}a\hbar q\right)\right. \\
&\left.+ \Delta_2^*\cos\left(\chi\left(1-\frac{v_d}{v_s}\right)\right)\cos B\cos\left(\frac{a}{2}\hbar q\right)\right] \\
&\times\coth\left[\Delta_1^*\cos\left(3\chi\left(1-\frac{v_d}{v_s}\right)\right)\sin A\sin\left(\frac{3}{2}a\hbar q\right)\right. \\
&\left.+ \Delta_2^*\cos\left(\chi\left(1-\frac{v_d}{v_s}\right)\right)\sin B\sin\left(\frac{a}{2}\hbar q\right)\right]\Bigg\}
\end{aligned} \tag{D.38}$$

Switching off the external electric field, we obtain

$$\Gamma(0) = \Gamma_o \left[\sinh\left\{ \Delta_1^* \sin\left(\frac{3}{2}a\hbar q\right) \sin A + \Delta_2^* \sin\left(\frac{a}{2}\hbar q\right) \sin B \right\} \right.$$
$$\left. \times \cosh\left\{ \Delta_1^* \cos\left(\frac{3}{2}a\hbar q\right) \cos A + \Delta_2^* \cos\left(\frac{a}{2}\hbar q\right) \cos B \right\} \right] \quad \text{(D.39)}$$

and

$$\Gamma_o = \frac{8\pi^3 n^* \Phi e^2 K^2 \tau \Theta(1-\alpha^2)}{\hbar\omega_q^2 \varepsilon^2 \sigma q v_s \Delta_1 I_o(\Delta_1^*) I_o(\Delta_2^*) \sqrt{1-\alpha^2}} \frac{2}{\sqrt{\pi}} \quad \text{(D.40)}$$

where Θ is defined to be the Heaviside step function.

$$v_d = \frac{eE_o \Delta_2 a^2 q}{\Omega}, \chi = \hbar\omega_q a/v_s, \alpha = \frac{\omega_q(1 - v_d/v_s)}{6\Delta_1 a q}$$

and v_d is the carrier drift velocity and v_s denotes acoustic wave velocity.

APPENDIX E

SOLUTION TO THE BOLTZMANN EQUATION

Electrons in equilibrium is defined by the Fermi-Dirac statistics given by the expression

$$f_o(\vec{p},\vec{r},t) = \frac{1}{e^{\theta}+1} \tag{E.1}$$

Let

$$\theta = [E_c(\vec{r},t) + E(p) - F_n(\vec{r},t)]/kT \tag{E.2}$$

Suppose that $f(\vec{r},\vec{p},t) = f_o(\vec{r},\vec{p},t) + f_1(\vec{p},\vec{r},t) + f^{'}(\vec{p},\vec{r},t)$, where $f_1(\vec{p},\vec{r},t)$ is a small perturbation and $f^{'}$ is the hot electron source distribution function. For $f^{'}(\vec{p},\vec{r},t) = 0$

$$\frac{\partial f(\vec{r},\vec{p},t)}{\partial t} + \vec{v}\cdot\frac{\partial f(\vec{r},\vec{p},t)}{\partial r} + F\cdot\frac{\partial f(\vec{r},\vec{p},t)}{\partial \vec{p}} = -\frac{f(\vec{r},\vec{p},t) - f_o(\vec{p})}{\tau} \tag{E.3}$$

$$\nabla_t(f_o+f_1) + v\cdot\nabla_r(f_o+f_1) + F\cdot\nabla_p(f_o+f_1) = -\frac{f_o+f_1-f_o}{\tau} \tag{E.4}$$

$$v\cdot\nabla_r(f_o+f_1) + F\cdot\nabla_p(f_o+f_1) = -\frac{f_1}{\tau} \tag{E.5}$$

$f_o(\vec{r},\vec{p},t) \gg f_1(\vec{p},\vec{r},t)$ and $\nabla_r f_o \gg f_1$

$$\vec{v}\cdot\nabla_r f_o(\vec{r},\vec{p},t) + F\cdot\nabla_p f_o(\vec{r},\vec{p},t) = -\frac{f_1(\vec{p},\vec{r},t)}{\tau} \tag{E.6}$$

Applying the chain rule

$$\vec{v}\cdot\frac{\partial f_o(\vec{r},\vec{p},t)}{\partial\theta}\cdot\nabla_r\theta + F\cdot\frac{\partial f(\vec{r},\vec{p},t)}{\partial\theta}\cdot\nabla_p\theta = -\frac{f_1(\vec{p},\vec{r},t)}{\tau} \tag{E.7}$$

Differentiating

$$\nabla_r\theta = \frac{[\nabla_r E_c(\vec{r}) - \nabla_r F_n(\vec{r})]}{kT} + [E_c(\vec{r}) + E(\vec{p}) - F_n(\vec{r})]\nabla_r\left(\frac{1}{kT(\vec{r})}\right) \tag{E.8}$$

also

$$\nabla_p\theta = \frac{\vec{v}}{kT(\vec{r})} \tag{E.9}$$

Putting E.4 and E.5 into E.3

$$\vec{v}\cdot\frac{\partial f_o(\vec{r},\vec{p},t)}{\partial\theta}\left\{\frac{[\nabla_r E_c(\vec{r}) - \nabla_r F_n(\vec{r})]}{kT} + [E_c(\vec{r}) + E(\vec{p}) - F_n(\vec{r})]\nabla_r\left(\frac{1}{kT(\vec{r})}\right)\right\} + F\frac{\partial f_o(\vec{r},\vec{p},t)}{\partial\theta}\frac{\vec{v}}{kT(\vec{r})} = -\frac{f_1(\vec{p},\vec{r},t)}{\tau} \tag{E.10}$$

$$\vec{v}\cdot\frac{\partial f_o(\vec{r},\vec{p},t)}{\partial\theta}\left\{\frac{[\nabla_r E_c(\vec{r}) - \nabla_r F_n(\vec{r})]}{kT} + [E_c(\vec{r}) - F_n(\vec{r})]\nabla_r\left(\frac{1}{kT(\vec{r})}\right)\right.$$

$$+\frac{F}{kT(\vec{r})}\Bigg\} = \frac{f_1(\vec{p},\vec{r},t)}{\tau} \quad \text{(E.11)}$$

$$\nabla_r E_c(\vec{r}) = -F$$

$$f_1(\vec{p},\vec{r},t) = -\frac{\vec{v}\tau}{kT}\cdot\frac{\partial f_o(\vec{r},\vec{p},t)}{\partial\theta}\Bigg\{-\frac{F}{kT(\vec{r})} - \nabla_r F_n(r) + [E_c(\vec{r}) + E(\vec{p}) - F_n(\vec{r})]\nabla_r\left(\frac{1}{kT(\vec{r})}\right) + \frac{F}{kT(\vec{r})}\Bigg\} \quad \text{(E.12)}$$

$$f_1(\vec{p},\vec{r},t) = -\frac{\vec{v}\tau}{kT}\cdot\frac{\partial f_o}{\partial\theta}\left\{[E_c(\vec{r}) + E(\vec{p}) - F_n(\vec{r})]\nabla_r\left(\frac{1}{kT(\vec{r})}\right) - \nabla_r F_n(r)\right\} \quad \text{(E.13)}$$

$$f_1 = -\frac{\vec{v}\tau}{kT}\cdot\frac{\partial f_o}{\partial\theta}\left\{kT[E_c(\vec{r}) + E(\vec{p}) - F_n(\vec{r})]\nabla_r\left(\frac{1}{T(\vec{r})}\right) - \nabla_r F_n(\vec{r})\right\} \quad \text{(E.14)}$$

$$f_1 = -\frac{\vec{v}\tau}{kT}\cdot\frac{\partial f_o}{\partial\theta}\left\{[E_c(\vec{r}) + E(\vec{p}) - F_n(\vec{r})]\left(\frac{\nabla_r T}{T}\right) - \nabla_r F_n(\vec{r})\right\} \quad \text{(E.15)}$$

$$f_1 = \frac{\tau}{kT}\left(-\frac{\partial f_o}{\partial\theta}\right)v\cdot\aleph \quad \text{(E.16)}$$

where $\aleph = -\nabla_r F_n(r) + [E_c(\vec{r}) + E(\vec{p}) - F_n(\vec{r})](\nabla_r T/T)$ is the generalized force. Substitute E.15 into $f = f_o + f_1$

$$f = f_o(\vec{p}) + \frac{\vec{v}\tau}{kT}\cdot\frac{\partial f_o(\vec{p})}{\partial\theta}\left\{[E_c(\vec{r}) + E(\vec{p}) - F_n(\vec{r})]\frac{\nabla_r T}{T} - \nabla_r F_n(r)\right\} \quad \text{(E.17)}$$

Let $\theta = [E_c(\vec{r}) + E(\vec{p}) - F_n(\vec{r})]/kT = \varepsilon(\vec{p})/kT \rightarrow kT\theta = \varepsilon(\vec{p})$

$$f(\vec{r},\vec{p},t) = f_o(\vec{r},\vec{p},t) + \frac{\vec{v}\tau}{kT}\cdot kT\frac{\partial f_o(\vec{p})}{\partial\varepsilon(\vec{p})}\left\{[E_c(\vec{r}) + E(\vec{p}) - F_n(\vec{r})]\frac{\nabla_r T}{T} - \nabla_r F_n(r)\right\} \quad \text{(E.18)}$$

$$f(\vec{r},\vec{p},t) = \tau^{-1}\int_0^{-\infty}\exp(-t/\tau)dt f_o(\vec{r},\vec{p},t) + \frac{\vec{v}\tau}{kT}\cdot kT\frac{\partial f_o(\vec{p})}{\partial\varepsilon(\vec{p})}\int_0^{-\infty}\exp(-t/\tau)dt\left\{[E_c(\vec{r}) + E(\vec{p}) - F_n(\vec{r})]\frac{\nabla_r T}{T} - \nabla_r F_n(r)\right\} \quad \text{(E.19)}$$

$$f(\vec{r},\vec{p},t) = \tau^{-1}\int_0^{-\infty}dt\exp(-t/\tau)f_o(\vec{p})$$

$$+\tau\int_0^{-\infty} dt\exp(-t/\tau)\left\{[E_c(\vec{r})+E(\vec{p})-F_n(\vec{r})]\frac{\nabla_r T}{T}-\nabla_r F_n(r)\right\}\vec{v}\cdot\frac{\partial f_o(\vec{p})}{\partial\varepsilon(\vec{p})} \tag{E.20}$$

$E_c(r) \ll E(\vec{p})$ and the quasi-fermi level $\nabla_r F_n = \nabla_r \mu$, where μ is the electrochemical potential.

$$\begin{aligned} f(\vec{r},\vec{p},t) &= \tau^{-1}\int_0^{-\infty} dt\exp(-t/\tau) f_o(\vec{p}) \\ &+\tau\int_0^{-\infty} dt\exp(-t/\tau)\left\{[E(\vec{p})-\mu]\frac{\nabla_r T}{T}-\nabla_r\mu\right\}\vec{v}\cdot\frac{\partial f_o(\vec{p})}{\partial\varepsilon(\vec{p})} \end{aligned} \tag{E.21}$$

Making the transformation $p \to \vec{p}-e\int_{t-t'}^{t}[E_o+Ecos\omega t']dt'$

$$\begin{aligned} f(\vec{r},\vec{p},t) &= \tau^{-1}\int_0^{-\infty} dt\exp(-t/\tau) f_o\left(\vec{p}-e\int_{t-t'}^{t}[E_o+Ecos\omega t']dt'\right) \\ &+\tau\int_0^{-\infty} dt\exp(-t/\tau)\left\{[E(\vec{p})-\mu]\frac{\nabla_r T}{T}-\nabla_r\mu\right\} \\ &\times\vec{v}(\vec{p})\cdot\frac{\partial f_o}{\partial\varepsilon(\vec{p})}(\vec{p}-e\int_{t-t'}^{t}[E_o+Ecos\omega t']dt') \end{aligned} \tag{E.22}$$

APPENDIX F

CARRIER CURRENT DENSITY OF FSWCNT

In the linear approximation of ∇T and $\nabla \mu$, the solution to the BTE is:

$$f(\vec{p}) = \tau^{-1}\int_0^\infty dt \exp\left(\frac{-t}{\tau}\right) f_o\left(\vec{p} - e\int_0^\infty [\vec{E}_o + \vec{E}\cos(\omega t')]dt'\right)$$
$$+\tau\int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt \left\{\left[\varepsilon(\vec{p} - e\int_0^\infty [\vec{E}_o + \vec{E}\cos(\omega t') - \mu])\right]\frac{\nabla T}{T} + \nabla\mu\right\}$$
$$\times v\left(\vec{p} - e\int_0^\infty [\vec{E}_o + \vec{E}\cos(\omega t')]dt'\right)\frac{\partial f_o}{\partial \varepsilon}(\vec{p} - e\int_{t-t'}^t [\vec{E}_o + \vec{E}\cos(\omega t')]dt') \tag{F.1}$$

The carrier current density is defined as:

$$\vec{J} = -e\sum_p v(\vec{p})f(\vec{p}) \tag{F.2}$$

Substitute F.1 into F.2 yields:

$$\vec{J} = e\tau^{-1}\int_0^\infty dt \exp\left(\frac{-t}{\tau}\right)\sum_p v(\vec{p})f_o\left(p - e\int_0^\infty [\vec{E}_o + E\cos(\omega t')]dt'\right)$$
$$+e\tau\int_0^\infty dt\exp\left(\frac{-t}{\tau}\right)\sum_p v(\vec{p})\left\{[\varepsilon(\vec{p} - e\int_0^\infty [\vec{E}_o + \vec{E}\cos(\omega t') - \mu])\frac{\nabla T}{T} + \nabla\mu\right\}$$
$$\times v\left(\vec{p} - e\int_0^\infty [\vec{E}_o + \vec{E}\cos(\omega t')]\right)\frac{\partial f_o}{\partial \varepsilon}(\vec{p} - e\int_0^\infty [\vec{E}_o + \vec{E}\cos(\omega t')]dt') \tag{F.3}$$

Making use of the transformation $p - e\int_0^\infty [\vec{E}_o + \vec{E}\cos(\omega t')]dt' \to p$ yields:

$$\vec{J} = e\tau^{-1}\int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt \sum_p v\left(\vec{p} - e\int_{t-t'}^t [\vec{E}_o + \vec{E}\cos(\omega t')]dt'\right) f_o(\vec{p})$$
$$+e\tau\int_0^\infty \exp(-t/\tau)dt\sum_p\left\{[\varepsilon(\vec{p}) - \mu]\frac{\nabla T}{T} + \nabla\mu\right\}$$
$$\times v(\vec{p})\frac{\partial f_o(\vec{p})}{\partial \varepsilon}v(\vec{p} - e\int_{t-t'}^t [\vec{E}_o + \vec{E}\cos(\omega t')]dt') \tag{F.4}$$

For a p-doped, FSWCNT, the dispersion law in the conduction band is obtained as:

$$\varepsilon(\vec{p}) = \varepsilon_o - 8\gamma_o\cos^3\frac{\sqrt{3}\vec{p}}{2\hbar} \tag{F.5}$$

Expanding the trigonometric identity yields

$$\varepsilon(\vec{p}) = \varepsilon_o - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} \tag{F.6}$$

where $u = \sqrt{3}/2\hbar$ and $w = 3\sqrt{3}/2\hbar$. Resolving the current density into $\vec{S}$ and $\vec{Z}$ components yields:

$$\vec{S} = e\tau^{-1}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt\sum_p v_s(\vec{p} - e\int_{t-t'}^t [\vec{E}_o + \vec{E}_s\cos(\omega t')]dt')f_o(\vec{p})$$
$$+ e\tau\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt\sum_p\left\{[\varepsilon(\vec{p}) - \mu]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times v_s(\vec{p})\frac{\partial f_o(\vec{p})}{\partial\varepsilon}v_s(\vec{p} - e\int_{t-t'}^t [\vec{E}_o + \vec{E}_s\cos(\omega t')]dt') \quad (F.7)$$

and similarly:

$$Z = e\tau^{-1}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt\sum_p v_z(\vec{p} - e\int_{t-t'}^t [\vec{E}_o + \vec{E}_z\cos(\omega t')]dt')f_o(\vec{p})$$
$$+ e\tau\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt\sum_p\left\{[\varepsilon(\vec{p}) - \mu]\frac{\nabla_z T}{T} + \nabla_z\mu\right\}$$
$$\times v_z(\vec{p})\frac{\partial f_o(\vec{p})}{\partial\varepsilon}v_z(\vec{p} - e\int_{t-t'}^t [\vec{E}_o + \vec{E}_z\cos(\omega t')]dt') \quad (F.8)$$

Making the transformation

$$\sum_p \to \frac{2}{(2\pi\hbar)^2}\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \quad (F.9)$$

$$\vec{S} = \frac{2e\tau^{-1}}{(2\pi\hbar)^2}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s$$
$$\times v_s(\vec{p} - e\int_{t-t'}^t [\vec{E}_o + \vec{E}_s\cos(\omega t')]dt')f_o(\vec{p})$$
$$+ \frac{2e\tau^{-1}}{(2\pi\hbar)^2}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s\left\{[\varepsilon(\vec{p}) - \mu]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times v_s(p)\frac{\partial f_o(\vec{p})}{\partial\varepsilon}v_s(\vec{p} - e\int_{t-t'}^t [\vec{E}_o + \vec{E}_s\cos(\omega t')]dt') \quad (F.10)$$

and

$$\vec{Z} = \frac{2e\tau^{-1}}{(2\pi\hbar)^2}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s$$
$$\times v_z(p - e\int_{t-t'}^t [\vec{E}_o + \vec{E}_z\cos(\omega t')]dt')f_o(\vec{p})$$
$$+ \frac{2e\tau^{-1}}{(2\pi\hbar)^2}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s\left\{[\varepsilon(\vec{p}) - \mu]\frac{\nabla_z T}{T} + \nabla_z\mu\right\}$$
$$\times v_z(p)\frac{\partial f_o(\vec{p})}{\partial\varepsilon}v_z(\vec{p} - e\int_{t-t'}^t [\vec{E}_o + \vec{E}_z\cos(\omega t')]dt') \quad (F.11)$$

respectively, where the integration is carried out within the first BZ $-\pi\hbar/b_s \leq \vec{p}_s \leq \pi\hbar/b_s$ and $-\pi\hbar/b_z \leq \vec{p}_z \leq \pi\hbar/b_z$ respectively.

$$v_s(\vec{p}) = \frac{\partial\varepsilon(\vec{p})}{\partial\vec{p}} = \frac{u\Delta_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}$$

$$v_s\left(\vec{p} - e\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right) = \frac{u\Delta_s b_s}{\hbar}\sin\left(\vec{p} - e\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right) \quad (F.12)$$

Expanding the expression:

$$v_s\left(\vec{p} - e\int_{t-t'}^{t}[\vec{E}_o + E\cos(\omega t')]dt'\right) = \frac{u\Delta_s b_s}{\hbar}\left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right) - \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)\right\} \quad (F.13)$$

Substituting F.13 into F.10

$$\vec{S} = -\frac{2e\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s \times\left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right) - \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)\right\}f_o(p) - \frac{2e\tau^{-1}}{(2\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s \times\left\{[\varepsilon(p)-\mu]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}v_s(p)\frac{\partial f_o(p)}{\partial\varepsilon} \times v_s(p - e\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt') \quad (F.14)$$

F.14 is too cumbersome to solve. Separating it into $\vec{S}_1$ and $\vec{S}_2$ without lost of generality yields:

$$\vec{S}_1 = \frac{2e\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s \times\left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right) - \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)\right\}f_o(\vec{p}) \quad (F.15)$$

and

$$\vec{S}_2 = \frac{2e\tau^{-1}}{(2\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\int_0^{\infty}\exp\left(\frac{-t}{\tau}\right)dt\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s$$
$$\times\left\{[\varepsilon_o - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \Delta_z\cos\frac{v\vec{p}_z b_z}{\hbar} - \mu]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times\sin\frac{u\vec{p}_s b_s}{\hbar}\frac{\partial f_o(p)}{\partial\varepsilon}\left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)\right.$$
$$\left. - \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)\right\} \quad (F.16)$$

where $\vec{S} = \vec{S}_1 + \vec{S}_2$. Considering, $\vec{S}_1$, $f_o(\vec{p})$ is given by:

$$f_o(\vec{p}) = A^{\dagger}\exp\left(\frac{-[\varepsilon_o - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \Delta_z\cos\frac{v\vec{p}_z b_z}{\hbar}] + \mu}{k_B T}\right) \quad (F.17)$$

where $A^{\dagger}$ is to be determined from the normalisation condition $\int f(\vec{p})d\vec{p} = n_o$.

$$n_o = \frac{2}{(2\pi\hbar)^2}\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z f_o(\vec{p}) \quad (F.18)$$

$$n_o = \frac{2A^{\dagger}}{(2\pi\hbar)^2}\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\exp\left(\frac{(-\varepsilon_o + \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar})}{k_B T} + \mu\right) \quad (F.19)$$

$$n_o = \frac{2A^{\dagger}}{(2\pi\hbar)^2}\exp\left(\frac{\mu - \varepsilon_o}{k_B T}\right)\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\exp\left(\frac{\Delta_z}{k_B T}\cos\frac{w\vec{p}_z b_z}{\hbar}\right)$$
$$\times\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\exp\left(\frac{\Delta_s}{k_B T}\cos\frac{u\vec{p}_s b_s}{\hbar}\right) \quad (F.20)$$

$$n_o = \frac{2A^{\dagger}}{(2\pi\hbar)^2}\exp\left(\frac{\mu - \varepsilon_o}{k_B T}\right)\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\exp\left(\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right)$$
$$\times\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\exp\left(\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}\right) \quad (F.21)$$

where $\Delta_s^* = \Delta_s/k_B T$ and $\Delta_z^* = \Delta_z/k_B T$. Changing the path for integration

$$Z_z = w\vec{p}_z b_z \qquad Z_s = u\vec{p}_s b_s$$
$$\frac{dZ_z}{d\vec{p}_z} = wb_z \qquad \frac{dZ_s}{d\vec{p}_s} = ub_s$$
$$\frac{dZ_z}{wb_z} = d\vec{p}_z \qquad \frac{dZ_s}{ub_s} = d\vec{p}_s$$

$$n_o = \frac{8A^{\dagger}}{(2\pi\hbar)^2 ub_s wb_z} \exp\left(\frac{\mu - \varepsilon_o}{k_B T}\right) \int_{-\pi}^{\pi} dZ_z \exp\left(\Delta_z^* \cos\frac{Z_z}{\hbar}\right) \times \int_{-\pi}^{\pi} dZ_s \exp\left(\Delta_s^* \cos\frac{Z_s}{\hbar}\right) \tag{F.22}$$

$$I_n(x) = \frac{1}{\pi}\int_0^{\pi} d\theta \cos n\theta \exp(x\cos\theta) \tag{F.23}$$

$$I_0(\Delta_s^*) = \frac{1}{\pi}\int_0^{\pi} dZ_z \exp(\Delta_s^* \cos Z_s) \tag{F.24}$$

$$n_o = \frac{2A^{\dagger}\hbar^2}{(\pi\hbar)^2 uwb_z b_s} \exp\left(\frac{\mu - \varepsilon_o}{k_B T}\right) \pi I_o(\Delta_s^*) \pi I_o(\Delta_z^*) \tag{F.25}$$

Making $A^{\dagger}$ the subject of F.25 yields:

$$A^{\dagger} = \frac{uwn_o b_s b_z}{2I_o(\Delta_s^*) I_o(\Delta_z^*)} \exp\left(\frac{\mu - \varepsilon_o}{k_B T}\right) \tag{F.26}$$

Substituting F.26 into F.17 yields:

$$f_o(p) = \frac{uwn_o b_s b_z}{2I_o(\Delta_s^*) I_o(\Delta_z^*)} \exp\left[\frac{\Delta_s}{k_B T}\cos\frac{u\vec{p}_s b_s}{\hbar} + \frac{\Delta_z}{k_B T}\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \tag{F.27}$$

$$f_o(p) = \frac{uwn_o b_s b_z}{2I_o(\Delta_s^*) I_o(\Delta_z^*)} \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \tag{F.28}$$

Substitute F.28 into F.15 yields:

$$\begin{aligned}\vec{S}_1 = \frac{2e\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{n_o uwb_s b_z}{2I_o(\Delta_z^*) I_o(\Delta_s^*)} \int_0^{\infty} \exp\left(\frac{-t}{\tau}\right) dt \\ \times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \Bigg\{ \sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right) \\ - \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)\Bigg\} \\ \times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{v\vec{p}_z b_z}{\hbar}\right]\end{aligned} \tag{F.29}$$

$$\begin{aligned}\vec{S}_1 = \frac{2e\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{n_o uwb_s b_z}{2I_o(\Delta_z^*) I_o(\Delta_s^*)} \int_0^{\infty} \exp\left(\frac{-t}{\tau}\right) dt \\ \times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \Bigg\{ \sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right) \\ \times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{v\vec{p}_z b_z}{\hbar}\right] \\ - \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right) \\ \times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{v\vec{p}_z b_z}{\hbar}\right]\Bigg\}\end{aligned} \tag{F.30}$$

Integrating over the first Brillouin zone, makes the odd functions zero i.e. $-\pi/b_s \leq \vec{p}_s \leq \pi/b_s$. Thus

$$\vec{S}_1 = -\frac{2e\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{n_o uwb_s b_z}{2I_o(\Delta_z^*)I_o(\Delta_s^*)}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right) \times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{v\vec{p}_z b_z}{\hbar}\right]\right\} \quad (F.31)$$

Rearranging the terms yields:

$$\vec{S}_1 = -\frac{2e\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{n_o uwb_s b_z}{2I_o(\Delta_z^*)I_o(\Delta_s^*)} \times \left\{\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt\sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right) \times \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s\cos\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}\right]\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\exp\left[\Delta_z^*\cos\frac{v\vec{p}_z b_z}{\hbar}\right]\right\} \quad (F.32)$$

Changing the intgration path

$$\vec{S}_1 = -\frac{2e\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{n_o uwb_s b_z}{2I_o(\Delta_z^*)I_o(\Delta_s^*)} \times \left\{\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt\sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right) \times \frac{1}{ub_s}\int_{-\pi/b_s}^{\pi/b_s} dZ_s\cos\frac{Z_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{Z_s}{\hbar}\right]\frac{1}{wb_z}\int_{-\pi/b_z}^{\pi/b_z} dZ_z\exp\left[\Delta_z^*\cos\frac{Z_z}{\hbar}\right]\right\} \quad (F.33)$$

$$\vec{S}_1 = -\frac{2e\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{n_o}{2I_o(\Delta_z^*)I_o(\Delta_s^*)} \times \left\{\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt\sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right) \times \frac{2}{\pi}\int_0^\pi dZ_s\cos\frac{Z_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{Z_s}{\hbar}\right]\frac{2}{\pi}\int_0^\pi dZ_z\exp\left[\Delta_z^*\cos\frac{Z_z}{\hbar}\right]\right\} \quad (F.34)$$

$$\vec{S}_1 = -\frac{e\tau^{-1}}{(\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{n_o}{I_o(\Delta_z^*)I_o(\Delta_s^*)} \times \left\{\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt\sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right) \times \frac{1}{\pi}\int_0^\pi dZ_s\cos\frac{Z_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{Z_s}{\hbar}\right]\frac{1}{\pi}\int_0^\pi dZ_z\exp\left[\Delta_z^*\cos\frac{Z_z}{\hbar}\right]\right\} \quad (F.35)$$

By definition

$$I_1(\Delta_s^*) = \frac{1}{\pi}\int_0^{\pi}\frac{dZ_s}{\hbar}\cos\frac{Z_s}{\hbar}\exp\left(\Delta_s^*\cos\frac{Z_s}{\hbar}\right) \tag{F.36}$$

and

$$I_0(\Delta_z^*) = \frac{1}{\pi}\int_0^{\pi}\frac{dZ_z}{\hbar}\exp\left(\Delta_z^*\cos\frac{Z_z}{\hbar}\right) \tag{F.37}$$

F.35 yields:

$$\begin{aligned}\vec{S}_1 = &-\frac{e\tau^{-1}}{(\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{n_o}{I_o(\Delta_z^*)I_o(\Delta_s^*)}\pi^2\hbar^2 I_1(\Delta_s^*)I_0(\Delta_z^*)\\ &\times\left\{\int_0^{\infty}\exp\left(\frac{-t}{\tau}\right)dt\sin\left(\frac{ub_se}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)\right.\\ &\left.\times\frac{1}{\pi}\int_0^{\pi}dZ_s\cos\frac{Z_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{Z_s}{\hbar}\right]\frac{1}{\pi}\int_0^{\pi}dZ_s\exp\left[\Delta_z^*\cos\frac{Z_z}{\hbar}\right]\right\}\end{aligned} \tag{F.38}$$

$$\begin{aligned}\vec{S}_1 = &-\frac{e\tau^{-1}n_ou\Delta_s b_s}{\hbar}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\\ &\times\left\{\int_0^{\infty}\exp\left(\frac{-t}{\tau}\right)dt\sin\left(\frac{ub_se}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)\right\}\end{aligned} \tag{F.39}$$

Making use o the identity:

$$\begin{aligned}&\int_0^{\infty}\exp\left(\frac{-t}{\tau}\right)\sin\left(\frac{ub_se}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)dt\\ &\qquad=\sum_{n=-\infty}J_n^2(\chi)\left[\frac{(ueb_s\vec{E}_o/\hbar+n\omega\hbar)\tau^2}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]\end{aligned} \tag{F.40}$$

$$\vec{S}_1 = -\frac{e\tau^{-1}n_ou\Delta_s b_s}{\hbar}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\sum_{n=-\infty}J_n^2(\chi)\left[\frac{(ueb_s\vec{E}_o/\hbar+n\omega\hbar)\tau^2}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right] \tag{F.41}$$

$$\begin{aligned}\vec{S}_1 = &-\frac{e^2\tau n_ou^2\Delta_s b_s^2}{\hbar^2}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\sum_{n=-\infty}J_n^2(\chi)\left[\frac{1}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]\\ &\times\left(\vec{E}_o+\frac{n\omega\hbar}{ueb_s}\right)\end{aligned} \tag{F.42}$$

Now let $\sigma_s(\vec{E})$ be

$$\sigma_s(\vec{E}) = -\frac{e^2\tau n_ou^2\Delta_s b_s^2}{\hbar^2}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\sum_{n=-\infty}J_n^2(\chi)\left[\frac{1}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right] \tag{F.43}$$

Let $\vec{E}_n = \vec{E}_o + n\omega\hbar/ueb_s$

$$\vec{S}_1 = -\sigma_s(\vec{E})\left(\vec{E}_o + \frac{n\omega\hbar}{ueb_s}\right) \tag{F.44}$$

$$\vec{S}_1 = -\sigma_s(\vec{E})\vec{E}_n \tag{F.45}$$

Solving for F.16

$$\begin{aligned}\vec{S}_2 = &-\frac{2e}{(2\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s\\ &\times\left\{\left[\varepsilon_o + \Delta_z\cos\frac{v\vec{p}_z b_z}{\hbar} + \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}\\ &\times\sin\frac{u\vec{p}_s b_s}{\hbar}\frac{\partial f_o(\vec{p})}{\partial\varepsilon}v_s(\vec{p} - e\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt')\end{aligned} \tag{F.46}$$

$$\frac{\partial f_o(\vec{p})}{\partial\varepsilon(\vec{p})} = -\frac{n_o uwb_s b_z}{2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \tag{F.47}$$

$$\begin{aligned}\vec{S}_2 = &-\frac{2e}{(2\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt\\ &\times\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s\left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}\\ &\times\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\ &\times\left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)\right.\\ &\left.-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)\right\}\end{aligned} \tag{F.48}$$

$$\begin{aligned}\vec{S}_2 = &-\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt\\ &\times\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s\cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)\\ &\times\left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}\\ &\times\sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\ &+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt\\ &\times\sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)\\ &\times\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s\left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.\end{aligned}$$

$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\bigg] \frac{\nabla_s T}{T} + \nabla_s \mu \bigg\}$$
$$\times \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\frac{u\vec{p}_s b_s}{\hbar} \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad \text{(F.49)}$$

Integrating over the first Brillouin zone, makes the odd functions zero i.e. $-\pi/b_s \leq \vec{p}_s \leq \pi/b_s$. Thus:

$$\vec{S}_2 = -\frac{8eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T} \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \cos\left(\frac{ub_s e}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')] dt'\right)$$
$$\times \left\{\left[\varepsilon_o - \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right] \frac{\nabla_s T}{T} + \nabla_s \mu\right\}$$
$$\times \sin^2\frac{u\vec{p}_s b_s}{\hbar} \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad \text{(F.50)}$$

$$\vec{S}_2 = -\frac{eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T} \int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')] dt'\right)$$
$$\times \left\{\left[\varepsilon_o - \Delta_z \cos\frac{v\vec{p}_z b_z}{\hbar} - \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right] \frac{\nabla_s T}{T} + \nabla_s \mu\right\}$$
$$\times \sin^2\frac{u\vec{p}_s b_s}{\hbar} \exp\left[\Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} + \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar}\right] \quad \text{(F.51)}$$

Changing the path of integration

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T} \int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \frac{1}{wb_z} \int_{-\pi/b_z}^{\pi/b_z} dZ_z \frac{1}{ub_s} \int_{-\pi/b_s}^{\pi/b_s} dZ_s \cos\left(\frac{ub_s e}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')] dt'\right)$$
$$\times \left\{\left[\varepsilon_o - \Delta_z \cos\frac{Z_z}{\hbar} - \Delta_s \cos\frac{Z_s}{\hbar} - \mu\right] \frac{\nabla_s T}{T} + \nabla_s \mu\right\}$$
$$\times \sin^2\frac{Z_s}{\hbar} \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar} + \Delta_z^* \cos\frac{Z_z}{\hbar}\right] \quad \text{(F.52)}$$

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$$\vec{S}_2 = -\frac{eu^2\Delta_s^2 b_s^2 n_o}{(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T} \int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt \frac{1}{\pi} \int_0^\pi dZ_z \frac{1}{\pi} \int_0^\pi dZ_s$$
$$\times \cos\left(\frac{ub_s e}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')] dt'\right)$$

$$\times \left\{ \left[\varepsilon_o - \Delta_z \cos\frac{Z_z}{\hbar} - \Delta_s \cos\frac{Z_s}{\hbar} - \mu \right] \frac{\nabla_s T}{T} + \nabla_s \mu \right\} \times \frac{1}{2}\left(1 - \cos\frac{2Z_s}{\hbar}\right) \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar} + \Delta_z^* \cos\frac{Z_z}{\hbar}\right] \quad \text{(F.53)}$$

$$\int_0^\infty \exp\left(\frac{-t}{\tau}\right) \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos(\omega t')]dt'\right) dt = \sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1 + (ueb_s\vec{E}_o/\hbar + n\omega\hbar)^2\tau^2}\right] \quad \text{(F.54)}$$

$$\vec{S}_2 = -\frac{eu^2\Delta_s^2 b_s^2 n_o}{(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1 + (ueb_s\vec{E}_o/\hbar + n\omega\hbar)^2\tau^2}\right] \times \frac{1}{\pi}\int_0^\pi dZ_z \frac{1}{\pi}\int_0^\pi dZ_s \left\{ \left[\varepsilon_o - \Delta_z \cos\frac{Z_z}{\hbar} - \Delta_s \cos\frac{Z_s}{\hbar} - \mu \right] \frac{\nabla_s T}{T} + \nabla_s \mu \right\} \times \frac{1}{2}\left(1 - \cos\frac{2Z_s}{\hbar}\right) \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar} + \Delta_z^* \cos\frac{Z_z}{\hbar}\right] \quad \text{(F.55)}$$

$$\begin{aligned}
\vec{S}_2 = &-\frac{eu^2\Delta_s^2 b_s^2 n_o}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \\
&\frac{1}{\pi}\int_0^\pi dZ_z \frac{1}{\pi}\int_0^\pi dZ_s \sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1 + (ueb_s\vec{E}_o/\hbar + n\omega\hbar)^2\tau^2}\right] \\
&\times \left\{ [\varepsilon_o - \mu] \frac{\nabla_s T}{T} + \nabla_s \mu \right\} \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar} + \Delta_z^* \cos\frac{Z_z}{\hbar}\right] \\
&+ \frac{eu^2\Delta_s^2 b_s^2 n_o}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\frac{1}{\pi}\int_0^\pi dZ_z \frac{1}{\pi}\int_0^\pi dZ_s \\
&\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1 + (ueb_s\vec{E}_o/\hbar + n\omega\hbar)^2\tau^2}\right] \\
&\times \left\{ [\varepsilon_o - \mu] \frac{\nabla_s T}{T} + \nabla_s \mu \right\} \\
&\times \left(\cos\frac{2Z_s}{\hbar}\right) \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar} + \Delta_z^* \cos\frac{Z_z}{\hbar}\right] \\
&+ \frac{eu^2\Delta_s^2 b_s^2 n_o v \Delta_s}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\frac{1}{\pi}\int_0^\pi dZ_z \frac{1}{\pi}\int_0^\pi dZ_s \\
&\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1 + (ueb_s\vec{E}_o/\hbar + n\omega\hbar)^2\tau^2}\right] \frac{\nabla_s T}{T} \\
&\times \left(\cos\frac{Z_s}{\hbar}\right) \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar} + \Delta_z^* \cos\frac{Z_z}{\hbar}\right] \\
&- \frac{eu^2\Delta_s^2 b_s^2 n_o v \Delta_s}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\frac{1}{\pi}\int_0^\pi dZ_z \frac{1}{\pi}\int_0^\pi dZ_s
\end{aligned}$$

$$\times \sum_{n=-\infty} J_n^2(\chi) \left[\frac{\tau}{1 + (ueb_s\vec{E}_o/\hbar + n\omega\hbar)^2\tau^2} \right] \frac{\nabla_s T}{T}$$
$$\times \left(\cos\frac{Z_s}{\hbar} \cos\frac{2Z_s}{\hbar} \right) \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar} + \Delta_z^* \cos\frac{Z_z}{\hbar} \right]$$
$$+ \frac{eu^2\Delta_s^2 b_s^2 n_o v\Delta_s}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \frac{1}{\pi}\int_0^\pi dZ_z \frac{1}{\pi}\int_0^\pi dZ_s$$
$$\times \sum_{n=-\infty} J_n^2(\chi) \left[\frac{\tau}{1 + (ueb_s\vec{E}_o/\hbar + n\omega\hbar)^2\tau^2} \right] \frac{\nabla_s T}{T}$$
$$\times \left(\cos\frac{Z_z}{\hbar} \right) \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar} + \Delta_z^* \cos\frac{Z_z}{\hbar} \right]$$
$$- \frac{eu^2\Delta_s^2 b_s^2 n_o v\Delta_s}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \frac{1}{\pi}\int_0^\pi dZ_z \frac{1}{\pi}\int_0^\pi dZ_s$$
$$\sum_{n=-\infty} J_n^2(\chi) \left[\frac{\tau}{1 + (ueb_s\vec{E}_o/\hbar + n\omega\hbar)^2\tau^2} \right] \frac{\nabla_s T}{T}$$
$$\times \left(\cos\frac{Z_z}{\hbar} \cos\frac{2Z_s}{\hbar} \right) \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar} + \Delta_z^* \cos\frac{Z_z}{\hbar} \right] \qquad \text{(F.56)}$$

Breaking down F.56 and solving yields:

$$\vec{S}_{21} = -\frac{eu^2\Delta_s^2 b_s^2 n_o}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \frac{1}{\pi}\int_0^\pi dZ_z \frac{1}{\pi}\int_0^\pi dZ_s$$
$$\times \sum_{n=-\infty} J_n^2(\chi) \left[\frac{\tau}{1 + (ueb_s\vec{E}_o/\hbar + n\omega\hbar)^2\tau^2} \right]$$
$$\times \left\{ [\varepsilon_o - \mu] \frac{\nabla_s T}{T} + \nabla_s \mu \right\} \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar} + \Delta_z^* \cos\frac{Z_z}{\hbar} \right] \qquad \text{(F.57)}$$

$$\vec{S}_{21} = -\frac{eu^2\Delta_s^2 b_s^2 n_o}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}$$
$$\times \sum_{n=-\infty} J_n^2(\chi) \left[\frac{\tau}{1 + (ueb_s\vec{E}_o/\hbar + n\omega\hbar)^2\tau^2} \right] \left\{ [\varepsilon_o - \mu] \frac{\nabla_s T}{T} + \nabla_s \mu \right\}$$
$$\times \frac{1}{\pi}\int_0^\pi dZ_z \left[\Delta_z \cos\frac{Z_z}{\hbar} \right] \frac{1}{\pi}\int_0^\pi dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar} \right] \qquad \text{(F.58)}$$

$$\vec{S}_{21} = -\frac{eu^2\Delta_s^2 b_s^2 n_o}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \hbar^2 I_o(\Delta_s^*) I_o(\Delta_z^*)$$
$$\times \sum_{n=-\infty} J_n^2(\chi) \left[\frac{\tau}{1 + (ueb_s\vec{E}_o/\hbar + n\omega\hbar)^2\tau^2} \right] \times \left\{ [\varepsilon_o - \mu] \frac{\nabla_s T}{T} + \nabla_s \mu \right\} \qquad \text{(F.59)}$$

$$\vec{S}_{21} = -\frac{eu^2\Delta_s^2 b_s^2 n_o}{2\hbar^2 k_B T} \sum_{n=-\infty} J_n^2(\chi) \left[\frac{\tau}{1 + (ueb_s\vec{E}_o/\hbar + n\omega\hbar)^2\tau^2} \right]$$

$$\times \left\{ [\varepsilon_o - \mu] \frac{\nabla_s T}{T} + \nabla_s \mu \right\} \quad (F.60)$$

$$\vec{S}_{22} = \frac{eu^2 \Delta_s^2 b_s^2 n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \times \sum_{n=-\infty} J_n^2(\chi) \left[\frac{\tau}{1 + (ueb_s \vec{E}_o/\hbar + n\omega\hbar)^2 \tau^2} \right] \left\{ [\varepsilon_o - \mu] \frac{\nabla_s T}{T} + \nabla_s \mu \right\} \times \frac{1}{\pi} \int_0^\pi dZ_z \exp\left[\Delta_z^* \cos \frac{Z_z}{\hbar} \right] \frac{1}{\pi} \int_0^\pi dZ_s \left(\cos \frac{2Z_s}{\hbar} \right) \exp\left[\Delta_s^* \cos \frac{Z_s}{\hbar} \right] \quad (F.61)$$

$$\vec{S}_{22} = \frac{eu^2 \Delta_s^2 b_s^2 n_o}{2(\hbar)^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \sum_{n=-\infty} J_n^2(\chi) \left[\frac{\tau}{1 + (ueb_s \vec{E}_o/\hbar + n\omega\hbar)^2 \tau^2} \right] \times \left\{ [\varepsilon_o - \mu] \frac{\nabla_s T}{T} + \nabla_s \mu \right\} I_2(\Delta_s^*) I_o(\Delta_z^*) \quad (F.62)$$

$$\vec{S}_{22} = \frac{eu^2 \Delta_s^2 b_s^2 n_o}{2(\hbar)^2 k_B T} \sum_{n=-\infty} J_n^2(\chi) \left[\frac{\tau}{1 + (ueb_s \vec{E}_o/\hbar + n\omega\hbar)^2 \tau^2} \right] \times \left\{ [\varepsilon_o - \mu] \frac{\nabla_s T}{T} + \nabla_s \mu \right\} \frac{I_2(\Delta_s^*)}{I_0(\Delta_s)} \quad (F.63)$$

where

$$I_2(\Delta_s^*) = \frac{1}{\pi} \int_0^\pi dZ_s \cos \frac{2Z_s}{\hbar} \exp\left(\Delta_s^* \cos \frac{Z_s}{\hbar} \right) \quad (F.64)$$

The second order modified Bessel function obey the folowing recurrence relation

$$I_{n+1}(x) = I_{n-1}(x) - \frac{2n}{x} I_n(x) \quad (F.65)$$

$$I_2(\Delta_s^*) = I_0(\Delta_s^*) - \frac{2}{\Delta_s^*} I_n(\Delta_s^*) \quad (F.66)$$

$$\vec{S}_{22} = \frac{eu^2 \Delta_s^2 b_s^2 n_o}{2(\hbar)^2 k_B T} \sum_{n=-\infty} J_n^2(\chi) \left[\frac{\tau}{1 + (ueb_s \vec{E}_o/\hbar + n\omega\hbar)^2 \tau^2} \right] \times \left\{ [\varepsilon_o - \mu] \frac{\nabla_s T}{T} + \nabla_s \mu \right\} \frac{I_0(\Delta_s^*) - \frac{2}{\Delta_s^*} I_1(\Delta_s^*)}{I_0(\Delta_s^*)} \quad (F.67)$$

$$\vec{S}_{22} = \frac{eu^2 \Delta_s^2 b_s^2 n_o}{2(\hbar)^2 k_B T} \sum_{n=-\infty} J_n^2(\chi) \left[\frac{\tau}{1 + (ueb_s \vec{E}_o/\hbar + n\omega\hbar)^2 \tau^2} \right] \times \left\{ [\varepsilon_o - \mu] \frac{\nabla_s T}{T} + \nabla_s \mu \right\} \left(1 - \frac{2}{\Delta_s^*} \frac{I_1(\Delta_s^*)}{I_0(\Delta_s^*)} \right) \quad (F.68)$$

$$\vec{S}_{22} = \frac{eu^2\Delta_s^2 b_s^2 n_o}{2(\hbar)^2 k_B T} \sum_{n=-\infty} J_n^2(\chi) \left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar + n\omega\hbar)^2\tau^2}\right] \times \left\{[\varepsilon_o - \mu]\frac{\nabla_s T}{T} + \nabla_s \mu\right\} - \frac{eu^2\Delta_s^2 b_s^2 n_o}{2(\hbar)^2 k_B T} \sum_{n=-\infty} J_n^2(\chi) \left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar + n\omega\hbar)^2\tau^2}\right] \times \left\{[\varepsilon_o - \mu]\frac{\nabla_s T}{T} + \nabla_s \mu\right\} \frac{2}{\Delta_s^*}\frac{I_1(\Delta_s^*)}{I_0(\Delta_s^*)} \quad (F.69)$$

$$\vec{S}_{23} = \frac{eu^2\Delta_s^2 b_s^2 n_o w\Delta_s}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \times \sum_{n=-\infty} J_n^2(\chi) \left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar + n\omega\hbar)^2\tau^2}\right] \frac{\nabla_s T}{T} \times \frac{1}{\pi}\int_0^\pi dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \frac{1}{\pi}\int_0^\pi dZ_s \left(\cos\frac{Z_s}{\hbar}\right) \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \quad (F.70)$$

$$\vec{S}_{23} = \frac{eu^2\Delta_s^2 b_s^2 n_o w\Delta_s}{2(\pi\hbar)^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \frac{\nabla_s T}{T} I_o(\Delta_z^*) I_1(\Delta_s^*) \times \sum_{n=-\infty} J_n^2(\chi) \left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar + n\omega\hbar)^2\tau^2}\right] \quad (F.71)$$

$$\vec{S}_{23} = \frac{eu^2\Delta_s^2 b_s^2 n_o w\Delta_s}{2\hbar^2 k_B T} \sum_{n=-\infty} J_n^2(\chi) \left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar + n\omega\hbar)^2\tau^2}\right] \frac{\nabla_s T}{T} \frac{I_1(\Delta_s^*)}{I_0(\Delta_s^*)} \quad (F.72)$$

$$\vec{S}_{24} = -\frac{eu^2\Delta_s^2 b_s^2 n_o w\Delta_s}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \times \sum_{n=-\infty} J_n^2(\chi) \left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar + n\omega\hbar)^2\tau^2}\right] \frac{\nabla_s T}{T} \times \frac{1}{\pi}\int_0^\pi dZ_z \frac{1}{\pi}\int_0^\pi dZ_s \left(\cos\frac{Z_s}{\hbar}\cos\frac{2Z_s}{\hbar}\right) \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar} + \Delta_s^* \cos\frac{Z_s}{\hbar}\right] \quad (F.73)$$

$$\vec{S}_{24} = -\frac{eu^2\Delta_s^2 b_s^2 n_o w\Delta_s}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \times \sum_{n=-\infty} J_n^2(\chi) \left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar + n\omega\hbar)^2\tau^2}\right] \frac{\nabla_s T}{T} \times \frac{1}{\pi}\int_0^\pi dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \frac{1}{\pi}\int_0^\pi dZ_s \left(\cos\frac{Z_s}{\hbar}\cos\frac{2Z_s}{\hbar}\right) \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \quad (F.74)$$

$$\vec{S}_{24} = -\frac{eu^2\Delta_s^2 b_s^2 n_o w\Delta_s}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\frac{\nabla_s T}{T} \times \sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(uebs_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]\frac{1}{\pi}\int_0^\pi dZ_z \exp\left[\Delta_z^*\cos\frac{Z_z}{\hbar}\right] \times \frac{1}{\pi}\int_0^\pi dZ_s\frac{1}{2}\left(\cos\frac{Z_s}{\hbar}+\cos\frac{3Z_s}{\hbar}\right)\exp\left[\Delta_s^*\cos\frac{Z_s}{\hbar}\right] \quad (F.75)$$

$$\vec{S}_{24} = -\frac{eu^2\Delta_s^2 b_s^2 n_o w\Delta_s}{4(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\frac{\nabla_s T}{T} \times \sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(uebs_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]\frac{1}{\pi}\int_0^\pi dZ_z \exp\left[\Delta_z^*\cos\frac{Z_z}{\hbar}\right] \times \frac{1}{\pi}\int_0^\pi dZ_s\frac{1}{2}\left(\cos\frac{Z_s}{\hbar}\right)\exp\left[\Delta_s^*\cos\frac{Z_s}{\hbar}\right] - \frac{eu^2\Delta_s^2 b_s^2 n_o v\Delta_s}{4(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\frac{\nabla_s T}{T}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(uebs_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right] \frac{1}{\pi}\int_0^\pi dZ_z \exp\left[\Delta_z^*\cos\frac{Z_z}{\hbar}\right]\frac{1}{\pi}\int_0^\pi dZ_s\left(\cos\frac{3Z_s}{\hbar}\right)\exp\left[\Delta_s^*\cos\frac{Z_s}{\hbar}\right] \quad (F.76)$$

$$\vec{S}_{24} = -\frac{eu^2\Delta_s^2 b_s^2 n_o w\Delta_s I_1(\Delta_s^*)I_o(\Delta_z^*)}{4\hbar^2 I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\frac{\nabla_s T}{T}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(uebs_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right] - \frac{eu^2\Delta_s^2 b_s^2 n_o}{4\hbar^2 I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\frac{\nabla_s T}{T} I_3(\Delta_s^*)I_o(\Delta_z^*)\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(uebs_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right] \quad (F.77)$$

$$\vec{S}_{24} = -\frac{eu^2\Delta_s^2 b_s^2 n_o w\Delta_s}{4\hbar^2 I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\frac{\nabla_s T}{T} \times \sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(uebs_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]\left\{\frac{I_1(\Delta_s^*)+I_3(\Delta_s^*)}{I_o(\Delta_s^*)}\right\} \quad (F.78)$$

where

$$I_2(\Delta_s^*) = \frac{1}{\pi}\int_0^\pi dZ_s\cos\frac{3Z_s}{\hbar}\exp\left(\Delta_s^*\cos\frac{Z_s}{\hbar}\right) \quad (F.79)$$

The third order modified Bessel function obey the folowing recurrence relation

$$I_{n+1}(x) = I_{n-1}(x) - \frac{2n}{x}I_n(x) \quad (F.80)$$

$$I_3(\Delta_s^*) = I_1(\Delta_s^*) - \frac{4}{\Delta_s^*}I_2(\Delta_s^*) \quad (F.81)$$

$$I_3(\Delta_s^*) = I_1(\Delta_s^*) - \frac{4}{\Delta_s^*}\left[I_0(\Delta_s^*) - \frac{4}{\Delta_s^*}I_1(\Delta_s^*)\right] \quad (F.82)$$

$$I_3(\Delta_s^*) = I_1(\Delta_s^*) - \frac{4}{\Delta_s^*}I_o(\Delta_s^*) - \frac{8}{\Delta_s^{*2}}I_1(\Delta_s^*) \quad \text{(F.83)}$$

Thus

$$\frac{I_1(\Delta_s^*)+I_3(\Delta_s^*)}{I_o(\Delta_s^*)} = \frac{I_1(\Delta_s^*)+I_1(\Delta_s^*)-\frac{4}{\Delta_s}I_o(\Delta_s^*)-\frac{8}{\Delta_s^{*2}}I_1(\Delta_s^*)}{I_o(\Delta_s^*)} \quad \text{(F.84)}$$

$$\frac{I_1(\Delta_s^*)+I_3(\Delta_s^*)}{I_o(\Delta_s^*)} = \frac{2I_1(\Delta_s^*)}{I_o(\Delta_s^*)} - \frac{4}{\Delta_s}\frac{I_o(\Delta_s^*)}{I_o(\Delta_s^*)} + \frac{8}{\Delta_s^{*2}}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} \quad \text{(F.85)}$$

$$\vec{S}_{24} = -\frac{eu^2\Delta_s^2 b_s^2 n_o w\Delta_s}{4\hbar^2 I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right] \times\left\{\frac{2I_1(\Delta_s^*)}{I_o(\Delta_s^*)} - \frac{4}{\Delta_s}\frac{I_o(\Delta_s^*)}{I_o(\Delta_s^*)} + \frac{8}{\Delta_s^{*2}}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right\}\frac{\nabla_s T}{T} \quad \text{(F.86)}$$

$$\vec{S}_{24} = -\frac{eu^2\Delta_s^2 b_s^2 n_o w\Delta_s}{2\hbar^2 k_B T}\frac{\nabla_s T}{T}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right] - \frac{eu^2\Delta_s^2 b_s^2 n_o w}{\hbar^2}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]\frac{\nabla_s T}{T} - \frac{2eu^2\Delta_s^2 b_s^2 n_o w}{\hbar^2\Delta_s}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]\frac{\nabla_s T}{T}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} \quad \text{(F.87)}$$

$$\vec{S}_{25} = \frac{eu^2\Delta_s^2 b_s^2 n_o w\Delta_z}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{\nabla_s T}{T}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right] \times\frac{1}{\pi}\int_0^\pi dZ_z\left(\cos\frac{Z_z}{\hbar}\right)\exp\left[\Delta_z^*\cos\frac{Z_z}{\hbar}\right]\frac{1}{\pi}\int_0^\pi dZ_s\exp\left[\Delta_s^*\cos\frac{Z_s}{\hbar}\right] \quad \text{(F.88)}$$

$$\vec{S}_{25} = \frac{eu^2\Delta_s^2 b_s^2 n_o u\Delta_z}{2\hbar^2 k_B T}\frac{\nabla_s T}{T}\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right] \quad \text{(F.89)}$$

$$\vec{S}_{26} = -\frac{eu^2\Delta_s^2 b_s^2 n_o w\Delta_s}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{\nabla_s T}{T}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right] \times\frac{1}{\pi}\int_0^\pi dZ_z\cos\frac{Z_z}{\hbar}\exp\left[\Delta_z^*\cos\frac{Z_z}{\hbar}\right]\times\frac{1}{\pi}\int_0^\pi dZ_s\cos\frac{2Z_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{Z_s}{\hbar}\right] \quad \text{(F.90)}$$

$$\vec{S}_{26} = -\frac{eu^2\Delta_s^2 b_s^2 n_o u\Delta_z}{2\hbar^2 k_B T}\frac{I_2(\Delta_s^*)I_1(\Delta_z^*)}{I_o(\Delta_s^*)I_o(\Delta_z^*)}\frac{\nabla_s T}{T}$$

$$\times \sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right] \tag{F.91}$$

$$\frac{I_2(\Delta_s^*)I_1(\Delta_z^*)}{I_o(\Delta_s^*)I_o(\Delta_z^*)} = \frac{\left[I_0(\Delta_s^*)-\frac{2}{\Delta_s^*}I_n(\Delta_s^*)\right]I_1(\Delta_z^*)}{I_o(\Delta_s^*)I_o(\Delta_z^*)}$$

$$\frac{I_2(\Delta_s^*)I_1(\Delta_z)}{I_o(\Delta_s^*)I_o(\Delta_z^*)} = \frac{I_o(\Delta_s^*)I_1(\Delta_z)}{I_o(\Delta_s^*)I_o(\Delta_z^*)} - \frac{2}{\Delta_s^*}\frac{I_1(\Delta_s^*)I_1(\Delta_z^*)}{I_o(\Delta_s^*)I_o(\Delta_z^*)}$$

$$\frac{I_2(\Delta_s^*)I_1(\Delta_z^*)}{I_o(\Delta_s^*)I_o(\Delta_z^*)} = \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} - \frac{2}{\Delta_s^*}\frac{I_1(\Delta_s^*)I_1(\Delta_z^*)}{I_o(\Delta_s^*)I_o(\Delta_z^*)}$$

$$\vec{S}_{26} = -\frac{eu^2\Delta_s^2b_s^2n_ou\Delta_z}{2\hbar^2k_BT}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]$$
$$\times\left\{\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} - \frac{2}{\Delta_s^*}\frac{I_1(\Delta_s^*)I_1(\Delta_z^*)}{I_o(\Delta_s^*)I_o(\Delta_z^*)}\right\}\frac{\nabla_sT}{T} \tag{F.92}$$

$$\vec{S}_{26} = -\frac{eu^2\Delta_s^2b_s^2n_ou\Delta_z}{2\hbar^2k_BT}\frac{\nabla_sT}{T}\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]$$
$$+\frac{eu^2\Delta_s^2b_s^2n_ou\Delta_z}{\hbar^2k_BT}\frac{1}{\Delta_s^*}\frac{I_1(\Delta_s^*)I_1(\Delta_z^*)}{I_o(\Delta_s^*)I_o(\Delta_z^*)}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]\frac{\nabla_sT}{T} \tag{F.93}$$

Adding all the terms in $\vec{S}_2$

$$\vec{S}_2 = -\frac{eu^2\Delta_s^2b_s^2n_o}{2\hbar^2k_BT}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]$$
$$\times\left\{[\varepsilon_o-\mu]\frac{\nabla_sT}{T}+\nabla_s\mu\right\}$$
$$+\frac{eu^2\Delta_s^2b_s^2n_o}{2\hbar^2k_BT}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]\left\{[\varepsilon_o-\mu]\frac{\nabla_sT}{T}+\nabla_s\mu\right\}$$
$$-\frac{eu^2\Delta_s^2b_s^2n_oI_1(\Delta_s^*)}{\hbar^2k_BTI_0(\Delta_s)}\frac{1}{\Delta_s^*}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]\left\{[\varepsilon_o-\mu]\frac{\nabla_sT}{T}+\nabla_s\mu\right\}$$
$$+\frac{eu^2\Delta_s^2b_s^2n_ov\Delta_s}{2\hbar^2k_BT}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]\frac{\nabla_sT}{T}\frac{I_1(\Delta_s^*)}{I_0(\Delta_s)}$$
$$-\frac{eu^2\Delta_s^2b_s^2n_ov\Delta_s}{2\hbar^2k_BT}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]\frac{\nabla_sT}{T}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}$$
$$+\frac{eu^2\Delta_s^2b_s^2n_ou}{\hbar^2}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]\frac{\nabla_sT}{T}$$
$$-\frac{2eu^2\Delta_s^2b_s^2n_o}{\hbar^2\Delta_s^*}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]\frac{\nabla_sT}{T}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}$$

$$+\frac{eu^2\Delta_s^2 b_s^2 n_o w\Delta_z}{2\hbar^2 k_B T}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]\frac{\nabla_s T}{T}\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}$$
$$-\frac{eu^2\Delta_s^2 b_s^2 n_o w\Delta_z}{2\hbar^2 k_B T}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]\frac{\nabla_s T}{T}\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}$$
$$+\frac{eu^2\Delta_s^2 b_s^2 n_o w\Delta_z}{\hbar^2 k_B T}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]\frac{\nabla_s T}{T}$$
$$\times\frac{1}{\Delta_s^*}\frac{I_1(\Delta_s^*)I_1(\Delta_z^*)}{I_o(\Delta_s^*)I_o(\Delta_z^*)} \quad \text{(F.94)}$$

Simplifying the expression

$$\vec{S}_2=-\frac{eu^2\Delta_s^2 b_s^2 n_o}{\hbar^2 k_B T}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]$$
$$\times\left\{[\varepsilon_o-\mu]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}\left(\frac{1}{\Delta_s^*}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right)$$
$$+\frac{eu^2\Delta_s^2 b_s^2 n_o}{\hbar^2}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]\frac{\nabla_s T}{T}$$
$$-\frac{2eu^2\Delta_s^2 b_s^2 n_o u}{\hbar^2\Delta_s^*}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]\frac{\nabla_s T}{T}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}$$
$$+\frac{eu^2\Delta_s^2 b_s^2 n_o w\Delta_z}{\hbar^2 k_B T}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]\frac{\nabla_s T}{T}$$
$$\times\frac{1}{\Delta_s^*}\frac{I_1(\Delta_s^*)I_1(\Delta_z^*)}{I_o(\Delta_s^*)I_o(\Delta_z^*)} \quad \text{(F.95)}$$

$$\vec{S}_2=-\frac{eu^2\Delta_s^2 b_s^2 n_o}{\hbar^2 k_B T}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]\left(\frac{1}{\Delta_s^*}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right)\nabla_s\mu$$
$$-\frac{eu^2\Delta_s^2 b_s^2 n_o}{\hbar^2 k_B T}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]\left(\frac{1}{\Delta_s}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right)[\varepsilon_o-\mu]\frac{\nabla_s T}{T}$$
$$+\frac{eu^2\Delta_s^2 b_s^2 n_o}{\hbar^2}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]\frac{\nabla_s T}{T}$$
$$-\frac{2eu^2\Delta_s^2 b_s^2 n_o u}{\hbar^2\Delta_s^*}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]\frac{\nabla_s T}{T}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}$$
$$+\frac{eu^2\Delta_s^2 b_s^2 n_o w\Delta_z}{\hbar^2 k_B T}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]\frac{\nabla_s T}{T}$$
$$\times\frac{1}{\Delta_s^*}\frac{I_1(\Delta_s^*)I_1(\Delta_z)}{I_o(\Delta_s^*)I_o(\Delta_z^*)} \quad \text{(F.96)}$$

$$\vec{S}_2=-\frac{eu^2\Delta_s b_s^2 n_o}{\hbar^2}\left(\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right)\nabla_s\mu\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]$$

$$-\frac{eu^2\Delta_s b_s^2 n_o}{\hbar^2}\left(\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right)[\varepsilon_o-\mu]\frac{\nabla_s T}{T}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]$$
$$+\frac{eu^2\Delta_s b_s^2 n_o}{\hbar^2}\Delta_s\frac{\nabla_s T}{T}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]$$
$$-\frac{2eu^2\Delta_s b_s^2 n_o w}{\hbar^2\Delta_s^*}\Delta_s\frac{\nabla_s T}{T}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]$$
$$+\frac{eu^2\Delta_s b_s^2 n_o w\Delta_z}{\hbar^2}\frac{\nabla_s T}{T}\frac{I_1(\Delta_s^*)I_1(\Delta_z)}{I_o(\Delta_s^*)I_o(\Delta_z^*)}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right] \quad \text{(F.97)}$$

$$\vec{S}_2=-\frac{e^2u^2\Delta_s b_s^2 n_o}{e\hbar^2}\left(\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right)\nabla_s\mu\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]$$
$$-\frac{e^2u^2\Delta_s b_s^2 n_o k}{\hbar^2 ke}\left(\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right)\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]$$
$$\times\left\{(\varepsilon_o-\mu)-\Delta_s\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+\frac{2\Delta_s}{\Delta_s^*}-\Delta_z\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right\}\frac{\nabla_s T}{T} \quad \text{(F.98)}$$

$$\vec{S}_2=-\frac{e^2u^2\Delta_s b_s^2 n_o}{\hbar^2}\left(\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right)\nabla_s\frac{\mu}{e}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]$$
$$-\frac{e^2u^2\Delta_s b_s^2 n_o}{\hbar^2}\frac{k_B}{e}\left(\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right)\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]$$
$$\times\left\{\left(\frac{\varepsilon_o-\mu}{k_B T}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right\}\nabla_s T \quad \text{(F.99)}$$

$$\vec{S}_2=-\frac{e^2u^2\Delta_s b_s^2 n_o}{\hbar^2}\left(\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right)\nabla_s\frac{\mu}{e}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]$$
$$-\frac{e^2u^2\Delta_s b_s^2 n_o}{\hbar^2}\frac{k_B}{e}\left(\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right)\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]$$
$$\times\left\{\left(\frac{\varepsilon_o-\mu}{k_B T}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right\}\nabla_s T \quad \text{(F.100)}$$

$$\vec{S}_2=-\frac{e^2u^2\tau\Delta_s b_s^2 n_o}{\hbar^2}\left(\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right)\nabla_s\frac{\mu}{e}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{1}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]$$
$$-\frac{e^2u^2\tau\Delta_s b_s^2 n_o}{\hbar^2}\frac{k_B}{e}\left(\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right)\sum_{n=-\infty} J_n^2(\chi)\left[\frac{1}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]$$
$$\times\left\{\left(\frac{\varepsilon_o-\mu}{k_B T}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right\}\nabla_s T \quad \text{(F.101)}$$

$$
\begin{aligned}
\vec{S}_2 = &-\sigma_s(\vec{E})\nabla_s \frac{\mu}{e} \\
&- \sigma_s(\vec{E})\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right) - \Delta_s\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} + 2 - \Delta_z\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right]\nabla_s T
\end{aligned} \quad (F.102)
$$

where $\sigma_s(\vec{E})$ is the conductivity. But $\vec{S} = \vec{S}_1 + \vec{S}_2$

$$
\begin{aligned}
S = &-\sigma_s(\vec{E})\left(\vec{E}_n + \nabla_s \frac{\mu}{e}\right) \\
&- \sigma_s(\vec{E})\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right) - \Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} + 2 - \Delta_z^*\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right]\nabla_s T
\end{aligned} \quad (F.103)
$$

$$
S = -\sigma_s(\vec{E})\vec{E}_{sn}^* - \sigma_s(\vec{E})\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right) - \Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} + 2 - \Delta_z^*\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right]\nabla_s T \quad (F.104)
$$

where $\vec{E}_{sn}^* = \vec{E}_n + \nabla_s\mu/e$. Similarly

$$
\vec{Z} = -\sigma_z(\vec{E})\vec{E}_{zn}^* - \sigma_z(\vec{E})\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right) - \Delta_z^*\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)} + 2 - \Delta_s^*\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right]\nabla_z T \quad (F.105)
$$

Resolving the current density into axial and circumferential components, the axial and circumferential components are given as: $\vec{J}_z = \vec{Z} + \vec{S}\sin\theta_h$ and $\vec{J}_s = \vec{S}\cos\theta_h$ respectively:

$$
\begin{aligned}
\vec{J}_z = &-\sigma_z(\vec{E})\vec{E}_{zn}^* - \sigma_s(\vec{E})\sin\theta_h\vec{E}_{sn}^* \\
&- \sigma_z(\vec{E})\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right) - \Delta_z^*\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)} + 2 - \Delta_s^*\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right]\nabla_z T \\
&- \sigma_s(\vec{E})\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right) - \Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} + 2 - \Delta_z^*\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right]\sin\theta_h\nabla_s T
\end{aligned} \quad (F.106)
$$

Utilising the relations $\vec{E}_s = \vec{E}_z\sin\theta_h$, $\nabla_s T = \nabla_z T\sin\theta_h$ and $\vec{E}_{sn}^* = \vec{E}_{zn}^*\sin\theta_h$

$$
\begin{aligned}
\vec{J}_z = &-\sigma_z(\vec{E})\vec{E}_{zn}^* - \sigma_s(\vec{E})\sin\theta_h\vec{E}_{sn}^* \\
&- \sigma_z(\vec{E})\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right) - \Delta_z^*\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)} + 2 - \Delta_s^*\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right]\nabla_z T \\
&- \sigma_s(\vec{E})\frac{k_B}{e}\sin^2\theta_h\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right) - \Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} + 2 - \Delta_z^*\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right]\nabla_z T
\end{aligned} \quad (F.107)
$$

$$
\begin{aligned}
\vec{J}_z = &-\sigma_z(\vec{E})\vec{E}_{zn}^* - \sigma_s(\vec{E})\sin^2\theta_h\vec{E}_{zn}^* \\
&- \sigma_z(\vec{E})\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right) - \Delta_z^*\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)} + 2 - \Delta_s^*\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right]\nabla_z T \\
&- \sigma_s(\vec{E})\frac{k_B}{e}\sin^2\theta_h\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right) - \Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} + 2 - \Delta_z^*\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right]\nabla_z T
\end{aligned} \quad (F.108)
$$

$$
\vec{J}_z = -(\sigma_z(\vec{E}) + \sigma_s(\vec{E})\sin^2\theta_h)\vec{E}_{zn}^*
$$

$$- \sigma_z(\vec{E})\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_z^*\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}+2-\Delta_s^*\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right]\nabla_zT$$
$$- \sigma_s(\vec{E})\frac{k_B}{e}\sin^2\theta_h\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right]\nabla_zT \quad (F.109)$$

$$\vec{J}_z = -(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)\nabla_z\left(\frac{\mu}{e}-\varphi\right)$$
$$- \sigma_z(\vec{E})\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_z^*\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}+2-\Delta_s^*\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right]\nabla_zT$$
$$- \sigma_s(\vec{E})\frac{k_B}{e}\sin^2\theta_h\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right]\nabla_zT \quad (F.110)$$

where

$$\vec{E}_{zn}^* = \nabla_z\left(\frac{\mu}{e}-\varphi\right)$$

and the circumferential component of the electron currrent density is given as:

$$\vec{J}_c = -\sigma_s(\vec{E})\cos\theta_h\vec{E}_{sn}^*$$
$$- \sigma_s(\vec{E})\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right]\nabla_sT\cos\theta_h \quad (F.111)$$

$$\vec{J}_c = -\sigma_s(\vec{E})\sin\theta_h\cos\theta_h\vec{E}_{zn}^*$$
$$- \sigma_s(\vec{E})\frac{k_B}{e}\sin\theta_h\cos\theta_h\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right]\nabla_zT \quad (F.112)$$

$$\vec{J}_c = -\sigma_s(\vec{E})\sin\theta_h\cos\theta_h\nabla_z\left(\frac{\mu}{e}-\varphi\right)$$
$$- \sigma_s(\vec{E})\frac{k_B}{e}\sin\theta_h\cos\theta_h\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right]\nabla_zT \quad (F.113)$$

Thus the circumferential σ_{cs} and axial σ_{zz} components of the electrical conductivity are given by the coeficients of the electric field E_{zn} as:

$$\sigma_{cs} = \sigma_s\sin\theta_h\cos\theta_h \qquad \sigma_{zz} = \sigma_z+\sigma_s\sin^2\theta_h$$

The differential thermoelectric power is defined as the ratio $\left|\frac{\vec{E}_{zn}^*}{\nabla_zT}\right|$ in an open circuit $\vec{J}=0$. Thus setting $\vec{J}_c$ to zero, the thermoelectric power α_{cz} along the circumferential direction is obtained as follows:

$$0 = -\sigma_s(\vec{E})\sin\theta_h\cos\theta_h\vec{E}_{zn}^*$$
$$- \sigma_s(\vec{E})\frac{k_B}{e}\sin\theta_h\cos\theta_h\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right]\nabla_zT \quad (F.114)$$

$$\left|\frac{\vec{E}^*_{zn}}{\nabla_z T}\right| = -\frac{\sigma_s(\vec{E})\frac{k_B}{e}\sin\theta_h\cos\theta_h\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta^*_s\frac{I_o(\Delta^*_s)}{I_1(\Delta^*_s)}+2-\Delta^*_z\frac{I_1(\Delta^*_z)}{I_o(\Delta^*_z)}\right]}{\sigma_s(\vec{E})\sin\theta_h\cos\theta_h} \quad (F.115)$$

$$\left|\frac{\vec{E}^*_{zn}}{\nabla_z T}\right| = -\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta^*_s\frac{I_o(\Delta^*_s)}{I_1(\Delta^*_s)}+2-\Delta^*_z\frac{I_1(\Delta^*_z)}{I_o(\Delta^*_z)}\right] \quad (F.116)$$

$$\alpha_{cz} = \left|\frac{\vec{E}^*_{zn}}{\nabla_z T}\right| = -\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta^*_s\frac{I_o(\Delta^*_s)}{I_1(\Delta^*_s)}+2-\Delta^*_z\frac{I_1(\Delta^*_z)}{I_o(\Delta^*_z)}\right] \quad (F.117)$$

Similarly, α_{zz} along the axial direction is obtained as follows $J_z = 0$

$$\begin{aligned} 0 = &-(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)\vec{E}^*_{zn} \\ &-\sigma_z(\vec{E})\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta^*_z\frac{I_o(\Delta^*_z)}{I_1(\Delta^*_z)}+2-\Delta^*_s\frac{I_1(\Delta^*_s)}{I_o(\Delta^*_s)}\right]\nabla_z T \\ &-\sigma_s(\vec{E})\frac{k_B}{e}\sin^2\theta_h\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta^*_s\frac{I_o(\Delta^*_s)}{I_1(\Delta^*_s)}+2-\Delta^*_z\frac{I_1(\Delta^*_z)}{I_o(\Delta^*_z)}\right]\nabla_z T \end{aligned} \quad (F.118)$$

$$\begin{aligned} \alpha_{zz} = \left|\frac{\vec{E}^*_{zn}}{\nabla_z T}\right| = & \\ &-\frac{\sigma_z(\vec{E})}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta^*_z\frac{I_o(\Delta^*_z)}{I_1(\Delta^*_z)}+2-\Delta^*_s\frac{I_1(\Delta^*_s)}{I_o(\Delta^*_s)}\right] \\ &-\frac{\sigma_s(\vec{E})\sin^2\theta_h}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta^*_s\frac{I_o(\Delta^*_s)}{I_1(\Delta^*_s)}+2-\Delta^*_z\frac{I_1(\Delta^*_z)}{I_o(\Delta^*_z)}\right] \end{aligned} \quad (F.119)$$

The electrical power factor along the circumferential $\mathscr{P}_{cs}$ and axial $\mathscr{P}_{zz}$ directions are given respectively as:

$$\mathscr{P}_{cs} = \alpha^2_{cz}\sigma_{cz} \qquad \mathscr{P}_{zz} = \alpha^2_{zz}\sigma_{zz}$$

The electrical power factor along the circumferential direction is given as:

$$\begin{aligned} \mathscr{P}_{cs} &= \alpha^2_{cz}\sigma_{cz} \\ &= \left\{\left(\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta^*_s\frac{I_o(\Delta^*_s)}{I_1(\Delta^*_s)}+2-\Delta^*_z\frac{I_1(\Delta_z)}{I_o(\Delta^*_z)}\right]\right)^2 \right. \\ &\left.\times\left(\frac{e^2\tau n_o u^2\Delta_s b_s^2}{\hbar^2}\frac{I_1(\Delta^*_s)}{I_o(\Delta^*_s)}\left[\frac{1}{1+\Omega_s^2\tau^2}\right]\right)\sin\theta_h\cos\theta_h\right\} \end{aligned} \quad (F.120)$$

where the output power density along the circumferential direction becomes;

$$\mathscr{Q}^{max}_{cs} = \left\{\left(\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta^*_s\frac{I_o(\Delta^*_s)}{I_1(\Delta^*_s)}+2-\Delta^*_z\frac{I_1(\Delta_z)}{I_o(\Delta^*_z)}\right]\right)^2\right.$$

$$\times\left(\frac{e^2\tau n_o u^2\Delta_s b_s^2}{\hbar^2}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\left[\frac{1}{1+\Omega_s^2\tau^2}\right]\right)\frac{\sin\theta_h\cos\theta_h(T_h-T_c)}{4h_\ell}\Bigg\} \quad \text{(F.121)}$$

Moreover, the axial component of the power factor yields:

$$\begin{aligned}
\mathscr{P}_{zz} &= \alpha_{zz}^2\sigma_{zz} \\
&= \Bigg\{\Bigg(\frac{\sigma_z(\vec{E})}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_z^*\frac{I_o(\Delta_z^*)}{I_1(\Delta_z)}+2-\Delta_s^*\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right] \\
&+\frac{\sigma_s(\vec{E})\sin^2\theta_h}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z)}{I_o(\Delta_z^*)}\right]\Bigg)^2 \\
&\times\frac{e^2\tau n_o}{\hbar^2}\left(w^2\Delta_z b_z^2\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left[\frac{1}{1+\Omega_z^2\tau^2}\right]+u^2\Delta_s b_s^2\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\left[\frac{1}{1+\Omega_s^2\tau^2}\right]\sin^2\theta_h\right)\Bigg\}
\end{aligned} \quad \text{(F.122)}$$

and similarly the output power density also gives;

$$\begin{aligned}
\mathscr{Q}_{zz}^{max} &= \Bigg\{\Bigg(\frac{\sigma_z(\vec{E})}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_z^*\frac{I_o(\Delta_z^*)}{I_1(\Delta_z)}+2-\Delta_s^*\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right] \\
&+\frac{\sigma_s(\vec{E})\sin^2\theta_h}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z)}{I_o(\Delta_z^*)}\right]\Bigg)^2 \\
&\times\frac{e^2\tau n_o(T_h-T_c)}{\hbar^2 4h_\ell}\left(w^2\Delta_z b_z^2\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left[\frac{1}{1+\Omega_z^2\tau^2}\right]+u^2\Delta_s b_s^2\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\left[\frac{1}{1+\Omega_s^2\tau^2}\right]\sin^2\theta_h\right)\Bigg\}
\end{aligned} \quad \text{(F.123)}$$

APPENDIX G

CARRIER THERMAL CURRENT DENSITY

In the linear approximation, ∇T and $\nabla \mu$, the solution to the Boltzmann equation is:

$$f(\vec{p},t) = \tau^{-1}\int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt f_o\left(\vec{p} - e\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right) + \int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt \left\{[\varepsilon(p)-\mu]\frac{\nabla T}{T} - \nabla\mu\right\} \times v(p)\cdot\frac{\partial f_o}{\partial\varepsilon}(\vec{p} - e\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt') \quad (G.1)$$

The electon thermal current density q is defined by

$$\vec{q} = \sum_p [\varepsilon(\vec{p}) - \mu] v(\vec{p}) f(\vec{p}) \quad (G.2)$$

$$\vec{q} = \tau^{-1}\int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt \sum_p [\varepsilon(\vec{p})-\mu]v(\vec{p}) f_o\left(\vec{p} - e\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right) + \int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt \sum_p [\varepsilon(\vec{p})-\mu]v(\vec{p}) \left\{[\varepsilon(\vec{p})-\mu]\frac{\nabla T}{T} - \nabla\mu\right\} \times v(p)\cdot\frac{\partial f_o}{\partial\varepsilon}(\vec{p} - e\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt') \quad (G.3)$$

Invoking the transform $\vec{p} - e\int_0^{\infty}[\vec{E}_o + \vec{E}\cos\omega t']dt' \rightarrow \vec{p}$ the equation becomes:

$$\vec{q} = \tau^{-1}\int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt \sum_p \left[\varepsilon\left(\vec{p} - e\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right) - \mu\right] \times v\left(\vec{p} - e\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right) f_o(\vec{p}) + \int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt \sum_p \left[\varepsilon\left(\vec{p} - e\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right) - \mu\right] \times \left\{[\varepsilon(\vec{p})-\mu]\frac{\nabla T}{T} + \nabla\mu\right\} v\left(\vec{p} - e\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\cdot\frac{\partial f_o(\vec{p})}{\partial\varepsilon} v(p) \quad (G.4)$$

Resolving the thermal current densities along the $\vec{S}^*$ and $\vec{Z}^*$ components:

$$\vec{Z}^* = \tau^{-1}\int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt \sum_p \left[\varepsilon\left(\vec{p} - e\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_z\cos\omega t']dt'\right) - \mu\right] \times \vec{v}_z\left(\vec{p} - e\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_z\cos\omega t']dt'\right) f_o(\vec{p})$$

$$+\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\sum_p\left[\varepsilon\left(\vec{p}-e\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_z\cos\omega t']dt'\right)-\mu\right]$$
$$\times\left\{[\varepsilon(\vec{p})-\mu]\frac{\nabla_z T}{T}+\nabla_z\mu\right\}\vec{v}_z\left(\vec{p}-e\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_z\cos\omega t']dt'\right)\cdot\frac{\partial f_o(\vec{p})}{\partial\varepsilon}\vec{v}_z(\vec{p}) \quad (G.5)$$

and

$$\vec{S}^* = \tau^{-1}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\sum_p\left[\varepsilon\left(\vec{p}-e\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)-\mu\right]$$
$$\times\vec{v}_s\left(\vec{p}-e\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)f_o(\vec{p})$$
$$+\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\sum_p\left[\varepsilon\left(\vec{p}-e\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)-\mu\right]$$
$$\times\left\{[\varepsilon(\vec{p})-\mu]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}\vec{v}_s\left(\vec{p}-e\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\cdot\frac{\partial f_o(\vec{p})}{\partial\varepsilon}\vec{v}_s(\vec{p}) \quad (G.6)$$

Making the transformation:

$$\sum_p \rightarrow \frac{2}{(2\pi\hbar)^2}\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z$$

$$\vec{Z}^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z f_o(p)$$
$$\times\left[\varepsilon\left(\vec{p}-e\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_z\cos\omega t']dt'\right)-\mu\right]\vec{v}_z\left(\vec{p}-e\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_z\cos\omega t']dt'\right)$$
$$+\frac{2}{(2\pi\hbar)^2}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z$$
$$\times\left[\varepsilon\left(\vec{p}-e\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_z\cos\omega t']dt'\right)-\mu\right]\left\{[\varepsilon(\vec{p})-\mu]\frac{\nabla_z T}{T}+\nabla_z\mu\right\}$$
$$\times\vec{v}_z\left(\vec{p}-e\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_z\cos\omega t']dt'\right)\cdot\frac{\partial f_o(\vec{p})}{\partial\varepsilon}\vec{v}_z(\vec{p}) \quad (G.7)$$

$$\vec{S}^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z$$
$$\left[\varepsilon\left(p-e\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)-\mu\right]\vec{v}_s\left(\vec{p}-e\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)f_o(\vec{p})$$
$$+\frac{2}{(2\pi\hbar)^2}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z$$
$$\times\left[\varepsilon\left(\vec{p}-e\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)-\mu\right]\left\{[\varepsilon(\vec{p})-\mu]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}$$

$$\times \vec{v}_s \left(\vec{p} - e \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right) \cdot \frac{\partial f_o(\vec{p})}{\partial \varepsilon} \vec{v}_s(\vec{p}) \quad \text{(G.8)}$$

where the integration is carried out in the first Brillouin zone. Using the band relation in F.6 and making a transformation yields:

$$\varepsilon \left(\vec{p} - e \int_{t-t'}^{t} [\vec{E}_o + \vec{E} \cos\omega t'] dt' \right) = \varepsilon_o - \Delta_s \cos \frac{ub_s}{\hbar} \left(\vec{p}_s - e \int_{t-t'}^{t} [\vec{E}_o + \vec{E} \cos\omega t'] dt' \right)$$
$$+ \Delta_z \cos \frac{wb_z}{\hbar} \left(\vec{p}_z - e \int_{t-t'}^{t} [\vec{E}_o - \vec{E} \cos\omega t'] dt' \right) \quad \text{(G.10)}$$

$$\varepsilon \left(\vec{p} - e \int_{t-t'}^{t} [\vec{E}_o + \vec{E} \cos\omega t'] dt' \right) =$$
$$\varepsilon_o - \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E} \cos\omega t'] dt' \right)$$
$$- \Delta_s \sin \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E} \cos\omega t'] dt' \right)$$
$$- \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \cos \left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E} \cos\omega t'] dt' \right)$$
$$- \Delta_z \sin \frac{w\vec{p}_z b_z}{\hbar} \sin \left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E} \cos\omega t'] dt' \right) \quad \text{(G.11)}$$

The electron miniband velocity along the $s-$coordinate is given as:

$$\vec{v}_s(\vec{p}) = \frac{\partial \varepsilon(p)}{\partial \vec{p}_s} = \frac{u\Delta_s b_s}{\hbar} \sin \frac{u\vec{p}_s b_s}{\hbar} \quad \text{(G.12)}$$

$$\vec{v}_s \left(\vec{p}_s - e \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right) =$$
$$\frac{u\Delta_s b_s}{\hbar} \sin \left(\vec{p}_s - e \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right) \quad \text{(G.13)}$$

$$\frac{u\Delta_s b_s}{\hbar} \sin \left(\vec{p}_s - e \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right) =$$
$$\frac{u\Delta_s b_s}{\hbar} \left\{ \sin \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right) \right.$$
$$\left. - \cos \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right) \right\} \quad \text{(G.14)}$$

$$\vec{S}^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2} \frac{u\Delta_s b_s}{\hbar} \int_0^{-\infty} \exp\left(\frac{-t}{\tau} \right) dt \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z$$
$$\times \left[\varepsilon_o - \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right) \right.$$

$$
\begin{aligned}
& - \Delta_s \sin \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \\
& - \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \cos \left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \\
& - \Delta_z \sin \frac{w\vec{p}_z b_z}{\hbar} \sin \left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) - \mu \Big] \\
& \times \Big\{ \sin \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \\
& - \cos \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \Big\} f_o(\vec{p}) \\
& + \frac{2}{(2\pi\hbar)^2} \frac{u^2 \Delta_s^2 b_s^2}{\hbar^2} \int_0^{-\infty} \exp\left(\frac{-t}{\tau} \right) dt \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \\
& \Big[\varepsilon_o - \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \\
& - \Delta_s \sin \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \\
& - \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \cos \left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \\
& - \Delta_z \sin \frac{w\vec{p}_z b_z}{\hbar} \sin \left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) - \mu \Big] \\
& \times \Big\{ \sin \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \\
& - \cos \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \Big\} \\
& \times \Big\{ [\varepsilon_o - \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} - \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} - \mu] \frac{\nabla_s T}{T} + \nabla_s \mu \Big\} \\
& \times \frac{\partial f_o(\vec{p})}{\partial \varepsilon} \sin \frac{u\vec{p}_s b_s}{\hbar} \qquad \text{(G.15)}
\end{aligned}
$$

Now G.15 is too long and cumbersome to solve. Splitting G.15 into two yields:

$$
\begin{aligned}
S_1^* = & \frac{2\tau^{-1}}{(2\pi\hbar)^2} \frac{u\Delta_s b_s}{\hbar} \int_0^{-\infty} \exp\left(\frac{-t}{\tau} \right) dt \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \\
& \Big[\varepsilon_o - \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \\
& - \Delta_s \sin \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \\
& - \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \cos \left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \\
& - \Delta_z \sin \frac{w\vec{p}_z b_z}{\hbar} \sin \left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) - \mu \Big] \\
& \times \Big\{ \sin \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)
\end{aligned}
$$

$$
-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\Bigg\} f_o(\vec{p}) \qquad \text{(G.16)}
$$

and

$$
\begin{aligned}
S_2^* = \frac{2}{(2\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z \\
\Bigg[\varepsilon_o-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
-\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
-\Delta_z\sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)-\mu\Bigg] \\
\times\Bigg\{\sin\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
-\cos\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\Bigg\} \\
\times\left\{[\varepsilon_o-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}-\mu]\frac{\nabla_s T}{T}+\nabla_s\mu\right\} \\
\times\frac{\partial f_o(\vec{p})}{\partial\varepsilon(\vec{p})}\sin\frac{u\vec{p}_s b_s}{\hbar} \qquad \text{(G.17)}
\end{aligned}
$$

Solving for G.16 explicitly

$$
\begin{aligned}
S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z \\
\Bigg[\varepsilon_o-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right) \\
-\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
-\Delta_z\sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)-\mu\Bigg] \\
\times\Bigg\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\Bigg\} \\
\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \qquad \text{(G.18)}
\end{aligned}
$$

Expanding the terms

$$
\begin{aligned}
S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\
\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\Bigg[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
-\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
-\Delta_z\sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg] \\
\times\Bigg\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\Bigg\} \quad \text{(G.19)}
\end{aligned}
$$

$$
\begin{aligned}
S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z \\
\times\Bigg[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
\times\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
-\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]
\end{aligned}
$$

$$
\begin{aligned}
&\times \sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&\times \sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&-\Delta_z\sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&\times \sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&+(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&-\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&-\Delta_z\sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\Bigg] \qquad (G.20)
\end{aligned}
$$

Setting the odd functions to zero

$$
\begin{aligned}
S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z \\
\times\Bigg[\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
\times\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
-(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
\times\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\times\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
\times\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\times\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
\times\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
-\Delta_z\sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
\times\exp\left[\Delta_s^*\sin\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
\times\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
+\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\frac{u\vec{p}_s b_s}{\hbar}\sin^2\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
\times\exp\left[\Delta_s^*\sin\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
+\Delta_z\sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
\times\cos\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
\times\exp\left[\Delta_s^*\sin\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg] \qquad \text{(G.21)}
\end{aligned}
$$

$$
\begin{aligned}
S_1^* = & \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z \\
& \times\left[\Delta_s\sin^2\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\right. \\
& \times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
& \times\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right) \\
& -(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
& \times\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right) \\
& -\Delta_s\cos^2\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right) \\
& \times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
& \times\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right) \\
& -\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right) \\
& \times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
& \left.\times\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\right] \qquad \text{(G.22)}
\end{aligned}
$$

$$
\begin{aligned}
S_1^* = & \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z \\
& \times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
& \times\left[-(\varepsilon_o-\mu)\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right. \\
& +\Delta_s\cos^2\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
& \times\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
& +\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
& \times\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
& -\Delta_s\sin^2\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)
\end{aligned}
$$

$$\times \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\Bigg] \quad \text{(G.23)}$$

Now changing the integration path

$$\begin{aligned}
S_1^* = & \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\
& \times\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
& \times\Bigg[-(\varepsilon_o-\mu)\cos\frac{Z_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
& +\Delta_s\cos^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
& \times\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right) \\
& +\Delta_z\cos\frac{Z_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
& \times\cos\frac{Z_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
& -\Delta_s\sin^2\frac{Z_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
& \times\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\Bigg] \quad \text{(G.24)}
\end{aligned}$$

$$\begin{aligned}
S_1^* = & \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\
& \times\frac{1}{ub_s}\int_{-\pi/b_s}^{\pi/b_s}dZ_s\frac{1}{wb_z}\int_{-\pi/b_z}^{\pi/b_z}dZ_z\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right] \\
& \times\Bigg[-(\varepsilon_o-\mu)\cos\frac{Z_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
& +\Delta_s\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\left(\cos^2\frac{Z_s}{\hbar}-\sin^2\frac{Z_s}{\hbar}\right) \\
& \times\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
& +\Delta_z\cos\frac{Z_z}{\hbar}\cos\frac{Z_s}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
& \times\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\Bigg] \quad \text{(G.25)}
\end{aligned}$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$

$$\times \frac{1}{ub_s}\int_{-\pi/b_s}^{\pi/b_s} dZ_s \frac{1}{wb_z}\int_{-\pi/b_z}^{\pi/b_z} dZ_z \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right]$$
$$\times \left[-(\varepsilon_o-\mu)\cos\frac{Z_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right.$$
$$+\frac{\Delta_s}{2}\cos\left(\frac{2ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\cos\frac{2Z_s}{\hbar}$$
$$\times \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$+\frac{\Delta_z}{2}\cos\frac{Z_z}{\hbar}\cos\frac{Z_s}{\hbar}\left[\sin\left\{\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right.\right.\right.$$
$$\left.\left.+\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right\}$$
$$-\sin\left\{\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right.\right.$$
$$\left.\left.\left.\left.-\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right\}\right]\right] \quad (G.26)$$

Solving for the terms in G.26

$$S_{11}^{*} = -\frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \frac{1}{ub_s}\int_{-\pi/b_s}^{\pi/b_s} dZ_s \frac{1}{wb_z}\int_{-\pi/b_z}^{\pi/b_z} dZ_z \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar} + \Delta_z^* \cos\frac{Z_z}{\hbar}\right]$$
$$\times(\varepsilon_o-\mu)\cos\frac{Z_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \quad (G.27)$$

$$S_{12}^{*} = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \frac{1}{ub_s}\int_{-\pi/b_s}^{\pi/b_s} dZ_s \frac{1}{wb_z}\int_{-\pi/b_z}^{\pi/b_z} dZ_z \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$\times\left[\frac{\Delta_s}{2}\sin\left(\frac{2ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right] \quad (G.28)$$

$$S_{13}^{*} = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \frac{1}{ub_s}\int_{-\pi/b_s}^{\pi/b_s} dZ_s \frac{1}{wb_z}\int_{-\pi/b_z}^{\pi/b_z} dZ_z \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right]$$
$$\times\left[\frac{\Delta_z}{2}\cos\frac{Z_z}{\hbar}\cos\frac{Z_s}{\hbar}\left[\sin\left\{\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right.\right.\right.\right.$$
$$\left.\left.\left.\left.+\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right\}\right]\right] \quad (G.29)$$

$$S^*_{14} = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{n_o}{2I_o(\Delta^*_s)I_o(\Delta^*_z)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\frac{1}{ub_s}\int_{-\pi/b_s}^{\pi/b_s}dZ_s\frac{1}{wb_z}\int_{-\pi/b_z}^{\pi/b_z}dZ_z\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]$$
$$\times\left[\frac{\Delta_z}{2}\cos\frac{Z_z}{\hbar}\cos\frac{Z_s}{\hbar}\left[\sin\left\{\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right.\right.\right.$$
$$\left.\left.\left.-\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right\}\right]\right] \quad (G.30)$$

From G.27

$$S^*_{11} = -\frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{n_o}{2I_o(\Delta^*_s)I_o(\Delta^*_z)}(\varepsilon_o-\mu)$$
$$\times\frac{2}{\pi}\int_0^{\pi}dZ_s\left[\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\right]\frac{2}{\pi}\int_0^{\pi}d\vec{p}_z\exp\left[\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$\times\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\cos\frac{Z_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \quad (G.31)$$

$$\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$=\sum_{n=-\infty}J_n^2(\chi)\left[\frac{(eb_s\vec{E}_o/\hbar+n\omega\hbar)\tau^2}{1+(eb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right] \quad (G.32)$$

For weak electric field

$$\sum_{n=-\infty}J_n^2(\chi)\left[\frac{(eb_s\vec{E}_o/\hbar+n\omega\hbar)\tau^2}{1+(eb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]$$
$$=\sum_{n=-\infty}J_n^2(\chi)[(eb_s\vec{E}_o+n\omega\hbar)\tau^2(1-0(eb_s\vec{E}_o+n\omega\hbar)^2)]$$
$$=\sum_{n=-\infty}J_n^2(\chi)[(eb_s\vec{E}_o+n\omega\hbar)\tau^2)]$$

$$I_1(\Delta^*_s)=\frac{1}{\pi}\int_0^{\pi}\frac{dZ_s}{\hbar}\cos\frac{Z_s}{\hbar}\exp\left(\Delta_s\cos\frac{Z_s}{\hbar}\right)$$
$$I_o(\Delta^*_s)=\frac{1}{\pi}\int_0^{\pi}\frac{dZ_s}{\hbar}\exp\left(\Delta_s\cos\frac{Z_s}{\hbar}\right)$$

$$S^*_{11} = -\frac{\tau^{-1}}{(\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{\hbar^2 n_o}{I_o(\Delta^*_s)I_o(\Delta^*_z)}(\varepsilon_o-\mu)I_1(\Delta^*_s)I_o(\Delta^*_z)$$
$$\times\sum_{n=-\infty}J_n^2(\chi)\left(\frac{eb_s\vec{E}_o}{\hbar}+n\omega\hbar\right)\tau^2 \quad (G.33)$$

$$S_{11}^{*} = -\frac{\tau u \Delta_s b_s}{\hbar}(\varepsilon_o - \mu) \sum_{n=-\infty} J_n^2(\chi) \left(\frac{e b_s \vec{E}_o}{\hbar} + n\omega\hbar \right) \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} \quad \text{(G.34)}$$

$$S_{12}^{*} = \frac{2\tau^{-1}}{(2\pi\hbar)^2} \frac{u\Delta_s b_s}{\hbar} \frac{n_o}{2I_o(\Delta_s^*)I_o(\Delta_z^*)} \frac{\Delta_s}{2} \int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt \frac{2}{\pi} \int_0^{\pi} dZ_s \frac{2}{\pi} \int_0^{\pi} dZ_z$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \left[\sin\left(\frac{2ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right] \quad \text{(G.35)}$$

$$S_{12}^{*} = \frac{\tau^{-1}}{(2\pi\hbar)^2} \frac{u\Delta_s b_s}{\hbar} \frac{n_o}{I_o(\Delta_s^*)I_o(\Delta_z^*)} \frac{\Delta_s}{2}$$
$$\int_0^{\pi} dZ_s \cos\frac{2Z_s}{\hbar} \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right]$$
$$\int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt \left[\sin\left(\frac{2ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right] \quad \text{(G.36)}$$

$$\int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt \sin\left(\frac{2ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right) =$$
$$\sum_{n=-\infty} J_n^2(\chi) \left[\frac{2(eb_s\vec{E}_o/\hbar + n\omega\hbar)\tau^2}{1 + 4(eb_s\vec{E}_o/\hbar + n\omega\hbar)^2\tau^2}\right]$$

For weak electric field

$$\sum_{n=-\infty} J_n^2(\chi) \left[\frac{2(eb_s\vec{E}_o/\hbar + n\omega\hbar)\tau^2}{1 + 4(eb_s\vec{E}_o/\hbar + n\omega\hbar)^2\tau^2}\right]$$
$$= \sum_{n=-\infty} J_n^2(\chi)[2(eb_s\vec{E}_o + n\omega\hbar)\tau^2(1 - 0(eb_s\vec{E}_o + n\omega\hbar)^2)]$$
$$= \sum_{n=-\infty} J_n^2(\chi)[2(eb_s\vec{E}_o + n\omega\hbar)\tau^2)]$$

$$S_{12}^{*} = \frac{\tau}{(\pi\hbar)^2} \frac{u\Delta_s b_s}{\hbar} \frac{n_o}{I_o(\Delta_s^*)I_o(\Delta_z^*)} \frac{\Delta_s}{2}$$
$$\times \sum_{n=-\infty} J_n^2(\chi) \left[2\left(\frac{eb_s\vec{E}_o}{\hbar} + n\omega\hbar\right)\right] I_2(\Delta_s^*)I_o(\Delta_z^*) \quad \text{(G.37)}$$

but

$$\frac{I_2(\Delta_s^*)}{I_o(\Delta_s^*)} = \frac{\left[I_o(\Delta_s^*) - \frac{2}{\Delta_s}I_1(\Delta_s^*)\right]}{I_o(\Delta_s^*)} = 1 - \frac{2}{\Delta_s}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} \quad \text{(G.38)}$$

$$S_{12}^{*} = \frac{\tau n_o u \Delta_s b_s}{\hbar} \frac{\Delta_s}{2} \sum_{n=-\infty} J_n^2(\chi) \left[2\left(\frac{eb_s\vec{E}_o}{\hbar} + n\omega\hbar\right)\right] \left(1 - \frac{2}{\Delta_s}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right) \quad \text{(G.39)}$$

$$S^*_{12} = \frac{\tau n_o u\Delta_s^2 b_s}{\hbar}\sum_{n=-\infty} J_n^2(\chi)\left[\left(\frac{eb_s\vec{E}_o}{\hbar}+n\omega\hbar\right)\right]\left(1-\frac{2}{\Delta_s}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right) \quad (G.40)$$

$$S^*_{13} = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\frac{\Delta_z}{2}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\frac{1}{ub_s}\int_{-\pi/b_s}^{\pi/b_s}dZ_s\frac{1}{wb_z}\int_{-\pi/b_z}^{\pi/b_z}dZ_z\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\cos\frac{Z_z}{\hbar}\cos\frac{Z_s}{\hbar}$$
$$\times\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'+\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \quad (G.41)$$

$$S^*_{13} = \frac{2\tau^{-1}}{(\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{n_o}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\frac{\Delta_z}{2}$$
$$\times\int_0^{\pi}dZ_s\cos\frac{Z_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{Z_s}{\hbar}\right]\int_0^{\pi}dZ_z\cos\frac{Z_z}{\hbar}\exp\left[\Delta_z^*\cos\frac{Z_z}{\hbar}\right]$$
$$\times\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\sin\left\{\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right.\right.$$
$$\left.\left.+\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right\} \quad (G.42)$$

$$S^*_{13} = \frac{2\tau^{-1}}{(\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{\hbar^2 n_o}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\frac{\Delta_z}{2}I_1(\Delta_s^*)I_1(\Delta_z)$$
$$\times\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\sin\left\{\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right.\right.$$
$$\left.\left.+\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right\} \quad (G.43)$$

$$\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\sin\left\{\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right.\right.$$
$$\left.\left.+\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right\}=$$
$$\sum_{n=-\infty}J_n^2(\chi)\left[\frac{[(eb_z\vec{E}_o/\hbar+n\omega\hbar)+(eb_s\vec{E}_o/\hbar+n\omega\hbar)]\tau^2}{1+[(eb_z\vec{E}_o/\hbar+n\omega\hbar)+(eb_s\vec{E}_o/\hbar+n\omega\hbar)]^2\tau^2}\right]$$

For weak electric field $(eb_s\vec{E}_o/\hbar+n\omega\hbar)^2 \ll 1$

$$\sum_{n=-\infty}J_n^2(\chi)\left[\frac{[(eb_z\vec{E}_o/\hbar+n\omega\hbar)+(eb_s\vec{E}_o/\hbar+n\omega\hbar)]\tau^2}{1+[(eb_z\vec{E}_o/\hbar+n\omega\hbar)+(eb_s\vec{E}_o/\hbar+n\omega\hbar)]^2\tau^2}\right]=$$
$$\sum_{n=-\infty}J_n^2(\chi)\left[\left(\frac{eb_z\vec{E}_o}{\hbar}+n\omega\hbar\right)+\left(\frac{eb_s\vec{E}_o}{\hbar}+n\omega\hbar\right)\right]\tau^2$$

$$\times\left[\left(\frac{eb_z\vec{E}_o}{\hbar}+n\omega\hbar\right)+\left(\frac{eb_s\vec{E}_o}{\hbar}+n\omega\hbar\right)\right]^2$$

$$=\sum_{n=-\infty} J_n^2(\chi)\left[\left(\frac{eb_z\vec{E}_o}{\hbar}+n\omega\hbar\right)+\left(\frac{eb_s\vec{E}_o}{\hbar}+n\omega\hbar\right)\right]\tau^2$$

$$S_{13}^{*}=\frac{\tau u\Delta_s b_s n_o}{\hbar}\frac{\Delta_z}{2}\frac{I_1(\Delta_s^*)I_1(\Delta_z)}{I_o(\Delta_s^*)I_o(\Delta_z^*)}\times\sum_{n=-\infty} J_n^2(\chi)\left[\left(\frac{eb_z\vec{E}_o}{\hbar}+n\omega\hbar\right)+\left(\frac{eb_s\vec{E}_o}{\hbar}+n\omega\hbar\right)\right] \tag{G.44}$$

$$S_{14}^{*}=-\frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\frac{\Delta_z}{2}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\frac{1}{ub_s}\int_{-\pi/b_s}^{\pi/b_s}dZ_s\frac{1}{wb_z}\int_{-\pi/b_z}^{\pi/b_z}dZ_z\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\cos\frac{Z_z}{\hbar}\cos\frac{Z_s}{\hbar}$$
$$\times\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'+\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \tag{G.45}$$

$$S_{14}^{*}=-\frac{2\tau^{-1}}{(\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{n_o}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\frac{\Delta_z}{2}$$
$$\times\int_0^{\pi}dZ_s\cos\frac{Z_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{Z_s}{\hbar}\right]\int_0^{\pi}dZ_z\cos\frac{Z_z}{\hbar}\exp\left[\Delta_z^*\cos\frac{Z_z}{\hbar}\right]$$
$$\times\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\sin\left\{\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right.\right.$$
$$\left.\left.-\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right\} \tag{G.46}$$

$$S_{14}^{*}=-\frac{2\tau^{-1}}{(\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{\hbar^2 n_o}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\frac{\Delta_z}{2}I_1(\Delta_s^*)I_1(\Delta_z^*)$$
$$\times\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\sin\left\{\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right.\right.$$
$$\left.\left.-\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right\} \tag{G.47}$$

$$\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\sin\left\{\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}\left[\vec{E}_o+\vec{E}_s\cos\omega t'\right]dt'\right.\right.$$
$$\left.\left.+\frac{ueb_s}{\hbar}\int_{t-t'}^{t}\left[\vec{E}_o+\vec{E}_s\cos\omega t'\right]dt'\right)\right\}=$$

$$\sum_{n=-\infty} J_n^2(\chi)\left[\frac{[(eb_z\vec{E}_o/\hbar+n\omega\hbar)+(eb_s\vec{E}_o/\hbar+n\omega\hbar)]\tau^2}{1+[(eb_z\vec{E}_o/\hbar+n\omega\hbar)+(eb_s\vec{E}_o/\hbar+n\omega\hbar)]^2\tau^2}\right]$$

For weak electric field $(eb_s\vec{E}_o/\hbar+n\omega\hbar)^2 \ll 1$

$$\sum_{n=-\infty} J_n^2(\chi)\left[\frac{[(eb_z\vec{E}_o/\hbar+n\omega\hbar)-(eb_s\vec{E}_o/\hbar+n\omega\hbar)]\tau^2}{1+[(eb_z\vec{E}_o/\hbar+n\omega\hbar)-(eb_s\vec{E}_o/\hbar+n\omega\hbar)]^2\tau^2}\right]=$$

$$\sum_{n=-\infty} J_n^2(\chi)\left[\left(\frac{eb_z\vec{E}_o}{\hbar}+n\omega\hbar\right)-\left(\frac{eb_s\vec{E}_o}{\hbar}+n\omega\hbar\right)\right]\tau^2 \times\left[\left(\frac{eb_z\vec{E}_o}{\hbar}+n\omega\hbar\right)-\left(\frac{eb_s\vec{E}_o}{\hbar}+n\omega\hbar\right)\right]^2$$

$$=\sum_{n=-\infty} J_n^2(\chi)\left[\left(\frac{eb_z\vec{E}_o}{\hbar}+n\omega\hbar\right)-\left(\frac{eb_s\vec{E}_o}{\hbar}+n\omega\hbar\right)\right]\tau^2$$

$$S_{14}^* = -\frac{\tau u\Delta_s b_s n_o}{\hbar}\frac{\Delta_z}{2}\frac{I_1(\Delta_s^*)I_1(\Delta_z)}{I_o(\Delta_s^*)I_o(\Delta_z^*)} \times\sum_{n=-\infty} J_n^2(\chi)\left[\left(\frac{eb_z\vec{E}_o}{\hbar}+n\omega\hbar\right)-\left(\frac{eb_s\vec{E}_o}{\hbar}+n\omega\hbar\right)\right] \quad \text{(G.48)}$$

Summing up the terms

$$S_1^* = -\frac{\tau u\Delta_s b_s n_o}{\hbar}(\varepsilon_o-\mu)\sum_{n=-\infty} J_n^2(\chi)\left(\frac{eb_s\vec{E}_o}{\hbar}+n\omega\hbar\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}$$
$$+\frac{\tau n_o u\Delta_s^2 b_s}{\hbar}\sum_{n=-\infty} J_n^2(\chi)\left[\left(\frac{eb_s\vec{E}_o}{\hbar}+n\omega\hbar\right)\right]\left(1-\frac{2}{\Delta_s}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right)$$
$$+\frac{\tau u\Delta_s\Delta_z b_s n_o}{2\hbar}\sum_{n=-\infty} J_n^2(\chi)\left[\left(\frac{eb_z\vec{E}_o}{\hbar}+n\omega\hbar\right)+\left(\frac{eb_s\vec{E}_o}{\hbar}+n\omega\hbar\right)\right]\frac{I_1(\Delta_s^*)I_1(\Delta_z)}{I_o(\Delta_s^*)I_o(\Delta_z^*)}$$
$$-\frac{\tau u\Delta_s\Delta_z b_s n_o}{2\hbar}\sum_{n=-\infty} J_n^2(\chi)\left[\left(\frac{eb_z\vec{E}_o}{\hbar}+n\omega\hbar\right)-\left(\frac{eb_s\vec{E}_o}{\hbar}+n\omega\hbar\right)\right]\frac{I_1(\Delta_s^*)I_1(\Delta_z)}{I_o(\Delta_s^*)I_o(\Delta_z^*)} \quad \text{(G.49)}$$

$$S_1^* = -\frac{\tau u\Delta_s b_s n_o}{\hbar}(\varepsilon_o-\mu)\sum_{n=-\infty} J_n^2(\chi)\left(\frac{eb_s\vec{E}_o}{\hbar}+n\omega\hbar\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}$$
$$+\frac{\tau n_o u\Delta_s^2 b_s}{\hbar}\sum_{n=-\infty} J_n^2(\chi)\left[\left(\frac{eb_s\vec{E}_o}{\hbar}+n\omega\hbar\right)\right]\left(1-\frac{2}{\Delta_s}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right)$$
$$+\frac{\tau u\Delta_s\Delta_z b_s n_o}{2\hbar}\sum_{n=-\infty} J_n^2(\chi)\left[\left(\frac{eb_z\vec{E}_o}{\hbar}+n\omega\hbar\right)+\left(\frac{eb_s\vec{E}_o}{\hbar}+n\omega\hbar\right)\right.$$

$$-\left(\frac{eb_z\vec{E}_o}{\hbar}+n\omega\hbar\right)-\left(\frac{eb_s\vec{E}_o}{\hbar}+n\omega\hbar\right)\Bigg]\frac{I_1(\Delta_s^*)I_1(\Delta_z)}{I_o(\Delta_s^*)I_o(\Delta_z^*)} \quad (G.50)$$

$$S_1^*=-\frac{\tau u\Delta_s b_s n_o}{\hbar}(\varepsilon_o-\mu)\sum_{n=-\infty}J_n^2(\chi)\left(\frac{eb_s\vec{E}_o}{\hbar}+n\omega\hbar\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}$$
$$+\frac{\tau n_o u\Delta_s^2 b_s}{\hbar}\sum_{n=-\infty}J_n^2(\chi)\left[\left(\frac{eb_s\vec{E}_o}{\hbar}+n\omega\hbar\right)\right]\left(1-\frac{2}{\Delta_s}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right)$$
$$+\frac{\tau u\Delta_s\Delta_z b_s n_o}{2\hbar}\sum_{n=-\infty}J_n^2(\chi)\left[\left(\frac{eb_s\vec{E}_o}{\hbar}+n\omega\hbar\right)+\left(\frac{eb_s\vec{E}_o}{\hbar}+n\omega\hbar\right)\right]\frac{I_1(\Delta_s^*)I_1(\Delta_z)}{I_o(\Delta_s^*)I_o(\Delta_z^*)} \quad (G.51)$$

$$S_1^*=-\frac{\tau u\Delta_s b_s n_o}{\hbar}(\varepsilon_o-\mu)\sum_{n=-\infty}J_n^2(\chi)\left(\frac{eb_s\vec{E}_o}{\hbar}+n\omega\hbar\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}$$
$$+\frac{\tau n_o u\Delta_s^2 b_s}{\hbar}\sum_{n=-\infty}J_n^2(\chi)\left[\left(\frac{eb_s\vec{E}_o}{\hbar}+n\omega\hbar\right)\right]\left(1-\frac{2}{\Delta_s}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right)$$
$$+\frac{\tau u\Delta_s\Delta_z b_s n_o}{2\hbar}2\sum_{n=-\infty}J_n^2(\chi)\left[\left(\frac{eb_s\vec{E}_o}{\hbar}+n\omega\hbar\right)\right]\frac{I_1(\Delta_s^*)I_1(\Delta_z)}{I_o(\Delta_s^*)I_o(\Delta_z^*)} \quad (G.52)$$

$$S_1^*=-\frac{\tau u\Delta_s b_s n_o}{\hbar}(\varepsilon_o-\mu)\sum_{n=-\infty}J_n^2(\chi)\left(\frac{eb_s\vec{E}_o}{\hbar}+n\omega\hbar\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}$$
$$+\frac{\tau n_o u\Delta_s^2 b_s}{\hbar}\sum_{n=-\infty}J_n^2(\chi)\left[\left(\frac{eb_s\vec{E}_o}{\hbar}+n\omega\hbar\right)\right]\left(1-\frac{2}{\Delta_s}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right)$$
$$+\frac{\tau u\Delta_s\Delta_z b_s n_o}{\hbar}\sum_{n=-\infty}J_n^2(\chi)\left[\left(\frac{eb_s\vec{E}_o}{\hbar}+n\omega\hbar\right)\right]\frac{I_1(\Delta_s^*)I_1(\Delta_z)}{I_o(\Delta_s^*)I_o(\Delta_z^*)} \quad (G.53)$$

$$S_1^*=\frac{\tau u\Delta_s b_s n_o}{\hbar}\sum_{n=-\infty}J_n^2(\chi)\left(\frac{eb_s\vec{E}_o}{\hbar}+n\omega\hbar\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}$$
$$\times\left\{-(\varepsilon_o-\mu)+\Delta_s\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s}+\Delta_z\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right)\right\} \quad (G.54)$$

$$S_1^*=\frac{e\tau u\Delta_s b_s^2 n_o}{\hbar^2}\sum_{n=-\infty}J_n^2(\chi)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\left(\vec{E}_o+\frac{n\omega\hbar}{eb_s}\right)$$
$$\times\left\{-(\varepsilon_o-\mu)+\Delta_s\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s}\right)+\Delta_z\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right\} \quad (G.55)$$

$$\sigma_s(\vec{E}) = \frac{e^2\tau u\Delta_s b_s^2 n_o}{\hbar^2}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} \tag{G.56}$$

$$S_1^* = -\sigma_s(\vec{E})\sum_{n=-\infty} J_n^2(\chi) \times \frac{1}{e}\left\{-(\varepsilon_o-\mu)+\Delta_s\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s}\right)+\Delta_z\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right\}\left(\vec{E}_o+\frac{n\omega\hbar}{eb_s}\right) \tag{G.57}$$

where $\vec{E}_n = \left(\vec{E}_o + n\omega\hbar/eb_s\right)$. Thus,

$$S_1^* = -\sigma_s(\vec{E})\sum_{n=-\infty} J_n^2(\chi) \times \frac{1}{e}\left\{-(\varepsilon_o-\mu)+\Delta_s\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s}\right)+\Delta_z\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right\}\vec{E}_n \tag{G.58}$$

Solving for G.17 explicitly

$$\begin{aligned} S_2^* = \frac{2}{(2\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s \\ \times\Bigg[\varepsilon_o-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\ -\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\ -\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\ -\Delta_z\sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)-\mu\Bigg] \\ \times\left\{\left[\varepsilon_o-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\} \\ \times\frac{\partial f_o}{\partial\varepsilon(p)}\sin\frac{u\vec{p}_s b_s}{\hbar}\Bigg\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\ -\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\Bigg\} \end{aligned} \tag{G.59}$$

$$\frac{\partial f_o(\vec{p})}{\partial\varepsilon(\vec{p})} = -\frac{n_o uwb_s b_z}{2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \tag{G.60}$$

Substituting G.60 into G.59 yields:

$$\begin{aligned} S_2^* = -\frac{2}{(2\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z \\ \int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\left[\varepsilon_o-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right. \\ -\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \end{aligned}$$

$$
\begin{aligned}
&- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
&- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) - \mu\Bigg] \\
&\times\left\{\left[\varepsilon_o - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\} \\
&\times \sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&\Bigg\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
&- \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\Bigg\}
\end{aligned}
\tag{G.61}
$$

Rearranging the terms

$$
\begin{aligned}
S_2^* = &-\frac{2}{(2\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_sb_z}{2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\
&\times\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&\times\Bigg[(\varepsilon_o-\mu) - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right) \\
&- \Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
&- \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
&- \Delta_z\sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\Bigg] \\
&\times\left\{\left[(\varepsilon_o-\mu) - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\} \\
&\times\sin\frac{u\vec{p}_s b_s}{\hbar}\Bigg\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
&- \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\Bigg\}
\end{aligned}
\tag{G.62}
$$

$$
\begin{aligned}
S_2^* = &-\frac{2}{(2\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_sb_z}{2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\
&\times\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&\times\Bigg[(\varepsilon_o-\mu)^2\frac{\nabla_s T}{T} - (\varepsilon_o-\mu)\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\frac{\nabla_s T}{T} \\
&- (\varepsilon_o-\mu)\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\frac{\nabla_s T}{T} + (\varepsilon_o-\mu)\nabla_s\mu
\end{aligned}
$$

$$
\begin{aligned}
&-(\varepsilon_o-\mu)\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\frac{\nabla_s T}{T}\\
&+\Delta_s^2\cos^2\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\frac{\nabla_s T}{T}\\
&+\Delta_s\Delta_z\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\frac{\nabla_s T}{T}\\
&-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\nabla_s\mu\\
&-(\varepsilon_o-\mu)\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\frac{\nabla_s T}{T}\\
&+\Delta_s^2\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\frac{\nabla_s T}{T}\\
&+\Delta_s\Delta_z\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\frac{\nabla_s T}{T}\\
&-\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\nabla_s\mu\\
&-(\varepsilon_o-\mu)\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\frac{\nabla_s T}{T}\\
&+\Delta_z^2\cos^2\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\frac{\nabla_s T}{T}\\
&+\Delta_s\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\frac{\nabla_s T}{T}\\
&-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\nabla_s\mu\\
&-(\varepsilon_o-\mu)\Delta_z\sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\frac{\nabla_s T}{T}\\
&+\Delta_z^2\sin\frac{w\vec{p}_z b_z}{\hbar}\cos\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\frac{\nabla_s T}{T}\\
&+\Delta_s\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\frac{\nabla_s T}{T}\\
&-\Delta_z\sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\nabla_s\mu\Bigg]\\
&\times\sin\frac{u\vec{p}_s b_s}{\hbar}\Bigg\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\qquad-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\Bigg\} \qquad (G.63)
\end{aligned}
$$

Setting the odd functions to zero yields

$$
\begin{aligned}
S_2^* = &-\frac{2}{(2\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\\
&\times\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]
\end{aligned}
$$

$$
\begin{aligned}
&\times \Bigg[(\varepsilon_o-\mu)^2\frac{\nabla_s T}{T} - (\varepsilon_o-\mu)\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\frac{\nabla_s T}{T} \\
&- (\varepsilon_o-\mu)\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\frac{\nabla_s T}{T} + (\varepsilon_o-\mu)\nabla_s\mu \\
&- (\varepsilon_o-\mu)\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \\
&+ \Delta_s^2\cos^2\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \\
&+ \Delta_s\Delta_z\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \\
&- \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\nabla_s\mu \\
&- (\varepsilon_o-\mu)\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \\
&+ \Delta_z^2\cos^2\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \\
&+ \Delta_s\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \\
&- \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\nabla_s\mu\Bigg] \\
&\times\sin^2\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \qquad \text{(G.64)}
\end{aligned}
$$

Changing the path of integration

$$
\begin{aligned}
S_2^* = &-\frac{2}{(2\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\
&\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right] \\
&\times\Bigg[(\varepsilon_o-\mu)^2\frac{\nabla_s T}{T} - (\varepsilon_o-\mu)\Delta_s\cos\frac{Z_s}{\hbar}\frac{\nabla_s T}{T} \\
&- (\varepsilon_o-\mu)\Delta_z\cos\frac{Z_z}{\hbar}\frac{\nabla_s T}{T} + (\varepsilon_o-\mu)\nabla_s\mu \\
&- (\varepsilon_o-\mu)\Delta_s\cos\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \\
&+ \Delta_s^2\cos^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \\
&+ \Delta_s\Delta_z\cos\frac{Z_s}{\hbar}\cos\frac{Z_z}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \\
&- \Delta_s\cos\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\nabla_s\mu \\
&- (\varepsilon_o-\mu)\Delta_z\cos\frac{Z_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\frac{\nabla_s T}{T}
\end{aligned}
$$

$$+ \Delta_z^2 \cos^2 \frac{Z_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\frac{\nabla_s T}{T}$$
$$+ \Delta_s \Delta_z \cos\frac{Z_z}{\hbar}\cos\frac{Z_s}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\frac{\nabla_s T}{T}$$
$$- \Delta_z \cos\frac{Z_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\nabla_s \mu\Bigg]$$
$$\times \sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \quad \text{(G.65)}$$

Solving for the terms in G.65

$$V_1 = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right]$$
$$\times (\varepsilon_o - \mu)^2 \sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad \text{(G.66)}$$

$$V_2 = +\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right](\varepsilon_o - \mu)\Delta_s \cos\frac{Z_s}{\hbar}$$
$$\times \sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad \text{(G.67)}$$

$$V_3 = +\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right](\varepsilon_o - \mu)\Delta_z \cos\frac{Z_z}{\hbar}$$
$$\times \sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad \text{(G.68)}$$

$$V_4 = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right](\varepsilon_o - \mu)$$
$$\times \sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad \text{(G.69)}$$

$$V_5 = +\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$

$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] (\varepsilon_o - \mu)\Delta_s \cos\frac{Z_s}{\hbar}$$

$$\times \sin^2\frac{Z_s}{\hbar} \cos^2\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad \text{(G.70)}$$

$$V_6 = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$

$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] \Delta_s^2 \cos^2\frac{Z_s}{\hbar}$$

$$\times \sin^2\frac{Z_s}{\hbar} \cos^2\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad \text{(G.71)}$$

$$V_7 = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$

$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] \Delta_s\Delta_z \cos\frac{Z_s}{\hbar}\cos\frac{Z_z}{\hbar}$$

$$\times \sin^2\frac{Z_s}{\hbar} \cos^2\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad \text{(G.72)}$$

$$V_8 = +\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$

$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] \Delta_s \cos\frac{Z_s}{\hbar}$$

$$\times \sin^2\frac{Z_s}{\hbar} \cos^2\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E} \cos\omega t']dt'\right)\nabla_s \mu \quad \text{(G.73)}$$

$$V_9 = +\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$

$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] (\varepsilon_o - \mu)\Delta_z \cos\frac{Z_z}{\hbar}$$

$$\times \sin^2\frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$

$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad \text{(G.74)}$$

$$V_{10} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$

$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] \Delta_z\Delta_s \cos\frac{Z_s}{\hbar}\cos\frac{Z_z}{\hbar}$$

$$\times \sin^2\frac{Z_s}{\hbar}\cos\left(\frac{uebs_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad \text{(G.75)}$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad \text{(G.76)}$$

$$V_{12} = +\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z\cos\frac{Z_z}{\hbar}$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\nabla_s\mu \quad \text{(G.77)}$$

Evaluating the terms

$$V_1 = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]$$
$$\times(\varepsilon_o-\mu)^2\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad \text{(G.78)}$$

$$\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) =$$
$$\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau^2}{1+[(eb_s\vec{E}_o/\hbar+n\omega\hbar)]^2\tau^2}\right]$$

For weak electric field $(eb_s\vec{E}_o/\hbar+n\omega\hbar)^2 \ll 1$. Thus,

$$\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau^2}{1+[(eb_s\vec{E}_o/\hbar+n\omega\hbar)]^2\tau^2}\right] =$$

$$\sum_{n=-\infty} J_n^2(\chi)[\tau(1-0[(eb_s\vec{E}_o/\hbar+n\omega\hbar)]^2)] = \sum_{n=-\infty} J_n^2(\chi)\tau$$

$$V_1 = -\frac{\tau}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}(\varepsilon_o-\mu)^2\sum_{n=-\infty} J_n^2(\chi)$$
$$\times\int_0^\pi dZ_z\left[\Delta_z\cos\frac{Z_z}{\hbar}\right]\int_0^\pi dZ_s\sin^2\frac{Z_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{Z_s}{\hbar}\right]\frac{\nabla_sT}{T} \quad \text{(G.79)}$$

$$V_1 = -\frac{\tau}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}(\varepsilon_o-\mu)^2\sum_{n=-\infty} J_n^2(\chi)$$
$$\times\int_0^\pi dZ_z\left[\Delta_z\cos\frac{Z_z}{\hbar}\right]\int_0^\pi dZ_s\left\{\frac{1}{2}\left(1-\cos\frac{2Z_s}{\hbar}\right)\right\}\exp\left[\Delta_s^*\cos\frac{Z_s}{\hbar}\right]\frac{\nabla_sT}{T}$$
$$\text{(G.80)}$$

$$V_1 = -\frac{\tau}{2(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}(\varepsilon_o-\mu)^2\sum_{n=-\infty} J_n^2(\chi)$$
$$\times\left\{\left(\int_0^\pi dZ_z\int_0^\pi dZ_s\exp\left[\Delta_s^*\cos\frac{Z_s}{\hbar}\right]\right.\right.$$
$$\left.\left.-\int_0^\pi dZ_z\exp\left[\Delta_z^*\cos\frac{Z_z}{\hbar}\right]\int_0^\pi dZ_s\cos\frac{2Z_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{Z_s}{\hbar}\right]\right)\right\}\frac{\nabla_sT}{T} \quad \text{(G.81)}$$

$$V_1 = -\frac{\tau}{2(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o\hbar^2}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}(\varepsilon_o-\mu)^2\sum_{n=-\infty} J_n^2(\chi)$$
$$\times[I_o(\Delta_s^*)I_o(\Delta_z^*)-I_2(\Delta_s^*)I_o(\Delta_z^*)]\frac{\nabla_sT}{T} \quad \text{(G.82)}$$

$$V_1 = -\frac{\tau}{2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o}{k_BT}(\varepsilon_o-\mu)^2\sum_{n=-\infty} J_n^2(\chi)\frac{\nabla_sT}{T}\left\{1-\frac{I_2(\Delta_s^*)}{I_o(\Delta_s^*)}\right\} \quad \text{(G.83)}$$

$$V_1 = -\frac{\tau}{2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o}{k_BT}(\varepsilon_o-\mu)^2\sum_{n=-\infty} J_n^2(\chi)\frac{\nabla_sT}{T}\left\{1-\frac{I_o(\Delta_s^*)-\frac{2}{\Delta_s^*}I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right\}$$
$$\text{(G.84)}$$

$$V_1 = -\frac{\tau}{2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o}{k_BT}(\varepsilon_o-\mu)^2\sum_{n=-\infty} J_n^2(\chi)\frac{\nabla_sT}{T}\left\{1-1+\frac{2}{\Delta_s^*}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right\} \quad \text{(G.85)}$$

$$V_1 = -\frac{\tau u^2\Delta_s b_s^2 n_o k}{\hbar^2}\frac{(\varepsilon_o-\mu)^2}{k_BT}\sum_{n=-\infty} J_n^2(\chi)\left\{\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right\}\nabla_sT \quad \text{(G.86)}$$

$$V_2 = +\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_sb_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\frac{1}{uwb_sb_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^\pi dZ_z\int_0^\pi dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right](\varepsilon_o-\mu)\Delta_s\cos\frac{Z_s}{\hbar}$$

$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{uebs_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.87)$$

$$V_2 = +\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o\Delta_s}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}(\varepsilon_o-\mu)\frac{\nabla_s T}{T}$$
$$\times\int_0^{\pi} dZ_z \exp\left[\Delta_z^*\cos\frac{Z_z}{\hbar}\right]\int_0^{\pi} dZ_s\cos\frac{Z_s}{\hbar}\sin^2\frac{Z_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{Z_s}{\hbar}\right]$$
$$\times\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\cos\left(\frac{uebs_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \quad (G.88)$$

Using the identities:

$$\sin^2 x = \frac{1}{2}(1-\cos 2x)$$
$$\cos x\cos y = \frac{1}{2}[\cos(x+y)+\cos(x-y)]$$

$$\cos\frac{Z_s}{\hbar}\sin^2\frac{Z_s}{\hbar} = \frac{1}{2}\cos\frac{Z_s}{\hbar}\left(1-\cos\frac{2Z_s}{\hbar}\right) = \frac{1}{2}\cos\frac{Z_s}{\hbar}-\frac{1}{2}\cos\frac{Z_s}{\hbar}\cos\frac{2Z_s}{\hbar}$$
$$= \frac{1}{2}\cos\frac{Z_s}{\hbar}-\frac{1}{4}\left(\cos\frac{Z_s}{\hbar}+\cos\frac{3Z_s}{\hbar}\right) = \frac{1}{2}\cos\frac{Z_s}{\hbar}-\frac{1}{4}\cos\frac{Z_s}{\hbar}-\frac{1}{4}\cos\frac{3Z_s}{\hbar}$$
$$= \frac{1}{4}\left(\cos\frac{Z_s}{\hbar}-\cos\frac{3Z_s}{\hbar}\right)$$

$$V_2 = +\frac{\tau}{4(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o\Delta_s}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}(\varepsilon_o-\mu)\sum_{n=-\infty} J_n^2(\chi)\frac{\nabla_s T}{T}$$
$$\times\int_0^{\pi} dZ_z \exp\left[\Delta_z^*\cos\frac{Z_z}{\hbar}\right]\int_0^{\pi} dZ_s\left(\cos\frac{Z_s}{\hbar}-\cos\frac{3Z_s}{\hbar}\right)\exp\left[\Delta_s^*\cos\frac{Z_s}{\hbar}\right] \quad (G.89)$$

$$V_2 = +\frac{\tau}{4(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o\Delta_s\hbar^2}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}(\varepsilon_o-\mu)\sum_{n=-\infty} J_n^2(\chi)\frac{\nabla_s T}{T}$$
$$\times[I_1(\Delta_s^*)-I_3(\Delta_s^*)I_o(\Delta_z^*)] \quad (G.90)$$

$$V_2 = +\frac{\tau u^2\Delta_s^2 b_s^2 n_o\Delta_s\hbar^2}{4(\hbar^2)\hbar^2 k_BT}(\varepsilon_o-\mu)\sum_{n=-\infty} J_n^2(\chi)\frac{\nabla_s T}{T}\left[\frac{I_1(\Delta_s^*)-I_3(\Delta_s^*)}{I_o(\Delta_s^*)}\right] \quad (G.91)$$

$$I_3(\Delta_s^*) = I_1(\Delta_s^*)-\frac{4}{\Delta_s^*}I_o(\Delta_s^*)+\frac{8}{\Delta_s^{*2}}I_1(\Delta_s^*)$$

Therefore

$$\left[\frac{I_1(\Delta_s^*)-I_3(\Delta_s^*)}{I_o(\Delta_s^*)}\right] = \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}-\frac{I_1(\Delta_s^*)-\frac{4}{\Delta_s^*}I_o(\Delta_s^*)+\frac{8}{\Delta_s^{*2}}I_1(\Delta_s^*)}{I_1(\Delta_s^*)}$$

$$= \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} - \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} + \frac{4I_o(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)} - \frac{8I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^{*2})} = \frac{4}{\Delta_s^*} - \frac{8I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^{*2})}$$

$$V_2 = +\frac{\tau u^2 \Delta_s^2 b_s^2 n_o \Delta_s}{4\hbar^2 k_B T}(\varepsilon_o - \mu) \sum_{n=-\infty} J_n^2(\chi) \frac{\nabla_s T}{T} \left(\frac{4}{\Delta_s^*} - \frac{8I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^{*2})} \right) \quad (G.92)$$

$$V_2 = +\frac{\tau u^2 \Delta_s^2 b_s^2 n_o k}{\hbar^2} \left(\frac{\varepsilon_o - \mu}{k_B T} \right) \sum_{n=-\infty} J_n^2(\chi) \left(1 - \frac{2I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)} \right) \nabla_s T \quad (G.93)$$

$$V_3 = +\frac{1}{(\pi\hbar)^2} \frac{u^2 \Delta_s^2 b_s^2}{\hbar^2} \frac{n_o u w b_s b_z}{I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \frac{1}{u w b_s b_z} \int_0^{-\infty} \exp\left(\frac{-t}{\tau} \right) dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar} \right] (\varepsilon_o - \mu) \Delta_z \cos\frac{Z_z}{\hbar}$$
$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t'] dt' \right) \frac{\nabla_s T}{T} \quad (G.94)$$

$$V_3 = +\frac{\tau u^2 \Delta_s^2 b_s^2}{(\pi\hbar)^2 \hbar^2} \frac{n_o \Delta_z}{I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} (\varepsilon_o - \mu) \sum_{n=-\infty} J_n^2(\chi) \frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \cos\frac{Z_z}{\hbar} \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar} \right] \int_0^{\pi} dZ_s \sin^2 \frac{Z_s}{\hbar} \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar} \right] \quad (G.95)$$

$$V_3 = +\frac{\tau u^2 \Delta_s^2 b_s^2}{(\pi\hbar)^2 \hbar^2} \frac{n_o \Delta_z}{I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} (\varepsilon_o - \mu) \sum_{n=-\infty} J_n^2(\chi) \frac{\nabla_s T}{T}$$
$$\times \frac{1}{2} \int_0^{\pi} dZ_z \cos\frac{Z_z}{\hbar} \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar} \right] \int_0^{\pi} dZ_s \left(1 - \cos\frac{2Z_s}{\hbar} \right) \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar} \right] \quad (G.96)$$

$$V_3 = +\frac{\tau u^2 \Delta_s^2 b_s^2}{2(\pi\hbar)^2 \hbar^2} \frac{n_o \Delta_z}{I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} (\varepsilon_o - \mu) \sum_{n=-\infty} J_n^2(\chi) \frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \cos\frac{Z_z}{\hbar} \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar} \right] \int_0^{\pi} dZ_s \left(1 - \cos\frac{2Z_s}{\hbar} \right) \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar} \right] \quad (G.97)$$

$$V_3 = +\frac{\tau u^2 \Delta_s^2 b_s^2 n_o \Delta_z}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} (\varepsilon_o - \mu) \sum_{n=-\infty} J_n^2(\chi) \frac{\nabla_s T}{T} [I_o(\Delta_s^*) - I_2(\Delta_s^*) I_1(\Delta_z^*)] \quad (G.98)$$

$$V_3 = +\frac{\tau u^2 \Delta_s^2 b_s^2 n_o \Delta_z}{2(\hbar)^2 \hbar^2 k_B T} (\varepsilon_o - \mu) \sum_{n=-\infty} J_n^2(\chi) \frac{\nabla_s T}{T} \left\{ \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} - \frac{I_2(\Delta_s^*)}{I_o(\Delta_s^*)} \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \right\} \quad (G.99)$$

where the integrals have been expressed in terms of modified Bessel functions. Using the recurrence relations:

$$\left\{\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} - \frac{I_2(\Delta_s^*)}{I_o(\Delta_s^*)}\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right\} = \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1 - \frac{I_2(\Delta_s^*)}{I_o(\Delta_s^*)}\right)$$

$$= \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1 - \frac{I_o(\Delta_s^*) - \frac{2}{\Delta_s^*}I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right) = \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1 - 1 + \frac{2}{\Delta_s^*}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right) = \frac{2}{\Delta_s^*}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}$$

$$V_3 = +\frac{\tau u^2 \Delta_s^2 b_s^2 n_o \Delta_z}{2(\hbar)^2 k_B T}(\varepsilon_o - \mu)\sum_{n=-\infty} J_n^2(\chi)\frac{\nabla_s T}{T}\frac{2}{\Delta_s^*}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \quad \text{(G.100)}$$

$$V_3 = +\frac{\tau u^2 \Delta_s b_s^2 n_o \Delta_z k}{(\hbar)^2}\left(\frac{\varepsilon_o - \mu}{k_B T}\right)\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\nabla_s T \quad \text{(G.101)}$$

$$V_4 = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o u w b_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{u w b_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi} dZ_z\int_0^{\pi} dZ_s \exp\left[\Delta_s\cos\frac{Z_s}{\hbar} + \Delta_z\cos\frac{Z_z}{\hbar}\right](\varepsilon_o - \mu)$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{u e b_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\nabla_s\mu \quad \text{(G.102)}$$

$$V_4 = -\frac{\tau u^2\Delta_s^2 b_s^2}{(\pi\hbar)^2\hbar^2}\frac{n_o}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\sum_{n=-\infty} J_n^2(\chi)\nabla_s\mu$$
$$\times(\varepsilon_o - \mu)\int_0^{\pi} dZ_z\exp\left[\Delta_z^*\cos\frac{Z_z}{\hbar}\right]\int_0^{\pi} dZ_s\sin^2\frac{Z_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{Z_s}{\hbar}\right] \quad \text{(G.103)}$$

$$V_4 = -\frac{\tau u^2\Delta_s^2 b_s^2}{(\pi\hbar)^2\hbar^2}\frac{n_o}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\sum_{n=-\infty} J_n^2(\chi)(\varepsilon_o - \mu)\nabla_s\mu$$
$$\times\int_0^{\pi} dZ_z\exp\left[\Delta_z^*\cos\frac{Z_z}{\hbar}\right]\frac{1}{2}\int_0^{\pi} dZ_s\left(1 - \cos\frac{2Z_s}{\hbar}\right)\exp\left[\Delta_s^*\cos\frac{Z_s}{\hbar}\right] \quad \text{(G.104)}$$

$$V_4 = -\frac{\tau u^2\Delta_s^2 b_s^2}{2\hbar^2}\frac{n_o}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\sum_{n=-\infty} J_n^2(\chi)(\varepsilon_o - \mu)\nabla_s\mu[I_o(\Delta_s^*) - I_2(\Delta_s^*)I_o(\Delta_z^*)] \quad \text{(G.105)}$$

$$V_4 = -\frac{\tau u^2\Delta_s^2 b_s^2}{2\hbar^2}\frac{n_o}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\sum_{n=-\infty} J_n^2(\chi)(\varepsilon_o - \mu)\nabla_s\mu[I_o(\Delta_s^*) - I_2(\Delta_s^*)I_o(\Delta_z^*)] \quad \text{(G.106)}$$

$$V_4 = -\frac{\tau u^2 \Delta_s^2 b_s^2}{2\hbar^2} \frac{n_o}{k_B T} \sum_{n=-\infty} J_n^2(\chi)(\varepsilon_o - \mu)\nabla_s \mu \left[1 - \frac{I_2(\Delta_s^*)}{I_o(\Delta_s^*)}\right] \tag{G.107}$$

$$\left[1 - \frac{I_2(\Delta_s^*)}{I_o(\Delta_s^*)}\right] = \left(1 - \frac{I_o(\Delta_s^*) - \frac{2}{\Delta_s^*} I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right) = \left(1 - 1 + \frac{2I_1(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)}\right) = \frac{2I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)}$$

$$V_4 = -\frac{\tau u^2 \Delta_s^2 b_s^2}{2\hbar^2} \frac{n_o}{k_B T} \sum_{n=-\infty} J_n^2(\chi)(\varepsilon_o - \mu)\nabla_s \mu \frac{2I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)} \tag{G.108}$$

$$V_4 = -\frac{\tau u^2 \Delta_s b_s^2 n_o}{\hbar^2}(\varepsilon_o - \mu) \sum_{n=-\infty} J_n^2(\chi) \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} \nabla_s \mu \tag{G.109}$$

$$\begin{aligned} V_5 = &+\frac{1}{(\pi\hbar)^2} \frac{u^2 \Delta_s^2 b_s^2}{\hbar^2} \frac{n_o u w b_s b_z}{I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \frac{1}{u w b_s b_z} \int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt \\ &\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] (\varepsilon_o - \mu)\Delta_s \cos\frac{Z_s}{\hbar} \\ &\times \sin^2\frac{Z_s}{\hbar} \cos^2\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t'] dt'\right) \frac{\nabla_s T}{T} \end{aligned} \tag{G.110}$$

$$\begin{aligned} &\int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt \cos^2\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t'] dt'\right) \\ &= \frac{1}{2} \int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt \left(1 + \cos\left(\frac{2ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t'] dt'\right)\right) \\ &= \frac{1}{2} \int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt + \frac{1}{2} \int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt \cos\left(\frac{2ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t'] dt'\right) \end{aligned}$$

For weak electric field

$$\begin{aligned} &= \frac{\tau}{2} + \frac{1}{2} \sum_{n=-\infty} J_n^2(\chi) \left[\frac{\tau}{1 + 4(e\vec{E}_o b_s/\hbar + n\omega\hbar)^2 \tau^2}\right] \\ &= \frac{1}{2}\left(\tau + \sum_{n=-\infty} J_n^2(\chi) \left[\tau \left(1 - 0\left(\frac{e\vec{E}_o b_s}{\hbar} + n\omega\hbar\right)^2\right)\right]\right) \\ &= \frac{1}{2}\left(\tau + \sum_{n=-\infty} J_n^2(\chi)\tau\right) = \frac{1}{2}\tau\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \end{aligned}$$

$$\begin{aligned} V_5 = &+\frac{\tau u^2 \Delta_s^2 b_s^2}{2(\pi\hbar)^2 \hbar^2} \frac{n_o}{I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} (\varepsilon_o - \mu) \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \frac{\nabla_s T}{T} \\ &\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \sin^2\frac{Z_s}{\hbar} \Delta_s \cos\frac{Z_s}{\hbar} \end{aligned} \tag{G.111}$$

$$V_5 = +\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s}{2(\pi\hbar)^2\hbar^2} \frac{n_o}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}(\varepsilon_o - \mu)\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^\pi dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right]\frac{1}{4}\int_0^\pi dZ_s\left(\cos\frac{3Z_s}{\hbar} - \cos\frac{Z_s}{\hbar}\right)\exp\left[\Delta_s^*\cos\frac{Z_s}{\hbar}\right] \quad (G.112)$$

$$V_5 = +\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s n_o(\varepsilon_o - \mu)}{8(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$[I_1(\Delta_s^*) - I_3(\Delta_s^*)I_o(\Delta_z^*)] \quad (G.113)$$

$$V_5 = +\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s n_o(\varepsilon_o - \mu)}{8\hbar^2 k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}\left\{\frac{I_1(\Delta_s^*) - I_3(\Delta_s^*)}{I_o(\Delta_s^*)}\right\} \quad (G.114)$$

$$V_5 = +\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s n_o(\varepsilon_o - \mu)}{8\hbar^2 k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}\left\{\frac{4}{\Delta_s^*} - \frac{8I_1(\Delta_s^*)}{\Delta_s^{*2} I_o(\Delta_s^*)}\right\} \quad (G.115)$$

$$V_5 = +\frac{\tau u^2 \Delta_s^2 b_s^2 n_o k}{2\hbar^2}\frac{(\varepsilon_o - \mu)}{k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\left\{1 - \frac{2I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)}\right\}\nabla_s T \quad (G.116)$$

$$V_6 = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o u w b_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{u w b_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^\pi dZ_z \int_0^\pi dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right]\Delta_s^2 \cos^2\frac{Z_s}{\hbar}$$
$$\times \sin^2\frac{Z_s}{\hbar}\cos^2\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.117)$$

$$V_6 = -\frac{u^2\Delta_s^2 b_s^2 \Delta_s^2 n_o}{(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^\pi dZ_z \exp\left[\Delta_z^*\cos\frac{Z_z}{\hbar}\right]\int_0^\pi dZ_s \exp\left[\Delta_s^*\cos\frac{Z_s}{\hbar}\right]\cos^2\frac{Z_s}{\hbar}\sin^2\frac{Z_s}{\hbar} \quad (G.118)$$

Evaluating the identity:

$$\cos^2\frac{Z_s}{\hbar}\sin^2\frac{Z_s}{\hbar} = \frac{1}{4}\sin^2\frac{2Z_s}{\hbar} = \frac{1}{4}\frac{1}{2}\left(1 - \cos\frac{4Z_s}{\hbar}\right) = \frac{1}{8}\left(1 - \cos\frac{4Z_s}{\hbar}\right)$$

$$V_6 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s^2 n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \frac{\nabla_s T}{T} \times \int_0^\pi dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \int_0^\pi dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{8}\left(1 - \cos\frac{4Z_s}{\hbar}\right) \quad \text{(G.119)}$$

$$V_6 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s^2 n_o}{16\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \frac{\nabla_s T}{T} \times [I_o(\Delta_s^*) - I_4(\Delta_s^*)] I_o(\Delta_z^*) \quad \text{(G.120)}$$

$$V_6 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s^2 n_o}{16\hbar^2 k_B T} \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \frac{\nabla_s T}{T} \frac{[I_o(\Delta_s^*) - I_4(\Delta_s^*)]}{I_o(\Delta_s^*)} \quad \text{(G.121)}$$

$$V_6 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s^2 n_o}{16\hbar^2 k_B T} \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \frac{\nabla_s T}{T} \left[1 - \frac{I_4(\Delta_s^*)}{I_o(\Delta_s^*)}\right] \quad \text{(G.122)}$$

The recurrence relation for I_4 is evaluated as:

$$I_4(\Delta_s^*) = I_2(\Delta_s^*) - \frac{6}{\Delta_s^*} I_3(\Delta_s^*) = I_o(\Delta_s^*) - \frac{2I_1(\Delta_s^*)}{\Delta_s^*} - \frac{6}{\Delta_s^*}\left(I_1(\Delta_s^*) - \frac{4I_o(\Delta_s^*)}{\Delta_s^*} - \frac{8I_1(\Delta_s^*)}{\Delta_s^{*2}}\right)$$
$$= I_o(\Delta_s^*) - \frac{8I_1(\Delta_s^*)}{\Delta_s^*} + \frac{24I_o(\Delta_s^*)}{\Delta_s^{*2}} - \frac{48I_1(\Delta_s^*)}{\Delta_s^{*3}}$$

Thus,

$$1 - \frac{I_4(\Delta_s^*)}{I_o(\Delta_s^*)} = 1 - 1 + \frac{8I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)} - \frac{24I_o(\Delta_s^*)}{\Delta_s^{*2} I_o(\Delta_s^*)} + \frac{48I_1(\Delta_s^*)}{\Delta_s^{*3} I_o(\Delta_s^*)}$$
$$= \frac{8}{\Delta_s^*}\left(\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} - \frac{3}{\Delta_s^*} + \frac{6I_1(\Delta_s^*)}{\Delta_s^{*2} I_o(\Delta_s^*)}\right)$$

Substituting the relation obtained into G.122 yields:

$$V_6 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s^2 n_o}{16\hbar^2 k_B T} \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \times \frac{8}{\Delta_s^*}\left(\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} - \frac{3}{\Delta_s^*} + \frac{6I_1(\Delta_s^*)}{\Delta_s^{*2} I_o(\Delta_s^*)}\right) \frac{\nabla_s T}{T} \quad \text{(G.123)}$$

$$V_6 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s^* k n_o}{2\hbar^2} \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \times \left(\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} - \frac{3}{\Delta_s^*} + \frac{6I_1(\Delta_s^*)}{\Delta_s^{*2} I_o(\Delta_s^*)}\right) \nabla_s T \quad \text{(G.124)}$$

$$V_7 = -\frac{1}{(\pi\hbar)^2} \frac{u^2 \Delta_s^2 b_s^2}{\hbar^2} \frac{n_o u w b_s b_z}{I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \frac{1}{u w b_s b_z} \int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt$$

$$\times \int_0^\pi dZ_z \int_0^\pi dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] \Delta_s \Delta_z \cos\frac{Z_s}{\hbar} \cos\frac{Z_z}{\hbar}$$
$$\times \sin^2\frac{Z_s}{\hbar} \cos^2\left(\frac{uebs_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.125)$$

$$V_7 = -\frac{u^2\Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}$$
$$\times \int_0^\pi dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^\pi dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \cos\frac{Z_s}{\hbar} \sin^2\frac{Z_s}{\hbar}$$
$$\times \int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt \cos^2\left(\frac{uebs_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.126)$$

$$V_7 = -\frac{\tau u^2\Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^\pi dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^\pi dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \quad (G.127)$$

$$V_7 = -\frac{\tau u^2\Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{8\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}[I_1(\Delta_s^*) - I_3(\Delta_s^*)] I_1(\Delta_z^*) \quad (G.128)$$

$$V_7 = -\frac{\tau u^2\Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{8\hbar^2 k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}\left[\frac{I_1(\Delta_s^*) - I_3(\Delta_s^*)}{I_o(\Delta_s^*)}\right]\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \quad (G.129)$$

$$V_7 = -\frac{\tau u^2\Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{8\hbar^2 k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}\left[\frac{4}{\Delta_s^*} - \frac{8 I_1(\Delta_s^*)}{\Delta_s^{*2} I_o(\Delta_s^*)}\right]\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \quad (G.130)$$

$$V_7 = -\frac{\tau u^2\Delta_s b_s^2 \Delta_s^* \Delta_z n_o k}{2\hbar^2}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\left[1 - \frac{2 I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)}\right]\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\nabla_s T \quad (G.131)$$

$$V_8 = +\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right) dt$$

$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] \Delta_s \cos\frac{Z_s}{\hbar}$$
$$\times \sin^2\frac{Z_s}{\hbar} \cos^2\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\nabla_s\mu \quad \text{(G.132)}$$

$$V_8 = +\frac{u^2\Delta_s^2 b_s^2 n_o \Delta_s}{(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \sin^2\frac{Z_s}{\hbar}\cos\frac{Z_s}{\hbar}$$
$$\times \int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt \cos^2\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\nabla_s\mu \quad \text{(G.133)}$$

$$V_8 = +\frac{u^2\Delta_s^2 b_s^2 n_o \Delta_s}{(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\nabla_s\mu$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \sin^2\frac{Z_s}{\hbar}\cos\frac{Z_s}{\hbar} \quad \text{(G.134)}$$

$$V_8 = +\frac{\tau u^2\Delta_s^2 b_s^2 \Delta_s n_o}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\nabla_s\mu$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \quad \text{(G.135)}$$

$$V_8 = +\frac{\tau u^2\Delta_s^2 b_s^2 \Delta_s n_o}{8\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\nabla_s\mu\,[I_1(\Delta_s^*) - I_3(\Delta_s^*)]\, I_1(\Delta_z^*) \quad \text{(G.136)}$$

$$V_8 = +\frac{\tau u^2\Delta_s^2 b_s^2 \Delta_s n_o}{8\hbar^2 k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\nabla_s\mu\left[\frac{I_1(\Delta_s^*) - I_3(\Delta_s^*)}{I_o(\Delta_s^*)}\right] \quad \text{(G.137)}$$

$$V_8 = +\frac{\tau u^2\Delta_s^2 b_s^2 \Delta_s n_o}{8\hbar^2 k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\nabla_s\mu\left[\frac{4}{\Delta_s^*} - \frac{8I_1(\Delta_s^*)}{\Delta_s^{*2} I_o(\Delta_s^*)}\right] \quad \text{(G.138)}$$

$$V_8 = +\frac{\tau u^2\Delta_s^2 b_s^2 n_o}{2\hbar^2}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\left[1 - \frac{2I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)}\right]\nabla_s\mu \quad \text{(G.139)}$$

$$V_9 = +\frac{u^2\Delta_s^2 b_s^2 \Delta_z n_o}{(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}(\varepsilon_o - \mu)\frac{\nabla_s T}{T}$$

$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \cos\frac{Z_z}{\hbar} \sin^2\frac{Z_s}{\hbar}$$
$$\times \int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right) \quad (G.140)$$

$$\int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$= \frac{1}{2}\left[\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt' + \frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\right.$$
$$\left.+\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt' - \frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\right]$$
$$= \frac{1}{2}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+[(e\vec{E}_ob_s/\hbar+n\omega\hbar)+(e\vec{E}_ob_s/\hbar+n\omega\hbar)]^2\tau^2}\right.$$
$$\left.+\frac{\tau}{1+[(e\vec{E}_ob_s/\hbar+n\omega\hbar)-(e\vec{E}_ob_s/\hbar+n\omega\hbar)]^2\tau^2}\right]$$
$$= \frac{1}{2}\sum_{n=-\infty} J_n^2(\chi)\left\{2\tau\left[1-0\left[\left(\frac{e\vec{E}_ob_s}{\hbar}+n\omega\hbar\right)+\left(\frac{e\vec{E}_ob_s}{\hbar}+n\omega\hbar\right)\right]^2\right.\right.$$
$$\left.\left.-0\left[\left(\frac{e\vec{E}_ob_s}{\hbar}+n\omega\hbar\right)-\left(\frac{e\vec{E}_ob_s}{\hbar}+n\omega\hbar\right)\right]^2\right]\right\}$$
$$= \sum_{n=-\infty} J_n^2(\chi)\tau$$

$$V_9 = +\frac{\tau u^2\Delta_s^2 b_s^2\Delta_z n_o}{(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}(\varepsilon_o-\mu)\frac{\nabla_sT}{T}\sum_{n=-\infty} J_n^2(\chi)\tau$$
$$\times \int_0^{\pi} dZ_z \cos\frac{Z_z}{\hbar}\exp\left[\Delta_z^*\cos\frac{Z_z}{\hbar}\right]\int_0^{\pi} dZ_s\exp\left[\Delta_s^*\cos\frac{Z_s}{\hbar}\right]\sin^2\frac{Z_s}{\hbar} \quad (G.141)$$

$$V_9 = +\frac{\tau u^2\Delta_s^2 b_s^2\Delta_z n_o}{(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}(\varepsilon_o-\mu)\frac{\nabla_sT}{T}\sum_{n=-\infty} J_n^2(\chi)\tau$$
$$\times \int_0^{\pi} dZ_z \cos\frac{Z_z}{\hbar}\exp\left[\Delta_z^*\cos\frac{Z_z}{\hbar}\right]\frac{1}{2}\int_0^{\pi} dZ_s\exp\left[\Delta_s^*\cos\frac{Z_s}{\hbar}\right]\left(1-\cos\frac{2Z_s}{\hbar}\right)$$
$$(G.142)$$

$$V_9 = +\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_z n_o}{2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}(\varepsilon_o - \mu)\frac{\nabla_s T}{T}\sum_{n=-\infty} J_n^2(\chi)[I_o(\Delta_s^*) - I_2(\Delta_s^*)] I_o(\Delta_z^*) \tag{G.143}$$

$$V_9 = +\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_z n_o}{2\hbar^2 k_B T}(\varepsilon_o - \mu)\frac{\nabla_s T}{T}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{I_o(\Delta_s^*) - I_2(\Delta_s^*)}{I_o(\Delta_s^*)}\right]\frac{I_o(\Delta_z^*)}{I_o(\Delta_z^*)} \tag{G.144}$$

$$V_9 = +\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_z n_o}{2\hbar^2 k_B T}(\varepsilon_o - \mu)\frac{\nabla_s T}{T}\sum_{n=-\infty} J_n^2(\chi)\left[1 - \frac{I_2(\Delta_s^*)}{I_o(\Delta_s^*)}\right]\frac{I_o(\Delta_z^*)}{I_o(\Delta_z^*)} \tag{G.145}$$

$$V_9 = +\frac{2\tau u^2 \Delta_s b_s^2 \Delta_z \Delta_s n_o}{2\hbar^2 \Delta_s^* k_B T}(\varepsilon_o - \mu)\frac{\nabla_s T}{T}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right]\frac{I_o(\Delta_z^*)}{I_o(\Delta_z^*)} \tag{G.146}$$

$$V_9 = +\frac{\tau u^2 \Delta_s b_s^2 \Delta_z n_o}{\hbar^2}(\varepsilon_o - \mu)\frac{\nabla_s T}{T}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right]\frac{I_o(\Delta_z^*)}{I_o(\Delta_z^*)} \tag{G.147}$$

$$V_9 = +\frac{\tau u^2 \Delta_s b_s^2 \Delta_z n_o k}{\hbar^2}\left(\frac{\varepsilon_o - \mu}{k_B T}\right)\sum_{n=-\infty} J_n^2(\chi)\left[\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right]\frac{I_o(\Delta_z^*)}{I_o(\Delta_z^*)}\nabla_s T \tag{G.148}$$

$$\begin{aligned} V_{10} = &-\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o u w b_s b_z}{I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\frac{1}{u w b_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\ &\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right]\Delta_z\Delta_s \cos\frac{Z_s}{\hbar}\cos\frac{Z_z}{\hbar} \\ &\times \sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right) \\ &\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \end{aligned} \tag{G.149}$$

$$\begin{aligned} V_{10} = &-\frac{u^2\Delta_s^2 b_s^2 \Delta_z \Delta_s n_o}{(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\frac{\nabla_s T}{T}\sum_{n=-\infty} J_n^2(\chi) \\ &\times \int_0^{\pi} dZ_z \cos\frac{Z_z}{\hbar}\exp\left[\Delta_z^*\cos\frac{Z_z}{\hbar}\right]\int_0^{\pi} dZ_s \exp\left[\Delta_s^*\cos\frac{Z_s}{\hbar}\right]\cos\frac{Z_s}{\hbar}\sin^2\frac{Z_s}{\hbar} \end{aligned} \tag{G.150}$$

$$V_{10} = -\frac{u^2\Delta_s^2 b_s^2 \Delta_z \Delta_s n_o}{(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\frac{\nabla_s T}{T}\sum_{n=-\infty} J_n^2(\chi)$$

$$\times \int_0^{\pi} dZ_z \cos\frac{Z_z}{\hbar} \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \cos\frac{Z_s}{\hbar} \sin^2\frac{Z_s}{\hbar} \tag{G.151}$$

$$V_{10} = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \sum_{n=-\infty} J_n^2(\chi) \frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \cos\frac{Z_z}{\hbar} \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.152}$$

$$V_{10} = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{4\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \sum_{n=-\infty} J_n^2(\chi) \frac{\nabla_s T}{T} [I_1(\Delta_s^*) - I_3(\Delta_s^*)] I_1(\Delta_z^*) \tag{G.153}$$

$$V_{10} = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{4\hbar^2 k_B T} \sum_{n=-\infty} J_n^2(\chi) \frac{\nabla_s T}{T} \left[\frac{I_1(\Delta_s^*) - I_3(\Delta_s^*)}{I_o(\Delta_s^*)}\right] \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \tag{G.154}$$

$$V_{10} = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{4\hbar^2 k_B T} \sum_{n=-\infty} J_n^2(\chi) \frac{\nabla_s T}{T} \left[\frac{4}{\Delta_s^*} - \frac{8 I_1(\Delta_s^*)}{\Delta_s^{*2} I_o(\Delta_s^*)}\right] \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \tag{G.155}$$

$$V_{10} = -\frac{\tau u^2 \Delta_s b_s^2 \Delta_s^* \Delta_z n_o k}{\hbar^2} \sum_{n=-\infty} J_n^2(\chi) \left[1 - \frac{2 I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)}\right] \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \nabla_s T \tag{G.156}$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2} \frac{u^2 \Delta_s^2 b_s^2}{\hbar^2} \frac{n_o u w b_s b_z}{I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \frac{1}{u w b_s b_z} \int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] \Delta_z^2 \cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2\frac{Z_s}{\hbar} \cos\left(\frac{u e b_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t'] dt'\right)$$
$$\times \cos\left(\frac{w e b_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t'] dt'\right) \frac{\nabla_s T}{T} \tag{G.157}$$

$$V_{11} = -\frac{u^2 \Delta_s^2 b_s^2 n_o \Delta_z^2}{(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos^2\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \sin^2\frac{Z_s}{\hbar}$$
$$\times \int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt \cos\left(\frac{u e b_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t'] dt'\right)$$
$$\times \cos\left(\frac{w e b_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t'] dt'\right) \tag{G.158}$$

Making use of the identity:

$$\cos^2\frac{Z_z}{\hbar} = \frac{1}{2}\left(1+\cos\frac{Z_z}{\hbar}\right) \qquad \sin^2\frac{Z_s}{\hbar} = \frac{1}{2}\left(1-\cos\frac{Z_s}{\hbar}\right)$$

$$V_{11} = -\frac{\tau u^2\Delta_s^2 b_s^2 n_o \Delta_z^2}{(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\sum_{n=-\infty} J_n^2(\chi)\frac{\nabla_s T}{T} \times \int_0^{\pi} dZ_z \exp\left[\Delta_z^*\cos\frac{Z_z}{\hbar}\right]\frac{1}{2}\left(1+\cos\frac{Z_s}{\hbar}\right) \times \int_0^{\pi} dZ_s \exp\left[\Delta_s^*\cos\frac{Z_s}{\hbar}\right]\frac{1}{2}\left(1-\cos\frac{Z_s}{\hbar}\right) \quad (G.159)$$

$$V_{11} = -\frac{\tau u^2\Delta_s^2 b_s^2 n_o \Delta_z^2}{4\hbar^2 I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\sum_{n=-\infty} J_n^2(\chi)\frac{\nabla_s T}{T} \times [I_1(\Delta_s^*)-I_2(\Delta_s^*)][I_o(\Delta_z^*)+I_2(\Delta_z^*)] \quad (G.160)$$

$$V_{11} = -\frac{\tau u^2\Delta_s^2 b_s^2 n_o \Delta_z^2}{4\hbar^2 k_B T}\sum_{n=-\infty} J_n^2(\chi)\frac{\nabla_s T}{T}\left(1-\frac{I_2(\Delta_s^*)}{I_o(\Delta_s^*)}\right)\left(1+\frac{I_2(\Delta_z^*)}{I_o(\Delta_z^*)}\right) \quad (G.161)$$

Evaluating the recurrence relation:

$$1+\frac{I_2(\Delta_z^*)}{I_o(\Delta_z^*)} = 1+\frac{I_o(\Delta_z^*)-\frac{2}{\Delta_z^*}I_1(\Delta_z^*)}{I_o(\Delta_z^*)} = 2-\frac{2}{\Delta_z^*}\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} = 2\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right)$$

$$V_{11} = -\frac{\tau u^2\Delta_s^2 b_s^2 n_o \Delta_z^2}{4\hbar^2 k_B T}\sum_{n=-\infty} J_n^2(\chi)\frac{\nabla_s T}{T}\frac{2}{\Delta_s^*}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}2\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right) \quad (G.162)$$

$$V_{11} = -\frac{\tau u^2\Delta_s^2 b_s^2 n_o \Delta_z^2}{\hbar^2 k_B T}\sum_{n=-\infty} J_n^2(\chi)\frac{\nabla_s T}{T}\frac{1}{\Delta_s^*}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right) \quad (G.163)$$

$$V_{11} = -\frac{\tau u^2\Delta_s b_s^2 n_o \Delta_z^* k}{\hbar^2 k_B T}\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right)\nabla_s T \quad (G.164)$$

$$V_{12} = +\frac{u^2\Delta_s^2 b_s^2 n_o \Delta_z}{(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\nabla_s\mu \times \int_0^{\pi} dZ_z \cos\frac{Z_z}{\hbar}\exp\left[\Delta_z^*\cos\frac{Z_z}{\hbar}\right]\int_0^{\pi} dZ_s \exp\left[\Delta_s^*\cos\frac{Z_s}{\hbar}\right]\sin^2\frac{Z_s}{\hbar} \times \int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right) \times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right) \quad (G.165)$$

$$V_{12} = + \frac{\tau u^2 \Delta_s^2 b_s^2 n_o \Delta_z}{(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \sum_{n=-\infty} J_n^2(\chi) \nabla_s \mu$$
$$\times \int_0^\pi dZ_z \cos\frac{Z_z}{\hbar} \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \int_0^\pi dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \left(1 - \cos\frac{2Z_s}{\hbar}\right) \tag{G.166}$$

$$V_{12} = + \frac{\tau u^2 \Delta_s^2 b_s^2 n_o \Delta_z}{2\hbar^2 k_B T} \sum_{n=-\infty} J_n^2(\chi) \nabla_s \mu \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \left(1 - \frac{I_2(\Delta_s^*)}{I_o(\Delta_s^*)}\right) \tag{G.167}$$

$$V_{12} = + \frac{\tau u^2 \Delta_s^2 b_s^2 n_o \Delta_z}{2\hbar^2 k_B T} \sum_{n=-\infty} J_n^2(\chi) \frac{2}{\Delta_s^*} \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \nabla_s \mu \tag{G.168}$$

$$V_{12} = + \frac{\tau u^2 \Delta_s b_s^2 n_o \Delta_z}{\hbar^2 k_B T} \sum_{n=-\infty} J_n^2(\chi) \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \nabla_s \mu \tag{G.169}$$

Summing the up $V_1....V_{12}$ yields:

$$\vec{S}_2^* = - \frac{\tau u^2 \Delta_s b_s^2 n_o k}{\hbar^2} \frac{(\varepsilon_o - \mu)^2}{k_B T} \sum_{n=-\infty} J_n^2(\chi) \left\{\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right\} \nabla_s T$$
$$+ \frac{\tau u^2 \Delta_s^2 b_s^2 n_o k}{\hbar^2} \left(\frac{\varepsilon_o - \mu}{k_B T}\right) \sum_{n=-\infty} J_n^2(\chi) \left(1 - \frac{2I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)}\right) \nabla_s T$$
$$+ \frac{\tau u^2 \Delta_s b_s^2 n_o \Delta_z k}{(\hbar)^2} \left(\frac{\varepsilon_o - \mu}{k_B T}\right) \sum_{n=-\infty} J_n^2(\chi) \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \nabla_s T$$
$$- \frac{\tau u^2 \Delta_s b_s^2 n_o}{\hbar^2} (\varepsilon_o - \mu) \sum_{n=-\infty} J_n^2(\chi) \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} \nabla_s \mu$$
$$+ \frac{\tau u^2 \Delta_s^2 b_s^2 n_o k}{2\hbar^2} \frac{(\varepsilon_o - \mu)}{k_B T} \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \left\{1 - \frac{2I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)}\right\} \nabla_s T$$
$$- \frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s^* k n_o}{2\hbar^2} \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \left(\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} - \frac{3}{\Delta_s^*} + \frac{6I_1(\Delta_s^*)}{\Delta_s^{*2} I_o(\Delta_s^*)}\right) \nabla_s T$$
$$- \frac{\tau u^2 \Delta_s b_s^2 \Delta_s^* \Delta_z n_o k}{2\hbar^2} \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \left[1 - \frac{2I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)}\right] \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \nabla_s T$$
$$+ \frac{\tau u^2 \Delta_s^2 b_s^2 n_o}{2\hbar^2} \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \left[1 - \frac{2I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)}\right] \nabla_s \mu$$
$$+ \frac{\tau u^2 \Delta_s b_s^2 \Delta_z n_o k}{\hbar^2} \left(\frac{\varepsilon_o - \mu}{k_B T}\right) \sum_{n=-\infty} J_n^2(\chi) \left[\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right] \frac{I_o(\Delta_z^*)}{I_o(\Delta_z^*)} \nabla_s T$$
$$- \frac{\tau u^2 \Delta_s b_s^2 \Delta_s^* \Delta_z n_o k}{\hbar^2} \sum_{n=-\infty} J_n^2(\chi) \left[1 - \frac{2I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)}\right] \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \nabla_s T$$
$$- \frac{\tau u^2 \Delta_s b_s^2 n_o \Delta_z^* k}{\hbar^2 k_B T} \sum_{n=-\infty} J_n^2(\chi) \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} \left(1 - \frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right) \nabla_s T$$
$$+ \frac{\tau u^2 \Delta_s b_s^2 n_o \Delta_z}{\hbar^2 k_B T} \sum_{n=-\infty} J_n^2(\chi) \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \nabla_s \mu \tag{G.170}$$

$$
\begin{aligned}
\vec{S}_2^* = &-\frac{\tau u^2\Delta_s b_s^2 n_o}{\hbar^2}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\Bigg\{(\varepsilon_o-\mu)\sum_{n=-\infty} J_n^2(\chi) \\
&-\frac{\Delta_s}{2}\left(1+\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)-\Delta_z\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\Bigg\}\nabla_s\mu \\
&-\frac{\tau u^2\Delta_s b_s^2 n_o k}{\hbar^2}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\Bigg\{\frac{(\varepsilon_o-\mu)^2}{k_BT}\sum_{n=-\infty} J_n^2(\chi) \\
&-\Delta_s\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\sum_{n=-\infty} J_n^2(\chi) \\
&-\Delta_z\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty} J_n^2(\chi) \\
&-\frac{\Delta_s}{2}\frac{(\varepsilon_o-\mu)}{k_BT}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+\sum_{n=-\infty} J_n^2(\chi)\right) \\
&+\frac{\Delta_s\Delta_s^*}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty} J_n^2(\chi)\right) \\
&+\frac{\Delta_z\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+\sum_{n=-\infty} J_n^2(\chi)\right) \\
&-\Delta_z\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty} J_n^2(\chi) \\
&+\Delta_z\Delta_s^*\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty} J_n^2(\chi) \\
&+\Delta_z\Delta_z^*\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right)\sum_{n=-\infty} J_n^2(\chi)\Bigg\}\nabla_s T
\end{aligned}
\tag{G.171}
$$

$$
\begin{aligned}
\vec{S}_2^* = &-\frac{\tau u^2\Delta_s b_s^2 n_o}{\hbar^2}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\Bigg\{(\varepsilon_o-\mu)\sum_{n=-\infty} J_n^2(\chi) \\
&-\frac{\Delta_s}{2}\left(1+\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)-\Delta_z\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\Bigg\}\nabla_s\mu \\
&-\frac{\tau u^2\Delta_s b_s^2 n_o k}{\hbar^2}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\Bigg\{\frac{(\varepsilon_o-\mu)^2}{k_BT}\sum_{n=-\infty} J_n^2(\chi) \\
&-2\Delta_z\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty} J_n^2(\chi) \\
&-\frac{\Delta_s}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right) \\
&+\frac{\Delta_s\Delta_s^*}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty} J_n^2(\chi)\right)
\end{aligned}
$$

$$+\frac{\Delta_z\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty}J_n^2(\chi)\right)$$

$$+\Delta_z\Delta_z^*\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right)\sum_{n=-\infty}J_n^2(\chi)\Bigg\}\nabla_s T \quad \text{(G.172)}$$

where in the presence of a weak field, $\sigma_s(\vec{E})$ is given as:

$$\sigma_s(\vec{E})=\frac{e^2\tau u^2\Delta_s b_s^2 n_o}{\hbar^2}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} \quad \text{(G.173)}$$

$$\vec{S}_2^*=-\sigma_s(\vec{E})\frac{1}{e}\Bigg\{(\varepsilon_o-\mu)\sum_{n=-\infty}J_n^2(\chi)$$

$$-\frac{\Delta_s}{2}\left(1+\sum_{n=-\infty}J_n^2(\chi)\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)-\Delta_z\sum_{n=-\infty}J_n^2(\chi)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\Bigg\}\nabla_s\frac{\mu}{e}$$

$$-\sigma_s(\vec{E})\frac{k}{e^2}\Bigg\{\frac{(\varepsilon_o-\mu)^2}{k_BT}\sum_{n=-\infty}J_n^2(\chi)-2\Delta_z\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty}J_n^2(\chi)$$

$$-\frac{\Delta_s}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty}J_n^2(\chi)\right)$$

$$+\frac{\Delta_s\Delta_s^*}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty}J_n^2(\chi)\right)$$

$$+\frac{\Delta_z\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty}J_n^2(\chi)\right)$$

$$+\Delta_z\Delta_z^*\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right)\sum_{n=-\infty}J_n^2(\chi)\Bigg\}\nabla_s T \quad \text{(G.174)}$$

Now, $\vec{S}^*=\vec{S}_1^*+\vec{S}_2^*$ which yields:

$$\vec{S}^*=-\sigma_s(\vec{E})\frac{1}{e}\sum_{n=-\infty}J_n^2(\chi)\left\{(\varepsilon_o-\mu)-\Delta_s\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)-\Delta_z\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right\}\vec{E}_n$$

$$-\sigma_s(\vec{E})\frac{1}{e}\Bigg\{(\varepsilon_o-\mu)\sum_{n=-\infty}J_n^2(\chi)-\frac{\Delta_s}{2}\left(1+\sum_{n=-\infty}J_n^2(\chi)\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)$$

$$-\Delta_z\sum_{n=-\infty}J_n^2(\chi)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\Bigg\}\nabla_s\frac{\mu}{e}$$

$$-\sigma_s(\vec{E})\frac{k}{e^2}\Bigg\{\frac{(\varepsilon_o-\mu)^2}{k_BT}\sum_{n=-\infty}J_n^2(\chi)$$

$$-\frac{\Delta_s}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty}J_n^2(\chi)\right)$$

$$
\begin{aligned}
&- 2\Delta_z \left(\frac{\varepsilon_o - \mu}{k_B T}\right) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \sum_{n=-\infty} J_n^2(\chi) \\
&+ \frac{\Delta_s \Delta_s^*}{2} \left(1 - \frac{3I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)} + \frac{6}{\Delta_s^{*2}}\right) \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \\
&+ \frac{\Delta_z \Delta_s^*}{2} \left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*}\right) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \left(1 + 3 \sum_{n=-\infty} J_n^2(\chi)\right) \\
&+ \Delta_z \Delta_z^* \left(1 - \frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right) \sum_{n=-\infty} J_n^2(\chi) \Bigg\} \nabla_s T \qquad \text{(G.175)}
\end{aligned}
$$

$$
\begin{aligned}
\vec{S}^* = &-\sigma_s(\vec{E}) \frac{1}{e} \Bigg\{ (\varepsilon_o - \mu) \sum_{n=-\infty} J_n^2(\chi) \\
&- \Delta_s \left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*}\right) \sum_{n=-\infty} J_n^2(\chi) - \Delta_z \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \sum_{n=-\infty} J_n^2(\chi) \Bigg\} \vec{E}_n \\
&- \sigma_s(\vec{E}) \frac{1}{e} \Bigg\{ (\varepsilon_o - \mu) \sum_{n=-\infty} J_n^2(\chi) \\
&- \frac{\Delta_s}{2} \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*}\right) \\
&- \Delta_z \sum_{n=-\infty} J_n^2(\chi) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \Bigg\} \nabla_s \frac{\mu}{e} \\
&- \sigma_s(\vec{E}) \frac{k}{e^2} \Bigg\{ \frac{(\varepsilon_o - \mu)^2}{k_B T} \sum_{n=-\infty} J_n^2(\chi) \\
&- \frac{\Delta_s}{2} \left(\frac{\varepsilon_o - \mu}{k_B T}\right) \left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*}\right) \left(1 + 3 \sum_{n=-\infty} J_n^2(\chi)\right) \\
&- 2\Delta_z \left(\frac{\varepsilon_o - \mu}{k_B T}\right) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \sum_{n=-\infty} J_n^2(\chi) \\
&+ \frac{\Delta_s \Delta_s^*}{2} \left(1 - \frac{3I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)} + \frac{6}{\Delta_s^{*2}}\right) \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \\
&+ \frac{\Delta_z \Delta_s^*}{2} \left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*}\right) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \left(1 + 3 \sum_{n=-\infty} J_n^2(\chi)\right) \\
&+ \Delta_z \Delta_z^* \left(1 - \frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right) \sum_{n=-\infty} J_n^2(\chi) \Bigg\} \nabla_s T \qquad \text{(G.176)}
\end{aligned}
$$

$$
\vec{S}^* = -\sigma_s(\vec{E}) \frac{1}{e} \Bigg\{ (\varepsilon_o - \mu) \sum_{n=-\infty} J_n^2(\chi)
$$

$$-\frac{\Delta_s}{2}\left(1+3\sum_{n=-\infty}J_n^2(\chi)\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)$$

$$-\Delta_z\sum_{n=-\infty}J_n^2(\chi)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\Bigg\}\left(\vec{E}_n+\nabla_s\frac{\mu}{e}\right)$$

$$-\sigma_s(\vec{E})\frac{k}{e^2}\Bigg\{\frac{(\varepsilon_o-\mu)^2}{k_BT}\sum_{n=-\infty}J_n^2(\chi)$$

$$-\frac{\Delta_s}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty}J_n^2(\chi)\right)$$

$$-2\Delta_z\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty}J_n^2(\chi)$$

$$+\frac{\Delta_s\Delta_s^*}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^*I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty}J_n^2(\chi)\right)$$

$$+\frac{\Delta_z\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty}J_n^2(\chi)\right)$$

$$+\Delta_z\Delta_z^*\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^*I_o(\Delta_z^*)}\right)\sum_{n=-\infty}J_n^2(\chi)\Bigg\}\nabla_sT \quad \text{(G.177)}$$

$$\vec{S}^*=-\sigma_s(\vec{E})\frac{1}{e}\Bigg\{(\varepsilon_o-\mu)\sum_{n=-\infty}J_n^2(\chi)$$

$$-\frac{\Delta_s}{2}\left(1+3\sum_{n=-\infty}J_n^2(\chi)\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)-\Delta_z\sum_{n=-\infty}J_n^2(\chi)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\Bigg\}\vec{E}_{sn}^*$$

$$-\sigma_s(\vec{E})\frac{k}{e^2}\Bigg\{\frac{(\varepsilon_o-\mu)^2}{k_BT}\sum_{n=-\infty}J_n^2(\chi)$$

$$-\frac{\Delta_s}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty}J_n^2(\chi)\right)$$

$$-2\Delta_z\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty}J_n^2(\chi)$$

$$+\frac{\Delta_s\Delta_s^*}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^*I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty}J_n^2(\chi)\right)$$

$$+\frac{\Delta_z\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty}J_n^2(\chi)\right)$$

$$+\Delta_z\Delta_z^*\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^*I_o(\Delta_z^*)}\right)\sum_{n=-\infty}J_n^2(\chi)\Bigg\}\nabla_sT \quad \text{(G.178)}$$

where $\vec{E}_{sn}^*=\vec{E}_n+\nabla_s\mu/e$. Following the same procedure above:

$$\vec{Z}^{*} = -\sigma_z(\vec{E})\frac{1}{e}\left\{(\varepsilon_o-\mu)\sum_{n=-\infty} J_n^2(\chi)\right.$$
$$-\frac{\Delta_z}{2}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)-\Delta_s\sum_{n=-\infty} J_n^2(\chi)\left.\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right\}\vec{E}_{zn}^{*}$$
$$-\sigma_z(\vec{E})\frac{k}{e^2}\left\{\frac{(\varepsilon_o-\mu)^2}{k_BT}\sum_{n=-\infty} J_n^2(\chi)\right.$$
$$-\frac{\Delta_z}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)$$
$$-2\Delta_s\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\sum_{n=-\infty} J_n^2(\chi)$$
$$+\frac{\Delta_z\Delta_z^*}{2}\left(1-\frac{3I_o(\Delta_z^*)}{\Delta_z^*I_1(\Delta_z^*)}+\frac{6}{\Delta_z^{*2}}\right)\left(1+\sum_{n=-\infty} J_n^2(\chi)\right)$$
$$+\frac{\Delta_s\Delta_z^*}{2}\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)$$
$$\left.+\Delta_s\Delta_s^*\left(1-\frac{I_1(\Delta_s^*)}{\Delta_s^*I_o(\Delta_s^*)}\right)\sum_{n=-\infty} J_n^2(\chi)\right\}\nabla_z T \quad \text{(G.179)}$$

The axial and circumferential components of the thermal current density are respectively given as:

$$\vec{q}_z = \vec{Z}^{*} + \vec{S}^{*}\sin\theta_h \qquad \vec{q}_c = \vec{S}^{*}\cos\theta_h$$

The axial current density yields:

$$\vec{q}_z = -\sigma_z(\vec{E})\frac{1}{e}\left\{(\varepsilon_o-\mu)\sum_{n=-\infty} J_n^2(\chi)\right.$$
$$-\frac{\Delta_z}{2}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)-\Delta_s\sum_{n=-\infty} J_n^2(\chi)\left.\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right\}\vec{E}_{zn}^{*}$$
$$-\sigma_z(\vec{E})\frac{k}{e^2}\left\{\frac{(\varepsilon_o-\mu)^2}{k_BT}\sum_{n=-\infty} J_n^2(\chi)\right.$$
$$-\frac{\Delta_z}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)$$
$$-2\Delta_s\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\sum_{n=-\infty} J_n^2(\chi)$$
$$+\frac{\Delta_z\Delta_z^*}{2}\left(1-\frac{3I_o(\Delta_z^*)}{\Delta_z^*I_1(\Delta_z^*)}+\frac{6}{\Delta_z^{*2}}\right)\left(1+\sum_{n=-\infty} J_n^2(\chi)\right)$$
$$+\frac{\Delta_s\Delta_z^*}{2}\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)$$

$$+\Delta_s\Delta_s^*\left(1-\frac{I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)}\right)\sum_{n=-\infty} J_n^2(\chi)\Bigg\}\nabla_z T$$

$$-\sigma_s(\vec{E})\frac{1}{e}\Bigg\{(\varepsilon_o-\mu)\sum_{n=-\infty} J_n^2(\chi)-\frac{\Delta_s}{2}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)$$

$$-\Delta_z\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\Bigg\}\vec{E}_{sn}^*\sin\theta_h$$

$$-\sigma_s(\vec{E})\frac{k}{e^2}\Bigg\{\frac{(\varepsilon_o-\mu)^2}{k_BT}\sum_{n=-\infty} J_n^2(\chi)$$

$$-\frac{\Delta_s}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)$$

$$-2\Delta_z\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty} J_n^2(\chi)$$

$$+\frac{\Delta_s\Delta_s^*}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty} J_n^2(\chi)\right)$$

$$+\frac{\Delta_z\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)$$

$$+\Delta_z\Delta_z^*\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right)\sum_{n=-\infty} J_n^2(\chi)\Bigg\}\nabla_s T\sin\theta_h \quad \text{(G.180)}$$

and the circumferential component gives:

$$\vec{q}_c=-\sigma_s(\vec{E})\frac{1}{e}\Bigg\{(\varepsilon_o-\mu)\sum_{n=-\infty} J_n^2(\chi)$$

$$-\frac{\Delta_s}{2}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)-\Delta_z\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\Bigg\}\vec{E}_{sn}^*$$

$$-\sigma_s(\vec{E})\frac{k}{e^2}\Bigg\{\frac{(\varepsilon_o-\mu)^2}{k_BT}\sum_{n=-\infty} J_n^2(\chi)$$

$$-\frac{\Delta_s}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)$$

$$-2\Delta_z\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty} J_n^2(\chi)$$

$$+\frac{\Delta_s\Delta_s^*}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty} J_n^2(\chi)\right)$$

$$+\frac{\Delta_z\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)$$

$$+\Delta_z\Delta_z^*\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right)\sum_{n=-\infty} J_n^2(\chi)\Bigg\}\nabla_s T\cos\theta_h \quad \text{(G.181)}$$

But $\vec{E}_s = \vec{E}_z \sin\theta_h$ and $\nabla_s T = \nabla_z T \sin\theta_h$. Hence:

$$
\begin{aligned}
\vec{q}_z = -\sigma_z(\vec{E})\frac{1}{e}\Bigg\{&(\varepsilon_o-\mu)\sum_{n=-\infty}J_n^2(\chi) \\
&-\frac{\Delta_z}{2}\left(1+3\sum_{n=-\infty}J_n^2(\chi)\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)-\Delta_s\sum_{n=-\infty}J_n^2(\chi)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\Bigg\}\vec{E}_{zn}^* \\
&-\sigma_z(\vec{E})\frac{k}{e^2}\Bigg\{\frac{(\varepsilon_o-\mu)^2}{k_BT}\sum_{n=-\infty}J_n^2(\chi) \\
&-\frac{\Delta_z}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\left(1+3\sum_{n=-\infty}J_n^2(\chi)\right) \\
&-2\Delta_s\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\sum_{n=-\infty}J_n^2(\chi) \\
&+\frac{\Delta_z\Delta_z^*}{2}\left(1-\frac{3I_o(\Delta_z^*)}{\Delta_z^*I_1(\Delta_z^*)}+\frac{6}{\Delta_z^{*2}}\right)\left(1+\sum_{n=-\infty}J_n^2(\chi)\right) \\
&+\frac{\Delta_s\Delta_z^*}{2}\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\left(1+3\sum_{n=-\infty}J_n^2(\chi)\right) \\
&+\Delta_s\Delta_s^*\left(1-\frac{I_1(\Delta_s^*)}{\Delta_s^*I_o(\Delta_s^*)}\right)\sum_{n=-\infty}J_n^2(\chi)\Bigg\}\nabla_zT \\
&-\sigma_s(\vec{E})\frac{1}{e}\Bigg\{(\varepsilon_o-\mu)\sum_{n=-\infty}J_n^2(\chi) \\
&-\frac{\Delta_s}{2}\left(1+3\sum_{n=-\infty}J_n^2(\chi)\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right) \\
&-\Delta_z\sum_{n=-\infty}J_n^2(\chi)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\Bigg\}\vec{E}_{zn}^*\sin^2\theta_h \\
&-\sigma_s(\vec{E})\frac{k}{e^2}\Bigg\{\frac{(\varepsilon_o-\mu)^2}{k_BT}\sum_{n=-\infty}J_n^2(\chi) \\
&-\frac{\Delta_s}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty}J_n^2(\chi)\right) \\
&-2\Delta_z\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty}J_n^2(\chi) \\
&+\frac{\Delta_s\Delta_s^*}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^*I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty}J_n^2(\chi)\right) \\
&+\frac{\Delta_z\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty}J_n^2(\chi)\right) \\
&+\Delta_z\Delta_z^*\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^*I_o(\Delta_z^*)}\right)\sum_{n=-\infty}J_n^2(\chi)\Bigg\}\nabla_zT\sin^2\theta_h \qquad \text{(G.182)}
\end{aligned}
$$

Multiplying through by $k_B T$

$$
\begin{aligned}
\vec{q}_z = -\frac{k_B T}{e} \Bigg\{ \sigma_z(\vec{E}) \Bigg[\left(\frac{\varepsilon_o - \mu}{k_B T}\right) \sum_{n=-\infty} J_n^2(\chi) \\
- \frac{\Delta_z^*}{2} \left(1 + 3 \sum_{n=-\infty} J_n^2(\chi)\right) \left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)} - \frac{2}{\Delta_z^*}\right) - \Delta_s^* \sum_{n=-\infty} J_n^2(\chi) \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} \Bigg] \\
+ \sigma_s(\vec{E}) \sin^2\theta_h \Bigg[\left(\frac{\varepsilon_o - \mu}{k_B T}\right) \sum_{n=-\infty} J_n^2(\chi) \\
- \frac{\Delta_s^*}{2} \left(1 + 3 \sum_{n=-\infty} J_n^2(\chi)\right) \left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*}\right) \\
- \Delta_z^* \sum_{n=-\infty} J_n^2(\chi) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \Bigg] \Bigg\} \vec{E}_{zn}^* \\
- \frac{k_B^2 T}{e^2} \Bigg\{ \sigma_z(\vec{E}) \Bigg[\left(\frac{\varepsilon_o - \mu}{k_B T}\right)^2 \sum_{n=-\infty} J_n^2(\chi) \\
- \frac{\Delta_z^*}{2} \left(\frac{\varepsilon_o - \mu}{k_B T}\right) \left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)} - \frac{2}{\Delta_z^*}\right) \left(1 + 3 \sum_{n=-\infty} J_n^2(\chi)\right) \\
- 2\Delta_s^* \left(\frac{\varepsilon_o - \mu}{k_B T}\right) \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} \sum_{n=-\infty} J_n^2(\chi) \\
+ \frac{\Delta_z^{*2}}{2} \left(1 - \frac{3 I_o(\Delta_z^*)}{\Delta_z^* I_1(\Delta_z^*)} + \frac{6}{\Delta_z^{*2}}\right) \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \\
+ \frac{\Delta_s^* \Delta_z^*}{2} \left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)} - \frac{2}{\Delta_z^*}\right) \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} \left(1 + 3 \sum_{n=-\infty} J_n^2(\chi)\right) \\
+ \Delta_s^{*2} \left(1 - \frac{I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)}\right) \sum_{n=-\infty} J_n^2(\chi) \Bigg] \\
+ \sigma_s(\vec{E}) \sin^2\theta_h \Bigg[\left(\frac{\varepsilon_o - \mu}{k_B T}\right)^2 \sum_{n=-\infty} J_n^2(\chi) \\
- \frac{\Delta_s^*}{2} \left(\frac{\varepsilon_o - \mu}{k_B T}\right) \left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*}\right) \left(1 + 3 \sum_{n=-\infty} J_n^2(\chi)\right) \\
- 2\Delta_z^* \left(\frac{\varepsilon_o - \mu}{k_B T}\right) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \sum_{n=-\infty} J_n^2(\chi) \\
+ \frac{\Delta_s^{*2}}{2} \left(1 - \frac{3 I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)} + \frac{6}{\Delta_s^{*2}}\right) \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \\
+ \frac{\Delta_z^* \Delta_s^*}{2} \left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*}\right) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \left(1 + 3 \sum_{n=-\infty} J_n^2(\chi)\right) \\
+ \Delta_z^{*2} \left(1 - \frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right) \sum_{n=-\infty} J_n^2(\chi) \Bigg] \Bigg\} \nabla_z T \quad \text{(G.183)}
\end{aligned}
$$

The circumferential current density is given as

$$
\begin{aligned}
\vec{q}_c = -\sigma_s(\vec{E})\frac{1}{e}\Bigg\{(\varepsilon_o-\mu)\sum_{n=-\infty} J_n^2(\chi) \\
-\frac{\Delta_s}{2}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)-\Delta_z\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\Bigg\}\vec{E}_{sn}^*\cos\theta_h \\
-\sigma_s(\vec{E})\frac{k}{e^2}\Bigg\{\frac{(\varepsilon_o-\mu)^2}{k_BT}\sum_{n=-\infty} J_n^2(\chi) \\
-\frac{\Delta_s}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right) \\
-2\Delta_z\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty} J_n^2(\chi) \\
+\frac{\Delta_s\Delta_s^*}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^*I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty} J_n^2(\chi)\right) \\
+\frac{\Delta_z\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right) \\
+\Delta_z\Delta_z^*\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^*I_o(\Delta_z^*)}\right)\sum_{n=-\infty} J_n^2(\chi)\Bigg\}\nabla_sT\cos\theta_h \qquad \text{(G.184)}
\end{aligned}
$$

$$
\begin{aligned}
\vec{q}_c = -\sigma_s(\vec{E})\frac{1}{e}\sin\theta_h\cos\theta_h\Bigg\{(\varepsilon_o-\mu)\sum_{n=-\infty} J_n^2(\chi) \\
-\frac{\Delta_s}{2}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)-\Delta_z\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\Bigg\}\vec{E}_{zn}^* \\
-\sigma_s(\vec{E})\frac{k}{e^2}\sin\theta_h\cos\theta_h\Bigg\{\frac{(\varepsilon_o-\mu)^2}{k_BT}\sum_{n=-\infty} J_n^2(\chi) \\
-\frac{\Delta_s}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right) \\
-2\Delta_z\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty} J_n^2(\chi) \\
+\frac{\Delta_s\Delta_s^*}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^*I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty} J_n^2(\chi)\right) \\
+\frac{\Delta_z\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right) \\
+\Delta_z\Delta_z^*\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^*I_o(\Delta_z^*)}\right)\sum_{n=-\infty} J_n^2(\chi)\Bigg\}\nabla_zT \qquad \text{(G.185)}
\end{aligned}
$$

Multiplying and dividing by k_BT

$$\begin{aligned}\vec{q}_c = -\sigma_s(\vec{E})\frac{k_BT}{e}\Bigg\{&\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\sum_{n=-\infty} J_n^2(\chi) \\ &-\frac{\Delta_s^*}{2}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right) \\ &-\Delta_z^*\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\Bigg\}\vec{E}_{zn}^*\sin\theta_h\cos\theta_h \\ &-\sigma_s(\vec{E})\frac{k_B^2T}{e^2}\Bigg\{\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty} J_n^2(\chi) \\ &-\frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right) \\ &-2\Delta_z^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty} J_n^2(\chi) \\ &+\frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^*I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty} J_n^2(\chi)\right) \\ &+\frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right) \\ &+\Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^*I_o(\Delta_z^*)}\right)\sum_{n=-\infty} J_n^2(\chi)\Bigg\}\nabla_zT\sin\theta_h\cos\theta_h \qquad \text{(G.186)}\end{aligned}$$

Let κ_{ec} and κ_{ez} be the circumferential and axial components of the electron thermal conductivity respectively

$$\begin{aligned}\kappa_{ec} = -\sigma_s(\vec{E})\frac{k_B^2T}{e^2}&\sin\theta_h\cos\theta_h \\ &\times\Bigg\{\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty} J_n^2(\chi) \\ &-\frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right) \\ &-2\Delta_z^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty} J_n^2(\chi) \\ &+\frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^*I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty} J_n^2(\chi)\right) \\ &+\frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right) \\ &+\Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^*I_o(\Delta_z^*)}\right)\sum_{n=-\infty} J_n^2(\chi)\Bigg\} \qquad \text{(G.187)}\end{aligned}$$

$$
\begin{aligned}
\kappa_{ez} = -\frac{k_B^2 T}{e^2} \Bigg\{ \sigma_z(\vec{E}) \Bigg[& \left(\frac{\varepsilon_o - \mu}{k_B T}\right)^2 \sum_{n=-\infty} J_n^2(\chi) \\
& - \frac{\Delta_z^*}{2} \left(\frac{\varepsilon_o - \mu}{k_B T}\right) \left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)} - \frac{2}{\Delta_z^*}\right) \left(1 + 3 \sum_{n=-\infty} J_n^2(\chi)\right) \\
& - 2\Delta_s^* \left(\frac{\varepsilon_o - \mu}{k_B T}\right) \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} \sum_{n=-\infty} J_n^2(\chi) \\
& + \frac{\Delta_z^{*2}}{2} \left(1 - \frac{3 I_o(\Delta_z^*)}{\Delta_z^* I_1(\Delta_z^*)} + \frac{6}{\Delta_z^{*2}}\right) \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \\
& + \frac{\Delta_s^* \Delta_z^*}{2} \left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)} - \frac{2}{\Delta_z^*}\right) \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} \left(1 + 3 \sum_{n=-\infty} J_n^2(\chi)\right) \\
& + \Delta_s^{*2} \left(1 - \frac{I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)}\right) \sum_{n=-\infty} J_n^2(\chi) \Bigg] \\
& + \sigma_s(\vec{E}) \sin^2\theta_h \Bigg[\left(\frac{\varepsilon_o - \mu}{k_B T}\right)^2 \sum_{n=-\infty} J_n^2(\chi) \\
& - \frac{\Delta_s^*}{2} \left(\frac{\varepsilon_o - \mu}{k_B T}\right) \left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*}\right) \left(1 + 3 \sum_{n=-\infty} J_n^2(\chi)\right) \\
& - 2\Delta_z^* \left(\frac{\varepsilon_o - \mu}{k_B T}\right) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \sum_{n=-\infty} J_n^2(\chi) \\
& + \frac{\Delta_s^{*2}}{2} \left(1 - \frac{3 I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)} + \frac{6}{\Delta_s^{*2}}\right) \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \\
& + \frac{\Delta_z^* \Delta_s^*}{2} \left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*}\right) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \left(1 + 3 \sum_{n=-\infty} J_n^2(\chi)\right) \\
& + \Delta_z^{*2} \left(1 - \frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right) \sum_{n=-\infty} J_n^2(\chi) \Bigg] \Bigg\}
\end{aligned} \tag{G.188}
$$

APPENDIX H

ENTROPY AND ENERGY DISSIPATED

Because the FSWCNT is such a small device, the physical mechanisms that operate within it are completely elastomeric. Mechanical work is done on the carriers travelling in the FSWCNT by an electric field in the presence of current. The scalar product $\vec{J}\cdot\vec{E}$ clearly equals the amount of work done per unit of time and volume. In the external contacts, this work is dissipated as heat. As a result, in the external homogeneous conductor, the quantity of heat evolved per unit time and volume is:

$$\vec{J}\cdot\vec{E} = \sigma E^2 = j^2/\sigma \tag{H.1}$$

This is Joule's law. The evolution of heat results in an increase in the entropy of the FSWCNT. When an amount of heat $dQ = \vec{J}\cdot\vec{E}dV$ is evolved, the entropy of the volume element dV increases by dQ/T. The rate of change of the total entropy of the material is therefore given as:

$$\frac{ds}{dt} = \int (\vec{J}\cdot\vec{E})dV \tag{H.2}$$

Since the entropy increases, its derivative must be positive. Putting $\vec{J} = \sigma\vec{E}$, shows that the conductivity is also positive. The symmetry of the kinetic coefficients gives a relation between the coefficient β and the coefficient α in

$$\vec{J} = \sigma(\vec{E} - \alpha\nabla T) \tag{H.3}$$

To derive this, the rate of change of the total entropy of the FSWCNT is calculated. The amount of heat evolved per unit time and volume is $-\nabla\cdot\mathbf{q}$. Hence;

$$\frac{ds}{dt} = -\int \frac{\nabla\cdot\mathbf{q}}{T}dV \tag{H.4}$$

Using the equation $div\vec{J} = 0$, yields;

$$\frac{\nabla\cdot\mathbf{q}}{T} = \frac{1}{T}\{div(\mathbf{q}-\phi\vec{J}) + div\phi\vec{J}\} = \frac{1}{T}div(\mathbf{q}-\phi\vec{J}) - \frac{\vec{J}\cdot\vec{E}}{T} \tag{H.5}$$

Integrating the first term by parts gives:

$$\frac{ds}{dt} = \int \frac{\vec{J}\cdot\vec{E}}{T}dV - \int \frac{(\mathbf{q}-\phi\vec{J})\cdot\nabla T}{T^2}dV \tag{H.6}$$

Considering $\vec{J}$ and $\mathbf{q}$, this quantity contains an amount $\phi\vec{J}$ resulting from the fact that each charged particle carries with it an energy $e\phi$. The difference $\mathbf{q}-\phi\vec{J}$, however, does not depend on the potential and can be written as a linear function of the gradients $\nabla\phi$ and ∇T, similarly to (H.3) for the current density. Presently this can be written as:

$$\vec{J} = \sigma T\frac{\vec{E}}{T} - \sigma\alpha\nabla T^2\frac{\nabla T}{T^2} \tag{H.7}$$

$$\mathbf{q} - \phi \vec{J} = \beta T \frac{\vec{E}}{T} - \gamma T^2 \frac{\nabla T}{T^2} \tag{H.8}$$

the coefficients $\sigma\alpha T^2$ and βT must be equal. Thus $\beta = \sigma\alpha T$, so that $\mathbf{q} - \phi\vec{J} = \sigma\alpha T\vec{E} - \gamma\nabla T$. Finally, expressing $\vec{E}$ in terms of $\vec{J}$ and ∇T by (H.3), we have the result

$$\mathbf{q} = (\phi + \alpha T)\vec{J} - \kappa\nabla T \tag{H.9}$$

where $\kappa = \gamma - T\alpha^2\sigma$ is simply the ordinary thermal conductivity, which gives the heat flux in the absence of an electric current. It should be pointed out that the condition that ds/dt should be positive places no new restriction on the thermoelectric coefficients. Substituting (H.3) and (H.9) in (H.6) gives:

$$\frac{ds}{dt} = \int \left(\frac{J^2}{\sigma T} + \frac{\kappa(\nabla T)^2}{T^2} \right) dV > 0 \tag{H.10}$$

whence the coefficients of the thermal and electrical conductivity must be positive. The rate of change of entropy along the axial direction is obtained as:

$$\frac{ds_{zz}}{dt} = \int \left(\frac{J_{zz}^2}{\sigma_{zz} T} + \frac{\kappa_{zz}(\nabla T)^2}{T^2} \right) dV > 0 \tag{H.11}$$

Similarly, i the circumferential direction

$$\frac{ds_{cz}}{dt} = \int \left(\frac{J_{cz}^2}{\sigma_{cz} T} + \frac{\kappa_{cz}(\nabla T)^2}{T^2} \right) dV > 0 \tag{H.12}$$

Let us consider the amount of heat $-div\mathbf{q}$ evolved per unit time and volume in the conductor. Taking the divergence of (H.6), we have

$$Q = -div\mathbf{q} = \nabla(\kappa\nabla T) + \vec{J}\cdot\vec{E} + \vec{J}\cdot\nabla(\alpha T) \tag{H.13}$$

or, substituting (H.3),

$$Q = \nabla(\kappa\nabla T) + \frac{J^2}{\sigma} - T\vec{J}\cdot\nabla\alpha \tag{H.14}$$

The first term on the right is the thermal conduction, and the second term is the Joule heat. The term of interest here is the third, which gives the thermoelectric effects. Thus, the workone done by the electric field on the axial component of the current density is given as:

$$Q_{zz} = \nabla(\kappa_{zz}\nabla T) + \frac{J_{zz}^2}{\sigma_{zz}} - T\vec{J}_{zz}\cdot\nabla\alpha_{zz} \tag{H.15}$$

$$Q_{zz} = \vec{\nabla}\left(-\frac{k_B^2 T}{e^2} \left\{ \sigma_z(\vec{E}) \left[\left(\frac{\varepsilon_o - \mu}{k_B T} \right)^2 \sum_{n=-\infty}^{\infty} J_n^2(\chi) \right.\right.\right.$$
$$\left. - \frac{\Delta_z^*}{2} \left(\frac{\varepsilon_o - \mu}{k_B T} \right) \left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)} - \frac{2}{\Delta_z^*} \right) \left(1 + 3 \sum_{n=-\infty}^{\infty} J_n^2(\chi) \right) \right.$$

$$
\begin{aligned}
& - 2\Delta_s^* \left(\frac{\varepsilon_o - \mu}{k_B T}\right) \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} \sum_{n=-\infty}^{\infty} J_n^2(\chi) \\
& + \frac{\Delta_z^{*2}}{2}\left(1 - \frac{3I_o(\Delta_z^*)}{\Delta_z^* I_1(\Delta_z^*)} + \frac{6}{\Delta_z^{*2}}\right)\left(1 + \sum_{n=-\infty}^{\infty} J_n^2(\chi)\right) \\
& + \frac{\Delta_s^*\Delta_z^*}{2}\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)} - \frac{2}{\Delta_z^*}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\left(1 + 3\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right) \\
& + \Delta_s^{*2}\left(1 - \frac{I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)}\right)\sum_{n=-\infty}^{\infty} J_n^2(\chi)\Bigg] \\
& + \sigma_s(\vec{E})\sin^2\theta_h \left[\left(\frac{\varepsilon_o - \mu}{k_B T}\right)^2 \sum_{n=-\infty}^{\infty} J_n^2(\chi)\right. \\
& - \frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o - \mu}{k_B T}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*}\right)\left(1 + 3\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right) \\
& - 2\Delta_z^* \left(\frac{\varepsilon_o - \mu}{k_B T}\right) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \sum_{n=-\infty}^{\infty} J_n^2(\chi) \\
& + \frac{\Delta_s^{*2}}{2}\left(1 - \frac{3I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)} + \frac{6}{\Delta_s^{*2}}\right)\left(1 + \sum_{n=-\infty}^{\infty} J_n^2(\chi)\right) \\
& + \frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1 + 3\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right) \\
& + \Delta_z^{*2}\left(1 - \frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right)\sum_{n=-\infty}^{\infty} J_n^2(\chi)\Bigg]\Bigg\}\vec{\nabla}T\Bigg) + \frac{j_{zz}^2}{\sigma_{zz}} \\
& + T\vec{J}_{zz}\cdot\vec{\nabla}\left(\frac{\sigma_z(\vec{E})}{(\sigma_z(\vec{E}) + \sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o - \mu}{k_B T}\right) - \Delta_z^*\frac{I_o(\Delta_z^*)}{I_1(\Delta_z)} + 2 - \Delta_s^*\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right]\right. \\
& \left. + \frac{\sigma_s(\vec{E})\sin^2\theta_h}{(\sigma_z(\vec{E}) + \sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o - \mu}{k_B T}\right) - \Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} + 2 - \Delta_z^*\frac{I_1(\Delta_z)}{I_o(\Delta_z^*)}\right]\right)
\end{aligned}
\tag{H.16}
$$

Similarly, the workdone in the circumferential direction is also given as:

$$
Q_{cz} = \vec{\nabla}(\vec{\kappa}_{cz}\vec{\nabla}T) + \frac{J_{cz}^2}{\sigma_{cz}} - T\vec{J}_{cz}\cdot\vec{\nabla}\alpha_{cz} \tag{H.17}
$$

$$
\begin{aligned}
Q_{cz} = \vec{\nabla}\Bigg(& -\sigma_s(\vec{E})\frac{k_B^2 T}{e^2}\sin\theta_h\cos\theta_h \left\{\left(\frac{\varepsilon_o - \mu}{k_B T}\right)^2 \sum_{n=-\infty}^{\infty} J_n^2(\chi)\right. \\
& - \frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o - \mu}{k_B T}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*}\right)\left(1 + 3\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right) \\
& - 2\Delta_z^* \left(\frac{\varepsilon_o - \mu}{k_B T}\right) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \sum_{n=-\infty}^{\infty} J_n^2(\chi)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right)\\
&+\frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right)\\
&+\Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right)\sum_{n=-\infty}^{\infty} J_n^2(\chi)\Bigg\}\vec{\nabla}T\Bigg)+\frac{J_{cz}^2}{\sigma_{cz}}\\
&-T\vec{J}_{cz}\cdot\vec{\nabla}\left(-\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z)}{I_o(\Delta_z^*)}\right]\right) \quad \text{(H.18)}
\end{aligned}
$$

APPENDIX I

PELTIER COEFFICIENT FOR FSWCNT

The analytical deductions to establish whether the FSWCNT obeys the Onsager relations is beyond the scope of this study. However, the Onsager relation is invoked to obtain the Peltier coeficient for FSWCNT for possible refridgeration applications. The electron current densities for the FSWCNT along the axial and circumferential directions are respectively, quoted as:

$$\vec{J}_c = -\sigma_s(\vec{E})\sin\theta_h\cos\theta_h\vec{E}^*_{zn} - \sigma_s(\vec{E})\frac{k_B}{e}\sin\theta_h\cos\theta_h\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right) - \Delta^*_s\frac{I_o(\Delta^*_s)}{I_1(\Delta^*_s)} + 2 - \Delta^*_z\frac{I_1(\Delta^*_z)}{I_o(\Delta^*_z)}\right]\nabla_z T \quad (I.1)$$

$$\vec{J}_z = -(\sigma_z(\vec{E}) + \sigma_s(\vec{E})\sin^2\theta_h)\vec{E}^*_{zn} - \sigma_z(\vec{E})\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right) - \Delta^*_z\frac{I_o(\Delta^*_z)}{I_1(\Delta^*_z)} + 2 - \Delta^*_s\frac{I_1(\Delta^*_s)}{I_o(\Delta^*_s)}\right]\nabla_z T - \sigma_s(\vec{E})\frac{k_B}{e}\sin^2\theta_h\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right) - \Delta^*_s\frac{I_o(\Delta^*_s)}{I_1(\Delta^*_s)} + 2 - \Delta^*_z\frac{I_1(\Delta^*_z)}{I_o(\Delta^*_z)}\right]\nabla_z T \quad (I.2)$$

In the compact form, this I.1 and I.2 takes the form;

$$\vec{J}_c = \sigma_{cz}\vec{E}^*_{zn} - \sigma_{cz}\alpha_{cz}\nabla_z T \quad (I.3)$$

$$\vec{J}_z = \sigma_{zz}\vec{E}^*_{zn} - \sigma_{zz}\alpha_{zz}\nabla_z T \quad (I.4)$$

Since the representation of $\mathbf{q}$ in terms of $\vec{E}^*_{zn}$ is not convenient when comparing theory with experiment, it becomes necessary to express $\mathbf{q}$ in termsof $\vec{J}$ and $\nabla_z T$. Hence, making the electric field the subject $\vec{E}^*_{zn}$ from the circumferential component in I.1 yields;

$$\vec{E}^*_{zn} = -\frac{\vec{J}_c}{\sigma_s(\vec{E})\sin\theta_h\cos\theta_h} - \frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right) - \Delta^*_s\frac{I_o(\Delta^*_s)}{I_1(\Delta^*_s)} + 2 - \Delta^*_z\frac{I_1(\Delta^*_z)}{I_o(\Delta^*_z)}\right]\nabla_z T \quad (I.5)$$

Substituting $\vec{E}^*_{zn}$ to $\mathbf{q}_c$ in G.186 yields;

$$\mathbf{q}_c = \sigma_s(\vec{E})\frac{k_BT}{e}\sin\theta_h\cos\theta_h\left\{\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\sum_{n=-\infty}J^2_n(\chi) - \frac{\Delta^*_s}{2}\left(1+3\sum_{n=-\infty}J^2_n(\chi)\right)\left(\frac{I_o(\Delta^*_s)}{I_1(\Delta^*_s)} - \frac{2}{\Delta^*_s}\right) - \Delta^*_z\sum_{n=-\infty}J^2_n(\chi)\frac{I_1(\Delta^*_z)}{I_o(\Delta^*_z)}\right\}\frac{j_c}{\sigma_s(\vec{E})\sin\theta_h\cos\theta_h}$$

$$+\sigma_s(\vec{E})\frac{k_BT}{e}\sin\theta_h\cos\theta_h\left\{\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\sum_{n=-\infty} J_n^2(\chi)\right.$$

$$-\frac{\Delta_s^*}{2}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)$$

$$\left.-\Delta_z^*\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right\}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right]\nabla_zT$$

$$-\sigma_s(\vec{E})\frac{k_B^2T}{e^2}\sin\theta_h\cos\theta_h\left\{\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty} J_n^2(\chi)\right.$$

$$-\frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)$$

$$-2\Delta_z^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty} J_n^2(\chi)$$

$$+\frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^*I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty} J_n^2(\chi)\right)$$

$$+\frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)$$

$$\left.+\Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^*I_o(\Delta_z^*)}\right)\sum_{n=-\infty} J_n^2(\chi)\right\}\nabla_zT \quad \text{(I.6)}$$

Simplifying yields:

$$\mathbf{q}_c=\frac{k_BT}{e}\left\{\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\sum_{n=-\infty} J_n^2(\chi)\right.$$

$$-\frac{\Delta_s^*}{2}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)$$

$$\left.-\Delta_z^*\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right\}j_c$$

$$-\sigma_s(\vec{E})\frac{k_B^2T}{e^2}\sin\theta_h\cos\theta_h\left\{\left\{\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty} J_n^2(\chi)\right.\right.$$

$$-\frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)$$

$$-2\Delta_z^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty} J_n^2(\chi)$$

$$+\frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^*I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty} J_n^2(\chi)\right)$$

$$+\frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)$$

$$+ \Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right)\sum_{n=-\infty} J_n^2(\chi)\Bigg\}$$

$$-\Bigg\{\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\sum_{n=-\infty} J_n^2(\chi)$$

$$-\frac{\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)$$

$$-\Delta_z^*\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\Bigg\}$$

$$\times\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right]\Bigg\}\nabla_z T \quad \text{(I.7)}$$

From I.2 $\vec{E}_{zn}^*$ is given as;

$$\vec{E}_{zn}^* = -\frac{\vec{J}_z}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}$$

$$-\frac{\sigma_z(\vec{E})}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_z^*\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}+2-\Delta_s^*\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right]\nabla_z T$$

$$-\frac{\sigma_s(\vec{E})}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\sin^2\theta_h\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right]\nabla_z T$$

(I.8)

Substituting $\vec{E}_{zn}^*$ into $\mathbf{q}_z$ in G.183;

$$\mathbf{q}_z = -\frac{k_BT}{e}\Bigg\{\sigma_z(\vec{E})\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\sum_{n=-\infty} J_n^2(\chi)\right.$$

$$\left.-\frac{\Delta_z^*}{2}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)-\Delta_s^*\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right]$$

$$+\sigma_s(\vec{E})\sin^2\theta_h\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\sum_{n=-\infty} J_n^2(\chi)\right.$$

$$-\frac{\Delta_s^*}{2}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)$$

$$\left.\left.-\Delta_z^*\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right]\right]\Bigg\}\frac{\vec{J}_z}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}$$

$$+\frac{k_B^2T}{e^2}\Bigg\{\sigma_z(\vec{E})\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\sum_{n=-\infty} J_n^2(\chi)\right.$$

$$\left.-\frac{\Delta_z^*}{2}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)-\Delta_s^*\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right]$$

$$+ \sigma_s(\vec{E}) \sin^2\theta_h \left[\left(\frac{\varepsilon_o - \mu}{k_B T} \right) \sum_{n=-\infty} J_n^2(\chi) \right.$$

$$\left. \left. - \frac{\Delta_s^*}{2} \left(1 + 3 \sum_{n=-\infty} J_n^2(\chi) \right) \left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*} \right) - \Delta_z^* \sum_{n=-\infty} J_n^2(\chi) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \right] \right\}$$

$$\times \left\{ \frac{\sigma_z(\vec{E})}{(\sigma_z(\vec{E}) + \sigma_s(\vec{E}) \sin^2\theta_h)} \left[\left(\frac{\varepsilon_o - \mu}{k_B T} \right) - \Delta_z^* \frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)} + 2 - \Delta_s^* \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} \right] \right.$$

$$\left. + \frac{\sigma_s(\vec{E})}{(\sigma_z(\vec{E}) + \sigma_s(\vec{E}) \sin^2\theta_h)} \sin^2\theta_h \left[\left(\frac{\varepsilon_o - \mu}{k_B T} \right) - \Delta_s^* \frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} + 2 - \Delta_z^* \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \right] \right\} \nabla_z T$$

$$- \frac{k_B^2 T}{e^2} \left\{ \sigma_z(\vec{E}) \left[\left(\frac{\varepsilon_o - \mu}{k_B T} \right)^2 \sum_{n=-\infty} J_n^2(\chi) \right. \right.$$

$$- \frac{\Delta_z^*}{2} \left(\frac{\varepsilon_o - \mu}{k_B T} \right) \left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)} - \frac{2}{\Delta_z^*} \right) \left(1 + 3 \sum_{n=-\infty} J_n^2(\chi) \right)$$

$$- 2\Delta_s^* \left(\frac{\varepsilon_o - \mu}{k_B T} \right) \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} \sum_{n=-\infty} J_n^2(\chi)$$

$$+ \frac{\Delta_z^{*2}}{2} \left(1 - \frac{3I_o(\Delta_z^*)}{\Delta_z^* I_1(\Delta_z^*)} + \frac{6}{\Delta_z^{*2}} \right) \left(1 + \sum_{n=-\infty} J_n^2(\chi) \right)$$

$$+ \frac{\Delta_s^* \Delta_z^*}{2} \left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)} - \frac{2}{\Delta_z^*} \right) \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} \left(1 + 3 \sum_{n=-\infty} J_n^2(\chi) \right)$$

$$\left. + \Delta_s^{*2} \left(1 - \frac{I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)} \right) \sum_{n=-\infty} J_n^2(\chi) \right]$$

$$+ \sigma_s(\vec{E}) \sin^2\theta_h \left[\left(\frac{\varepsilon_o - \mu}{k_B T} \right)^2 \sum_{n=-\infty} J_n^2(\chi) \right.$$

$$- \frac{\Delta_s^*}{2} \left(\frac{\varepsilon_o - \mu}{k_B T} \right) \left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*} \right) \left(1 + 3 \sum_{n=-\infty} J_n^2(\chi) \right)$$

$$- 2\Delta_z^* \left(\frac{\varepsilon_o - \mu}{k_B T} \right) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \sum_{n=-\infty} J_n^2(\chi)$$

$$+ \frac{\Delta_s^{*2}}{2} \left(1 - \frac{3I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)} + \frac{6}{\Delta_s^{*2}} \right) \left(1 + \sum_{n=-\infty} J_n^2(\chi) \right)$$

$$+ \frac{\Delta_z^* \Delta_s^*}{2} \left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*} \right) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \left(1 + 3 \sum_{n=-\infty} J_n^2(\chi) \right)$$

$$\left. \left. + \Delta_z^{*2} \left(1 - \frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)} \right) \sum_{n=-\infty} J_n^2(\chi) \right] \right\} \nabla_z T \quad \text{(I.9)}$$

Simplifying further yields:

$$\mathbf{q}_z = \frac{k_B T}{e} \left\{ \frac{\sigma_z(\vec{E})}{(\sigma_z(\vec{E}) + \sigma_s(\vec{E}) \sin^2\theta_h)} \left[\left(\frac{\varepsilon_o - \mu}{k_B T} \right) \sum_{n=-\infty} J_n^2(\chi) \right. \right.$$

$$
\begin{aligned}
& -\frac{\Delta_z^*}{2}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)-\Delta_s^*\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\Bigg] \\
& +\frac{\sigma_s(\vec{E})\sin^2\theta_h}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\Bigg[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\sum_{n=-\infty} J_n^2(\chi) \\
& -\frac{\Delta_s^*}{2}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right) \\
& -\Delta_z^*\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\Bigg]\Bigg\}\vec{J}_z \\
& -\frac{k_B^2T}{e^2}\Bigg\{\sigma_z(\vec{E})\Bigg[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty} J_n^2(\chi) \\
& -\frac{\Delta_z^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right) \\
& -2\Delta_s^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\sum_{n=-\infty} J_n^2(\chi) \\
& +\frac{\Delta_z^{*2}}{2}\left(1-\frac{3I_o(\Delta_z^*)}{\Delta_z^* I_1(\Delta_z^*)}+\frac{6}{\Delta_z^{*2}}\right)\left(1+\sum_{n=-\infty} J_n^2(\chi)\right) \\
& +\frac{\Delta_s^*\Delta_z^*}{2}\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right) \\
& +\Delta_s^{*2}\left(1-\frac{I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)}\right)\sum_{n=-\infty} J_n^2(\chi)\Bigg] \\
& +\sigma_s(\vec{E})\sin^2\theta_h\Bigg[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty} J_n^2(\chi) \\
& -\frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right) \\
& -2\Delta_z^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty} J_n^2(\chi) \\
& +\frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty} J_n^2(\chi)\right) \\
& +\frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right) \\
& +\Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right)\sum_{n=-\infty} J_n^2(\chi)\Bigg]\Bigg\}\nabla_z T \\
& +\frac{k_B^2T}{e^2}\Bigg\{\sigma_z(\vec{E})\Bigg[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\sum_{n=-\infty} J_n^2(\chi) \\
& -\frac{\Delta_z^*}{2}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)-\Delta_s^*\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\Bigg]
\end{aligned}
$$

$$+ \sigma_s(\vec{E})\sin^2\theta_h \left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right) \sum_{n=-\infty} J_n^2(\chi) - \frac{\Delta_s^*}{2}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right) - \Delta_z^* \sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right]\Bigg\}$$

$$\times \left\{ \frac{\sigma_z(\vec{E})}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_z^*\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}+2-\Delta_s^*\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right] + \frac{\sigma_s(\vec{E})\sin^2\theta_h}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right]\right\}\nabla_z T \tag{I.10}$$

$$\mathbf{q}_z = \frac{k_BT}{e}\Bigg\{ \frac{\sigma_z(\vec{E})}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\sum_{n=-\infty} J_n^2(\chi) - \frac{\Delta_z^*}{2}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right) - \Delta_s^*\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right]$$

$$+ \frac{\sigma_s(\vec{E})\sin^2\theta_h}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\sum_{n=-\infty} J_n^2(\chi) - \frac{\Delta_s^*}{2}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right) - \Delta_z^*\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right]\Bigg\}\vec{J}_z$$

$$- \frac{k_B^2T}{e^2}\Bigg\{\sigma_z(\vec{E})\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty} J_n^2(\chi) - \frac{\Delta_z^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right) - 2\Delta_s^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\sum_{n=-\infty} J_n^2(\chi)\right.$$

$$+ \frac{\Delta_z^{*2}}{2}\left(1-\frac{3I_o(\Delta_z^*)}{\Delta_z^* I_1(\Delta_z^*)}+\frac{6}{\Delta_z^{*2}}\right)\left(1+\sum_{n=-\infty} J_n^2(\chi)\right) + \frac{\Delta_s^*\Delta_z^*}{2}\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right) \left. + \Delta_s^{*2}\left(1-\frac{I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)}\right)\sum_{n=-\infty} J_n^2(\chi)\right]$$

$$
\begin{aligned}
&+ \sigma_s(\vec{E})\sin^2\theta_h \Bigg[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2 \sum_{n=-\infty} J_n^2(\chi) \\
&- \frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right) \\
&- 2\Delta_z^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty} J_n^2(\chi) \\
&+ \frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty} J_n^2(\chi)\right) \\
&+ \frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right) \\
&+ \Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right)\sum_{n=-\infty} J_n^2(\chi)\Bigg] \\
&+ (\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)\Bigg\{\frac{\sigma_z(\vec{E})}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\Bigg[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\sum_{n=-\infty} J_n^2(\chi) \\
&- \frac{\Delta_z^*}{2}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right) \\
&- \Delta_s^*\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\Bigg]\Bigg\} \\
&+ \frac{\sigma_s(\vec{E})\sin^2\theta_h}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\Bigg[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\sum_{n=-\infty} J_n^2(\chi) \\
&- \frac{\Delta_s^*}{2}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right) \\
&- \Delta_z^*\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\Bigg] \\
&\times\Bigg\{\frac{\sigma_z(\vec{E})}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_z^*\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}+2-\Delta_s^*\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right] \\
&+ \frac{\sigma_s(\vec{E})\sin^2\theta_h}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right]\Bigg\}\Bigg\}\nabla_z T
\end{aligned} \tag{I.11}
$$

I.7 and I.11 are in the form of the Onsager relations quoted as:

$$\mathbf{q}_c = \Pi_{cz}\vec{J}_c - \kappa_{cz}\nabla_z T \tag{I.12}$$

$$\mathbf{q}_z = \Pi_{zz}\vec{J}_z - \kappa_{zz}\nabla_z T \tag{I.13}$$

where κ is the electron thermal conductivity when the carrier current density $\vec{J}$ is zero and Π is the Peltier coefficient is given as $\Pi = \alpha T$. As usual α is the thermopower. Comparing the equations, the circumferential component of the

Peltier coefficient as:

$$\Pi_{cz} = \frac{k_B}{e}\left\{\left(\frac{\varepsilon_o - \mu}{k_B T}\right)\sum_{n=-\infty} J_n^2(\chi) - \frac{\Delta_s^*}{2}\left(1 + 3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*}\right) - \Delta_z^*\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right\}T \quad (I.14)$$

and the axial component is obtained as;

$$\Pi_{zz} = \frac{k_B}{e}\left\{\frac{\sigma_z(\vec{E})}{(\sigma_z(\vec{E}) + \sigma_s(\vec{E})\sin^2\theta_h)}\left[\left(\frac{\varepsilon_o - \mu}{k_B T}\right)\sum_{n=-\infty} J_n^2(\chi) - \frac{\Delta_z^*}{2}\left(1 + 3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)} - \frac{2}{\Delta_z^*}\right) - \Delta_s^*\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right] + \frac{\sigma_s(\vec{E})\sin^2\theta_h}{(\sigma_z(\vec{E}) + \sigma_s(\vec{E})\sin^2\theta_h)}\left[\left(\frac{\varepsilon_o - \mu}{k_B T}\right)\sum_{n=-\infty} J_n^2(\chi) - \frac{\Delta_s^*}{2}\left(1 + 3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*}\right) - \Delta_z^*\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right]\right\}T \quad (I.15)$$

Comparing I.12 and I.13, the thermal conductivity is obtained again as:

$$\kappa_{cz} = \sigma_s(\vec{E})\frac{k_B^2 T}{e^2}\sin\theta_h\cos\theta_h\left[\left\{\left(\frac{\varepsilon_o - \mu}{k_B T}\right)^2\sum_{n=-\infty} J_n^2(\chi) - \frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o - \mu}{k_B T}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*}\right)\left(1 + 3\sum_{n=-\infty} J_n^2(\chi)\right) - 2\Delta_z^*\left(\frac{\varepsilon_o - \mu}{k_B T}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty} J_n^2(\chi) + \frac{\Delta_s^{*2}}{2}\left(1 - \frac{3I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)} + \frac{6}{\Delta_s^{*2}}\right)\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) + \frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1 + 3\sum_{n=-\infty} J_n^2(\chi)\right) + \Delta_z^{*2}\left(1 - \frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right)\sum_{n=-\infty} J_n^2(\chi)\right\} - \left\{\left(\frac{\varepsilon_o - \mu}{k_B T}\right)\sum_{n=-\infty} J_n^2(\chi) - \frac{\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*}\right)\left(1 + 3\sum_{n=-\infty} J_n^2(\chi)\right)\right.$$

$$- \Delta_z^* \sum_{n=-\infty} J_n^2(\chi) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \Bigg\}$$
$$\times \left[\left(\frac{\varepsilon_o - \mu}{k_B T} \right) - \Delta_s \frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} + 2 - \Delta_z^* \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \right] \Bigg] \quad \text{(I.16)}$$

$$\kappa_{zz} = \frac{k_B^2 T}{e^2} \Bigg\{ \sigma_z(\vec{E}) \Bigg[\left(\frac{\varepsilon_o - \mu}{k_B T} \right)^2 \sum_{n=-\infty} J_n^2(\chi)$$
$$- \frac{\Delta_z^*}{2} \left(\frac{\varepsilon_o - \mu}{k_B T} \right) \left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)} - \frac{2}{\Delta_z^*} \right) \left(1 + 3 \sum_{n=-\infty} J_n^2(\chi) \right)$$
$$- 2\Delta_s^* \left(\frac{\varepsilon_o - \mu}{k_B T} \right) \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} \sum_{n=-\infty} J_n^2(\chi)$$
$$+ \frac{\Delta_z^{*2}}{2} \left(1 - \frac{3 I_o(\Delta_z^*)}{\Delta_z^* I_1(\Delta_z^*)} + \frac{6}{\Delta_z^{*2}} \right) \left(1 + \sum_{n=-\infty} J_n^2(\chi) \right)$$
$$+ \frac{\Delta_s^* \Delta_z^*}{2} \left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)} - \frac{2}{\Delta_z^*} \right) \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} \left(1 + 3 \sum_{n=-\infty} J_n^2(\chi) \right)$$
$$+ \Delta_s^{*2} \left(1 - \frac{I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)} \right) \sum_{n=-\infty} J_n^2(\chi) \Bigg]$$
$$+ \sigma_s(\vec{E}) \sin^2 \theta_h \Bigg[\left(\frac{\varepsilon_o - \mu}{k_B T} \right)^2 \sum_{n=-\infty} J_n^2(\chi)$$
$$- \frac{\Delta_s^*}{2} \left(\frac{\varepsilon_o - \mu}{k_B T} \right) \left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*} \right) \left(1 + 3 \sum_{n=-\infty} J_n^2(\chi) \right)$$
$$- 2\Delta_z^* \left(\frac{\varepsilon_o - \mu}{k_B T} \right) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \sum_{n=-\infty} J_n^2(\chi)$$
$$+ \frac{\Delta_s^{*2}}{2} \left(1 - \frac{3 I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)} + \frac{6}{\Delta_s^{*2}} \right) \left(1 + \sum_{n=-\infty} J_n^2(\chi) \right)$$
$$+ \frac{\Delta_z^* \Delta_s^*}{2} \left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*} \right) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \left(1 + 3 \sum_{n=-\infty} J_n^2(\chi) \right)$$
$$+ \Delta_z^{*2} \left(1 - \frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)} \right) \sum_{n=-\infty} J_n^2(\chi) \Bigg]$$
$$+ (\sigma_z(\vec{E}) + \sigma_s(\vec{E}) \sin^2 \theta_h) \Bigg\{ \frac{\sigma_z(\vec{E})}{\sigma_z(\vec{E}) + \sigma_s(\vec{E}) \sin^2 \theta_h} \Bigg[\left(\frac{\varepsilon_o - \mu}{k_B T} \right) \sum_{n=-\infty} J_n^2(\chi)$$
$$- \frac{\Delta_z^*}{2} \left(1 + 3 \sum_{n=-\infty} J_n^2(\chi) \right) \left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)} - \frac{2}{\Delta_z^*} \right) - \Delta_s^* \sum_{n=-\infty} J_n^2(\chi) \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} \Bigg]$$
$$+ \frac{\sigma_s(\vec{E}) \sin^2 \theta_h}{\sigma_z(\vec{E}) + \sigma_s(\vec{E}) \sin^2 \theta_h} \Bigg[\left(\frac{\varepsilon_o - \mu}{k_B T} \right) \sum_{n=-\infty} J_n^2(\chi) \Bigg]$$

$$
\begin{aligned}
& - \frac{\Delta_s^*}{2}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)-\Delta_z^*\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\Bigg\} \\
& \times\Bigg\{\frac{\sigma_z(\vec{E})}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_z^*\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}+2-\Delta_s^*\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right] \\
& +\frac{\sigma_s(\vec{E})\sin^2\theta_h}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right]\Bigg\}\Bigg\}\nabla_z T
\end{aligned}
\tag{I.17}
$$

Transport of heat and charge is commonly estimated using the measured σ using the Wiedemann-Franz law $\kappa = T\sigma L$, where L is the Lorentz number. The Lorentz number along the circumferential direction L_{cz} is given as:

$$
\begin{aligned}
L_{cz} = \frac{k_B^2T}{e^2}\Bigg[\Bigg\{&\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty} J_n^2(\chi) \\
& -\frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right) \\
& -2\Delta_z^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty} J_n^2(\chi) \\
& +\frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^*I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty} J_n^2(\chi)\right) \\
& +\frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right) \\
& +\Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^*I_o(\Delta_z^*)}\right)\sum_{n=-\infty} J_n^2(\chi)\Bigg\} \\
& -\Bigg\{\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\sum_{n=-\infty} J_n^2(\chi) \\
& -\frac{\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right) \\
& -\Delta_z^*\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\Bigg\} \\
& \times\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right]\Bigg]
\end{aligned}
\tag{I.18}
$$

and the axial Lorentz number is given as:

$$
\begin{aligned}
L_{zz} = \frac{k_B^2T}{e^2(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\Bigg\{\sigma_z(\vec{E})\Bigg[&\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty} J_n^2(\chi) \\
& -\frac{\Delta_z^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)
\end{aligned}
$$

$$
\begin{aligned}
&- 2\Delta_s^* \left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\sum_{n=-\infty} J_n^2(\chi) \\
&+ \frac{\Delta_z^{*2}}{2}\left(1-\frac{3I_o(\Delta_z^*)}{\Delta_z^* I_1(\Delta_z^*)}+\frac{6}{\Delta_z^{*2}}\right)\left(1+\sum_{n=-\infty} J_n^2(\chi)\right) \\
&+ \frac{\Delta_s^*\Delta_z^*}{2}\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right) \\
&+ \Delta_s^{*2}\left(1-\frac{I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)}\right)\sum_{n=-\infty} J_n^2(\chi)\Bigg] \\
&+ \sigma_s(\vec{E})\sin^2\theta_h\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty} J_n^2(\chi)\right. \\
&- \frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right) \\
&- 2\Delta_z^* \left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty} J_n^2(\chi) \\
&+ \frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty} J_n^2(\chi)\right) \\
&+ \frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right) \\
&+ \Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right)\sum_{n=-\infty} J_n^2(\chi)\Bigg] \\
&+ (\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)\left\{\frac{\sigma_z(\vec{E})}{\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\sum_{n=-\infty} J_n^2(\chi)\right.\right. \\
&- \frac{\Delta_z^*}{2}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)-\Delta_s^*\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\Bigg] \\
&+ \frac{\sigma_s(\vec{E})\sin^2\theta_h}{\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\sum_{n=-\infty} J_n^2(\chi)\right] \\
&- \frac{\Delta_s^*}{2}\left(1+3\sum_{n=-\infty} J_n^2(\chi)\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)-\Delta_z^*\sum_{n=-\infty} J_n^2(\chi)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\Bigg\} \\
&\times\left\{\frac{\sigma_z(\vec{E})}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_z^*\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}+2-\Delta_s^*\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right]\right. \\
&+ \left.\left.\frac{\sigma_s(\vec{E})\sin^2\theta_h}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right]\right\}\right\}\nabla_z T
\end{aligned}
\tag{I.19}
$$

APPENDIX J

THERMOELECTRIC TRANSPORT MATRIX

When the FSWCNT is subjected to simultaneous thermal and electric potential gradients, leading in the presence of coupled thermal and electrical currents, a number of TE effects may arise. The following general expressions are obtained assuming a reasonable first approximation and a linear dependence between the electrical $\mathbf{j}$ and thermal $\mathbf{q}$ current densities on one side, and the electrical potential $\nabla\phi$ and temperature ∇T gradients that originate them on the other:

$$\mathbf{j} = -(\mathbf{L}_{11}\nabla\phi + \mathbf{L}_{11}\nabla T) \tag{J.1}$$

$$\mathbf{q} = -(\mathbf{L}_{21}\nabla\phi + \mathbf{L}_{22}\nabla T) \tag{J.2}$$

In the case of FSWCNT with anisotropic physical characteristics, the coefficients $\mathbf{j}$ are tensors. For FSWCNT with a high structural symmetry degree and thus isotropic behavior, these tensor magnitudes are reduced to scalar numbers. The negative sign is used to describe the phenomenological behaviour found for thermal (Fourier's law) and electrical (Ohm's law) currents. According to J.1 and J.2, the present densities $\mathbf{j}$ and $\mathbf{q}$ can be stated in a unified manner by using the matrix expression:

$$\begin{pmatrix} \mathbf{j} \\ \mathbf{q} \end{pmatrix} = \begin{pmatrix} \mathbf{L}_{11} & \mathbf{L}_{12} \\ \mathbf{L}_{21} & \mathbf{L}_{22} \end{pmatrix} \begin{pmatrix} \nabla\phi \\ \nabla T \end{pmatrix} \tag{J.3}$$

which can be written in a more compact vectorial form as:

$$\mathbf{J} = -\tilde{L}\nabla U \tag{J.4}$$

where $\tilde{L}$ is the TE transport matrix tensor, $\mathbf{J} \equiv (\mathbf{j},\mathbf{q})^t$ is the current vector, and $U \equiv (\phi,T)^t$ is the vector transposition, and the superscript t indicates vector transposition. We can demonstrate that, despite their conceptual simplicity, the transport matrix elements L_{ij} are not accessible to direct measurement by remembering the important discoveries. TE effects are instead naturally characterized in terms of a number of transport coefficients, such as the thermal conductivity $kappa$, the electrical conductivity $\sigma = \rho^{-1}$, and the mutually related Seebeck, α, and Peltier, Π. As a result, it is straightforward to express the transport matrix elements L_{ij} in terms of these transport coefficients. To this goal, consider the following experimental settings:

- The FSWCNT is kept at constant temperature ($\nabla T \equiv 0$) and an electrical current $\mathbf{j}$ is generated by applying an external voltage $\nabla\phi$. Taking the Ohm's law into consideration $\mathbf{j} = -\sigma\nabla\phi$, J.3 gives:

$$\sigma(T) = L_{11} \tag{J.5}$$

- Electric current cannot flow through the FSWCNT because it is electrically insulated ($\mathbf{j} = 0$) and a thermal gradient ∇T is applied to generate the Seebeck potential $\nabla\phi = -\alpha\nabla T$. Hence, from J.3 one gets:

$$S(T) = L_{12}L_{11}^{-1} \tag{J.6}$$

- The FSWCNT is kept at constant temperature ($\nabla T \equiv 0$) as an electrical current $\mathbf{j}$ flows through the FSWCNT. Due to the Peltier effect, the presence of a thermal current density which is proportional to the electric current is observed, that is $q = \Pi\mathbf{j}$, so that from J.3 one obtains:

$$\Pi(T) = L_{21}L_{11}^{-1} \tag{J.7}$$

- The FSWCNT is electrically insulated to prevent any electric current from flowing through it ($\mathbf{j} = 0$) while a thermal gradient ∇T is maintained. According to Fourier's law, the measured heat current density is given as: $\mathbf{q} = \kappa\nabla T$, so that from J.3 one obtains:

$$\kappa(T) = L_{22} - L_{12}L_{21}L_{11}^{-1} \tag{J.8}$$

By a proper combination the nested equations given in J.5-J.8 and taking note of the first Kelvin relation $\Pi = \alpha T$, J.3 can be finally expressed in the form:

$$\begin{pmatrix} \mathbf{j} \\ \mathbf{q} \end{pmatrix} = \begin{pmatrix} \sigma & \sigma\alpha \\ \sigma\alpha T & \sigma\alpha^2 T \end{pmatrix} \begin{pmatrix} -\nabla\phi \\ -\nabla T \end{pmatrix} \tag{J.9}$$

Thus, the transport coefficients $\sigma(T)$, $\kappa(T)$, and $\alpha(T)$ completely determines the TE transport matrix describing the relations between currents and gradients linearly. In the limit where $\alpha = 0$, the transport matrix is diagonal and $\mathbf{j}$ and $\mathbf{q}$ are completely separated from one other. However, the thermopower in the nondiagonal terms of the TE transport matrix, determines the coupled transport of charge and heat through the FSWCNT. However, the TE transport matrix in J.9 simplifies when $\kappa \to 0$. This analytical result shows that FSWCNTs exhibiting low thermal conductivity is of interest in TE studies.

APPENDIX K

PUBLICATION(S)

- [1] Sekyi-Arthur, D., Mensah, S. Y., K.W. Adu, Mensah, N. G., Dompreh, K. A., and Edziah, R. Tunable power factor of fluorine-doped carbon nanotube. J. Appl. Phys. 128, 244301 (2020). doi: 10.1063/5.0031326
- [2] Sekyi-Arthur, D., Mensah, S.Y., Adu, K.W., Dompreh, K.A., Edziah, R.Mensah, N.G., and Jebuni-Adanu, C.(2020). Induced Hall-like current by Acoustic Phonons in Semiconductor Fluorinated Carbon Nanotube . World Journal of Condensed Matter Physics, 10, 71-87.https://doi.org/10.4236/wjcmp.2020. 102005
- [3] Sekyi-Arthur, D., Mensah, S.Y., Adu, K.W., Dompreh, K.A., Edziah, R. and Mensah, N.G. (2020). Acoustoelectric Effect in Fluorinated Carbon Nanotube in the Absence of External Electric Field. World Journal of Condensed Matter Physics, 10, 1-11.https://doi.org/10.4236/wjcmp.2020. 101001
- [4] Sekyi-Arthur, D., Mensah, S.Y., Adu, K.W., Dompreh, K.A., Edziah, R., Mensah, N.G., and Jebuni-Adanu, C. (2020). Semiconductor Fluorinated Carbon Nanotube as a Low Voltage Current Amplifier Acoustic Device. World Journal of Condensed Matter Physics, 10, 12-26.https://doi.org/ 10.4236/wjcmp.2020.101002
- [5] Sekyi-Arthur, D., S. Y. Mensah, N. G. Mensah, K. A. Dompreh, and R. Edziah. Absorption of Acoustic Phonons in Fluorinated Carbon Nanotube with Double Periodic Band. Phonons in Low Dimensional Structures (pp.129-142). InTech. http://dx.doi.org/10.5772/intechopen.78231 (2018)
- [6] Sekyi-Arthur, D., Mensah, S.Y., Adu, K.W., Dompreh, K.A., Edziah, R. and Mensah, N.G. (2021). Fluorine Doped Single Walled Carbon Nanotube for a SASER Device. World Journal of Condensed Matter Physics. ID: 4800486 (accepted)
- [7] Sekyi-Arthur,D.; Mensah, S.Y.; Dompreh, K.A.; Adu, K.W.;Edziah, R; Mensah, N.G. Giant Thermoelectric Figure of Merit in Fluorine Doped Single Walled Carbon Nanotube. Nanomaterials 2021, 1,0.https://doi.org/10.20944/preprints202106.0070.v1

APPENDIX A

ACOUSTODYNAMICS IN FSWCNT

The energy dispersion for the semiconductor FSWCNT is give as:

$$\varepsilon(\vec{p}_z) = \varepsilon_o + \Xi_n \gamma_o \cos^{2N-1}(a\vec{p}_z) \tag{A.1}$$

For $N = 2$, the dispersion relation yields:

$$\varepsilon(\vec{p}_z) = \varepsilon_o + 8\gamma_0 \cos^3(a\vec{p}_z) \tag{A.2}$$

A.2 is expanded as:

$$\varepsilon(\vec{p}) = \varepsilon_o + \Delta_1 \cos(3a\vec{p}_z) + \Delta_2 \cos(a\vec{p}_z) \tag{A.3}$$

where $\Delta_1 = 2\gamma_o$ and $\Delta_2 = 6\gamma_o$. The acoustoelectric current density is defined as:

$$\vec{j}^{AE} = -e \sum_{n,n'} \int U^{ac}_{n,n'} \Psi_i(\vec{p}_z) d^2 p_z. \tag{A.4}$$

Here $\Psi_i(\vec{p})$ is the solution to the BTE in the absence of a magnetic field. $U^{ac}_{n,n'}$ is the electron-phonon interaction and is given as:

$$U^{ac}_{n,n'} =$$
$$\frac{2\pi\vec{\Phi}}{\omega_q v_s} \sum_{n,n'} \{ |G_{\vec{p}_z - \hbar\vec{q}, \vec{p}_z}|^2 [f(\varepsilon_n(\vec{p}_z - \hbar\vec{q})) - f(\varepsilon_n(\vec{p}_z))] \delta(\varepsilon_n(\vec{p}_z - \hbar\vec{q}) - \varepsilon_n(\vec{p}_z) + \hbar\omega_q)$$
$$+ |G_{\vec{p}_z + \hbar\vec{q}, \vec{p}_z}|^2 [f(\varepsilon_{n'}(\vec{p}_z + \hbar\vec{q})) - f(\varepsilon_{n'}(\vec{p}_z))] \delta(\varepsilon_{n'}(\vec{p}_z + \hbar\vec{q}) - \varepsilon_{n'}(\vec{p}_z) - \hbar\omega_q) \} \tag{A.5}$$

Employing the principle of detailed balance and denoting $\vec{p}_z^{\,\prime} = \vec{p}_z \pm \hbar\vec{q}$, yields the condition:

The condition $\vec{p}_z^{\,\prime} = \vec{p}_z \pm \hbar\vec{q}$ is obtained by applying the principle of detailed balance and indicating $vecp'z = vecpzpmhbarvecq$.

$$|G_{p',p}|^2 = |G_{p,p'}|^2 \tag{A.6}$$

and the electron-phonon interaction's matrix element is provided as

$$|G_{p',p}| = \frac{4\pi e \mathscr{K}}{\sqrt{2\rho\omega_q \varepsilon}} \tag{A.7}$$

where $\mathscr{K}$ represents the piezoelectric modulus, ε represents the lattice dielectric constant, and ρ represents the FSWCNT density. After that, the AE current density adopts the following shape:

$$\vec{j}_z^{AE} = -\frac{2e}{(2\pi\hbar)^2} \frac{2\pi\vec{\Phi}}{\omega_q v_s} \sum_{n,n'} |G_{p_z', p_z}|^2 [f(\varepsilon_{n'}(\vec{p}_z)) - f(\varepsilon_{n'}(\vec{p}_z + \hbar q))]$$

$$\times [\Psi_i(p_z+\hbar q)-\Psi_i(p_z)]\delta(\varepsilon_{n'}(\vec{p}_z+\hbar q)-\varepsilon_{n'}(\vec{p}_z)-\hbar\omega_q)d^2\vec{p}_z \quad (A.8)$$

where, $\Psi_i(p_z)=l_i(\vec{p}_z)$ is the electron mean free path written as:

$$l_z=\tau v_z \quad (A.9)$$

$$v_z=\frac{\partial\varepsilon(\vec{p})}{\partial\vec{p}} \quad (A.10)$$

Substituting A.9 and A.10 into A.8 yields:

$$\vec{j}_z^{AE}=-\frac{16e^3\pi\vec{\Phi}\mathscr{K}^2\tau}{2\varepsilon^2\omega_q^2 v_s\rho\hbar^2}\sum_{n,n'}[f(\varepsilon_{n'}(\vec{p}_z))-f(\varepsilon_{n'}(\vec{p}_z+\hbar\vec{q}))]$$
$$\times [v_z(p_z+\hbar\vec{q})-v_z(\vec{p})]\delta(\varepsilon_{n'}(\vec{p}_z+\hbar\vec{q})-\varepsilon_{n'}(\vec{p}_z)-\hbar\omega_q)dp_z \quad (A.11)$$

The electron distribution function is given by the shifted fermi-dirac distribution as

$$f_o(p)=\frac{1}{1+\exp[(\varepsilon(\vec{p})-\mu)/k_BT]} \quad (A.12)$$

When A.12 is substituted for A.11, the result is an equation with the term $\mathscr{F}_{1/2}$, which represents the Fermi-Dirac integral of order $1/2$ and is expressed as:

$$\vec{j}_z^{AE}=-\frac{16e^3\pi\vec{\Phi}\mathscr{K}^2\tau}{2\varepsilon^2\omega_q^2 v_s\rho\hbar^2}\sum_{n,n'}[\mathscr{F}_{1/2}(\varepsilon_{n'}(\vec{p}_z))-\mathscr{F}_{1/2}(\varepsilon_{n'}(\vec{p}_z+\hbar\vec{q}))]$$
$$\times [v_z(p_z+\hbar\vec{q})-v_z(p_z)]\delta(\varepsilon_{n'}(\vec{p}_z+\hbar q)-\varepsilon_{n'}(\vec{p}_z)-\hbar\omega_q)d\vec{p} \quad (A.13)$$

The Fermi-Dirac integral $\mathscr{F}_{1/2}$, is given as:

$$\mathscr{F}_{1/2}=\frac{1}{\Gamma(1/2)}\int_0^\infty\frac{\eta_f^{1/2}d\eta}{1+\exp(\eta-\eta_f)} \quad (A.14)$$

$(\mu-\varepsilon_c)/k_BT\equiv\eta_f$ and $\Gamma(1/2)$ is the Gamma function of the order $1/2$. For non-degenerate electron gas, A.12 becomes:

$$f_o(\vec{p})=A^\dagger\exp\left(-\frac{\varepsilon(\vec{p})-\mu}{k_BT}\right) \quad (A.15)$$

Solving the Boltzmann equation in the $\tau-$ approximation yields the electron distribution function in the presence of a constant electric field, $\vec{E}(t)$:

$$f(\vec{p}_z)=\int_0^\infty\frac{dt}{\tau}\exp(-t/\tau)f_o(\vec{p}_z-\vec{p}') \quad (A.16)$$

where $A^\dagger$ is the normalization constant that may be calculated using the normalization condition $\int f(\vec{p})d\vec{p}=n_o$ as follows:

$$A^\dagger=\frac{3n_oa^2}{2I_o(\Delta_1^*)I_o(\Delta_2^*)}\exp\left(\frac{\varepsilon_o-\mu}{k_BT}\right) \quad (A.17)$$

The dispersion relation yields the following electron velocity:

$$v_z(\vec{p}_z) = -[3a\Delta_1 \sin(3a\vec{p}_z) + a\Delta_2 \sin(a\vec{p}_z)] \tag{A.18}$$

Substituting A.14-A.17 into A.11, the current density is obtained as

$$\begin{aligned}
\vec{j}_z^{AE} &= \frac{4A^\dagger \pi \vec{\Phi} e^3 \mathscr{K}^2 \tau \Theta(1-\alpha^2)}{\hbar^3 \omega_q^2 \varepsilon^2 v_s \rho a q \sqrt{1-\alpha^2}} \int_0^\infty \exp\left(\frac{dt'}{\tau}\right) \\
&\left\{ \sinh\left[\Delta_1^* \cos(3ea\vec{E}t') \sin A \sin\left(\frac{3}{2}a\hbar q\right) + \Delta_2^* \cos(ea\vec{E}t') \sin B \sin\left(\frac{a}{2}\hbar q\right)\right]\right. \\
&\times \sinh\left[\Delta_1^* \cos(3ea\vec{E}t') \cos A \cos\left(\frac{3}{2}a\hbar q\right) + \Delta_2^* \cos(ea\vec{E}t') \cos B \cos\left(\frac{a}{2}\hbar q\right)\right] \\
&- 4\left(\Delta_2^* \sin(ea\vec{E}t') \cos B \sin\left(\frac{a}{2}\hbar q\right) + \Delta_1^* \cos A \sin(3ea\vec{E}t') \sin\left(\frac{3}{2}a\hbar q\right)\right. \\
&\left. + \Delta_1^* \Delta_2^* \sin(ea\vec{E}t') \sin(3ea\vec{E}t') \cos A \cos B \sin\left(\frac{a}{2}\hbar q\right) \sin\left(\frac{3}{2}a\hbar q\right)\right) \\
&\times \cosh\left[\Delta_1^* \cos(3ea\vec{E}t') \cos A \cos\left(\frac{3}{2}a\hbar q\right) + \Delta_2^* \cos(ea\vec{E}t') \cos B \cos\left(\frac{a}{2}\hbar q\right)\right] \\
&\left. \times \cosh\left[\Delta_1^* \cos(3ea\vec{E}t') \sin A \sin\left(\frac{3}{2}a\hbar q\right) + \Delta_2^* \cos(ea\vec{E}t') \sin B \sin\left(\frac{a}{2}\hbar q\right)\right]\right\}
\end{aligned} \tag{D.19}$$

where $\Delta_1^* = \Delta_1/k_B T$, and $\Delta_2^* = \Delta_2/k_B T$. Switching off the external electric field ($\vec{E} = 0$), A.19 reduces to:

$$\begin{aligned}
\vec{j}_z^{FSWCNT} &= \frac{4A^\dagger \pi \vec{\Phi} e^3 \mathscr{K}^2 \tau \Theta(1-\alpha^2)}{\hbar^3 \omega_q^2 \varepsilon^2 v_s \rho a q \sqrt{1-\alpha^2}} \\
&\times \left[\sinh\left\{\Delta_1^* \sin\left(\frac{3}{2}a\hbar q\right) \sin A + \Delta_2^* \sin\left(\frac{a}{2}\hbar q\right) \sin B\right\}\right. \\
&\left. \times \sinh\left\{\Delta_1^* \cos\left(\frac{3}{2}a\hbar q\right) \cos A + \Delta_2^* \cos\left(\frac{a}{2}\hbar q\right) \cos B\right\}\right]
\end{aligned} \tag{A.20}$$

Simplifying A.19 gives;

$$\begin{aligned}
\vec{j}_z^{FSWCNT} = \vec{j}_o &\left[\sinh\left\{\Delta_1^* \sin\left(\frac{3}{2}a\hbar q\right) \sin A + \Delta_2^* \sin\left(\frac{a}{2}\hbar q\right) \sin B\right\}\right. \\
&\left. \times \sinh\left\{\Delta_1^* \cos\left(\frac{3}{2}a\hbar q\right) \cos A + \Delta_2^* \cos\left(\frac{a}{2}\hbar q\right) \cos B\right\}\right]
\end{aligned} \tag{A.21}$$

where

$$\vec{j}_o = \frac{6A^\dagger \pi \vec{\Phi} e^3 \mathscr{K}^2 a \tau \Theta(1-\alpha^2)}{\hbar^3 \omega_q^2 \varepsilon v_s \rho q \sqrt{1-\alpha^2}}, \qquad \alpha = \frac{\omega_q}{12\gamma_o a q}$$

and

$$A = \frac{3}{4}\sin^{-1}\left(\frac{\omega_q}{12\gamma_o aq}\right), \quad B = \frac{1}{4}\sin^{-1}\left(\frac{\omega_q}{12\gamma_o aq}\right)$$

and Θ is a Heaviside step function.

The same approach is used for the FSWCNT to compare the results with those of undoped SWCNT. Using the $\vec{p}_z$ orbital's tight-binding energy dispersion, which is given as:

$$\varepsilon(\vec{p}_z) = \pm\gamma_o\left(1 - 2\cos\left(\frac{\vec{p}_z\sqrt{3}b}{2\hbar}\right)\right) \tag{A.24}$$

From A.22, the acoustocurrent density in undoped SWCNT determined by deformation potential is given by;

$$\vec{j}_z^{SWCNT} = \vec{j}_o^{SWCNT}\sinh\left\{\frac{\hbar\omega_q}{k_BT}\right\}\sinh\left\{\Delta^*\sqrt{1-\alpha^2}\cos\left(\frac{a\hbar q}{2}\right)\right\} \tag{A.25}$$

where

$$\vec{j}_o^{SWCNT} = \frac{\Lambda^2 q^2 e\tau n_o a\Theta(1-\alpha_o^2)}{\pi\hbar^2\omega_q^2 v_s\rho\sin(a\hbar q/2)I_o(\Delta^*)\sqrt{1-\alpha_o^2}} \tag{A.26}$$

and

$$\alpha_o = \frac{\hbar\omega_q}{4\gamma_o\sin(a\hbar q/2)}, \quad \Delta = 4\gamma_o, \quad \Delta^* = \Delta/k_BT$$

APPENDIX B

LOW VOLTAGE-CURRENT ACOUSTIC DEVICE

The FSWCNT energy band is quoted as:

$$\varepsilon(\vec{p}_z) = \varepsilon_o + \Delta_1 \cos(3a\vec{p}_z) + \Delta_2 \cos(a\vec{p}_z) \tag{B.1}$$

where p_o, is the momentum within the first BZ i.e $-\pi/a \leq p_o \leq \pi/a$, with $\Delta_1 = 2\Delta$, and $\Delta_2 = 6\Delta$.
Following the procedure in Appendix A, in the presence of an external electric field, yields

$$\begin{aligned}
j_z^{AE} = & \frac{4A^{\dagger}\pi\vec{\Phi}e^3\mathscr{K}^2\Theta(1-\alpha^2)}{\hbar^3\omega_q^2\varepsilon^2\rho aq\sqrt{1-\alpha^2}} \int_0^{\infty} \exp\left(-\frac{dt'}{\tau}\right) \\
& \times \left\{ \sinh\left[\Delta_1^* \cos(3ea\vec{E}t') \sin A \sin\left(\frac{3}{2}a\hbar q\right) + \Delta_2^* \cos(ea\vec{E}t') \sin B \sin\left(\frac{a}{2}\hbar q\right)\right] \right. \\
& \times \sinh\left[\Delta_1^* \cos(3ea\vec{E}t') \cos A \cos\left(\frac{3}{2}a\hbar q\right) + \Delta_2^* \cos(ea\vec{E}t') \cos B \cos\left(\frac{a}{2}\hbar q\right)\right] \\
& - 4\left(\Delta_2^* \sin(ea\vec{E}t') \cos B \sin\left(\frac{a}{2}\hbar q\right) + \Delta_1^* \cos A \sin(3ea\vec{E}t') \sin\left(\frac{3}{2}a\hbar q\right)\right. \\
& \left. + \Delta_1^*\Delta_2^* \sin(p'a) \sin(3ea\vec{E}t') \cos A \cos B \sin\left(\frac{a}{2}\hbar q\right) \sin\left(\frac{3}{2}a\hbar q\right)\right) \\
& \times \cosh\left[\Delta_1^* \cos(3ea\vec{E}t') \cos A \cos\left(\frac{3}{2}a\hbar q\right) + \Delta_2^* \cos(ea\vec{E}t') \cos B \cos\left(\frac{a}{2}\hbar q\right)\right] \\
& \left. \times \cosh\left[\Delta_1^* \cos(3ea\vec{E}t') \sin A \sin\left(\frac{3}{2}a\hbar q\right) + \Delta_2^* \cos(ea\vec{E}t') \sin B \sin\left(\frac{a}{2}\hbar q\right)\right] \right\}
\end{aligned} \tag{B.1}$$

Simplifying B.1 yields:

$$\begin{aligned}
\vec{j}_z^{AE} = \vec{j}_z^{AE}(0) & \left\{ 1 - 4\left(\Delta_2^* \sin\left(\chi\left(1-\frac{v_d}{v_s}\right)\right) \cos B \sin\left(\frac{a}{2}\hbar q\right)\right.\right. \\
& \left. + \Delta_1^* \cos A \sin\left(3\chi\left(1-\frac{v_d}{v_s}\right)\right) \sin\left(\frac{3}{2}a\hbar q\right)\right) \\
& \times \coth\left[\Delta_1^* \cos\left(3\chi\left(1-\frac{v_d}{v_s}\right)\right) \cos A \cos\left(\frac{3}{2}a\hbar q\right)\right. \\
& \left. + \Delta_2^* \cos\left(\chi\left(1-\frac{v_d}{v_s}\right)\right) \cos B \cos\left(\frac{a}{2}\hbar q\right)\right] \\
& \times \coth\left[\Delta_1^* \cos\left(3\chi\left(1-\frac{v_d}{v_s}\right)\right) \sin A \sin\left(\frac{3}{2}a\hbar q\right)\right. \\
& \left.\left. + \Delta_2^* \cos\left(\chi\left(1-\frac{v_d}{v_s}\right)\right) \sin B \sin\left(\frac{a}{2}\hbar q\right)\right]\right\}
\end{aligned} \tag{B.2}$$

where $\vec{j}_z^{AE}(0)$ denotes the acoustoelectric current density in the absence of an external electric field and is defined as:

$$\vec{j}_z^{AE}(0) = \vec{j}_o \left[\sinh\left\{ \Delta_1^* \sin\left(\frac{3}{2}a\hbar q\right) \sin A + \Delta_2^* \sin\left(\frac{a}{2}\hbar q\right) \sin B \right\} \right. \\ \left. \times \sinh\left\{ \Delta_1^* \cos\left(\frac{3}{2}a\hbar q\right) \cos A + \Delta_2^* \cos\left(\frac{a}{2}\hbar q\right) \cos B \right\} \right] \quad \text{(B.22)}$$

and $\alpha = \omega_q(1 - v_d/v_s)/12\gamma_o aq$ with

$$\vec{j}_o = \frac{4A^\dagger \pi \vec{\Phi} e^3 \mathscr{K}^2 \tau \Theta (1-\alpha^2)}{\hbar^3 \omega_q^2 \varepsilon^2 \rho a q \sqrt{1-\alpha^2}} \qquad \chi = \hbar \omega_q (1 - v_d/v_s) a / v_s$$

APPENDIX C

HALL-LIKE CURRENT INDUCED BY ACOUSTIC PHONONS

The current density for such an orientation is given as:

$$\vec{j} = \frac{2e}{(2\pi\hbar)^2} \sum_{n,n'} \int U^{ac}_{n,n'} \Psi_i(\vec{p},\vec{H}) d^2\vec{p}_z \tag{C.2}$$

$$\vec{v}\frac{\partial \Psi_i}{\partial \vec{p}} + \vec{W}_p\{\Psi\} = \vec{v}_i \tag{C.3}$$

Ψ_i is the root of the kinetic equation given as:

$$\frac{e}{c}\left(\vec{v}\times\vec{H}\right)\frac{\partial \Psi_i}{\partial \vec{p}} + \vec{W}_p\{\Psi\} = \vec{v}_i \tag{C.5}$$

where $\vec{v}_i$ is the electron velocity and $\vec{W}_p\{...\} = (\partial f/\partial\varepsilon...)^{-1}\vec{W}_p(\partial f/\partial\varepsilon)$. The operator $\vec{W}_{\vec{p}}$ is a Hermitian operator and it is the collision operator describing the relaxation of the nonequilibrium distribution of the electron. Assuming τ to be constant, the collision operator has the form $\vec{W}_{\vec{p}} = 1/\tau$. Assuming a solution to C.5 as:

$$\Psi_i = \Psi_i^0 + \Psi_i^1 + \Psi_i^2 + ... \tag{C.6}$$

Substituting C.6 into C.5 and solving by the method of iteration to obtain the zero approximation in the absence of the magnetic field ($\vec{H} = 0$) as:

$$\Psi_i^0 = v_i\tau \tag{C.7}$$

Similarly, the first approximation yields:

$$\Psi_i^1 = -\frac{\tau^2 e}{mc}(\vec{v}\times\vec{H})_i \tag{C.8}$$

and $i = x,y,z$. Substituting C.7 and C.8 into C.3 and using the principle of detailed balance, i.e $|G_{p',p}|^2 = |G_{p,p'}|^2$, the net current density is obtaied as:

$$\begin{aligned}\vec{j}_i = &\frac{2e}{(2\pi\hbar)^2}\frac{2\pi\vec{\Phi}}{\omega_q v_s}\sum_{n,n'}\int |G_{\vec{p}_z+\hbar q,\vec{p}_z}|^2[f(\varepsilon_{n'}(\vec{p}_z)) - f(\varepsilon_{n'}(\vec{p}_z+\hbar\vec{q}))]\\ &\times[\Psi_i(\vec{p}_z+\hbar\vec{q}) - \Psi_i(\vec{p}_z)]\delta(\varepsilon_{n'}(\vec{p}_z+\hbar\vec{q}) - \varepsilon_{n'}(\vec{p}_z) - \hbar\omega_q)d^2\vec{p}_z\\ &-\frac{2e}{(2\pi\hbar)^2}\frac{2\pi\vec{\Phi}}{\omega_q v_s}\frac{e\tau^2}{mc}\sum_{n,n'}\int |G_{\vec{p}_z+\hbar\vec{q},\vec{p}_z}|^2[f(\varepsilon_{n'}(\vec{p}_z)) - f(\varepsilon_{n'}(\vec{p}_z+\hbar\vec{q}))]\\ &\times[\Psi_i(\vec{p}_z+\hbar\vec{q}) - \Psi_i(\vec{p}_z)]\delta(\varepsilon_{n'}(\vec{p}_z+\hbar\vec{q}) - \varepsilon_{n'}(\vec{p}_z) - \hbar\omega_q)d^2\vec{p}_z\end{aligned} \tag{C.9}$$

Substituting A.7, A.9 and A.10 into C.9 yields;

$$\vec{j} = -\frac{2\mathscr{K}^2\pi\vec{\Phi}e^3\tau\Theta(1-\alpha^2)}{\hbar^3\omega_q^2\varepsilon\rho a\sqrt{1-\alpha^2}}\sum_{n,n'}\int[f(\varepsilon_{n'}(\vec{p}_z)) - f(\varepsilon_{n'}(\vec{p}_z+\hbar\vec{q}))]$$

$$\times [v_z(\vec{p}_z+\hbar\vec{q})-v_z(\vec{p}_z)]\delta(\varepsilon_{n'}(\vec{p}_z+\hbar\vec{q})-\varepsilon_{n'}(\vec{p}_z)-\hbar\omega_q)d\vec{p}_z$$
$$-\frac{2\mathscr{K}^2\pi\vec{\Phi}e^4\tau^2\Theta(1-\alpha^2)}{\hbar^3\omega_q^2\varepsilon\rho a\sqrt{1-\alpha^2}mc}\sum_{n,n'}\int[f(\varepsilon_{n'}(\vec{p}_z))-f(\varepsilon_{n'}(\vec{p}_z+\hbar\vec{q}))]$$
$$\times[(\vec{v}(\vec{p}_z+\hbar\vec{q})\times\vec{H})-(\vec{v}(\vec{p}_z)\times\vec{H})]\delta(\varepsilon_{n'}(\vec{p}_z+\hbar q)-\varepsilon_{n'}(\vec{p}_z)-\hbar\omega_q)d^2\vec{p}_z \tag{C.13}$$

Similarly, the AME current density in C.13 is obtained after some cumbersome calculations as:

$$\vec{j}_y^{AME}=-\frac{2A^\dagger\mathscr{K}^2\pi\vec{\Phi}e^3\tau^2\Theta(1-\alpha^2)\vec{\Omega}}{\hbar^3\omega_q^2\varepsilon\rho a\sqrt{1-\alpha^2}}\int_0^\infty\exp\left(-\frac{dt'}{\tau}\right)$$
$$\times\left\{\sinh\left[\Delta_1^*\cos(3eaEt')\sin A\sin\left(\frac{3}{2}a\hbar q\right)+\Delta_2^*\cos(eaEt')\sin B\sin\left(\frac{a}{2}\hbar q\right)\right]\right.$$
$$\times\sinh\left[\Delta_1^*\cos(3eaEt')\cos A\cos\left(\frac{3}{2}a\hbar q\right)+\Delta_2^*\cos(eaEt')\cos B\cos\left(\frac{a}{2}\hbar q\right)\right]$$
$$-4\left(\Delta_2^*\sin(eaEt')\cos B\sin\left(\frac{a}{2}\hbar q\right)+\Delta_1^*\cos A\sin(3eaEt')\sin\left(\frac{3}{2}a\hbar q\right)\right.$$
$$\left.+\Delta_1^*\Delta_2^*\sin(eaEt')\sin(3eaEt')\cos A\cos B\sin\left(\frac{a}{2}\hbar q\right)\sin\left(\frac{3}{2}a\hbar q\right)\right)$$
$$\times\cosh\left[\Delta_1^*\cos(3eaEt')\cos A\cos\left(\frac{3}{2}a\hbar q\right)+\Delta_2^*\cos(eaEt')\cos B\cos\left(\frac{a}{2}\hbar q\right)\right]$$
$$\left.\times\cosh\left[\Delta_1^*\cos(3eaEt')\sin A\sin\left(\frac{3}{2}a\hbar q\right)+\Delta_2^*\cos(eaEt')\sin B\sin\left(\frac{a}{2}\hbar q\right)\right]\right\} \tag{C.17}$$

where $\vec{\Omega}=\mu\vec{H}/\hbar c$, $\Delta_1^*=\Delta_1/k_BT$, $\Delta_2^*=\Delta_2/k_BT$. Simplifying further yields:

$$\vec{j}_y^{AME}=\vec{j}_y(0)\left\{1-4\left(\Delta_2^*\sin\left(\chi\left(1-\frac{v_d}{v_s}\right)\right)\cos B\sin\left(\frac{a}{2}\hbar q\right)\right.\right.$$
$$\left.+\Delta_1^*\cos A\sin\left(3\chi\left(1-\frac{v_d}{v_s}\right)\right)\sin\left(\frac{3}{2}a\hbar q\right)\right)$$
$$\times\coth\left[\Delta_1^*\cos\left(3\chi\left(1-\frac{v_d}{v_s}\right)\right)\cos A\cos\left(\frac{3}{2}a\hbar q\right)\right.$$
$$\left.+\Delta_2^*\cos\left(\chi\left(1-\frac{v_d}{v_s}\right)\right)\cos B\cos\left(\frac{a}{2}\hbar q\right)\right]$$
$$\times\coth\left[\Delta_1^*\cos\left(3\chi\left(1-\frac{v_d}{v_s}\right)\right)\sin A\sin\left(\frac{3}{2}a\hbar q\right)\right.$$
$$\left.\left.+\Delta_2^*\cos\left(\chi\left(1-\frac{v_d}{v_s}\right)\right)\sin B\sin\left(\frac{a}{2}\hbar q\right)\right]\right\} \tag{C.18}$$

Switching off the external electric field from C.18 yields:

$$\vec{j}_y(0)=\vec{j}_{oy}\left[\sinh\left\{\Delta_1^*\sin\left(\frac{3}{2}a\hbar q\right)\sin A+\Delta_2^*\sin\left(\frac{a}{2}\hbar q\right)\sin B\right\}\right.$$

$$\times \sinh\left\{\Delta_1^* \cos\left(\frac{3}{2}a\hbar q\right)\cos A + \Delta_2^* \cos\left(\frac{a}{2}\hbar q\right)\cos B\right\}\Bigg] \quad \text{(C.19)}$$

where

$$\vec{j}_{oy} = -\frac{4e^3 A^\dagger \mathscr{K}^2 \pi \vec{\Phi} \tau^2 \Theta(1-\alpha^2)}{\hbar^4 \omega_q^2 v_s \varepsilon \rho a q \sqrt{1-\alpha^2}} \frac{\mu \vec{H}}{c} \quad \text{(C.20)}$$

and

$$A = \frac{3}{4}\sin^{-1}\left(\frac{\omega_q}{12\Delta a q}\right), B = \frac{1}{4}\sin^{-1}\left(\frac{\omega_q}{12\Delta a q}\right)$$

The following relation exists between the attenuation coefficient Γ_{abs} and E_{SAME}:

$$\Gamma_{abs}\vec{\Phi} = \frac{n_o e \vec{E}_{SAME}}{\Omega} \quad \text{(C.21)}$$

Thus, the Hall-like (surface acoustomagnetoelectric) field $\vec{E}_{SAME}$ yields:

$$\vec{E}_{SAME} = \frac{2A^\dagger \pi \vec{\Phi}^2 K^2 \Theta(1-\alpha^2)}{3\hbar^2 \omega_q^2 \rho v_s \Delta \varepsilon a q \sqrt{1-\alpha^2} n_o e}\left(\frac{\mu H}{\hbar c}\right)$$
$$\times \left[\sinh\left\{\Delta_1^* \sin\left(\frac{3}{2}a\hbar q\right)\sin A + \Delta_2^* \sin\left(\frac{a}{2}\hbar q\right)\sin B\right\}\right.$$
$$\left.\times \cosh\left\{\Delta_1^* \cos\left(\frac{3}{2}a\hbar q\right)\cos A + \Delta_2^* \cos\left(\frac{a}{2}\hbar q\right)\cos B\right\}\right] \quad \text{(C.22)}$$

The drift velocity of the electrons yields:

$$\vec{v}_d = \mu \vec{E}_{SAME} = \frac{2A^\dagger \pi \vec{\Phi}^2 K^2 \Theta(1-\alpha^2)}{3\hbar^2 \omega_q^2 \rho v_s \Delta \varepsilon a q \sqrt{1-\alpha^2} n_o e}\left(\frac{\mu^2 H}{\hbar c}\right)$$
$$\times \left[\sinh\left\{\Delta_1^* \sin\left(\frac{3}{2}a\hbar q\right)\sin A + \Delta_2^* \sin\left(\frac{a}{2}\hbar q\right)\sin B\right\}\right.$$
$$\left.\times \cosh\left\{\Delta_1^* \cos\left(\frac{3}{2}a\hbar q\right)\cos A + \Delta_2^* \cos\left(\frac{a}{2}\hbar q\right)\cos B\right\}\right] \quad \text{(C.23)}$$

APPENDIX D

HIGH-FREQUENCY AMPLIFICATION OF ACOUSTIC PHONONS

Employing the Hamiltonian of the electron-phonon system for FSWCNT in the second quantization formalism as:

$$H = \sum_{p,\nu} \varepsilon^{(\nu)}(\vec{p}) \left(\vec{p} - \frac{e}{c}\vec{A}(t)\right) a_p^{(\nu+)} a_\nu^{(\nu)} + \sum_k \omega_k b_k^+ b_k \dots$$
$$+ \frac{1}{\sqrt{N}} \sum_{p,k} \sum_{\nu\nu' n} c_k m_{\nu\nu'}(k_z) a_p^{(\nu+)} a_{p-k+ng}^{\nu'} (b_k^+ + b_{-k}) \quad \text{(D.2)}$$

$a_p^{(+)}$ and a_p are the creation and annihilation operators of an electron with quasi-momentum p in the νth miniband respectively, and b_k^+ and b_k are the phonon creation and annihilation operators, respectively. N is the number of FSWCNT periods, $g = (0, 0, 2\pi/d)$ is the FSWCNT reciprocal vector, and $m_{\nu\nu'}$ is given as:

$$m_{\nu\nu'}(k_z) = \int \varphi_{\nu'}^*(z)\varphi_\nu(z) e^{ik_z} dz \quad \text{(D.3)}$$

where $\varphi_\nu(z)$ is the wavefunction of the νth state in one of the one-dimensional mini band from which the FSWCNT potential is formed. Proceeding to calculate the attenuation (or amplification) coefficient, equation of motion for the phonons from the Heisenberg formalism is given as:

$$i\frac{\partial}{\partial t}\langle b_q \rangle_t = \langle [b_q, H] \rangle_t = \omega_q \langle b_q \rangle_t + \frac{1}{\sqrt{N}} G_{-q} \sum_p m_{\nu\nu'}(-q) \langle a_p^{(\nu+)} a_{p+qng}^{(\nu)} \rangle_t \quad \text{(D.4)}$$

Again for $\langle a_p^{(\nu+)} a_{p+qng}^{(\nu)} \rangle_t$ yields:

$$i\frac{\partial}{\partial t}\langle a_p^{(\nu+)} a_{p+qng}^{(\nu)} \rangle_t = (\varepsilon_{p+q}^{\nu'} - \varepsilon_p^{\nu}) \langle a_p^{(\nu+)} a_{p+qng}^{(\nu')} \rangle_t$$
$$+ \frac{1}{N} \sum_p \sum_{\nu'\nu n} G_{k'k} \Big[M_{\nu'\nu''}(k_z) \langle a_p^{(\nu+)} a_{p+q-k+(n+n')g}^{(\nu')} (b_k + b_{-k}^+)_t \rangle_t$$
$$- M_{\nu''\nu}(q_z) \langle a_{p+k-n'g}^{(\nu)} a_{p+q+ng}^{(\nu')} (b_k + b_{-k}^+)_t \rangle_t \Big] \quad \text{(D.5)}$$

Solving D.5 within the initial condition $\langle a_p^{(\nu)} a_{p+qng}^{(\nu+)} \rangle_{t=\infty} = 0$ and substituting into D.4, we obtain

$$i\frac{\partial}{\partial t}\langle b_q \rangle_t = \omega_q \langle b_q \rangle_t - I \sum_p \sum_{\nu'\nu n} G_{-q} M_{\nu\nu'}(-q)$$
$$\times \int_{-\infty}^{t} e^{\int_{t'}^{t}(\varepsilon_{p+q} - \varepsilon_p)dt''} dt' \sum_{\nu'\nu''} G_k [M_{\nu'\nu''}(k_z) \langle a_p^{(\nu+)} a_{p+q-k+(n'+n)g}^{(\nu'')}$$
$$\times (b_k + b_{-k}^+)_t \rangle \tau' - M_{\nu''\nu}(k_z) \langle a_{p+k-ng}^{(\nu+)} a_p + q + (n'+n) g^{(\nu'')} (b_k + b_{-k}^+)) \rangle_{t'}] \quad \text{(D.6)}$$

For weak electron-phonon interaction and considered as a perturbation, RHS of D.6 is decoupled using

$$\langle a_p^{(\nu+)} a_{p'}^{(\nu')} b_k \rangle = \delta_{kk'} \delta_{\nu\nu'} \langle b_k \rangle_k n_p^\nu \tag{D.7}$$

where $n_p^\nu = \langle a_p^{(\nu+)} a_{p'}^{(\nu')} \rangle_t$ is the electron distribution function. We further obtain

$$\begin{aligned}\frac{\partial}{\partial t}\langle b_q\rangle_t + i\omega_q\langle b_q\rangle_t = &\sum_p \sum_{\nu'\nu n} G_{-q}G_{q+nq}M_{\nu'\nu}(-q_z)M_{\nu'\nu}(-q_z+ng_z)[n_p^{\nu'} - n_{p+q}^{\nu'}] \\ &\times \int_{-\infty}^{t} dt'(\langle b_{q+ng}\rangle_{t'} + \langle b^+_{-q-ng}\rangle_{t'})\exp\Big[i(\varepsilon_p^\nu - \varepsilon_{p+q}^{\nu'})(t-t') \\ &- \frac{3eE_oa\Delta_1}{\Omega}\{\sin(\vec{p}_s+\vec{q})a - \sin\vec{p}_s a\}(\sin\Omega t - \sin\Omega t') \\ &- \frac{eE_oa\Delta_2}{\Omega}\{\sin(\vec{p}_z+\vec{q})a - \sin\vec{p}_z a\}(\sin\Omega t - \sin\Omega t')\Big]\end{aligned} \tag{D.8}$$

Employing planar conditions

$$\begin{aligned}\frac{\partial}{\partial t}\langle b_q\rangle_t + i\omega_q\langle b_q\rangle_t = &\sum_p \sum_{\nu'\nu n} G_{-q}G_{q+nq}M_{\nu'\nu}(-q_z)M_{\nu'\nu}(-q_z+ng_z)[n_p^{\nu'} - n_{p+q}^{\nu'}] \\ &\times \int_{-\infty}^{t} dt'(\langle b_{q+ng}\rangle_{t'} + \langle b^+_{-q-ng}\rangle_{t'})\exp\Big[i(\varepsilon_p^\nu - \varepsilon_{p+q}^{\nu'})(t-t') \\ &- \frac{eE_oa\Delta_2}{\Omega}\{\sin(p_z+q)a - \sin p_z a\}(\sin\Omega t - \sin\Omega t')\Big]\end{aligned} \tag{D.9}$$

Eliminating $\langle b^+_{-q-ng}\rangle_t$ and taking the Fourier transform of the component

$$B_q(\omega) = \int_{-\infty}^{\infty} \langle b_q\rangle_t \exp(i\omega t)dt \tag{D.10}$$

yields

$$\begin{aligned}(\omega-\omega_q)B_q(\omega) = &\sum_{k=-\infty}\sum_{\nu'\nu} G_{-q}G_{q+ng}M_{\nu\nu'}(-q_z)M_{\nu'\nu}(q_z-ng_z) \\ &\times \frac{2\omega_{q+ng}}{\omega+\omega_{q+ng}-\Omega k} B_q(\omega-k)M_k(q,\omega)\end{aligned} \tag{D.11}$$

where

$$M_k(q,\omega) = \sum_{\ell=-\infty} J_\ell(\xi)J_{\ell+k}(\xi)\Pi(q,\omega+\ell\Omega) \tag{D.12}$$

$$\Pi_{\nu\nu'}(q,\omega+\ell\Omega) = \sum_p \frac{n_p^\nu - n_{p+q}^{\nu'}}{\varepsilon_{p+q}^{\nu'} - \varepsilon_p^\nu - \omega_q} \tag{D.13}$$

and

$$\xi = \frac{eE_oa\Delta_2}{\Omega^2}[\sin(\vec{p}_z+\vec{q})a - \sin(\vec{p}_z a)] \tag{D.14}$$

Subsequently, we shall confine ourselves to the case when the electron gas is non-degenerate and only the lowest miniband is filled, i.e. $n_p^\nu = n_p$. We shall also limit ourselves to $k = 0$, since when $k \neq 0$, the summation give terms of higher perturbation. Then for $n = 0$ we obtain from D.11

$$\omega^2 - \omega_q^2 - 2\omega_q^2 G_q^2 M_o(q,\omega) = 0 \tag{D.15}$$

This simplifies to

$$\omega - \omega_q - G_q^2 \sum_{\ell=-\infty}^{\infty} J_\ell^2(\xi)\Pi_q^0(\omega_q - \Omega\ell) = 0 \tag{D.16}$$

Then the phonon amplification in the presence of the electromagnetic wave will be given by

$$\Gamma(\omega) = -Im\omega = \sum_{\ell=-\infty}^{\infty} J_\ell^2(\xi)\Gamma_q^0(\omega_q + \ell\Omega) \tag{D.17}$$

where

$$\Gamma_q^0(\omega) = \pi G_q^2 \sum_p U_{\nu,\nu'}^{ac} \tag{D.18}$$

$U_{\nu,\nu'}^{ac}$ is the electron-phonon interaction expressed as:

$$U_{\nu,\nu'}^{ac} = \sum_{\nu,\nu'} \{|G_{p-\hbar q,p}|^2[n(\vec{p}-\hbar q) - n(p)]\delta(\varepsilon_\nu(\vec{p}-\hbar\vec{q}) - \varepsilon_\nu(p) + \hbar\omega_q - \xi)$$
$$+ |G_{\vec{p}+\hbar\vec{q},\vec{p}}|^2[n(\vec{p}+\hbar\vec{q}) - n(\vec{p})]\delta(\varepsilon_{\nu'}(\vec{p}+\hbar\vec{q}) - \varepsilon_{\nu'}(\vec{p}) - \hbar\omega_q + \xi)\} \tag{D.19}$$

i.e. the imaginary part of D.17 is the polarization vector. For large values, the Bessel function $J_\ell(\xi)$ is small except when the order is equal to the argument. From D.16 we obtain

$$\xi = \frac{eE_o a^2 \Delta_2 q}{\Omega^2} \tag{D.20}$$

Suming over $|\ell|$ by employing the approximation as in ref. [6]

$$\sum_{\ell=-\infty} J_\ell^2(\xi)\delta(E - \ell\Omega) \approx \frac{1}{2}[\delta(E-\xi) + \delta(E+\xi)] \tag{D.21}$$

where $E = \varepsilon(\vec{p}+\hbar\vec{q}) - \varepsilon(\vec{p}) - \hbar\omega_q$. $G(\vec{p}\pm\hbar\vec{q},\vec{p})$ is the matrix element of the electron-phonon interaction. Letting $\vec{p}' = \vec{p} \pm \hbar\vec{q}$ scattering into/out of a state p'/p is the same hence

$$|G_{p',p}|^2 = |G_{p,p'}|^2 \tag{D.22}$$

The attenuation or amplification coefficient becomes;

$$\Gamma_q(\omega) = \frac{\pi}{2}\sum_p |G_{p',p}|^2[n(p) - n(p+\hbar q)]$$
$$\times \{\delta(\varepsilon_\nu(p+\hbar q) - \varepsilon_\nu(p) - \hbar\omega_q - \xi) + \delta(\varepsilon_\nu(p+\hbar q) - \varepsilon_\nu(p) + \hbar\omega_q + \xi)\} \tag{D.23}$$

The first $\delta-$function in D.23 corresponds to the emission and the second to the absorption of phonons. The number of photons absorbed or emitted is the same order of magnitude as the ratio of the classical oscillatory energy of the electron to that of the phonon

$$\ell = \frac{2e^2E_o^2/(e\tau\Omega^2/\mu)}{\Omega} \tag{D.24}$$

Multiple photon absorption or emission processes ($\ell \gg 1$) are valid for laser fields where

$$E_o \gg \left(\frac{\tau\Omega^3}{2e\mu}\right)^{1/2} \tag{D.25}$$

For low electron temperature, and for $kT \ll \xi$ the emission term is neglible compared to the absorption term. This is justified provided $\Delta_2 \gg kT$.
D.24 becomes

$$\Gamma_q(\omega) = \frac{\pi|G_{p',p}|^2}{2}\sum_n [n(\vec{p}) - n(\vec{p}+\hbar\vec{q})] \times \delta(\varepsilon_v(\vec{p}+\hbar\vec{q}) - \varepsilon_v(\vec{p}) - \hbar\omega_q + \xi)d^2\vec{p} \tag{D.26}$$

Changing the summation to an integral due to the continuity of the carrier states as a function of the momentum p, as a result we obtain

$$\Gamma_q(\omega) = \frac{2\pi\Phi|G_{p',p}|^2}{\omega_q v_s}\int [n(\vec{p}) - n(\vec{p}+\hbar\vec{q})] \times \delta(\varepsilon_v(\vec{p}+\hbar\vec{q}) - \varepsilon_v(\vec{p}) - \hbar\omega_q + \xi)d^2\vec{p} \tag{D.27}$$

Employing cylindrical coordinates in relation to the tubular geometry of FSWCNT yields:

$$\begin{aligned}\Gamma(\omega) = \Gamma(0)\Bigg\{1 - 4\bigg(&\Delta_2^*\sin\left(\chi\left(1-\frac{v_d}{v_s}\right)\right)\cos B\sin\left(\frac{a}{2}\hbar q\right) \\ &+ \Delta_1^*\cos A\sin\left(3\chi\left(1-\frac{v_d}{v_s}\right)\right)\sin\left(\frac{3}{2}a\hbar q\right)\bigg) \\ &\times\tanh\left[\Delta_1^*\cos\left(3\chi\left(1-\frac{v_d}{v_s}\right)\right)\cos A\cos\left(\frac{3}{2}a\hbar q\right)\right. \\ &\left.+ \Delta_2^*\cos\left(\chi\left(1-\frac{v_d}{v_s}\right)\right)\cos B\cos\left(\frac{a}{2}\hbar q\right)\right] \\ &\times\coth\left[\Delta_1^*\cos\left(3\chi\left(1-\frac{v_d}{v_s}\right)\right)\sin A\sin\left(\frac{3}{2}a\hbar q\right)\right. \\ &\left.+ \Delta_2^*\cos\left(\chi\left(1-\frac{v_d}{v_s}\right)\right)\sin B\sin\left(\frac{a}{2}\hbar q\right)\right]\Bigg\}\end{aligned} \tag{D.38}$$

Switching off the external electric field, we obtain

$$\Gamma(0) = \Gamma_o\left[\sinh\left\{\Delta_1^*\sin\left(\frac{3}{2}a\hbar q\right)\sin A + \Delta_2^*\sin\left(\frac{a}{2}\hbar q\right)\sin B\right\}\right.$$

$$\times \cosh\left\{\Delta_1^* \cos\left(\frac{3}{2}a\hbar q\right)\cos A + \Delta_2^* \cos\left(\frac{a}{2}\hbar q\right)\cos B\right\}\Bigg] \quad \text{(D.39)}$$

and

$$\Gamma_o = \frac{8\pi^3 n^* \Phi e^2 K^2 \tau \Theta(1-\alpha^2)}{\hbar\omega_q^2 \varepsilon^2 \sigma q v_s \Delta_1 I_o(\Delta_1^*) I_o(\Delta_2^*)\sqrt{1-\alpha^2}} \frac{2}{\sqrt{\pi}} \quad \text{(D.40)}$$

where Θ is defined to be the Heaviside step function.

$$v_d = \frac{eE_o\Delta_2 a^2 q}{\Omega}, \chi = \hbar\omega_q a/v_s, \alpha = \frac{\omega_q(1 - v_d/v_s)}{6\Delta_1 aq}$$

and v_d is the carrier drift velocity and v_s denotes acoustic wave velocity.

APPENDIX E

SOLUTION TO THE BOLTZMANN EQUATION

Electrons in equilibrium is defined by the Fermi-Dirac statistics given by the expression

$$f_o(\vec{p},\vec{r},t) = \frac{1}{e^{\theta}+1} \tag{E.1}$$

Let

$$\theta = [E_c(\vec{r},t) + E(p) - F_n(\vec{r},t)]/kT \tag{E.2}$$

Suppose that $f(\vec{r},\vec{p},t) = f_o(\vec{r},\vec{p},t) + f_1(\vec{p},\vec{r},t) + f^{'}(\vec{p},\vec{r},t)$, where $f_1(\vec{p},\vec{r},t)$ is a small perturbation and $f^{'}$ is the hot electron source distribution function. For $f^{'}(\vec{p},\vec{r},t) = 0$

$$\frac{\partial f(\vec{r},\vec{p},t)}{\partial t} + \vec{v}\cdot\frac{\partial f(\vec{r},\vec{p},t)}{\partial r} + F\cdot\frac{\partial f(\vec{r},\vec{p},t)}{\partial \vec{p}} = -\frac{f(\vec{r},\vec{p},t) - f_o(\vec{p})}{\tau} \tag{E.3}$$

$$\nabla_t(f_o+f_1) + v\cdot\nabla_r(f_o+f_1) + F\cdot\nabla_p(f_o+f_1) = -\frac{f_o+f_1-f_o}{\tau} \tag{E.4}$$

$$v\cdot\nabla_r(f_o+f_1) + F\cdot\nabla_p(f_o+f_1) = -\frac{f_1}{\tau} \tag{E.5}$$

$f_o(\vec{r},\vec{p},t) \gg f_1(\vec{p},\vec{r},t)$ and $\nabla_r f_o \gg f_1$

$$\vec{v}\cdot\nabla_r f_o(\vec{r},\vec{p},t) + F\cdot\nabla_p f_o(\vec{r},\vec{p},t) = -\frac{f_1(\vec{p},\vec{r},t)}{\tau} \tag{E.6}$$

Applying the chain rule

$$\vec{v}\cdot\frac{\partial f_o(\vec{r},\vec{p},t)}{\partial\theta}\cdot\nabla_r\theta + F\cdot\frac{\partial f(\vec{r},\vec{p},t)}{\partial\theta}\cdot\nabla_p\theta = -\frac{f_1(\vec{p},\vec{r},t)}{\tau} \tag{E.7}$$

Differentiating

$$\nabla_r\theta = \frac{[\nabla_r E_c(\vec{r}) - \nabla_r F_n(\vec{r})]}{kT} + [E_c(\vec{r}) + E(\vec{p}) - F_n(\vec{r})]\nabla_r\left(\frac{1}{kT(\vec{r})}\right) \tag{E.8}$$

also

$$\nabla_p\theta = \frac{\vec{v}}{kT(\vec{r})} \tag{E.9}$$

Putting E.4 and E.5 into E.3

$$\vec{v}\cdot\frac{\partial f_o(\vec{r},\vec{p},t)}{\partial\theta}\left\{\frac{[\nabla_r E_c(\vec{r}) - \nabla_r F_n(\vec{r})]}{kT} + [E_c(\vec{r}) + E(\vec{p}) - F_n(\vec{r})]\nabla_r\left(\frac{1}{kT(\vec{r})}\right)\right\} + F\frac{\partial f_o(\vec{r},\vec{p},t)}{\partial\theta}\frac{\vec{v}}{kT(\vec{r})} = -\frac{f_1(\vec{p},\vec{r},t)}{\tau} \tag{E.10}$$

$$\vec{v}\cdot\frac{\partial f_o(\vec{r},\vec{p},t)}{\partial\theta}\left\{\frac{[\nabla_r E_c(\vec{r}) - \nabla_r F_n(\vec{r})]}{kT} + [E_c(\vec{r}) - F_n(\vec{r})]\nabla_r\left(\frac{1}{kT(\vec{r})}\right)\right.$$

$$+ \frac{F}{kT(\vec{r})} \Bigg\} = \frac{f_1(\vec{p},\vec{r},t)}{\tau} \quad \text{(E.11)}$$

$$\nabla_r E_c(\vec{r}) = -F$$

$$f_1(\vec{p},\vec{r},t) = -\frac{\vec{v}\tau}{kT} \cdot \frac{\partial f_o(\vec{r},\vec{p},t)}{\partial \theta} \Bigg\{ -\frac{F}{kT(\vec{r})} - \nabla_r F_n(r) + [E_c(\vec{r}) + E(\vec{p}) - F_n(\vec{r})]\nabla_r \left(\frac{1}{kT(\vec{r})}\right) + \frac{F}{kT(\vec{r})} \Bigg\} \quad \text{(E.12)}$$

$$f_1(\vec{p},\vec{r},t) = -\frac{\vec{v}\tau}{kT} \cdot \frac{\partial f_o}{\partial \theta} \left\{ [E_c(\vec{r}) + E(\vec{p}) - F_n(\vec{r})]\nabla_r \left(\frac{1}{kT(\vec{r})}\right) - \nabla_r F_n(r) \right\} \quad \text{(E.13)}$$

$$f_1 = -\frac{\vec{v}\tau}{kT} \cdot \frac{\partial f_o}{\partial \theta} \left\{ kT[E_c(\vec{r}) + E(\vec{p}) - F_n(\vec{r})]\nabla_r \left(\frac{1}{T(\vec{r})}\right) - \nabla_r F_n(\vec{r}) \right\} \quad \text{(E.14)}$$

$$f_1 = -\frac{\vec{v}\tau}{kT} \cdot \frac{\partial f_o}{\partial \theta} \left\{ [E_c(\vec{r}) + E(\vec{p}) - F_n(\vec{r})]\left(\frac{\nabla_r T}{T}\right) - \nabla_r F_n(\vec{r}) \right\} \quad \text{(E.15)}$$

$$f_1 = \frac{\tau}{kT}\left(-\frac{\partial f_o}{\partial \theta}\right) v \cdot \aleph \quad \text{(E.16)}$$

where $\aleph = -\nabla_r F_n(r) + [E_c(\vec{r}) + E(\vec{p}) - F_n(\vec{r})]\,(\nabla_r T/T)$ is the generalized force. Substitute E.15 into $f = f_o + f_1$

$$f = f_o(\vec{p}) + \frac{\vec{v}\tau}{kT} \cdot \frac{\partial f_o(\vec{p})}{\partial \theta} \left\{ [E_c(\vec{r}) + E(\vec{p}) - F_n(\vec{r})]\frac{\nabla_r T}{T} - \nabla_r F_n(r) \right\} \quad \text{(E.17)}$$

Let $\theta = [E_c(\vec{r}) + E(\vec{p}) - F_n(\vec{r})]/kT = \varepsilon(\vec{p})/kT \rightarrow kT\theta = \varepsilon(\vec{p})$

$$f(\vec{r},\vec{p},t) = f_o(\vec{r},\vec{p},t) + \frac{\vec{v}\tau}{kT} \cdot kT\frac{\partial f_o(\vec{p})}{\partial \varepsilon(\vec{p})} \left\{ [E_c(\vec{r}) + E(\vec{p}) - F_n(\vec{r})]\frac{\nabla_r T}{T} - \nabla_r F_n(r) \right\} \quad \text{(E.18)}$$

$$f(\vec{r},\vec{p},t) = \tau^{-1}\int_0^{-\infty} \exp(-t/\tau)dt\, f_o(\vec{r},\vec{p},t) + \frac{\vec{v}\tau}{kT} \cdot kT\frac{\partial f_o(\vec{p})}{\partial \varepsilon(\vec{p})} \int_0^{-\infty} \exp(-t/\tau)dt \left\{ [E_c(\vec{r}) + E(\vec{p}) - F_n(\vec{r})]\frac{\nabla_r T}{T} - \nabla_r F_n(r) \right\} \quad \text{(E.19)}$$

$$f(\vec{r},\vec{p},t) = \tau^{-1}\int_0^{-\infty} dt \exp(-t/\tau) f_o(\vec{p})$$

$$+\tau\int_0^{-\infty} dt\exp(-t/\tau)\left\{[E_c(\vec{r})+E(\vec{p})-F_n(\vec{r})]\frac{\nabla_r T}{T}-\nabla_r F_n(r)\right\}\vec{v}\cdot\frac{\partial f_o(\vec{p})}{\partial\varepsilon(\vec{p})} \tag{E.20}$$

$E_c(r) \ll E(\vec{p})$ and the quasi-fermi level $\nabla_r F_n = \nabla_r \mu$, where μ is the electrochemical potential.

$$\begin{aligned} f(\vec{r},\vec{p},t) &= \tau^{-1}\int_0^{-\infty} dt\exp(-t/\tau)f_o(\vec{p}) \\ &+\tau\int_0^{-\infty} dt\exp(-t/\tau)\left\{[E(\vec{p})-\mu]\frac{\nabla_r T}{T}-\nabla_r\mu\right\}\vec{v}\cdot\frac{\partial f_o(\vec{p})}{\partial\varepsilon(\vec{p})} \end{aligned} \tag{E.21}$$

Making the transformation $p \to \vec{p} - e\int_{t-t'}^{t}[E_o + Ecos\omega t^{'}]dt^{'}$

$$\begin{aligned} f(\vec{r},\vec{p},t) &= \tau^{-1}\int_0^{-\infty} dt\exp(-t/\tau)f_o\left(\vec{p}-e\int_{t-t'}^{t}[E_o+Ecos\omega t^{'}]dt^{'}\right) \\ &+\tau\int_0^{-\infty} dt\exp(-t/\tau)\left\{[E(\vec{p})-\mu]\frac{\nabla_r T}{T}-\nabla_r\mu\right\} \\ &\times\vec{v}(\vec{p})\cdot\frac{\partial f_o}{\partial\varepsilon(\vec{p})}(\vec{p}-e\int_{t-t'}^{t}[E_o+Ecos\omega t^{'}]dt^{'}) \end{aligned} \tag{E.22}$$

APPENDIX F

CARRIER CURRENT DENSITY OF FSWCNT

In the linear approximation of ∇T and $\nabla\mu$, the solution to the BTE is:

$$f(\vec{p}) = \tau^{-1}\int_0^\infty dt \exp\left(\frac{-t}{\tau}\right) f_o\left(\vec{p} - e\int_0^\infty [\vec{E}_o + \vec{E}\cos(\omega t')]dt'\right)$$
$$+\tau\int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt\left\{\left[\varepsilon(\vec{p} - e\int_0^\infty [\vec{E}_o + \vec{E}\cos(\omega t') - \mu])\right]\frac{\nabla T}{T} + \nabla\mu\right\}$$
$$\times v\left(\vec{p} - e\int_0^\infty [\vec{E}_o + \vec{E}\cos(\omega t')]dt'\right)\frac{\partial f_o}{\partial\varepsilon}(\vec{p} - e\int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos(\omega t')]dt') \quad \text{(F.1)}$$

The carrier current density is defined as:

$$\vec{J} = -e\sum_p v(\vec{p}) f(\vec{p}) \quad \text{(F.2)}$$

Substitute F.1 into F.2 yields:

$$\vec{J} = e\tau^{-1}\int_0^\infty dt \exp\left(\frac{-t}{\tau}\right)\sum_p v(\vec{p}) f_o\left(p - e\int_0^\infty [\vec{E}_o + E\cos(\omega t')]dt'\right)$$
$$+e\tau\int_0^\infty dt \exp\left(\frac{-t}{\tau}\right)\sum_p v(\vec{p})\left\{[\varepsilon(\vec{p} - e\int_0^\infty [\vec{E}_o + \vec{E}\cos(\omega t') - \mu])\frac{\nabla T}{T} + \nabla\mu\right\}$$
$$\times v\left(\vec{p} - e\int_0^\infty [\vec{E}_o + \vec{E}\cos(\omega t')]\right)\frac{\partial f_o}{\partial\varepsilon}(\vec{p} - e\int_0^\infty [\vec{E}_o + \vec{E}\cos(\omega t')]dt') \quad \text{(F.3)}$$

Making the transformation $p - e\int_0^\infty [\vec{E}_o + \vec{E}\cos(\omega t')]dt' \rightarrow p$ yields:

$$\vec{J} = e\tau^{-1}\int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt\sum_p v\left(\vec{p} - e\int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos(\omega t')]dt'\right) f_o(\vec{p})$$
$$+e\tau\int_0^\infty \exp(-t/\tau)dt\sum_p\left\{[\varepsilon(\vec{p}) - \mu]\frac{\nabla T}{T} + \nabla\mu\right\}$$
$$\times v(\vec{p})\frac{\partial f_o(\vec{p})}{\partial\varepsilon} v(\vec{p} - e\int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos(\omega t')]dt') \quad \text{(F.4)}$$

For a p-doped, semiconductor FSWCNT, the dispersion law in the conduction band is obtained as:

$$\varepsilon(\vec{p}) = \varepsilon_o - 8\gamma_o\cos^3\frac{\sqrt{3}\vec{p}}{2\hbar} \quad \text{(F.5)}$$

Expanding the trigonometric identity yields

$$\varepsilon(\vec{p}) = \varepsilon_o - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} \quad \text{(F.6)}$$

where $u = \sqrt{3}/2\hbar$ and $w = 3\sqrt{3}/2\hbar$. Resolving the current density into $\vec{S}$ and $\vec{Z}$ components yields:

$$\vec{S} = e\tau^{-1}\int_0^{\infty} \exp\left(\frac{-t}{\tau}\right) dt \sum_p v_s(\vec{p} - e\int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')]dt') f_o(\vec{p})$$
$$+ e\tau \int_0^{\infty} \exp\left(\frac{-t}{\tau}\right) dt \sum_p \left\{ [\varepsilon(\vec{p}) - \mu]\frac{\nabla_s T}{T} + \nabla_s \mu \right\}$$
$$\times v_s(\vec{p})\frac{\partial f_o(\vec{p})}{\partial \varepsilon} v_s(\vec{p} - e\int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')]dt') \quad \text{(F.7)}$$

and similarly:

$$Z = e\tau^{-1}\int_0^{\infty} \exp\left(\frac{-t}{\tau}\right) dt \sum_p v_z(\vec{p} - e\int_{t-t'}^{t} [\vec{E}_o + \vec{E}_z \cos(\omega t')]dt') f_o(\vec{p})$$
$$+ e\tau \int_0^{\infty} \exp\left(\frac{-t}{\tau}\right) dt \sum_p \left\{ [\varepsilon(\vec{p}) - \mu]\frac{\nabla_z T}{T} + \nabla_z \mu \right\}$$
$$\times v_z(\vec{p})\frac{\partial f_o(\vec{p})}{\partial \varepsilon} v_z(\vec{p} - e\int_{t-t'}^{t} [\vec{E}_o + \vec{E}_z \cos(\omega t')]dt') \quad \text{(F.8)}$$

Making the transformation

$$\sum_p \rightarrow \frac{2}{(2\pi\hbar)^2}\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \quad \text{(F.9)}$$

$$\vec{S} = \frac{2e\tau^{-1}}{(2\pi\hbar)^2}\int_0^{\infty} \exp\left(\frac{-t}{\tau}\right) dt \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s$$
$$\times v_s(\vec{p} - e\int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')]dt') f_o(\vec{p})$$
$$+ \frac{2e\tau^{-1}}{(2\pi\hbar)^2}\int_0^{\infty} \exp\left(\frac{-t}{\tau}\right) dt \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{ [\varepsilon(\vec{p}) - \mu]\frac{\nabla_s T}{T} + \nabla_s \mu \right\}$$
$$\times v_s(p)\frac{\partial f_o(\vec{p})}{\partial \varepsilon} v_s(\vec{p} - e\int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')]dt') \quad \text{(F.10)}$$

and

$$\vec{Z} = \frac{2e\tau^{-1}}{(2\pi\hbar)^2}\int_0^{\infty} \exp\left(\frac{-t}{\tau}\right) dt \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s$$
$$\times v_z(p - e\int_{t-t'}^{t} [\vec{E}_o + \vec{E}_z \cos(\omega t')]dt') f_o(\vec{p})$$
$$+ \frac{2e\tau^{-1}}{(2\pi\hbar)^2}\int_0^{\infty} \exp\left(\frac{-t}{\tau}\right) dt \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{ [\varepsilon(\vec{p}) - \mu]\frac{\nabla_z T}{T} + \nabla_z \mu \right\}$$
$$\times v_z(p)\frac{\partial f_o(\vec{p})}{\partial \varepsilon} v_z(\vec{p} - e\int_{t-t'}^{t} [\vec{E}_o + \vec{E}_z \cos(\omega t')]dt') \quad \text{(F.11)}$$

respectively, where the integration is carried out within the first brillouin zone $-\pi\hbar/b_s \leq \vec{p}_s \leq \pi\hbar/b_s$ and $-\pi\hbar/b_z \leq \vec{p}_z \leq \pi\hbar/b_z$ respectively.

$$v_s(\vec{p}) = \frac{\partial\varepsilon(\vec{p})}{\partial\vec{p}} = \frac{u\Delta_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}$$
$$v_s\left(\vec{p} - e\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t^{'})]dt^{'}\right) =$$
$$\frac{u\Delta_s b_s}{\hbar}\sin\left(\vec{p} - e\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t^{'})]dt^{'}\right) \quad \text{(F.12)}$$

Expanding the expression:

$$v_s\left(\vec{p} - e\int_{t-t'}^{t}[\vec{E}_o + E\cos(\omega t^{'})]dt^{'}\right) =$$
$$\frac{u\Delta_s b_s}{\hbar}\left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t^{'})]dt^{'}\right)\right.$$
$$\left. - \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t^{'})]dt^{'}\right)\right\} \quad \text{(F.13)}$$

Substituting F.13 into F.10

$$\vec{S} = -\frac{2e\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\int_0^{\infty}\exp\left(\frac{-t}{\tau}\right)dt\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s$$
$$\times\left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t^{'})]dt^{'}\right)\right.$$
$$\left. - \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t^{'})]dt^{'}\right)\right\} f_o(p)$$
$$-\frac{2e\tau^{-1}}{(2\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\int_0^{\infty}\exp\left(\frac{-t}{\tau}\right)dt\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s$$
$$\times\left\{[\varepsilon(p) - \mu]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}v_s(p)\frac{\partial f_o(p)}{\partial\varepsilon}$$
$$\times v_s(p - e\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t^{'})]dt^{'}) \quad \text{(F.14)}$$

F.14 is too cumbersome to solve. Separating it into $\vec{S}_1$ and $\vec{S}_2$ without lost of generality yields:

$$\vec{S}_1 = \frac{2e\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\int_0^{\infty}\exp\left(\frac{-t}{\tau}\right)dt\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s$$
$$\times\left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t^{'})]dt^{'}\right)\right.$$
$$\left. - \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t^{'})]dt^{'}\right)\right\} f_o(\vec{p}) \quad \text{(F.15)}$$

and

$$\vec{S}_2 = \frac{2e\tau^{-1}}{(2\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s$$
$$\times\left\{[\varepsilon_o - \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} - \Delta_z \cos\frac{v\vec{p}_z b_z}{\hbar} - \mu]\frac{\nabla_s T}{T} + \nabla_s \mu\right\}$$
$$\times \sin\frac{u\vec{p}_s b_s}{\hbar}\frac{\partial f_o(p)}{\partial \varepsilon}\left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)\right.$$
$$\left. - \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)\right\} \quad \text{(F.16)}$$

where $\vec{S} = \vec{S}_1 + \vec{S}_2$. Considering, $\vec{S}_1$, $f_o(\vec{p})$ is given by:

$$f_o(\vec{p}) = A^\dagger \exp\left(\frac{-[\varepsilon_o - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \Delta_z\cos\frac{v\vec{p}_z b_z}{\hbar}] + \mu}{k_B T}\right) \quad \text{(F.17)}$$

where $A^\dagger$ is to be determined from the normalisation condition $\int f(\vec{p})d\vec{p} = n_o$.

$$n_o = \frac{2}{(2\pi\hbar)^2}\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z f_o(\vec{p}) \quad \text{(F.18)}$$

$$n_o = \frac{2A^\dagger}{(2\pi\hbar)^2}\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \exp\left(\frac{(-\varepsilon_o + \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar})}{k_B T} + \mu\right) \quad \text{(F.19)}$$

$$n_o = \frac{2A^\dagger}{(2\pi\hbar)^2}\exp\left(\frac{\mu - \varepsilon_o}{k_B T}\right)\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \exp\left(\frac{\Delta_z}{k_B T}\cos\frac{w\vec{p}_z b_z}{\hbar}\right)$$
$$\times \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \exp\left(\frac{\Delta_s}{k_B T}\cos\frac{u\vec{p}_s b_s}{\hbar}\right) \quad \text{(F.20)}$$

$$n_o = \frac{2A^\dagger}{(2\pi\hbar)^2}\exp\left(\frac{\mu - \varepsilon_o}{k_B T}\right)\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \exp\left(\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right)$$
$$\times \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \exp\left(\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}\right) \quad \text{(F.21)}$$

where $\Delta_s^* = \Delta_s/k_B T$ and $\Delta_z^* = \Delta_z/k_B T$. Changing the path for integration

$$Z_z = w\vec{p}_z b_z \qquad Z_s = u\vec{p}_s b_s$$
$$\frac{dZ_z}{d\vec{p}_z} = wb_z \qquad \frac{dZ_s}{d\vec{p}_s} = ub_s$$
$$\frac{dZ_z}{wb_z} = d\vec{p}_z \qquad \frac{dZ_s}{ub_s} = d\vec{p}_s$$

$$n_o = \frac{8A^{\dagger}}{(2\pi\hbar)^2 ub_s wb_z} \exp\left(\frac{\mu - \varepsilon_o}{k_B T}\right) \int_{-\pi}^{\pi} dZ_z \exp\left(\Delta_z^* \cos\frac{Z_z}{\hbar}\right) \times \int_{-\pi}^{\pi} dZ_s \exp\left(\Delta_s^* \cos\frac{Z_s}{\hbar}\right) \quad \text{(F.22)}$$

$$I_n(x) = \frac{1}{\pi}\int_0^{\pi} d\theta \cos n\theta \exp(x\cos\theta) \quad \text{(F.23)}$$

$$I_0(\Delta_s^*) = \frac{1}{\pi}\int_0^{\pi} dZ_z \exp(\Delta_s^* \cos Z_s) \quad \text{(F.24)}$$

$$n_o = \frac{2A^{\dagger}\hbar^2}{(\pi\hbar)^2 uwb_z b_s} \exp\left(\frac{\mu - \varepsilon_o}{k_B T}\right) \pi I_o(\Delta_s^*) \pi I_o(\Delta_z^*) \quad \text{(F.25)}$$

Making $A^{\dagger}$ the subject of F.25 yields:

$$A^{\dagger} = \frac{uwn_o b_s b_z}{2I_o(\Delta_s^*) I_o(\Delta_z^*)} \exp\left(\frac{\mu - \varepsilon_o}{k_B T}\right) \quad \text{(F.26)}$$

Substituting F.26 into F.17 yields:

$$f_o(p) = \frac{uwn_o b_s b_z}{2I_o(\Delta_s^*) I_o(\Delta_z^*)} \exp\left[\frac{\Delta_s}{k_B T}\cos\frac{u\vec{p}_s b_s}{\hbar} + \frac{\Delta_z}{k_B T}\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad \text{(F.27)}$$

$$f_o(p) = \frac{uwn_o b_s b_z}{2I_o(\Delta_s^*) I_o(\Delta_z^*)} \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad \text{(F.28)}$$

Substitute F.28 into F.15 yields:

$$\begin{aligned} \vec{S}_1 = {} & \frac{2e\tau^{-1}}{(2\pi\hbar)^2} \frac{u\Delta_s b_s}{\hbar} \frac{n_o uwb_s b_z}{2I_o(\Delta_z^*) I_o(\Delta_s^*)} \int_0^{\infty} \exp\left(\frac{-t}{\tau}\right) dt \\ & \times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{ \sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')] dt'\right) \right. \\ & \left. - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')] dt'\right) \right\} \\ & \times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{v\vec{p}_z b_z}{\hbar}\right] \end{aligned} \quad \text{(F.29)}$$

$$\begin{aligned} \vec{S}_1 = {} & \frac{2e\tau^{-1}}{(2\pi\hbar)^2} \frac{u\Delta_s b_s}{\hbar} \frac{n_o uwb_s b_z}{2I_o(\Delta_z^*) I_o(\Delta_s^*)} \int_0^{\infty} \exp\left(\frac{-t}{\tau}\right) dt \\ & \times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{ \sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')] dt'\right) \right. \\ & \times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{v\vec{p}_z b_z}{\hbar}\right] \\ & - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')] dt'\right) \\ & \left. \times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{v\vec{p}_z b_z}{\hbar}\right] \right\} \end{aligned} \quad \text{(F.30)}$$

Integrating over the first Brillouin zone, makes the odd functions zero i.e. $-\pi/b_s \leq \vec{p}_s \leq \pi/b_s$. Thus

$$\vec{S}_1 = -\frac{2e\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{n_o uwb_s b_z}{2I_o(\Delta_z^*)I_o(\Delta_s^*)}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right) \times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{v\vec{p}_z b_z}{\hbar}\right]\right\} \quad (F.31)$$

Rearranging the terms yields:

$$\vec{S}_1 = -\frac{2e\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{n_o uwb_s b_z}{2I_o(\Delta_z^*)I_o(\Delta_s^*)} \times \left\{\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right) \times \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}\right]\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \exp\left[\Delta_z^*\cos\frac{v\vec{p}_z b_z}{\hbar}\right]\right\} \quad (F.32)$$

Changing the intgration path

$$\vec{S}_1 = -\frac{2e\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{n_o uwb_s b_z}{2I_o(\Delta_z^*)I_o(\Delta_s^*)} \times \left\{\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right) \times \frac{1}{ub_s}\int_{-\pi/b_s}^{\pi/b_s} dZ_s \cos\frac{Z_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{Z_s}{\hbar}\right]\frac{1}{wb_z}\int_{-\pi/b_z}^{\pi/b_z} dZ_z \exp\left[\Delta_z^*\cos\frac{Z_z}{\hbar}\right]\right\} \quad (F.33)$$

$$\vec{S}_1 = -\frac{2e\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{n_o}{2I_o(\Delta_z^*)I_o(\Delta_s^*)} \times \left\{\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right) \times \frac{2}{\pi}\int_0^\pi dZ_s \cos\frac{Z_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{Z_s}{\hbar}\right]\frac{2}{\pi}\int_0^\pi dZ_z \exp\left[\Delta_z^*\cos\frac{Z_z}{\hbar}\right]\right\} \quad (F.34)$$

$$\vec{S}_1 = -\frac{e\tau^{-1}}{(\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{n_o}{I_o(\Delta_z^*)I_o(\Delta_s^*)} \times \left\{\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right) \times \frac{1}{\pi}\int_0^\pi dZ_s \cos\frac{Z_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{Z_s}{\hbar}\right]\frac{1}{\pi}\int_0^\pi dZ_z \exp\left[\Delta_z^*\cos\frac{Z_z}{\hbar}\right]\right\} \quad (F.35)$$

By definition

$$I_1(\Delta_s^*) = \frac{1}{\pi}\int_0^{\pi}\frac{dZ_s}{\hbar}\cos\frac{Z_s}{\hbar}\exp\left(\Delta_s^*\cos\frac{Z_s}{\hbar}\right) \quad \text{(F.36)}$$

and

$$I_0(\Delta_z^*) = \frac{1}{\pi}\int_0^{\pi}\frac{dZ_z}{\hbar}\exp\left(\Delta_z^*\cos\frac{Z_z}{\hbar}\right) \quad \text{(F.37)}$$

F.35 yields:

$$\begin{aligned}\vec{S}_1 = &-\frac{e\tau^{-1}}{(\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{n_o}{I_o(\Delta_z^*)I_o(\Delta_s^*)}\pi^2\hbar^2 I_1(\Delta_s^*)I_0(\Delta_z^*)\\ &\times\left\{\int_0^{\infty}\exp\left(\frac{-t}{\tau}\right)dt\sin\left(\frac{ub_se}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)\right.\\ &\left.\times\frac{1}{\pi}\int_0^{\pi}dZ_s\cos\frac{Z_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{Z_s}{\hbar}\right]\frac{1}{\pi}\int_0^{\pi}dZ_s\exp\left[\Delta_z^*\cos\frac{Z_z}{\hbar}\right]\right\}\end{aligned} \quad \text{(F.38)}$$

$$\begin{aligned}\vec{S}_1 = &-\frac{e\tau^{-1}n_ou\Delta_s b_s}{\hbar}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\\ &\times\left\{\int_0^{\infty}\exp\left(\frac{-t}{\tau}\right)dt\sin\left(\frac{ub_se}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)\right\}\end{aligned} \quad \text{(F.39)}$$

Making use o the identity:

$$\begin{aligned}&\int_0^{\infty}\exp\left(\frac{-t}{\tau}\right)\sin\left(\frac{ub_se}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)dt\\ &\qquad = \sum_{n=-\infty} J_n^2(\chi)\left[\frac{(ueb_s\vec{E}_o/\hbar+n\omega\hbar)\tau^2}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]\end{aligned} \quad \text{(F.40)}$$

$$\vec{S}_1 = -\frac{e\tau^{-1}n_ou\Delta_s b_s}{\hbar}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{(ueb_s\vec{E}_o/\hbar+n\omega\hbar)\tau^2}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right] \quad \text{(F.41)}$$

$$\begin{aligned}\vec{S}_1 = &-\frac{e^2\tau n_o u^2\Delta_s b_s^2}{\hbar^2}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{1}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]\\ &\times\left(\vec{E}_o+\frac{n\omega\hbar}{ueb_s}\right)\end{aligned} \quad \text{(F.42)}$$

Now let $\sigma_s(\vec{E})$ be

$$\sigma_s(\vec{E}) = -\frac{e^2\tau n_o u^2\Delta_s b_s^2}{\hbar^2}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{1}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right] \quad \text{(F.43)}$$

Let $\vec{E}_n = \vec{E}_o + n\omega\hbar/ueb_s$

$$\vec{S}_1 = -\sigma_s(\vec{E})\left(\vec{E}_o + \frac{n\omega\hbar}{ueb_s}\right) \tag{F.44}$$

$$\vec{S}_1 = -\sigma_s(\vec{E})\vec{E}_n \tag{F.45}$$

Solving for F.16

$$\begin{aligned}\vec{S}_2 = &-\frac{2e}{(2\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \\ &\times\left\{\left[\varepsilon_o + \Delta_z\cos\frac{v\vec{p}_z b_z}{\hbar} + \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\} \\ &\times\sin\frac{u\vec{p}_s b_s}{\hbar}\frac{\partial f_o(\vec{p})}{\partial\varepsilon}v_s(\vec{p} - e\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt')\end{aligned} \tag{F.46}$$

$$\frac{\partial f_o(\vec{p})}{\partial\varepsilon(\vec{p})} = -\frac{n_o uwb_s b_z}{2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \tag{F.47}$$

$$\begin{aligned}\vec{S}_2 = &-\frac{2e}{(2\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\ &\times\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s\left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\} \\ &\times\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\ &\times\left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)\right. \\ &\left.-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)\right\}\end{aligned} \tag{F.48}$$

$$\begin{aligned}\vec{S}_2 = &-\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\ &\times\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s\cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right) \\ &\times\left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\} \\ &\times\sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\ &+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\ &\times\sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right) \\ &\times\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s\left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.\end{aligned}$$

$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\Big]\frac{\nabla_s T}{T} + \nabla_s \mu\Big\}$$
$$\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad (F.49)$$

Thus:

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times\left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$+ \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \Big\{\Big[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}$$
$$- \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\Big]\frac{\nabla_s T}{T} + \nabla_s\mu\Big\}$$
$$\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad (F.50)$$

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times\left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$+ \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \Big\{\Big[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}$$
$$- \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\Big]\frac{\nabla_s T}{T} + \nabla_s\mu\Big\}$$

$$\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \tag{F.51}$$

Changing the path of integration

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s\cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}$$
$$\times\sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.$$
$$\left.\left.-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}$$
$$\times\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \tag{F.52}$$

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s\cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}$$
$$\times\sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.$$
$$\left.\left.-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}$$

$$\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad (F.53)$$

$$\begin{aligned}\vec{S}_2 = &-\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt\\
&\times\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s\cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)\\
&\times\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}\\
&\times\sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt\\
&\times\sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)\\
&\times\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.\\
&\left.\left.-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}\\
&\times\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad (F.54)\end{aligned}$$

$$\begin{aligned}\vec{S}_2 = &-\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt\\
&\times\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s\cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)\\
&\times\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}\\
&\times\sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt\\
&\times\sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)\\
&\times\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.\\
&\left.\left.-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}\\
&\times\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad (F.55)\end{aligned}$$

$$\vec{S}_2 = -\frac{eu^2\Delta_s^2 b_s^2 n_o}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}$$
$$\frac{1}{\pi}\int_0^{\pi} dZ_z \frac{1}{\pi}\int_0^{\pi} dZ_s \sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]$$
$$\times\left\{[\varepsilon_o-\mu]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}\exp\left[\Delta_s^*\cos\frac{Z_s}{\hbar}+\Delta_z^*\cos\frac{Z_z}{\hbar}\right]$$
$$+\frac{eu^2\Delta_s^2 b_s^2 n_o}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{\pi}\int_0^{\pi} dZ_z \frac{1}{\pi}\int_0^{\pi} dZ_s$$
$$\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]$$
$$\times\left\{[\varepsilon_o-\mu]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}$$
$$\times\left(\cos\frac{2Z_s}{\hbar}\right)\exp\left[\Delta_s^*\cos\frac{Z_s}{\hbar}+\Delta_z^*\cos\frac{Z_z}{\hbar}\right]$$
$$+\frac{eu^2\Delta_s^2 b_s^2 n_o v\Delta_s}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{\pi}\int_0^{\pi} dZ_z \frac{1}{\pi}\int_0^{\pi} dZ_s$$
$$\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]\frac{\nabla_s T}{T}$$
$$\times\left(\cos\frac{Z_s}{\hbar}\right)\exp\left[\Delta_s^*\cos\frac{Z_s}{\hbar}+\Delta_z^*\cos\frac{Z_z}{\hbar}\right]$$
$$-\frac{eu^2\Delta_s^2 b_s^2 n_o v\Delta_s}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{\pi}\int_0^{\pi} dZ_z \frac{1}{\pi}\int_0^{\pi} dZ_s$$
$$\times\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]\frac{\nabla_s T}{T}$$
$$\times\left(\cos\frac{Z_s}{\hbar}\cos\frac{2Z_s}{\hbar}\right)\exp\left[\Delta_s^*\cos\frac{Z_s}{\hbar}+\Delta_z^*\cos\frac{Z_z}{\hbar}\right]$$
$$+\frac{eu^2\Delta_s^2 b_s^2 n_o v\Delta_s}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{\pi}\int_0^{\pi} dZ_z \frac{1}{\pi}\int_0^{\pi} dZ_s$$
$$\times\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]\frac{\nabla_s T}{T}$$
$$\times\left(\cos\frac{Z_z}{\hbar}\right)\exp\left[\Delta_s^*\cos\frac{Z_s}{\hbar}+\Delta_z^*\cos\frac{Z_z}{\hbar}\right]$$
$$-\frac{eu^2\Delta_s^2 b_s^2 n_o v\Delta_s}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{\pi}\int_0^{\pi} dZ_z \frac{1}{\pi}\int_0^{\pi} dZ_s$$
$$\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]\frac{\nabla_s T}{T}$$
$$\times\left(\cos\frac{Z_z}{\hbar}\cos\frac{2Z_s}{\hbar}\right)\exp\left[\Delta_s^*\cos\frac{Z_s}{\hbar}+\Delta_z^*\cos\frac{Z_z}{\hbar}\right] \quad \text{(F.56)}$$

Breaking down F.56 and solving yields:

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s\cos\left(\frac{ub_se}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_zb_z}{\hbar}-\Delta_s\cos\frac{u\vec{p}_sb_s}{\hbar}-\mu\right]\frac{\nabla_sT}{T}+\nabla_s\mu\right\}$$
$$\times\sin^2\frac{u\vec{p}_sb_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_sb_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_zb_z}{\hbar}\right]$$
$$+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\sin\left(\frac{ub_se}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_zb_z}{\hbar}\right.\right.$$
$$\left.\left.-\Delta_s\cos\frac{u\vec{p}_sb_s}{\hbar}-\mu\right]\frac{\nabla_sT}{T}+\nabla_s\mu\right\}$$
$$\times\cos\frac{u\vec{p}_sb_s}{\hbar}\sin\frac{u\vec{p}_sb_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_sb_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_zb_z}{\hbar}\right] \quad \text{(F.57)}$$

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s\cos\left(\frac{ub_se}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_zb_z}{\hbar}-\Delta_s\cos\frac{u\vec{p}_sb_s}{\hbar}-\mu\right]\frac{\nabla_sT}{T}+\nabla_s\mu\right\}$$
$$\times\sin^2\frac{u\vec{p}_sb_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_sb_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_zb_z}{\hbar}\right]$$
$$+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\sin\left(\frac{ub_se}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_zb_z}{\hbar}\right.\right.$$
$$\left.\left.-\Delta_s\cos\frac{u\vec{p}_sb_s}{\hbar}-\mu\right]\frac{\nabla_sT}{T}+\nabla_s\mu\right\}$$
$$\times\cos\frac{u\vec{p}_sb_s}{\hbar}\sin\frac{u\vec{p}_sb_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_sb_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_zb_z}{\hbar}\right] \quad \text{(F.58)}$$

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$

$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t^{'})]dt^{'}\right)$$
$$\times \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$+ \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t^{'})]dt^{'}\right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.$$
$$\left.\left. - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad (F.59)$$

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t^{'})]dt^{'}\right)$$
$$\times \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$+ \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t^{'})]dt^{'}\right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.$$
$$\left.\left. - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad (F.60)$$

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t^{'})]dt^{'}\right)$$

$$\times\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_zb_z}{\hbar}-\Delta_s\cos\frac{u\vec{p}_sb_s}{\hbar}-\mu\right]\frac{\nabla_sT}{T}+\nabla_s\mu\right\}$$
$$\times\sin^2\frac{u\vec{p}_sb_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_sb_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_zb_z}{\hbar}\right]$$
$$+\frac{2eu^2\Delta_s^2b_s^2n_ouwb_sb_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\sin\left(\frac{ub_se}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_zb_z}{\hbar}\right.\right.$$
$$\left.\left.-\Delta_s\cos\frac{u\vec{p}_sb_s}{\hbar}-\mu\right]\frac{\nabla_sT}{T}+\nabla_s\mu\right\}$$
$$\times\cos\frac{u\vec{p}_sb_s}{\hbar}\sin\frac{u\vec{p}_sb_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_sb_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_zb_z}{\hbar}\right]\quad\text{(F.61)}$$

$$\vec{S}_2=-\frac{2eu^2\Delta_s^2b_s^2n_ouwb_sb_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\cos\left(\frac{ub_se}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_zb_z}{\hbar}-\Delta_s\cos\frac{u\vec{p}_sb_s}{\hbar}-\mu\right]\frac{\nabla_sT}{T}+\nabla_s\mu\right\}$$
$$\times\sin^2\frac{u\vec{p}_sb_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_sb_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_zb_z}{\hbar}\right]$$
$$+\frac{2eu^2\Delta_s^2b_s^2n_ouwb_sb_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\sin\left(\frac{ub_se}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_zb_z}{\hbar}\right.\right.$$
$$\left.\left.-\Delta_s\cos\frac{u\vec{p}_sb_s}{\hbar}-\mu\right]\frac{\nabla_sT}{T}+\nabla_s\mu\right\}$$
$$\times\cos\frac{u\vec{p}_sb_s}{\hbar}\sin\frac{u\vec{p}_sb_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_sb_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_zb_z}{\hbar}\right]\quad\text{(F.62)}$$

$$\vec{S}_2=-\frac{2eu^2\Delta_s^2b_s^2n_ouwb_sb_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\cos\left(\frac{ub_se}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_zb_z}{\hbar}-\Delta_s\cos\frac{u\vec{p}_sb_s}{\hbar}-\mu\right]\frac{\nabla_sT}{T}+\nabla_s\mu\right\}$$

$$\times \sin^2 \frac{u\vec{p}_s b_s}{\hbar} \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$+ \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.$$
$$\left.\left. - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad \text{(F.63)}$$

where

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$+ \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.$$
$$\left.\left. - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad \text{(F.64)}$$

The second order modified Bessel function obey the folowing recurrence relation

$$I_{n+1}(x) = I_{n-1}(x) - \frac{2n}{x}I_n(x) \quad \text{(F.65)}$$

$$I_2(\Delta_s^*) = I_0(\Delta_s^*) - \frac{2}{\Delta_s^*}I_n(\Delta_s^*) \quad \text{(F.66)}$$

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$

$$
\begin{aligned}
&\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)\\
&\times\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}\\
&\times\sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt\\
&\times\sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)\\
&\times\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.\\
&\left.\left.-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}\\
&\times\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \qquad \text{(F.67)}
\end{aligned}
$$

$$
\begin{aligned}
\vec{S}_2 = &-\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt\\
&\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)\\
&\times\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}\\
&\times\sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt\\
&\times\sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)\\
&\times\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.\\
&\left.\left.-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}\\
&\times\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \qquad \text{(F.68)}
\end{aligned}
$$

$$
\begin{aligned}
\vec{S}_2 = &-\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt\\
&\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)
\end{aligned}
$$

$$\times \left\{ \left[\varepsilon_o - \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} - \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} - \mu \right] \frac{\nabla_s T}{T} + \nabla_s \mu \right\}$$
$$\times \sin^2 \frac{u\vec{p}_s b_s}{\hbar} \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$+ \frac{2eu^2 \Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2 \hbar^2 2I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \sin \left(\frac{ub_s e}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')] dt' \right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{ \left[\varepsilon_o - \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \right.\right.$$
$$\left.\left. - \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} - \mu \right] \frac{\nabla_s T}{T} + \nabla_s \mu \right\}$$
$$\times \cos \frac{u\vec{p}_s b_s}{\hbar} \sin \frac{u\vec{p}_s b_s}{\hbar} \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \quad \text{(F.69)}$$

$$\vec{S}_2 = - \frac{2eu^2 \Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2 \hbar^2 2I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos \left(\frac{ub_s e}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')] dt' \right)$$
$$\times \left\{ \left[\varepsilon_o - \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} - \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} - \mu \right] \frac{\nabla_s T}{T} + \nabla_s \mu \right\}$$
$$\times \sin^2 \frac{u\vec{p}_s b_s}{\hbar} \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$+ \frac{2eu^2 \Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2 \hbar^2 2I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \sin \left(\frac{ub_s e}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')] dt' \right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{ \left[\varepsilon_o - \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \right.\right.$$
$$\left.\left. - \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} - \mu \right] \frac{\nabla_s T}{T} + \nabla_s \mu \right\}$$
$$\times \cos \frac{u\vec{p}_s b_s}{\hbar} \sin \frac{u\vec{p}_s b_s}{\hbar} \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \quad \text{(F.70)}$$

$$\vec{S}_2 = - \frac{2eu^2 \Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2 \hbar^2 2I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos \left(\frac{ub_s e}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')] dt' \right)$$
$$\times \left\{ \left[\varepsilon_o - \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} - \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} - \mu \right] \frac{\nabla_s T}{T} + \nabla_s \mu \right\}$$

$$\times \sin^2 \frac{u\vec{p}_s b_s}{\hbar} \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$+ \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T} \int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')] dt'\right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o - \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar}\right.\right.$$
$$\left.\left. - \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} - \mu\right] \frac{\nabla_s T}{T} + \nabla_s \mu\right\}$$
$$\times \cos \frac{u\vec{p}_s b_s}{\hbar} \sin \frac{u\vec{p}_s b_s}{\hbar} \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar}\right] \quad \text{(F.71)}$$

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T} \int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')] dt'\right)$$
$$\times \left\{\left[\varepsilon_o - \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} - \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} - \mu\right] \frac{\nabla_s T}{T} + \nabla_s \mu\right\}$$
$$\times \sin^2 \frac{u\vec{p}_s b_s}{\hbar} \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$+ \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T} \int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')] dt'\right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o - \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar}\right.\right.$$
$$\left.\left. - \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} - \mu\right] \frac{\nabla_s T}{T} + \nabla_s \mu\right\}$$
$$\times \cos \frac{u\vec{p}_s b_s}{\hbar} \sin \frac{u\vec{p}_s b_s}{\hbar} \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar}\right] \quad \text{(F.72)}$$

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T} \int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')] dt'\right)$$
$$\times \left\{\left[\varepsilon_o - \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} - \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} - \mu\right] \frac{\nabla_s T}{T} + \nabla_s \mu\right\}$$
$$\times \sin^2 \frac{u\vec{p}_s b_s}{\hbar} \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$
\begin{aligned}
&+\frac{2eu^2\Delta_s^2b_s^2n_ouwb_sb_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
&\times \sin\left(\frac{ub_se}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right) \\
&\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_zb_z}{\hbar}\right.\right. \\
&\left.\left.-\Delta_s\cos\frac{u\vec{p}_sb_s}{\hbar}-\mu\right]\frac{\nabla_sT}{T}+\nabla_s\mu\right\} \\
&\times \cos\frac{u\vec{p}_sb_s}{\hbar}\sin\frac{u\vec{p}_sb_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_sb_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_zb_z}{\hbar}\right] \qquad \text{(F.73)}
\end{aligned}
$$

$$
\begin{aligned}
\vec{S}_2 = &-\frac{2eu^2\Delta_s^2b_s^2n_ouwb_sb_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
&\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_se}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right) \\
&\times \left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_zb_z}{\hbar}-\Delta_s\cos\frac{u\vec{p}_sb_s}{\hbar}-\mu\right]\frac{\nabla_sT}{T}+\nabla_s\mu\right\} \\
&\times \sin^2\frac{u\vec{p}_sb_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_sb_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_zb_z}{\hbar}\right] \\
&+\frac{2eu^2\Delta_s^2b_s^2n_ouwb_sb_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
&\times \sin\left(\frac{ub_se}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right) \\
&\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_zb_z}{\hbar}\right.\right. \\
&\left.\left.-\Delta_s\cos\frac{u\vec{p}_sb_s}{\hbar}-\mu\right]\frac{\nabla_sT}{T}+\nabla_s\mu\right\} \\
&\times \cos\frac{u\vec{p}_sb_s}{\hbar}\sin\frac{u\vec{p}_sb_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_sb_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_zb_z}{\hbar}\right] \qquad \text{(F.74)}
\end{aligned}
$$

$$
\begin{aligned}
\vec{S}_2 = &-\frac{2eu^2\Delta_s^2b_s^2n_ouwb_sb_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
&\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_se}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right) \\
&\times \left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_zb_z}{\hbar}-\Delta_s\cos\frac{u\vec{p}_sb_s}{\hbar}-\mu\right]\frac{\nabla_sT}{T}+\nabla_s\mu\right\} \\
&\times \sin^2\frac{u\vec{p}_sb_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_sb_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_zb_z}{\hbar}\right] \\
&+\frac{2eu^2\Delta_s^2b_s^2n_ouwb_sb_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt
\end{aligned}
$$

$$\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t^{'})]dt^{'}\right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}$$
$$\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad (F.75)$$

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t^{'})]dt^{'}\right)$$
$$\times \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}$$
$$\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t^{'})]dt^{'}\right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}$$
$$\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad (F.76)$$

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t^{'})]dt^{'}\right)$$
$$\times \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}$$
$$\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t^{'})]dt^{'}\right)$$

$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{ \left[\varepsilon_o - \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} - \mu \right] \frac{\nabla_s T}{T} + \nabla_s \mu \right\} \times \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\frac{u\vec{p}_s b_s}{\hbar} \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right] \quad (F.77)$$

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T} \int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt \times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')] dt' \right) \times \left\{ \left[\varepsilon_o - \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} - \mu \right] \frac{\nabla_s T}{T} + \nabla_s \mu \right\} \times \sin^2\frac{u\vec{p}_s b_s}{\hbar} \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right] + \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T} \int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt \times \sin\left(\frac{ub_s e}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')] dt' \right) \times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{ \left[\varepsilon_o - \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} - \mu \right] \frac{\nabla_s T}{T} + \nabla_s \mu \right\} \times \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\frac{u\vec{p}_s b_s}{\hbar} \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right] \quad (F.78)$$

$$I_2(\Delta_s^*) = \frac{1}{\pi} \int_0^\pi dZ_s \cos\frac{3Z_s}{\hbar} \exp\left(\Delta_s^* \cos\frac{Z_s}{\hbar} \right) \quad (F.79)$$

The third order modified Bessel function obey the folowing recurrence relation

$$I_{n+1}(x) = I_{n-1}(x) - \frac{2n}{x} I_n(x) \quad (F.80)$$

$$I_3(\Delta_s^*) = I_1(\Delta_s^*) - \frac{4}{\Delta_s^*} I_2(\Delta_s^*) \quad (F.81)$$

$$I_3(\Delta_s^*) = I_1(\Delta_s^*) - \frac{4}{\Delta_s^*}\left[I_0(\Delta_s^*) - \frac{4}{\Delta_s^*} I_1(\Delta_s^*) \right] \quad (F.82)$$

$$I_3(\Delta_s^*) = I_1(\Delta_s^*) - \frac{4}{\Delta_s^*} I_o(\Delta_s^*) - \frac{8}{\Delta_s^{*2}} I_1(\Delta_s^*) \quad (F.83)$$

Thus

$$\frac{I_1(\Delta_s^*) + I_3(\Delta_s^*)}{I_o(\Delta_s^*)} = \frac{I_1(\Delta_s^*) + I_1(\Delta_s^*) - \frac{4}{\Delta_s} I_o(\Delta_s^*) - \frac{8}{\Delta_s^{*2}} I_1(\Delta_s^*)}{I_o(\Delta_s^*)} \quad (F.84)$$

$$\frac{I_1(\Delta_s^*)+I_3(\Delta_s^*)}{I_o(\Delta_s^*)} = \frac{2I_1(\Delta_s^*)}{I_o(\Delta_s^*)} - \frac{4}{\Delta_s}\frac{I_o(\Delta_s^*)}{I_o(\Delta_s^*)} + \frac{8}{\Delta_s^{*2}}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} \tag{F.85}$$

$$\begin{aligned}
\vec{S}_2 = &-\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
&\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_se}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right) \\
&\times \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_zb_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_sb_s}{\hbar} - \mu\right]\frac{\nabla_sT}{T} + \nabla_s\mu\right\} \\
&\times \sin^2\frac{u\vec{p}_sb_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_sb_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_zb_z}{\hbar}\right] \\
&+ \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
&\times \sin\left(\frac{ub_se}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right) \\
&\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_zb_z}{\hbar}\right.\right. \\
&\left.\left. - \Delta_s\cos\frac{u\vec{p}_sb_s}{\hbar} - \mu\right]\frac{\nabla_sT}{T} + \nabla_s\mu\right\} \\
&\times \cos\frac{u\vec{p}_sb_s}{\hbar}\sin\frac{u\vec{p}_sb_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_sb_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_zb_z}{\hbar}\right]
\end{aligned} \tag{F.86}$$

$$\begin{aligned}
\vec{S}_2 = &-\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
&\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_se}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right) \\
&\times \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_zb_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_sb_s}{\hbar} - \mu\right]\frac{\nabla_sT}{T} + \nabla_s\mu\right\} \\
&\times \sin^2\frac{u\vec{p}_sb_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_sb_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_zb_z}{\hbar}\right] \\
&+ \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
&\times \sin\left(\frac{ub_se}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right) \\
&\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_zb_z}{\hbar}\right.\right. \\
&\left.\left. - \Delta_s\cos\frac{u\vec{p}_sb_s}{\hbar} - \mu\right]\frac{\nabla_sT}{T} + \nabla_s\mu\right\} \\
&\times \cos\frac{u\vec{p}_sb_s}{\hbar}\sin\frac{u\vec{p}_sb_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_sb_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_zb_z}{\hbar}\right]
\end{aligned} \tag{F.87}$$

$$
\begin{aligned}
\vec{S}_2 = &-\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
&\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right) \\
&\times \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\} \\
&\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&+ \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
&\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right) \\
&\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right. \\
&\left.\left. - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\} \\
&\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]
\end{aligned} \tag{F.88}
$$

$$
\begin{aligned}
\vec{S}_2 = &-\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
&\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right) \\
&\times \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\} \\
&\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&+ \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
&\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right) \\
&\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right. \\
&\left.\left. - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\} \\
&\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]
\end{aligned} \tag{F.89}
$$

$$
\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt
$$

$$
\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)
\times \left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}
\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]
+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt
\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)
\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}
-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}
\times\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad (F.90)
$$

$$
\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt
\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)
\times \left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}
\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]
+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt
\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)
\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}
-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}
\times\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad (F.91)
$$

$$
\frac{I_2(\Delta_s^*)I_1(\Delta_z^*)}{I_o(\Delta_s^*)I_o(\Delta_z^*)} = \frac{\left[I_0(\Delta_s^*)-\frac{2}{\Delta_s^*}I_n(\Delta_s^*)\right]I_1(\Delta_z^*)}{I_o(\Delta_s^*)I_o(\Delta_z^*)}
$$

$$
\frac{I_2(\Delta_s^*)I_1(\Delta_z)}{I_o(\Delta_s^*)I_o(\Delta_z^*)} = \frac{I_o(\Delta_s^*)I_1(\Delta_z)}{I_o(\Delta_s^*)I_o(\Delta_z^*)} - \frac{2}{\Delta_s^*}\frac{I_1(\Delta_s^*)I_1(\Delta_z^*)}{I_o(\Delta_s^*)I_o(\Delta_z^*)}
$$

$$\frac{I_2(\Delta_s^*)I_1(\Delta_z^*)}{I_o(\Delta_s^*)I_o(\Delta_z^*)} = \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} - \frac{2}{\Delta_s^*}\frac{I_1(\Delta_s^*)I_1(\Delta_z^*)}{I_o(\Delta_s^*)I_o(\Delta_z^*)}$$

$$\begin{aligned}
\vec{S}_2 = &-\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
&\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right) \\
&\times \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\} \\
&\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&+ \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
&\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right) \\
&\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right. \\
&\left.\left. - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\} \\
&\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]
\end{aligned} \quad (F.92)$$

$$\begin{aligned}
\vec{S}_2 = &-\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
&\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right) \\
&\times \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\} \\
&\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&+ \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
&\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right) \\
&\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right. \\
&\left.\left. - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\} \\
&\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]
\end{aligned} \quad (F.93)$$

Adding all the terms in $\vec{S}_2$

$$\vec{S}_2 = -\frac{eu^2\Delta_s^2 b_s^2 n_o}{2\hbar^2 k_B T}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right] \times\left\{[\varepsilon_o-\mu]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}$$
$$+\frac{eu^2\Delta_s^2 b_s^2 n_o}{2\hbar^2 k_B T}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]\left\{[\varepsilon_o-\mu]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}$$
$$-\frac{eu^2\Delta_s^2 b_s^2 n_o I_1(\Delta_s^*)}{\hbar^2 k_B T I_0(\Delta_s)}\frac{1}{\Delta_s^*}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]\left\{[\varepsilon_o-\mu]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}$$
$$+\frac{eu^2\Delta_s^2 b_s^2 n_o v\Delta_s}{2\hbar^2 k_B T}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]\frac{\nabla_s T}{T}\frac{I_1(\Delta_s^*)}{I_0(\Delta_s)}$$
$$-\frac{eu^2\Delta_s^2 b_s^2 n_o v\Delta_s}{2\hbar^2 k_B T}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]\frac{\nabla_s T}{T}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}$$
$$+\frac{eu^2\Delta_s^2 b_s^2 n_o u}{\hbar^2}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]\frac{\nabla_s T}{T}$$
$$-\frac{2eu^2\Delta_s^2 b_s^2 n_o}{\hbar^2\Delta_s^*}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]\frac{\nabla_s T}{T}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}$$
$$+\frac{eu^2\Delta_s^2 b_s^2 n_o w\Delta_z}{2\hbar^2 k_B T}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]\frac{\nabla_s T}{T}\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}$$
$$-\frac{eu^2\Delta_s^2 b_s^2 n_o w\Delta_z}{2\hbar^2 k_B T}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]\frac{\nabla_s T}{T}\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}$$
$$+\frac{eu^2\Delta_s^2 b_s^2 n_o w\Delta_z}{\hbar^2 k_B T}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+(ueb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]\frac{\nabla_s T}{T} \times\frac{1}{\Delta_s^*}\frac{I_1(\Delta_s^*)I_1(\Delta_z^*)}{I_o(\Delta_s^*)I_o(\Delta_z^*)} \quad (F.94)$$

Simplifying the expression

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \times\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s\cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}$$
$$\times\sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \times\sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$

$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{ \left[\varepsilon_o - \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} - \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} - \mu \right] \frac{\nabla_s T}{T} + \nabla_s \mu \right\}$$
$$\times \cos \frac{u\vec{p}_s b_s}{\hbar} \sin \frac{u\vec{p}_s b_s}{\hbar} \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \quad \text{(F.95)}$$

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T} \int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')] dt'\right)$$
$$\times \left\{ \left[\varepsilon_o - \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} - \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} - \mu \right] \frac{\nabla_s T}{T} + \nabla_s \mu \right\}$$
$$\times \sin^2 \frac{u\vec{p}_s b_s}{\hbar} \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$+ \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T} \int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')] dt'\right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{ \left[\varepsilon_o - \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} - \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} - \mu \right] \frac{\nabla_s T}{T} + \nabla_s \mu \right\}$$
$$\times \cos \frac{u\vec{p}_s b_s}{\hbar} \sin \frac{u\vec{p}_s b_s}{\hbar} \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \quad \text{(F.96)}$$

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T} \int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')] dt'\right)$$
$$\times \left\{ \left[\varepsilon_o - \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} - \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} - \mu \right] \frac{\nabla_s T}{T} + \nabla_s \mu \right\}$$
$$\times \sin^2 \frac{u\vec{p}_s b_s}{\hbar} \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$+ \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T} \int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')] dt'\right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{ \left[\varepsilon_o - \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \right.\right.$$

$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\bigg]\frac{\nabla_s T}{T} + \nabla_s\mu\bigg\}$$
$$\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad \text{(F.97)}$$

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$+ \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \bigg\{\bigg[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}$$
$$- \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\bigg]\frac{\nabla_s T}{T} + \nabla_s\mu\bigg\}$$
$$\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad \text{(F.98)}$$

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$+ \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \bigg\{\bigg[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}$$
$$- \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\bigg]\frac{\nabla_s T}{T} + \nabla_s\mu\bigg\}$$

$$\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad (F.99)$$

$$\begin{aligned}
\vec{S}_2 = &-\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
&\times\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right) \\
&\times\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\} \\
&\times\sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
&\times\sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right) \\
&\times\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right. \\
&\left.\left.-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\} \\
&\times\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad (F.100)
\end{aligned}$$

$$\begin{aligned}
\vec{S}_2 = &-\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
&\times\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right) \\
&\times\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\} \\
&\times\sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
&\times\sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right) \\
&\times\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right. \\
&\left.\left.-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\} \\
&\times\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad (F.101)
\end{aligned}$$

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$+ \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.$$
$$\left.\left. - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad \text{(F.102)}$$

where $\sigma_s(\vec{E})$ is the conductivity. But $\vec{S} = \vec{S}_1 + \vec{S}_2$

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$+ \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.$$
$$\left.\left. - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad \text{(F.103)}$$

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$

$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos(\omega t')]dt'\right)$$
$$\times \left\{\left[\varepsilon_o - \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s \mu\right\}$$
$$\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$+ \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos(\omega t')]dt'\right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o - \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.$$
$$\left.\left. - \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s \mu\right\}$$
$$\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad \text{(F.104)}$$

where $\vec{E}_{sn}^* = \vec{E}_n + \nabla_s \mu / e$. Similarly

$$\vec{Z} = -\sigma_z(\vec{E})\vec{E}_{zn}^* - \sigma_z(\vec{E})\frac{k_B}{e}\left[\left(\frac{\varepsilon_o - \mu}{k_B T}\right) - \Delta_z^* \frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)} + 2 - \Delta_s^* \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right]\nabla_z T \quad \text{(F.105)}$$

Resolving the current density into axial and circumferential components, the axial and circumferential components are given as: $\vec{J}_z = \vec{Z} + \vec{S}\sin\theta_h$ and $\vec{J}_s = \vec{S}\cos\theta_h$ respectively:

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos(\omega t')]dt'\right)$$
$$\times \left\{\left[\varepsilon_o - \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s \mu\right\}$$
$$\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$+ \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos(\omega t')]dt'\right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o - \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.$$
$$\left.\left. - \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s \mu\right\}$$
$$\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad \text{(F.106)}$$

Utilising the relations $\vec{E}_s = \vec{E}_z \sin\theta_h$, $\nabla_s T = \nabla_z T \sin\theta_h$ and $\vec{E}^*_{sn} = \vec{E}^*_{zn} \sin\theta_h$

$$\begin{aligned}
\vec{S}_2 = &-\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
&\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right) \\
&\times \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\} \\
&\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&+ \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
&\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right) \\
&\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right. \\
&\left.\left. - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\} \\
&\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \qquad (F.107)
\end{aligned}$$

$$\begin{aligned}
\vec{S}_2 = &-\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
&\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right) \\
&\times \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\} \\
&\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&+ \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
&\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right) \\
&\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right. \\
&\left.\left. - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\} \\
&\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \qquad (F.108)
\end{aligned}$$

$$
\begin{aligned}
\vec{S}_2 = & -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
& \times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right) \\
& \times \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\} \\
& \times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
& + \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
& \times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right) \\
& \times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right. \\
& \left.\left. - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\} \\
& \times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]
\end{aligned} \tag{F.109}
$$

$$
\begin{aligned}
\vec{S}_2 = & -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
& \times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right) \\
& \times \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\} \\
& \times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
& + \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
& \times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right) \\
& \times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right. \\
& \left.\left. - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\} \\
& \times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]
\end{aligned} \tag{F.110}
$$

where

$$\vec{E}_{zn}^* = \nabla_z\left(\frac{\mu}{e} - \varphi\right)$$

and the circumferential component of the electron currrent density is given as:

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times\left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.$$
$$\left.\left. - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad \text{(F.111)}$$

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times\left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.$$
$$\left.\left. - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad \text{(F.112)}$$

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$+ \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.$$
$$\left.\left. - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad \text{(F.113)}$$

Thus the circumferential σ_{cs} and axial σ_{zz} components of the electrical conductivity are given by the coeficients of the electric field E_{zn} as:

$$\sigma_{cs} = \sigma_s \sin\theta_h \cos\theta_h \qquad \sigma_{zz} = \sigma_z + \sigma_s \sin^2\theta_h$$

The differential thermoelectric power is defined as the ratio $\left|\frac{\vec{E}_{zn}^*}{\nabla_z T}\right|$ in an open circuit $\vec{J} = 0$. Thus setting $\vec{J}_c$ to zero, the thermoelectric power α_{cz} along the circumferential direction is obtained as follows:

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$+ \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.$$

$$- \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} - \mu \Bigg] \frac{\nabla_s T}{T} + \nabla_s \mu \Bigg\}$$
$$\times \cos \frac{u\vec{p}_s b_s}{\hbar} \sin \frac{u\vec{p}_s b_s}{\hbar} \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \quad \text{(F.114)}$$

$$\vec{S}_2 = -\frac{2eu^2 \Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2 \hbar^2 2I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')] dt'\right)$$
$$\times \left\{ \left[\varepsilon_o - \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} - \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} - \mu \right] \frac{\nabla_s T}{T} + \nabla_s \mu \right\}$$
$$\times \sin^2 \frac{u\vec{p}_s b_s}{\hbar} \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$+ \frac{2eu^2 \Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2 \hbar^2 2I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')] dt'\right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \Bigg\{ \Bigg[\varepsilon_o - \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar}$$
$$- \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} - \mu \Bigg] \frac{\nabla_s T}{T} + \nabla_s \mu \Bigg\}$$
$$\times \cos \frac{u\vec{p}_s b_s}{\hbar} \sin \frac{u\vec{p}_s b_s}{\hbar} \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \quad \text{(F.115)}$$

$$\vec{S}_2 = -\frac{2eu^2 \Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2 \hbar^2 2I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')] dt'\right)$$
$$\times \left\{ \left[\varepsilon_o - \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} - \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} - \mu \right] \frac{\nabla_s T}{T} + \nabla_s \mu \right\}$$
$$\times \sin^2 \frac{u\vec{p}_s b_s}{\hbar} \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$+ \frac{2eu^2 \Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2 \hbar^2 2I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')] dt'\right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \Bigg\{ \Bigg[\varepsilon_o - \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar}$$
$$- \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} - \mu \Bigg] \frac{\nabla_s T}{T} + \nabla_s \mu \Bigg\}$$

$$\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad (F.116)$$

$$\alpha_{cz} = \left|\frac{\vec{E}_{zn}^*}{\nabla_z T}\right| \quad (F.117)$$

Similarly, α_{zz} along the axial direction is obtained as follows $J_z = 0$

$$\begin{aligned}
\vec{S}_2 = &-\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
&\times\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s\cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right) \\
&\times\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\} \\
&\times\sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
&\times\sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right) \\
&\times\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right. \\
&\left.\left.-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\} \\
&\times\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad (F.118)
\end{aligned}$$

$$\begin{aligned}
\vec{S}_2 = &-\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
&\times\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s\cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right) \\
&\times\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\} \\
&\times\sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
&\times\sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right) \\
&\times\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right. \\
&\left.\left.-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}
\end{aligned}$$

$$\times\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad (F.119)$$

The electrical power factor along the circumferential $\mathscr{P}_{cs}$ and axial $\mathscr{P}_{zz}$ directions are given respectively as:

$$\mathscr{P}_{cs}=\alpha_{cz}^2\sigma_{cz} \qquad \mathscr{P}_{zz}=\alpha_{zz}^2\sigma_{zz}$$

The electrical power factor along the circumferential direction is given as:

$$\begin{aligned}\vec{S}_2 = &-\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt\\
&\times\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)\\
&\times\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}\\
&\times\sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt\\
&\times\sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)\\
&\times\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.\\
&\left.\left.-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}\\
&\times\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad (F.120)\end{aligned}$$

where the output power density along the circumferential direction becomes;

$$\begin{aligned}\vec{S}_2 = &-\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt\\
&\times\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)\\
&\times\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}\\
&\times\sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt\\
&\times\sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)\end{aligned}$$

$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{ \left[\varepsilon_o - \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} - \mu \right] \frac{\nabla_s T}{T} + \nabla_s \mu \right\}$$
$$\times \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\frac{u\vec{p}_s b_s}{\hbar} \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right] \quad \text{(F.121)}$$

Moreover, the axial component of the power factor yields:

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T} \int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')]dt'\right)$$
$$\times \left\{ \left[\varepsilon_o - \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} - \mu \right] \frac{\nabla_s T}{T} + \nabla_s \mu \right\}$$
$$\times \sin^2\frac{u\vec{p}_s b_s}{\hbar} \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$+ \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T} \int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')]dt'\right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{ \left[\varepsilon_o - \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} - \mu \right] \frac{\nabla_s T}{T} + \nabla_s \mu \right\}$$
$$\times \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\frac{u\vec{p}_s b_s}{\hbar} \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right] \quad \text{(F.122)}$$

and similarly the output power density also gives;

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T} \int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')]dt'\right)$$
$$\times \left\{ \left[\varepsilon_o - \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} - \mu \right] \frac{\nabla_s T}{T} + \nabla_s \mu \right\}$$
$$\times \sin^2\frac{u\vec{p}_s b_s}{\hbar} \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$+ \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T} \int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')]dt'\right)$$

$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{ \left[\varepsilon_o - \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} - \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} - \mu \right] \frac{\nabla_s T}{T} + \nabla_s \mu \right\} \times \cos \frac{u\vec{p}_s b_s}{\hbar} \sin \frac{u\vec{p}_s b_s}{\hbar} \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \quad \text{(F.123)}$$

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T} \int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt \times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')] dt'\right) \times \left\{ \left[\varepsilon_o - \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} - \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} - \mu \right] \frac{\nabla_s T}{T} + \nabla_s \mu \right\} \times \sin^2 \frac{u\vec{p}_s b_s}{\hbar} \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] + \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T} \int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt \times \sin\left(\frac{ub_s e}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')] dt'\right) \times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{ \left[\varepsilon_o - \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} - \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} - \mu \right] \frac{\nabla_s T}{T} + \nabla_s \mu \right\} \times \cos \frac{u\vec{p}_s b_s}{\hbar} \sin \frac{u\vec{p}_s b_s}{\hbar} \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \quad \text{(F.100)}$$

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T} \int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt \times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')] dt'\right) \times \left\{ \left[\varepsilon_o - \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} - \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} - \mu \right] \frac{\nabla_s T}{T} + \nabla_s \mu \right\} \times \sin^2 \frac{u\vec{p}_s b_s}{\hbar} \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] + \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T} \int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt \times \sin\left(\frac{ub_s e}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')] dt'\right) \times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{ \left[\varepsilon_o - \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \right.\right.$$

$$-\Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\Big]\frac{\nabla_s T}{T} + \nabla_s \mu\Big\}$$
$$\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad \text{(F.101)}$$

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times\left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s\left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.$$
$$\left.\left.-\Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s \mu\right\}$$
$$\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad \text{(F.102)}$$

where $\sigma_s(\vec{E})$ is the conductivity. But $\vec{S} = \vec{S}_1 + \vec{S}_2$

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times\left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s\left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.$$
$$\left.\left.-\Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s \mu\right\}$$

$$\times\cos\frac{u\vec{p}_sb_s}{\hbar}\sin\frac{u\vec{p}_sb_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_sb_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_zb_z}{\hbar}\right] \quad (F.103)$$

$$\vec{S}_2=-\frac{2eu^2\Delta_s^2b_s^2n_ouwb_sb_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty\exp\left(\frac{-t}{\tau}\right)dt\times\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\cos\left(\frac{ub_se}{\hbar}\int_{t-t'}^{t}[\vec{E}_o \quad (F.104)$$

where $\vec{E}^*_{sn}=\vec{E}_n+\nabla_s\mu/e$. Similarly

$$\vec{Z}=-\sigma_z(\vec{E})\vec{E}^*_{zn}-\sigma_z(\vec{E})\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_z^*\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}+2-\Delta_s^*\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right]\nabla_zT \quad (F.105)$$

Resolving the current density into axial and circumferential components, the axial and circumferential components are given as: $\vec{J}_z=\vec{Z}+\vec{S}\sin\theta_h$ and $\vec{J}_s=\vec{S}\cos\theta_h$ respectively:

$$\begin{aligned}\vec{S}_2=&-\frac{2eu^2\Delta_s^2b_s^2n_ouwb_sb_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty\exp\left(\frac{-t}{\tau}\right)dt\\&\times\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\cos\left(\frac{ub_se}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)\\&\times\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_zb_z}{\hbar}-\Delta_s\cos\frac{u\vec{p}_sb_s}{\hbar}-\mu\right]\frac{\nabla_sT}{T}+\nabla_s\mu\right\}\\&\times\sin^2\frac{u\vec{p}_sb_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_sb_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_zb_z}{\hbar}\right]\\&+\frac{2eu^2\Delta_s^2b_s^2n_ouwb_sb_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty\exp\left(\frac{-t}{\tau}\right)dt\\&\times\sin\left(\frac{ub_se}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)\\&\times\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_zb_z}{\hbar}\right.\right.\\&\left.\left.-\Delta_s\cos\frac{u\vec{p}_sb_s}{\hbar}-\mu\right]\frac{\nabla_sT}{T}+\nabla_s\mu\right\}\\&\times\cos\frac{u\vec{p}_sb_s}{\hbar}\sin\frac{u\vec{p}_sb_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_sb_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_zb_z}{\hbar}\right]\end{aligned} \quad (F.106)$$

Utilising the relations $\vec{E}_s=\vec{E}_z\sin\theta_h$, $\nabla_sT=\nabla_zT\sin\theta_h$ and $\vec{E}^*_{sn}=\vec{E}^*_{zn}\sin\theta_h$

$$\begin{aligned}\vec{S}_2=&-\frac{2eu^2\Delta_s^2b_s^2n_ouwb_sb_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty\exp\left(\frac{-t}{\tau}\right)dt\\&\times\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\cos\left(\frac{ub_se}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)\\&\times\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_zb_z}{\hbar}-\Delta_s\cos\frac{u\vec{p}_sb_s}{\hbar}-\mu\right]\frac{\nabla_sT}{T}+\nabla_s\mu\right\}\\&\times\sin^2\frac{u\vec{p}_sb_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_sb_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_zb_z}{\hbar}\right]\end{aligned}$$

$$
\begin{aligned}
&+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
&\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right) \\
&\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right. \\
&\left.\left. - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\} \\
&\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]
\end{aligned} \quad \text{(F.107)}
$$

$$
\begin{aligned}
\vec{S}_2 = &-\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
&\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right) \\
&\times \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\} \\
&\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
&\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right) \\
&\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right. \\
&\left.\left. - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\} \\
&\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]
\end{aligned} \quad \text{(F.108)}
$$

$$
\begin{aligned}
\vec{S}_2 = &-\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
&\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right) \\
&\times \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\} \\
&\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt
\end{aligned}
$$

$$\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.$$
$$\left.\left. - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad \text{(F.109)}$$

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$+ \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.$$
$$\left.\left. - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad \text{(F.110)}$$

where

$$\vec{E}_{zn}^* = \nabla_z\left(\frac{\mu}{e} - \varphi\right)$$

and the circumferential component of the electron currrent density is given as:

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$+ \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$

$$\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t^{'})]dt^{'}\right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.$$
$$\left.\left.-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}$$
$$\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad \text{(F.111)}$$

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t^{'})]dt^{'}\right)$$
$$\times \left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}$$
$$\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t^{'})]dt^{'}\right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.$$
$$\left.\left.-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}$$
$$\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad \text{(F.112)}$$

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t^{'})]dt^{'}\right)$$
$$\times \left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}$$
$$\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t^{'})]dt^{'}\right)$$

$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \Bigg\{ \Bigg[\varepsilon_o - \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} - \mu \Bigg] \frac{\nabla_s T}{T} + \nabla_s \mu \Bigg\} \times \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\frac{u\vec{p}_s b_s}{\hbar} \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad \text{(F.113)}$$

Thus the circumferential σ_{cs} and axial σ_{zz} components of the electrical conductivity are given by the coeficients of the electric field E_{zn} as:

$$\sigma_{cs} = \sigma_s \sin\theta_h \cos\theta_h \qquad \sigma_{zz} = \sigma_z + \sigma_s \sin^2\theta_h$$

The differential thermoelectric power is defined as the ratio $\left|\frac{\vec{E}^*_{zn}}{\nabla_z T}\right|$ in an open circuit $\vec{J} = 0$. Thus setting $\vec{J}_c$ to zero, the thermoelectric power α_{cz} along the circumferential direction is obtained as follows:

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right) \times \Bigg\{\Bigg[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\Bigg]\frac{\nabla_s T}{T} + \nabla_s\mu\Bigg\} \times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] + \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right) \times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \Bigg\{\Bigg[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\Bigg]\frac{\nabla_s T}{T} + \nabla_s\mu\Bigg\} \times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad \text{(F.114)}$$

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right) \times \Bigg\{\Bigg[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\Bigg]\frac{\nabla_s T}{T} + \nabla_s\mu\Bigg\} \times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.$$
$$\left.\left.-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}$$
$$\times\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad (F.115)$$

$$\vec{S}_2=-\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt\times\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o \quad (F.116)$$

$$\alpha_{cz}=\left|\frac{\vec{E}_{zn}^*}{\nabla_z T}\right| \quad (F.117)$$

Similarly, α_{zz} along the axial direction is obtained as follows $J_z=0$

$$\vec{S}_2=-\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}$$
$$\times\sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.$$
$$\left.\left.-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}$$
$$\times\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad (F.118)$$

$$\vec{S}_2=-\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$

$$\times \left\{ \left[\varepsilon_o - \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} - \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} - \mu \right] \frac{\nabla_s T}{T} + \nabla_s \mu \right\}$$
$$\times \sin^2 \frac{u\vec{p}_s b_s}{\hbar} \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$+ \frac{2eu^2 \Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2 \hbar^2 2I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')] dt'\right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{ \left[\varepsilon_o - \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \right.\right.$$
$$\left.\left. - \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} - \mu \right] \frac{\nabla_s T}{T} + \nabla_s \mu \right\}$$
$$\times \cos \frac{u\vec{p}_s b_s}{\hbar} \sin \frac{u\vec{p}_s b_s}{\hbar} \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \quad \text{(F.119)}$$

The electrical power factor along the circumferential $\mathscr{P}_{cs}$ and axial $\mathscr{P}_{zz}$ directions are given respectively as:

$$\mathscr{P}_{cs} = \alpha_{cz}^2 \sigma_{cz} \qquad \mathscr{P}_{zz} = \alpha_{zz}^2 \sigma_{zz}$$

The electrical power factor along the circumferential direction is given as:

$$\vec{S}_2 = -\frac{2eu^2 \Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2 \hbar^2 2I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')] dt'\right)$$
$$\times \left\{ \left[\varepsilon_o - \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} - \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} - \mu \right] \frac{\nabla_s T}{T} + \nabla_s \mu \right\}$$
$$\times \sin^2 \frac{u\vec{p}_s b_s}{\hbar} \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$+ \frac{2eu^2 \Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2 \hbar^2 2I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')] dt'\right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{ \left[\varepsilon_o - \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \right.\right.$$
$$\left.\left. - \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} - \mu \right] \frac{\nabla_s T}{T} + \nabla_s \mu \right\}$$
$$\times \cos \frac{u\vec{p}_s b_s}{\hbar} \sin \frac{u\vec{p}_s b_s}{\hbar} \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \quad \text{(F.120)}$$

where the output power density along the circumferential direction becomes;

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}$$
$$\times\sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.$$
$$\left.\left.-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}$$
$$\times\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad \text{(F.121)}$$

Moreover, the axial component of the power factor yields:

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}$$
$$\times\sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.$$
$$\left.\left.-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}$$
$$\times\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad \text{(F.122)}$$

and similarly the output power density also gives;

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t^{'})]dt^{'}\right)$$
$$\times \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$+ \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t^{'})]dt^{'}\right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.$$
$$\left.\left. - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad \text{(F.123)}$$

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t^{'})]dt^{'}\right)$$
$$\times \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$+ \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t^{'})]dt^{'}\right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.$$
$$\left.\left. - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad \text{(F.100)}$$

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$

$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$+ \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.$$
$$\left.\left. - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad (F.101)$$

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$+ \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.$$
$$\left.\left. - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad (F.102)$$

where $\sigma_s(\vec{E})$ is the conductivity. But $\vec{S} = \vec{S}_1 + \vec{S}_2$

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$

$$\times \left\{ \left[\varepsilon_o - \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} - \mu \right] \frac{\nabla_s T}{T} + \nabla_s \mu \right\}$$
$$\times \sin^2 \frac{u\vec{p}_s b_s}{\hbar} \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$+ \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T} \int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')] dt' \right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{ \left[\varepsilon_o - \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \right.\right.$$
$$\left.\left. - \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} - \mu \right] \frac{\nabla_s T}{T} + \nabla_s \mu \right\}$$
$$\times \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\frac{u\vec{p}_s b_s}{\hbar} \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right] \quad (F.103)$$

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T} \int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt \times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar} \int_{t-t'}^{t} [\vec{E}_o \right. \quad (F.104)$$

where $\vec{E}_{sn}^* = \vec{E}_n + \nabla_s \mu / e$. Similarly

$$\vec{Z} = -\sigma_z(\vec{E})\vec{E}_{zn}^* - \sigma_z(\vec{E}) \frac{k_B}{e} \left[\left(\frac{\varepsilon_o - \mu}{k_B T} \right) - \Delta_z^* \frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)} + 2 - \Delta_s^* \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} \right] \nabla_z T \quad (F.105)$$

Resolving the current density into axial and circumferential components, the axial and circumferential components are given as: $\vec{J}_z = \vec{Z} + \vec{S}\sin\theta_h$ and $\vec{J}_s = \vec{S}\cos\theta_h$ respectively:

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T} \int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')] dt' \right)$$
$$\times \left\{ \left[\varepsilon_o - \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} - \mu \right] \frac{\nabla_s T}{T} + \nabla_s \mu \right\}$$
$$\times \sin^2 \frac{u\vec{p}_s b_s}{\hbar} \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$+ \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T} \int_0^\infty \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos(\omega t')] dt' \right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{ \left[\varepsilon_o - \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \right.\right.$$
$$\left.\left. - \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} - \mu \right] \frac{\nabla_s T}{T} + \nabla_s \mu \right\}$$

$$\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad \text{(F.106)}$$

Utilising the relations $\vec{E}_s = \vec{E}_z \sin\theta_h$, $\nabla_s T = \nabla_z T \sin\theta_h$ and $\vec{E}_{sn}^* = \vec{E}_{zn}^* \sin\theta_h$

$$\begin{aligned}
\vec{S}_2 = &-\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
&\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right) \\
&\times \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\} \\
&\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
&\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right) \\
&\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right. \\
&\left.\left. - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\} \\
&\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad \text{(F.107)}
\end{aligned}$$

$$\begin{aligned}
\vec{S}_2 = &-\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
&\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right) \\
&\times \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\} \\
&\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
&\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right) \\
&\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right. \\
&\left.\left. - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\} \\
&\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad \text{(F.108)}
\end{aligned}$$

$$
\begin{aligned}
\vec{S}_2 = &-\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
&\times\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right) \\
&\times\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\} \\
&\times\sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
&\times\sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right) \\
&\times\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right. \\
&\left.\left.-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\} \\
&\times\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]
\end{aligned} \tag{F.109}
$$

$$
\begin{aligned}
\vec{S}_2 = &-\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
&\times\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right) \\
&\times\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\} \\
&\times\sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \\
&\times\sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right) \\
&\times\int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right. \\
&\left.\left.-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\} \\
&\times\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]
\end{aligned} \tag{F.110}
$$

where

$$\vec{E}_{zn}^* = \nabla_z\left(\frac{\mu}{e}-\varphi\right)$$

and the circumferential component of the electron currrent density is given as:

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$+ \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.$$
$$\left.\left. - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad \text{(F.111)}$$

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$+ \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.$$
$$\left.\left. - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad \text{(F.112)}$$

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\cos\left(\frac{ub_se}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_zb_z}{\hbar}-\Delta_s\cos\frac{u\vec{p}_sb_s}{\hbar}-\mu\right]\frac{\nabla_sT}{T}+\nabla_s\mu\right\}$$
$$\times\sin^2\frac{u\vec{p}_sb_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_sb_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_zb_z}{\hbar}\right]$$
$$+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\sin\left(\frac{ub_se}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_zb_z}{\hbar}\right.\right.$$
$$\left.\left.-\Delta_s\cos\frac{u\vec{p}_sb_s}{\hbar}-\mu\right]\frac{\nabla_sT}{T}+\nabla_s\mu\right\}$$
$$\times\cos\frac{u\vec{p}_sb_s}{\hbar}\sin\frac{u\vec{p}_sb_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_sb_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_zb_z}{\hbar}\right] \quad (F.113)$$

Thus the circumferential σ_{cs} and axial σ_{zz} components of the electrical conductivity are given by the coeficients of the electric field E_{zn} as:

$$\sigma_{cs}=\sigma_s\sin\theta_h\cos\theta_h \qquad \sigma_{zz}=\sigma_z+\sigma_s\sin^2\theta_h$$

The differential thermoelectric power is defined as the ratio $\left|\frac{\vec{E}_{zn}^*}{\nabla_zT}\right|$ in an open circuit $\vec{J}=0$. Thus setting $\vec{J}_c$ to zero, the thermoelectric power α_{cz} along the circumferential direction is obtained as follows:

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\cos\left(\frac{ub_se}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_zb_z}{\hbar}-\Delta_s\cos\frac{u\vec{p}_sb_s}{\hbar}-\mu\right]\frac{\nabla_sT}{T}+\nabla_s\mu\right\}$$
$$\times\sin^2\frac{u\vec{p}_sb_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_sb_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_zb_z}{\hbar}\right]$$
$$+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\sin\left(\frac{ub_se}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_zb_z}{\hbar}\right.\right.$$

$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\Big] \frac{\nabla_s T}{T} + \nabla_s \mu\Big\}$$

$$\times \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\frac{u\vec{p}_s b_s}{\hbar} \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad (F.114)$$

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$

$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)$$

$$\times \left\{\left[\varepsilon_o - \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s \mu\right\}$$

$$\times \sin^2\frac{u\vec{p}_s b_s}{\hbar} \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$+ \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$

$$\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)$$

$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \Big\{\Big[\varepsilon_o - \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar}$$

$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\Big] \frac{\nabla_s T}{T} + \nabla_s \mu\Big\}$$

$$\times \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\frac{u\vec{p}_s b_s}{\hbar} \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad (F.115)$$

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt \times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o\right. \quad (F.116)$$

$$\alpha_{cz} = \left|\frac{\vec{E}_{zn}^*}{\nabla_z T}\right| \quad (F.117)$$

Similarly, α_{zz} along the axial direction is obtained as follows $J_z = 0$

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$

$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos(\omega t')]dt'\right)$$

$$\times \left\{\left[\varepsilon_o - \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s \mu\right\}$$

$$\times \sin^2\frac{u\vec{p}_s b_s}{\hbar} \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$+ \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$

$$\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o - \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.$$
$$\left.\left. - \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s \mu\right\}$$
$$\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad \text{(F.118)}$$

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$+ \frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \left\{\left[\varepsilon_o - \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.$$
$$\left.\left. - \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s \mu\right\}$$
$$\times \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad \text{(F.119)}$$

The electrical power factor along the circumferential $\mathscr{P}_{cs}$ and axial $\mathscr{P}_{zz}$ directions are given respectively as:

$$\mathscr{P}_{cs} = \alpha_{cz}^2\sigma_{cz} \qquad \mathscr{P}_{zz} = \alpha_{zz}^2\sigma_{zz}$$

The electrical power factor along the circumferential direction is given as:

$$\vec{S}_2 = -\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times \left\{\left[\varepsilon_o - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \mu\right]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times \sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.$$
$$\left.\left.-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}$$
$$\times\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad (F.120)$$

where the output power density along the circumferential direction becomes;

$$\vec{S}_2=-\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}$$
$$\times\sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.$$
$$\left.\left.-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}$$
$$\times\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad (F.121)$$

Moreover, the axial component of the power factor yields:

$$\vec{S}_2=-\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t')]dt'\right)$$
$$\times\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}$$
$$\times\sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t^{'})]dt^{'}\right)$$
$$\times\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.$$
$$\left.\left.-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}$$
$$\times\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad \text{(F.122)}$$

and similarly the output power density also gives;

$$\vec{S}_2=-\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\cos\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t^{'})]dt^{'}\right)$$
$$\times\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}$$
$$\times\sin^2\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$+\frac{2eu^2\Delta_s^2 b_s^2 n_o uwb_s b_z}{(2\pi\hbar)^2\hbar^2 2I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\int_0^\infty \exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \sin\left(\frac{ub_s e}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos(\omega t^{'})]dt^{'}\right)$$
$$\times\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\left\{\left[\varepsilon_o-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\right.\right.$$
$$\left.\left.-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}-\mu\right]\frac{\nabla_s T}{T}+\nabla_s\mu\right\}$$
$$\times\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\frac{u\vec{p}_s b_s}{\hbar}\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad \text{(F.123)}$$

APPENDIX G

CARRIER THERMAL CURRENT DENSITY

In the linear approximation, ∇T and $\nabla\mu$, the solution to the Boltzmann equation is:

$$f(\vec{p},t) = \tau^{-1}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\, f_o\left(\vec{p}-e\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right) + \int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\left\{[\varepsilon(p)-\mu]\frac{\nabla T}{T}-\nabla\mu\right\} \times v(p)\cdot\frac{\partial f_o}{\partial\varepsilon}(\vec{p}-e\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt') \quad (G.1)$$

The electon thermal current density q is defined by

$$\vec{q} = \sum_p[\varepsilon(\vec{p})-\mu]v(\vec{p})f(\vec{p}) \quad (G.2)$$

$$\vec{q} = \tau^{-1}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\sum_p[\varepsilon(\vec{p})-\mu]v(\vec{p})f_o\left(\vec{p}-e\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right) + \int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\sum_p[\varepsilon(\vec{p})-\mu]v(\vec{p})\left\{[\varepsilon(\vec{p})-\mu]\frac{\nabla T}{T}-\nabla\mu\right\} \times v(p)\cdot\frac{\partial f_o}{\partial\varepsilon}(\vec{p}-e\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt') \quad (G.3)$$

Invoking the transform $\vec{p}-e\int_0^{\infty}[\vec{E}_o+\vec{E}\cos\omega t']dt' \to \vec{p}$ the equation becomes:

$$\vec{q} = \tau^{-1}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\sum_p\left[\varepsilon\left(\vec{p}-e\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)-\mu\right] \times v\left(\vec{p}-e\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)f_o(\vec{p}) + \int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\sum_p\left[\varepsilon\left(\vec{p}-e\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)-\mu\right] \times\left\{[\varepsilon(\vec{p})-\mu]\frac{\nabla T}{T}+\nabla\mu\right\}v\left(\vec{p}-e\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\cdot\frac{\partial f_o(\vec{p})}{\partial\varepsilon}v(p) \quad (G.4)$$

Resolving the thermal current densities along the $\vec{S}^*$ and $\vec{Z}^*$ components:

$$\vec{Z}^* = \tau^{-1}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\sum_p\left[\varepsilon\left(\vec{p}-e\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_z\cos\omega t']dt'\right)-\mu\right] \times\vec{v}_z\left(\vec{p}-e\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_z\cos\omega t']dt'\right)f_o(\vec{p})$$

$$+\int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt \sum_p \left[\varepsilon\left(\vec{p} - e\int_{t-t'}^{t} [\vec{E}_o + \vec{E}_z \cos\omega t']dt'\right) - \mu\right]$$
$$\times\left\{[\varepsilon(\vec{p}) - \mu]\frac{\nabla_z T}{T} + \nabla_z \mu\right\} \vec{v}_z\left(\vec{p} - e\int_{t-t'}^{t} [\vec{E}_o + \vec{E}_z \cos\omega t']dt'\right) \cdot \frac{\partial f_o(\vec{p})}{\partial\varepsilon} \vec{v}_z(\vec{p}) \quad \text{(G.5)}$$

and

$$\vec{S}^* = \tau^{-1}\int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt \sum_p \left[\varepsilon\left(\vec{p} - e\int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) - \mu\right]$$
$$\times \vec{v}_s\left(\vec{p} - e\int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) f_o(\vec{p})$$
$$+\int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt \sum_p \left[\varepsilon\left(\vec{p} - e\int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) - \mu\right]$$
$$\times\left\{[\varepsilon(\vec{p}) - \mu]\frac{\nabla_s T}{T} + \nabla_s \mu\right\} \vec{v}_s\left(\vec{p} - e\int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \cdot \frac{\partial f_o(\vec{p})}{\partial\varepsilon} \vec{v}_s(\vec{p}) \quad \text{(G.6)}$$

Making the transformation:

$$\sum_p \to \frac{2}{(2\pi\hbar)^2}\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z$$

$$\vec{Z}^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z f_o(p)$$
$$\times\left[\varepsilon\left(\vec{p} - e\int_{t-t'}^{t} [\vec{E}_o + \vec{E}_z \cos\omega t']dt'\right) - \mu\right] \vec{v}_z\left(\vec{p} - e\int_{t-t'}^{t} [\vec{E}_o + \vec{E}_z \cos\omega t']dt'\right)$$
$$+\frac{2}{(2\pi\hbar)^2}\int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z$$
$$\times\left[\varepsilon\left(\vec{p} - e\int_{t-t'}^{t} [\vec{E}_o + \vec{E}_z \cos\omega t']dt'\right) - \mu\right]\left\{[\varepsilon(\vec{p}) - \mu]\frac{\nabla_z T}{T} + \nabla_z \mu\right\}$$
$$\times \vec{v}_z\left(\vec{p} - e\int_{t-t'}^{t} [\vec{E}_o + \vec{E}_z \cos\omega t']dt'\right) \cdot \frac{\partial f_o(\vec{p})}{\partial\varepsilon} \vec{v}_z(\vec{p}) \quad \text{(G.7)}$$

$$\vec{S}^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z$$
$$\left[\varepsilon\left(p - e\int_{t-t'}^{t} [\vec{E}_o + \vec{E} \cos\omega t']dt'\right) - \mu\right] \vec{v}_s\left(\vec{p} - e\int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) f_o(\vec{p})$$
$$+\frac{2}{(2\pi\hbar)^2}\int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z$$
$$\times\left[\varepsilon\left(\vec{p} - e\int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) - \mu\right]\left\{[\varepsilon(\vec{p}) - \mu]\frac{\nabla_s T}{T} + \nabla_s \mu\right\}$$

$$\times \vec{v}_s \left(\vec{p} - e \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \cdot \frac{\partial f_o(\vec{p})}{\partial \varepsilon} \vec{v}_s(\vec{p}) \quad \text{(G.8)}$$

where the integration is carried out in the first Brillouin zone. Using the band relation in F.6 and making a transformation yields:

$$\varepsilon \left(\vec{p} - e \int_{t-t'}^{t} [\vec{E}_o + \vec{E} \cos \omega t'] dt' \right) = \varepsilon_o - \Delta_s \cos \frac{ub_s}{\hbar} \left(\vec{p}_s - e \int_{t-t'}^{t} [\vec{E}_o + \vec{E} \cos \omega t'] dt' \right)$$
$$+ \Delta_z \cos \frac{wb_z}{\hbar} \left(\vec{p}_z - e \int_{t-t'}^{t} [\vec{E}_o - \vec{E} \cos \omega t'] dt' \right) \quad \text{(G.10)}$$

$$\varepsilon \left(\vec{p} - e \int_{t-t'}^{t} [\vec{E}_o + \vec{E} \cos \omega t'] dt' \right) =$$
$$\varepsilon_o - \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E} \cos \omega t'] dt' \right)$$
$$- \Delta_s \sin \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E} \cos \omega t'] dt' \right)$$
$$- \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \cos \left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E} \cos \omega t'] dt' \right)$$
$$- \Delta_z \sin \frac{w\vec{p}_z b_z}{\hbar} \sin \left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E} \cos \omega t'] dt' \right) \quad \text{(G.11)}$$

The electron miniband velocity along the $s-$coordinate is given as:

$$\vec{v}_s(\vec{p}) = \frac{\partial \varepsilon(p)}{\partial \vec{p}_s} = \frac{u\Delta_s b_s}{\hbar} \sin \frac{u\vec{p}_s b_s}{\hbar} \quad \text{(G.12)}$$

$$\vec{v}_s \left(\vec{p}_s - e \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) =$$
$$\frac{u\Delta_s b_s}{\hbar} \sin \left(\vec{p}_s - e \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \quad \text{(G.13)}$$

$$\frac{u\Delta_s b_s}{\hbar} \sin \left(\vec{p}_s - e \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) =$$
$$\frac{u\Delta_s b_s}{\hbar} \left\{ \sin \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \right.$$
$$\left. - \cos \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \right\} \quad \text{(G.14)}$$

$$\vec{S}^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2} \frac{u\Delta_s b_s}{\hbar} \int_0^{-\infty} \exp\left(\frac{-t}{\tau} \right) dt \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z$$
$$\times \left[\varepsilon_o - \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \right.$$

$$- \Delta_s \sin \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$

$$- \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \cos \left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$

$$- \Delta_z \sin \frac{w\vec{p}_z b_z}{\hbar} \sin \left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) - \mu \Bigg]$$

$$\times \left\{ \sin \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \right.$$

$$\left. - \cos \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \right\} f_o(\vec{p})$$

$$+ \frac{2}{(2\pi\hbar)^2} \frac{u^2 \Delta_s^2 b_s^2}{\hbar^2} \int_0^{-\infty} \exp\left(\frac{-t}{\tau} \right) dt \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z$$

$$\Bigg[\varepsilon_o - \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$

$$- \Delta_s \sin \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$

$$- \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \cos \left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$

$$- \Delta_z \sin \frac{w\vec{p}_z b_z}{\hbar} \sin \left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) - \mu \Bigg]$$

$$\times \left\{ \sin \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \right.$$

$$\left. - \cos \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \right\}$$

$$\times \left\{ [\varepsilon_o - \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} - \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} - \mu] \frac{\nabla_s T}{T} + \nabla_s \mu \right\}$$

$$\times \frac{\partial f_o(\vec{p})}{\partial \varepsilon} \sin \frac{u\vec{p}_s b_s}{\hbar} \quad \text{(G.15)}$$

Now G.15 is too long and cumbersome to solve. Splitting G.15 into two yields:

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2} \frac{u\Delta_s b_s}{\hbar} \int_0^{-\infty} \exp\left(\frac{-t}{\tau} \right) dt \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z$$

$$\Bigg[\varepsilon_o - \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$

$$- \Delta_s \sin \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$

$$- \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \cos \left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$

$$- \Delta_z \sin \frac{w\vec{p}_z b_z}{\hbar} \sin \left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) - \mu \Bigg]$$

$$\times \left\{ \sin \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \right.$$

$$- \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\Bigg\} f_o(\vec{p}) \quad (G.16)$$

and

$$S_2^* = \frac{2}{(2\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z$$
$$\Bigg[\varepsilon_o - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$- \Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$- \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$- \Delta_z\sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) - \mu\Bigg]$$
$$\times\Bigg\{\sin\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$- \cos\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\Bigg\}$$
$$\times\left\{[\varepsilon_o - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar} - \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar} - \mu]\frac{\nabla_s T}{T} + \nabla_s\mu\right\}$$
$$\times\frac{\partial f_o(\vec{p})}{\partial\varepsilon(\vec{p})}\sin\frac{u\vec{p}_s b_s}{\hbar} \quad (G.17)$$

Solving for G.16 explicitly

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z$$
$$\Bigg[\varepsilon_o - \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$- \Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$- \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$- \Delta_z\sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) - \mu\Bigg]$$
$$\times\Bigg\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$- \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\Bigg\}$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \quad (G.18)$$

Expanding the terms

$$
\begin{aligned}
S_1^* = & \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\
& \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right. \\
& -\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
& \times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
& -\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
& \times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
& -\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
& \times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
& -\Delta_z\sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
& \left.\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right] \\
& \times\left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right. \\
& \left.-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right\} \qquad \text{(G.19)}
\end{aligned}
$$

$$
\begin{aligned}
S_1^* = & \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\
& \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right. \\
& -\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
& \times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
& -\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
& \times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
& -\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
& \times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]
\end{aligned}
$$

$$
\begin{aligned}
&- \Delta_z \sin \frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t^{'}]dt^{'}\right) \\
&\times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg] \\
&\times \left\{\sin \frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t^{'}]dt^{'}\right)\right. \\
&\left. - \cos \frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t^{'}]dt^{'}\right)\right\} \qquad \text{(G.20)}
\end{aligned}
$$

Setting the odd functions to zero

$$
\begin{aligned}
S_1^* = &\frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\
&\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t^{'}]dt^{'}\right) \\
&\times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_s \sin \frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t^{'}]dt^{'}\right) \\
&\times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t^{'}]dt^{'}\right) \\
&\times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_z \sin \frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t^{'}]dt^{'}\right) \\
&\times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg] \\
&\times \left\{\sin \frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t^{'}]dt^{'}\right)\right. \\
&\left. - \cos \frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t^{'}]dt^{'}\right)\right\} \qquad \text{(G.21)}
\end{aligned}
$$

$$
\begin{aligned}
S_1^* = &\frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\
&\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar}\right]\right. \\
&- \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t^{'}]dt^{'}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right) \\
&\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right) \\
&\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right) \\
&\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg] \\
&\times \Bigg\{\sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right) \\
&- \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)\Bigg\} \qquad \text{(G.22)}
\end{aligned}
$$

$$
\begin{aligned}
S_1^* = &\frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\
&\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right) \\
&\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right) \\
&\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right) \\
&\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right) \\
&\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg] \\
&\times \Bigg\{\sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)
\end{aligned}
$$

$$- \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\Bigg\} \quad (G.23)$$

Now changing the integration path

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\Bigg[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_z\sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times\Bigg\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\Bigg\} \quad (G.24)$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\Bigg[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$- \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \cos \left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\times \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_z \sin \frac{w\vec{p}_z b_z}{\hbar} \sin \left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\times \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \Bigg]$$
$$\times \left\{ \sin \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \right.$$
$$\left. - \cos \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \right\} \quad (G.25)$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2} \frac{u\Delta_s b_s}{\hbar} \frac{uwn_o b_s b_z}{2I_o(\Delta_s^*) I_o(\Delta_z^*)} \int_0^{-\infty} \exp\left(\frac{-t}{\tau} \right) dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu) \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \right.$$
$$- \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\times \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_s \sin \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\times \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \cos \left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\times \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_z \sin \frac{w\vec{p}_z b_z}{\hbar} \sin \left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\times \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \Bigg]$$
$$\times \left\{ \sin \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \right.$$
$$\left. - \cos \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \right\} \quad (G.26)$$

Solving for the terms in G.26

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2} \frac{u\Delta_s b_s}{\hbar} \frac{uwn_o b_s b_z}{2I_o(\Delta_s^*) I_o(\Delta_z^*)} \int_0^{-\infty} \exp\left(\frac{-t}{\tau} \right) dt$$

$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o - \mu) \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right]$$

$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right)$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right]$$

$$- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right)$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right]$$

$$- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right)$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right]$$

$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right)$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right] \Bigg]$$

$$\times \Bigg\{ \sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right)$$

$$- \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right) \Bigg\} \tag{G.27}$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2} \frac{u\Delta_s b_s}{\hbar} \frac{uwn_o b_s b_z}{2I_o(\Delta_s^*) I_o(\Delta_z^*)} \int_0^{-\infty} \exp\left(\frac{-t}{\tau} \right) dt$$

$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o - \mu) \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right]$$

$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right)$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right]$$

$$- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right)$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right]$$

$$- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right)$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right]$$

$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right)$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)\right.$$
$$\left. - \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)\right\} \quad (G.28)$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right.$$
$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)\right.$$
$$\left. - \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)\right\} \quad (G.29)$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right.$$
$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)$$

$$
\begin{aligned}
&\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
&\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
&\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg] \\
&\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right. \\
&\left. - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \qquad \text{(G.30)}
\end{aligned}
$$

From G.27

$$
\begin{aligned}
S_1^* = &\frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\
&\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right. \\
&- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
&\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
&\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
&\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
&\left.\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right] \\
&\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right. \\
&\left. - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \qquad \text{(G.31)}
\end{aligned}
$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_z\sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times\Bigg\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\Bigg\} \qquad (G.32)$$

For weak electric field

$$\sum_{n=-\infty} J_n^2(\chi)\left[\frac{(eb_s\vec{E}_o/\hbar+n\omega\hbar)\tau^2}{1+(eb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]$$
$$=\sum_{n=-\infty} J_n^2(\chi)[(eb_s\vec{E}_o+n\omega\hbar)\tau^2(1-0(eb_s\vec{E}_o+n\omega\hbar)^2)]$$
$$=\sum_{n=-\infty} J_n^2(\chi)[(eb_s\vec{E}_o+n\omega\hbar)\tau^2)]$$

$$I_1(\Delta_s^*)=\frac{1}{\pi}\int_0^{\pi}\frac{dZ_s}{\hbar}\cos\frac{Z_s}{\hbar}\exp\left(\Delta_s\cos\frac{Z_s}{\hbar}\right)$$
$$I_o(\Delta_s^*)=\frac{1}{\pi}\int_0^{\pi}\frac{dZ_s}{\hbar}\exp\left(\Delta_s\cos\frac{Z_s}{\hbar}\right)$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right.$$

$$
\begin{aligned}
& - \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
& \times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
& - \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
& \times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
& - \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
& \times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
& - \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
& \times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]\\
& \times \Bigg\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
& - \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\Bigg\} \qquad \text{(G.33)}
\end{aligned}
$$

$$
\begin{aligned}
S_1^* = & \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\\
& \int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\Bigg[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
& - \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
& \times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
& - \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
& \times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
& - \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
& \times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
& - \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
& \times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]
\end{aligned}
$$

$$\times \left\{ \sin \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) - \cos \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \right\} \quad (G.34)$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2} \frac{u\Delta_s b_s}{\hbar} \frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)} \int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu) \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \right.$$
$$- \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_s \sin \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \cos \left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_z \sin \frac{w\vec{p}_z b_z}{\hbar} \sin \left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\left. \times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \right]$$
$$\times \left\{ \sin \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) - \cos \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \right\} \quad (G.35)$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2} \frac{u\Delta_s b_s}{\hbar} \frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)} \int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu) \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \right.$$
$$- \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_s \sin \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$

$$- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times\left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right.$$
$$\left.- \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right\} \tag{G.36}$$

$$\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\sin\left(\frac{2ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)=$$
$$\sum_{n=-\infty} J_n^2(\chi)\left[\frac{2(eb_s\vec{E}_o/\hbar+n\omega\hbar)\tau^2}{1+4(eb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]$$

For weak electric field

$$\sum_{n=-\infty} J_n^2(\chi)\left[\frac{2(eb_s\vec{E}_o/\hbar+n\omega\hbar)\tau^2}{1+4(eb_s\vec{E}_o/\hbar+n\omega\hbar)^2\tau^2}\right]$$
$$=\sum_{n=-\infty} J_n^2(\chi)[2(eb_s\vec{E}_o+n\omega\hbar)\tau^2(1-0(eb_s\vec{E}_o+n\omega\hbar)^2)]$$
$$=\sum_{n=-\infty} J_n^2(\chi)[2(eb_s\vec{E}_o+n\omega\hbar)\tau^2)]$$

$$S_1^*=\frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\left[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right.$$
$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$
\begin{aligned}
&- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right) \\
&\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg] \\
&\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)\right. \\
&\left. - \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)\right\} \qquad (G.37)
\end{aligned}
$$

but

$$
\begin{aligned}
S_1^* = &\frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\
&\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right. \\
&- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right) \\
&\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right) \\
&\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right) \\
&\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right) \\
&\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg] \\
&\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)\right. \\
&\left. - \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)\right\} \qquad (G.38)
\end{aligned}
$$

$$
\begin{aligned}
S_1^* = &\frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\
&\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right. \\
&- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)
\end{aligned}
$$

$$
\begin{aligned}
&\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
&\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
&\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
&\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg] \\
&\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right. \\
&\left.- \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \qquad \text{(G.39)}
\end{aligned}
$$

$$
\begin{aligned}
S_1^* = &\frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\
&\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
&\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
&\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
&\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
&\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg] \\
&\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right.
\end{aligned}
$$

$$- \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\Bigg\} \quad (G.40)$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_z\sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times\Bigg\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\Bigg\} \quad (G.41)$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$

$$
\begin{aligned}
&\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
&\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg] \\
&\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right. \\
&\left. - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \qquad \text{(G.42)}
\end{aligned}
$$

$$
\begin{aligned}
S_1^* = {}& \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\
&\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
&\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
&\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
&\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
&\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg] \\
&\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right. \\
&\left. - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \qquad \text{(G.43)}
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \sin\left\{\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right.\right. \\
&\left.\left. + \frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} =
\end{aligned}
$$

$$\sum_{n=-\infty} J_n^2(\chi)\left[\frac{[(eb_z\vec{E}_o/\hbar+n\omega\hbar)+(eb_s\vec{E}_o/\hbar+n\omega\hbar)]\tau^2}{1+[(eb_z\vec{E}_o/\hbar+n\omega\hbar)+(eb_s\vec{E}_o/\hbar+n\omega\hbar)]^2\tau^2}\right]$$

For weak electric field $(eb_s\vec{E}_o/\hbar+n\omega\hbar)^2 \ll 1$

$$\sum_{n=-\infty} J_n^2(\chi)\left[\frac{[(eb_z\vec{E}_o/\hbar+n\omega\hbar)+(eb_s\vec{E}_o/\hbar+n\omega\hbar)]\tau^2}{1+[(eb_z\vec{E}_o/\hbar+n\omega\hbar)+(eb_s\vec{E}_o/\hbar+n\omega\hbar)]^2\tau^2}\right] =$$

$$\sum_{n=-\infty} J_n^2(\chi)\left[\left(\frac{eb_z\vec{E}_o}{\hbar}+n\omega\hbar\right)+\left(\frac{eb_s\vec{E}_o}{\hbar}+n\omega\hbar\right)\right]\tau^2 \times\left[\left(\frac{eb_z\vec{E}_o}{\hbar}+n\omega\hbar\right)+\left(\frac{eb_s\vec{E}_o}{\hbar}+n\omega\hbar\right)\right]^2$$

$$=\sum_{n=-\infty} J_n^2(\chi)\left[\left(\frac{eb_z\vec{E}_o}{\hbar}+n\omega\hbar\right)+\left(\frac{eb_s\vec{E}_o}{\hbar}+n\omega\hbar\right)\right]\tau^2$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_ob_sb_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\left[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_sb_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_zb_z}{\hbar}\right]\right.$$
$$-\Delta_s\cos\frac{u\vec{p}_sb_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_sb_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_zb_z}{\hbar}\right]$$
$$-\Delta_s\sin\frac{u\vec{p}_sb_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_sb_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_zb_z}{\hbar}\right]$$
$$-\Delta_z\cos\frac{w\vec{p}_zb_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_sb_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_zb_z}{\hbar}\right]$$
$$-\Delta_z\sin\frac{w\vec{p}_zb_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\left.\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_sb_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_zb_z}{\hbar}\right]\right]$$
$$\times\left\{\sin\frac{u\vec{p}_sb_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right.$$
$$\left.-\cos\frac{u\vec{p}_sb_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right\} \quad (G.44)$$

$$
\begin{aligned}
S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2} \frac{u\Delta_s b_s}{\hbar} \frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)} \int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt \\
\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu) \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \right. \\
- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
\left. \times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \right] \\
\times \left\{ \sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \right. \\
\left. - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \right\} \quad \text{(G.45)}
\end{aligned}
$$

$$
\begin{aligned}
S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2} \frac{u\Delta_s b_s}{\hbar} \frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)} \int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt \\
\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu) \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \right. \\
- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]
\end{aligned}
$$

$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)\right.$$
$$\left. - \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)\right\} \quad (G.46)$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)\right.$$
$$\left. - \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)\right\} \quad (G.47)$$

$$\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\,\sin\left\{\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}\left[\vec{E}_o + \vec{E}_s\cos\omega t'\right]dt'\right.\right.$$
$$\left.\left. + \frac{ueb_s}{\hbar}\int_{t-t'}^{t}\left[\vec{E}_o + \vec{E}_s\cos\omega t'\right]dt'\right)\right\} =$$
$$\sum_{n=-\infty} J_n^2(\chi)\left[\frac{[(eb_z\vec{E}_o/\hbar + n\omega\hbar) + (eb_s\vec{E}_o/\hbar + n\omega\hbar)]\tau^2}{1 + [(eb_z\vec{E}_o/\hbar + n\omega\hbar) + (eb_s\vec{E}_o/\hbar + n\omega\hbar)]^2\tau^2}\right]$$

For weak electric field $(eb_s\vec{E}_o/\hbar + n\omega\hbar)^2 \ll 1$

$$
\begin{aligned}
S_1^* = &\frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\
&\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right. \\
&-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
&\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&-\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
&\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
&\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&-\Delta_z\sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
&\left.\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right] \\
&\times\left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right. \\
&\left.-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
S_1^* = &\frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\
&\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right. \\
&-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
&\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&-\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
&\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)
\end{aligned}
$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right.$$
$$\left. - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \quad (G.48)$$

Summing up the terms

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right.$$
$$\left. - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \quad (G.49)$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$- \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_s \sin \frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \sin \frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right.$$
$$\left. - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \quad \text{(G.50)}$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_s \sin \frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \sin \frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$

$$\times \left\{ \sin \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) - \cos \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \right\} \quad (G.51)$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2} \frac{u\Delta_s b_s}{\hbar} \frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)} \int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu) \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \right.$$
$$- \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_s \sin \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \cos \left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_z \sin \frac{w\vec{p}_z b_z}{\hbar} \sin \left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\left. \times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \right]$$
$$\times \left\{ \sin \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) - \cos \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \right\} \quad (G.52)$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2} \frac{u\Delta_s b_s}{\hbar} \frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)} \int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu) \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \right.$$
$$- \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_s \sin \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$

$$
\begin{aligned}
&- \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&- \Delta_z \sin \frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]\\
&\times\left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right.\\
&\left.-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right\} \qquad (G.53)
\end{aligned}
$$

$$
\begin{aligned}
S_1^* = &\frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\\
&\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\left[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right.\\
&- \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&- \Delta_s \sin \frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&- \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&- \Delta_z \sin \frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]\\
&\times\left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right.\\
&\left.-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right\} \qquad (G.54)
\end{aligned}
$$

$$
\begin{aligned}
S_1^* = &\frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\\
&\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\left[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right.
\end{aligned}
$$

$$
\begin{aligned}
& - \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
& \times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
& - \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
& \times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
& - \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
& \times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
& - \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
& \times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]\\
& \times\left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right.\\
& \left. - \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right\} \qquad (G.55)
\end{aligned}
$$

$$\sigma_s(\vec{E}) = \qquad (G.56)$$

$$
\begin{aligned}
S_1^* = & \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\\
& \int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\left[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right.\\
& - \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
& \times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
& - \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
& \times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
& - \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
& \times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
& - \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
& \times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]
\end{aligned}
$$

$$\times \left\{ \sin \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) - \cos \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \right\} \quad (G.57)$$

where $\vec{E}_n = \left(\vec{E}_o + n\omega\hbar / eb_s \right)$. Thus,

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2} \frac{u\Delta_s b_s}{\hbar} \frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)} \int_0^{-\infty} \exp\left(\frac{-t}{\tau} \right) dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu) \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \right.$$
$$- \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_s \sin \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \cos \left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_z \sin \frac{w\vec{p}_z b_z}{\hbar} \sin \left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\left. \times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \right]$$
$$\times \left\{ \sin \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) - \cos \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \right\} \quad (G.58)$$

Solving for G.17 explicitly

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2} \frac{u\Delta_s b_s}{\hbar} \frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)} \int_0^{-\infty} \exp\left(\frac{-t}{\tau} \right) dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu) \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \right.$$
$$- \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$

$$
\begin{aligned}
&- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
&\quad\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
&\quad\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
&\quad\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg] \\
&\times\Bigg\{\sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
&\quad - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\Bigg\} \qquad \text{(G.59)}
\end{aligned}
$$

$$
\frac{\partial f_o(\vec{p})}{\partial \varepsilon(\vec{p})} = -\frac{n_o uwb_s b_z}{2I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T} \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \qquad \text{(G.60)}
$$

Substituting G.60 into G.59 yields:

$$
\begin{aligned}
S_1^* = &\frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\
&\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right. \\
&- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
&\quad\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
&\quad\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
&\quad\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
&\quad\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg] \\
&\times\Bigg\{\sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)
\end{aligned}
$$

$$-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\Bigg\} \quad \text{(G.61)}$$

Rearranging the terms

$$\begin{aligned}
S_1^* = {} & \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\
& \int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\left[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right. \\
& -\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
& \times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
& -\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
& \times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
& -\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
& \times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
& -\Delta_z\sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
& \left.\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right] \\
& \times\left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right. \\
& \left.-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right\} \quad \text{(G.62)}
\end{aligned}$$

$$\begin{aligned}
S_1^* = {} & \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\
& \int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\left[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right. \\
& -\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
& \times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
& -\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
& \times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]
\end{aligned}$$

$$
\begin{aligned}
&- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
&\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
&\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg] \\
&\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right. \\
&\left. - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \qquad (G.63)
\end{aligned}
$$

Setting the odd functions to zero yields

$$
\begin{aligned}
S_1^* = &\frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\
&\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right. \\
&- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
&\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
&\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
&\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
&\left.\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right] \\
&\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right. \\
&\left. - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \qquad (G.64)
\end{aligned}
$$

Changing the path of integration

$$
S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt
$$

$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu) \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right] \right.$$
$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t']dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t']dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t']dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t']dt' \right)$$
$$\left. \times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right] \right]$$
$$\times \left\{ \sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t']dt' \right) \right.$$
$$\left. - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t']dt' \right) \right\} \quad \text{(G.65)}$$

Solving for the terms in G.65

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2} \frac{u\Delta_s b_s}{\hbar} \frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)} \int_0^{-\infty} \exp\left(\frac{-t}{\tau} \right) dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu) \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right] \right.$$
$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t']dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t']dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t']dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t']dt' \right)$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$

$$\times \left\{ \sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right.$$

$$\left. - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \quad \text{(G.66)}$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$

$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right.$$

$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$

$$\times \left\{ \sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right.$$

$$\left. - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \quad \text{(G.67)}$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$

$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right.$$

$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$

$$\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right.$$

$$\left. - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \quad (G.68)$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$

$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right.$$

$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$

$$\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right.$$

$$\left. - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \quad (G.69)$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$

$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right.$$
$$\left. - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \tag{G.70}$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$

$$\times \left\{ \sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right.$$

$$\left. - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \quad (G.71)$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$

$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right.$$

$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$

$$\times \left\{ \sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right.$$

$$\left. - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \quad (G.72)$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$

$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right.$$

$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right.$$
$$\left. - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \tag{G.73}$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right.$$
$$\left. - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \tag{G.74}$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$

$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu) \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right] \right.$$
$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right)$$
$$\left. \times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right] \right]$$
$$\times \left\{ \sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right) \right.$$
$$\left. - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right) \right\} \quad \text{(G.75)}$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2} \frac{u\Delta_s b_s}{\hbar} \frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)} \int_0^{-\infty} \exp\left(\frac{-t}{\tau} \right) dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu) \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right] \right.$$
$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right)$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$

$$\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right.$$

$$\left. - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \quad \text{(G.76)}$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right)dt$$

$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$

$$\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right.$$

$$\left. - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \quad \text{(G.77)}$$

Evaluating the terms

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right)dt$$

$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$- \Delta_s \sin \frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \sin \frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times \left\{ \sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right.$$
$$\left. - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \quad (G.78)$$

$$\int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) =$$
$$\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau^2}{1+[(eb_s\vec{E}_o/\hbar + n\omega\hbar)]^2\tau^2}\right]$$

For weak electric field $(eb_s\vec{E}_o/\hbar + n\omega\hbar)^2 \ll 1$. Thus,

$$\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau^2}{1+[(eb_s\vec{E}_o/\hbar + n\omega\hbar)]^2\tau^2}\right] =$$
$$\sum_{n=-\infty} J_n^2(\chi)[\tau(1 - 0[(eb_s\vec{E}_o/\hbar + n\omega\hbar)]^2)] = \sum_{n=-\infty} J_n^2(\chi)\tau$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right.$$
$$- \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_s \sin \frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$
\begin{aligned}
&- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]\\
&\times\left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right.\\
&\left.-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right\} \qquad (G.79)
\end{aligned}
$$

$$
\begin{aligned}
S_1^* = &\frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\\
&\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\left[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right.\\
&- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\left.\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right]\\
&\times\left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right.\\
&\left.-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right\} \qquad (G.80)
\end{aligned}
$$

$$
\begin{aligned}
S_1^* = &\frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\\
&\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\left[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right.
\end{aligned}
$$

$$- \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\times \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_s \sin \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\times \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \cos \left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\times \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_z \sin \frac{w\vec{p}_z b_z}{\hbar} \sin \left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\times \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \Bigg]$$
$$\times \Bigg\{ \sin \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$- \cos \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \Bigg\} \quad \text{(G.81)}$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2} \frac{u\Delta_s b_s}{\hbar} \frac{uwn_o b_s b_z}{2I_o(\Delta_s^*) I_o(\Delta_z^*)} \int_0^{-\infty} \exp\left(\frac{-t}{\tau} \right) dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o - \mu) \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\times \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_s \sin \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\times \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \cos \left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\times \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_z \sin \frac{w\vec{p}_z b_z}{\hbar} \sin \left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\times \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \Bigg]$$

$$\times \left\{ \sin \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) - \cos \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \right\} \quad (G.82)$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2} \frac{u\Delta_s b_s}{\hbar} \frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)} \int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu) \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \right.$$
$$- \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_s \sin \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \cos \left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_z \sin \frac{w\vec{p}_z b_z}{\hbar} \sin \left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\left. \times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \right]$$
$$\times \left\{ \sin \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) - \cos \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \right\} \quad (G.83)$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2} \frac{u\Delta_s b_s}{\hbar} \frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)} \int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu) \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \right.$$
$$- \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_s \sin \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$

$$
\begin{aligned}
&- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]\\
&\times\left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right.\\
&\left.- \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right\} \qquad (G.84)
\end{aligned}
$$

$$
\begin{aligned}
S_1^* =& \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\\
&\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\left[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right.\\
&- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]\\
&\times\left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right.\\
&\left.- \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right\} \qquad (G.85)
\end{aligned}
$$

$$
\begin{aligned}
S_1^* =& \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\\
&\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\left[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right.
\end{aligned}
$$

$$
\begin{aligned}
& - \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \\
& \times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \\
& - \Delta_s \sin \frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \\
& \times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \\
& - \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \\
& \times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \\
& - \Delta_z \sin \frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \\
& \times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \Bigg] \\
& \times \left\{ \sin \frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \right. \\
& \left. - \cos \frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \right\} \qquad (G.86)
\end{aligned}
$$

$$
\begin{aligned}
S_1^* = & \frac{2\tau^{-1}}{(2\pi\hbar)^2} \frac{u\Delta_s b_s}{\hbar} \frac{uwn_o b_s b_z}{2I_o(\Delta_s^*) I_o(\Delta_z^*)} \int_0^{-\infty} \exp\left(\frac{-t}{\tau} \right) dt \\
& \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o - \mu) \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \\
& - \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \\
& \times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \\
& - \Delta_s \sin \frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \\
& \times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \\
& - \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \\
& \times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \\
& - \Delta_z \sin \frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \\
& \times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \Bigg]
\end{aligned}
$$

$$\times \left\{ \sin \frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right) \right.$$
$$\left. - \cos \frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right) \right\} \quad \text{(G.87)}$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2} \frac{u\Delta_s b_s}{\hbar} \frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)} \int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu) \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right] \right.$$
$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right)$$
$$\left. \times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right] \right]$$
$$\times \left\{ \sin \frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right) \right.$$
$$\left. - \cos \frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right) \right\} \quad \text{(G.88)}$$

Using the identities:

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos x \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$$

$$\cos\frac{Z_s}{\hbar}\sin^2\frac{Z_s}{\hbar} = \frac{1}{2}\cos\frac{Z_s}{\hbar}\left(1 - \cos\frac{2Z_s}{\hbar}\right) = \frac{1}{2}\cos\frac{Z_s}{\hbar} - \frac{1}{2}\cos\frac{Z_s}{\hbar}\cos\frac{2Z_s}{\hbar}$$
$$= \frac{1}{2}\cos\frac{Z_s}{\hbar} - \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} + \cos\frac{3Z_s}{\hbar}\right) = \frac{1}{2}\cos\frac{Z_s}{\hbar} - \frac{1}{4}\cos\frac{Z_s}{\hbar} - \frac{1}{4}\cos\frac{3Z_s}{\hbar}$$
$$= \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right)$$

$$
\begin{aligned}
S_1^* = & \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\
& \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
& -\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
& \times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
& -\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
& \times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
& -\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
& \times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
& -\Delta_z\sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
& \times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg] \\
& \times\Bigg\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
& -\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\Bigg\} \qquad \text{(G.89)}
\end{aligned}
$$

$$
\begin{aligned}
S_1^* = & \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\
& \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
& -\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
& \times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
& -\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
& \times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
& -\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
& \times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]
\end{aligned}
$$

$$
\begin{aligned}
&- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]\\
&\times\left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right.\\
&\left.-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right\} \qquad \text{(G.90)}
\end{aligned}
$$

$$
\begin{aligned}
S_1^* &= \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\\
&\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\left[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right.\\
&- \Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&- \Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&- \Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&- \Delta_z\sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]\\
&\times\left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right.\\
&\left.-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right\} \qquad \text{(G.91)}
\end{aligned}
$$

$$
I_3(\Delta_s^*) = I_1(\Delta_s^*) - \frac{4}{\Delta_s^*}I_o(\Delta_s^*) + \frac{8}{\Delta_s^{*2}}I_1(\Delta_s^*)
$$

Therefore

$$
\begin{aligned}
&\left[\frac{I_1(\Delta_s^*)-I_3(\Delta_s^*)}{I_o(\Delta_s^*)}\right] = \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} - \frac{I_1(\Delta_s^*)-\frac{4}{\Delta_s^*}I_o(\Delta_s^*)+\frac{8}{\Delta_s^{*2}}I_1(\Delta_s^*)}{I_1(\Delta_s^*)}\\
&= \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} - \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} + \frac{4I_o(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)} - \frac{8I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^{*2})} = \frac{4}{\Delta_s^*} - \frac{8I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^{*2})}
\end{aligned}
$$

$$V_2 = +\frac{\tau u^2\Delta_s^2 b_s^2 n_o \Delta_s}{4\hbar^2 k_B T}(\varepsilon_o - \mu)\sum_{n=-\infty} J_n^2(\chi)\frac{\nabla_s T}{T} \tag{G.92}$$

$$V_2 = +\frac{\tau u^2\Delta_s^2 b_s^2 n_o k}{\hbar^2}\left(\frac{\varepsilon_o - \mu}{k_B T}\right)\sum_{n=-\infty} J_n^2(\chi)\nabla_s T \tag{G.93}$$

$$\begin{aligned}
S_1^* = {} & \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\
& \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o - \mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
& - \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right) \\
& \times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
& - \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right) \\
& \times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
& - \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right) \\
& \times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
& - \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right) \\
& \times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg] \\
& \times \Bigg\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right) \\
& - \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)\Bigg\}
\end{aligned} \tag{G.94}$$

$$\begin{aligned}
S_1^* = {} & \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\
& \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o - \mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
& - \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right) \\
& \times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
& - \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)
\end{aligned}$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times \Bigg\{\sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)$$
$$- \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)\Bigg\} \quad (G.95)$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times \Bigg\{\sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)$$
$$- \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)\Bigg\} \quad (G.96)$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$

$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu) \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right] \right.$$
$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right)$$
$$\left. \times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right] \right]$$
$$\times \left\{ \sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right) \right.$$
$$\left. - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right) \right\} \quad \text{(G.97)}$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2} \frac{u\Delta_s b_s}{\hbar} \frac{uwn_o b_s b_z}{2I_o(\Delta_s^*) I_o(\Delta_z^*)} \int_0^{-\infty} \exp\left(\frac{-t}{\tau} \right) dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu) \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right] \right.$$
$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right)$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$

$$\times \left\{ \sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \quad \text{(G.98)}$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times \left\{ \sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \quad \text{(G.99)}$$

where the integrals have been expressed in terms of modified Bessel functions. Using the recurrence relations:

$$\left\{\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} - \frac{I_2(\Delta_s^*)}{I_o(\Delta_s^*)}\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\right\} = \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1 - \frac{I_2(\Delta_s^*)}{I_o(\Delta_s^*)}\right)$$
$$= \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1 - \frac{I_o(\Delta_s^*) - \frac{2}{\Delta_s^*}I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right) = \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1 - 1 + \frac{2}{\Delta_s^*}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right) = \frac{2}{\Delta_s^*}\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$

$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_z\sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times\left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right.$$
$$\left.-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right\} \quad \text{(G.100)}$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_z\sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right.$$
$$\left. - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \quad \text{(G.101)}$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right.$$
$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right.$$
$$\left. - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \quad \text{(G.102)}$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right.$$
$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right.$$
$$\left. - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \quad \text{(G.103)}$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right.$$
$$\left. - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \quad \text{(G.104)}$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$

$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu) \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right] \right.$$
$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right)$$
$$\left. \times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right] \right]$$
$$\times \left\{ \sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right) \right.$$
$$\left. - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right) \right\} \quad \text{(G.105)}$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2} \frac{u\Delta_s b_s}{\hbar} \frac{uwn_o b_s b_z}{2I_o(\Delta_s^*) I_o(\Delta_z^*)} \int_0^{-\infty} \exp\left(\frac{-t}{\tau} \right) dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu) \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right] \right.$$
$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos\omega t'] dt' \right)$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right.$$
$$\left. - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \quad (G.106)$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right.$$
$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right.$$
$$\left. - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \quad (G.107)$$

$$\left[1 - \frac{I_2(\Delta_s^*)}{I_o(\Delta_s^*)}\right] = \left(1 - \frac{I_o(\Delta_s^*) - \frac{2}{\Delta_s^*}I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right)$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right.$$
$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right.$$
$$\left. - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \quad \text{(G.108)}$$

$$V_4 = -\frac{\tau u^2 \Delta_s b_s^2 n_o}{\hbar^2}(\varepsilon_o - \mu) \sum_{n=-\infty} J_n^2(\chi) \quad \text{(G.109)}$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right.$$
$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\left.\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right]$$

$$\times \left\{ \sin \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) - \cos \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \right\} \quad (G.110)$$

$$\int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt \cos^2 \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E} \cos \omega t'] dt' \right) =$$

For weak electric field

$$= \frac{\tau}{2} + \frac{1}{2} \sum_{n=-\infty} J_n^2(\chi) \left[\frac{\tau}{1 + 4(e\vec{E}_o b_s/\hbar + n\omega\hbar)^2 \tau^2} \right] =$$

$$V_5 = + \frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s n_o (\varepsilon_o - \mu)}{8(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \left(1 + \sum_{n=-\infty} J_n^2(\chi) \right) \frac{\nabla_s T}{T} [I_1(\Delta_s^*) - I_3(\Delta_s^*) I_o(\Delta_z^*)] \quad (G.111)$$

$$V_5 = + \frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s n_o (\varepsilon_o - \mu)}{8(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \left(1 + \sum_{n=-\infty} J_n^2(\chi) \right) \frac{\nabla_s T}{T} [I_1(\Delta_s^*) - I_3(\Delta_s^*) I_o(\Delta_z^*)] \quad (G.112)$$

$$V_5 = + \frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s n_o (\varepsilon_o - \mu)}{8(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \left(1 + \sum_{n=-\infty} J_n^2(\chi) \right) \frac{\nabla_s T}{T} [I_1(\Delta_s^*) - I_3(\Delta_s^*) I_o(\Delta_z^*)] \quad (G.113)$$

$$V_5 = + \frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s n_o (\varepsilon_o - \mu)}{8(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \left(1 + \sum_{n=-\infty} J_n^2(\chi) \right) \frac{\nabla_s T}{T} [I_1(\Delta_s^*) - I_3(\Delta_s^*) I_o(\Delta_z^*)] \quad (G.114)$$

$$V_5 = + \frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s n_o (\varepsilon_o - \mu)}{8(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \left(1 + \sum_{n=-\infty} J_n^2(\chi) \right) \frac{\nabla_s T}{T} [I_1(\Delta_s^*) - I_3(\Delta_s^*) I_o(\Delta_z^*)] \quad (G.115)$$

$$V_5 = + \frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s n_o (\varepsilon_o - \mu)}{8(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \left(1 + \sum_{n=-\infty} J_n^2(\chi) \right) \frac{\nabla_s T}{T}$$

$$\mathrm{I}_1(\Delta_s^*) - I_3(\Delta_s^*) I_o(\Delta_z^*)$$

(G.116)

$$V_5 = +\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s n_o(\varepsilon_o - \mu)}{8(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$[I_1(\Delta_s^*) - I_3(\Delta_s^*) I_o(\Delta_z^*)] \quad (G.117)$$

$$V_5 = +\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s n_o(\varepsilon_o - \mu)}{8(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$[I_1(\Delta_s^*) - I_3(\Delta_s^*) I_o(\Delta_z^*)] \quad (G.118)$$

Evaluating the identity:

$$\cos^2\frac{Z_s}{\hbar}\sin^2\frac{Z_s}{\hbar} = \frac{1}{4}\sin^2\frac{2Z_s}{\hbar} =$$

$$V_6 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s^2 n_o}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{8}\left(1 - \cos\frac{4Z_s}{\hbar}\right) \quad (G.119)$$

$$V_6 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s^2 n_o}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{8}\left(1 - \cos\frac{4Z_s}{\hbar}\right) \quad (G.120)$$

$$V_6 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s^2 n_o}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{8}\left(1 - \cos\frac{4Z_s}{\hbar}\right) \quad (G.121)$$

$$V_6 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s^2 n_o}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{8}\left(1 - \cos\frac{4Z_s}{\hbar}\right) \quad (G.122)$$

The recurrence relation for I_4 is evaluated as:

$$I_4(\Delta_s^*) = I_2(\Delta_s^*) - \frac{6}{\Delta_s^*} I_3(\Delta_s^*) = I_o(\Delta_s^*) - \frac{2I_1(\Delta_s^*)}{\Delta_s^*} - \frac{6}{\Delta_s^*}\left(I_1(\Delta_s^*) - \frac{4I_o(\Delta_s^*)}{\Delta_s^*} - \frac{8I_1(\Delta_s^*)}{\Delta_s^{*2}}\right)$$

$$= I_o(\Delta_s^*) - \frac{8I_1(\Delta_s^*)}{\Delta_s^*} + \frac{24I_o(\Delta_s^*)}{\Delta_s^{*2}} - \frac{48I_1(\Delta_s^*)}{\Delta_s^{*3}}$$

Thus,

$$1 - \frac{I_4(\Delta_s^*)}{I_o(\Delta_s^*)} = 1 - 1 + \frac{8I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)} - \frac{24I_o(\Delta_s^*)}{\Delta_s^{*2} I_o(\Delta_s^*)} + \frac{48I_1(\Delta_s^*)}{\Delta_s^{*3} I_o(\Delta_s^*)}$$
$$= \frac{8}{\Delta_s^*}\left(\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} - \frac{3}{\Delta_s^*} + \frac{6I_1(\Delta_s^*)}{\Delta_s^{*2} I_o(\Delta_s^*)}\right)$$

Substituting the relation obtained into G.122 yields:

$$V_6 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s^2 n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^\pi dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \int_0^\pi dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{8}\left(1 - \cos\frac{4Z_s}{\hbar}\right) \quad \text{(G.123)}$$

$$V_6 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s^2 n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^\pi dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \int_0^\pi dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{8}\left(1 - \cos\frac{4Z_s}{\hbar}\right) \quad \text{(G.124)}$$

$$V_6 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s^2 n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^\pi dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \int_0^\pi dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{8}\left(1 - \cos\frac{4Z_s}{\hbar}\right) \quad \text{(G.125)}$$

$$V_6 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s^2 n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^\pi dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \int_0^\pi dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{8}\left(1 - \cos\frac{4Z_s}{\hbar}\right) \quad \text{(G.126)}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^\pi dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^\pi dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \quad \text{(G.127)}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$

$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.128}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.129}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.130}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.131}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.132}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.133}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$

$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.134}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.135}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.136}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.137}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.138}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.139}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$

$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.140}$$

$$\int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right) \times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$

$$= \frac{1}{2}\left[\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt' + \frac{web_z}{\hbar}\int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right) + \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt' - \frac{web_z}{\hbar}\int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\right]$$

$$= \frac{1}{2}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1 + [(e\vec{E}_o b_s/\hbar + n\omega\hbar) + (e\vec{E}_o b_s/\hbar + n\omega\hbar)]^2\tau^2} + \frac{\tau}{1 + [(e\vec{E}_o b_s/\hbar + n\omega\hbar) - (e\vec{E}_o b_s/\hbar + n\omega\hbar)]^2\tau^2}\right]$$

$$= \frac{1}{2}\sum_{n=-\infty} J_n^2(\chi)\left\{2\tau\left[1 - 0\left[\left(\frac{e\vec{E}_o b_s}{\hbar} + n\omega\hbar\right) + \left(\frac{e\vec{E}_o b_s}{\hbar} + n\omega\hbar\right)\right]^2 - 0\left[\left(\frac{e\vec{E}_o b_s}{\hbar} + n\omega\hbar\right) - \left(\frac{e\vec{E}_o b_s}{\hbar} + n\omega\hbar\right)\right]^2\right]\right\}$$

$$= \sum_{n=-\infty} J_n^2(\chi)\tau$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.141}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.142}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$

$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.143}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.144}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.145}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.146}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.147}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.148}$$

$$V_{10} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] \Delta_z \Delta_s \cos\frac{Z_s}{\hbar} \cos\frac{Z_z}{\hbar}$$

$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right) \frac{\nabla_s T}{T} \quad \text{(G.149)}$$

$$V_{10} = -\frac{1}{(\pi\hbar)^2} \frac{u^2 \Delta_s^2 b_s^2}{\hbar^2} \frac{n_o uwb_s b_z}{I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \frac{1}{uwb_s b_z} \int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] \Delta_z \Delta_s \cos\frac{Z_s}{\hbar} \cos\frac{Z_z}{\hbar}$$
$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right) \frac{\nabla_s T}{T} \quad \text{(G.150)}$$

$$V_{10} = -\frac{1}{(\pi\hbar)^2} \frac{u^2 \Delta_s^2 b_s^2}{\hbar^2} \frac{n_o uwb_s b_z}{I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \frac{1}{uwb_s b_z} \int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] \Delta_z \Delta_s \cos\frac{Z_s}{\hbar} \cos\frac{Z_z}{\hbar}$$
$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right) \frac{\nabla_s T}{T} \quad \text{(G.151)}$$

$$V_{10} = -\frac{1}{(\pi\hbar)^2} \frac{u^2 \Delta_s^2 b_s^2}{\hbar^2} \frac{n_o uwb_s b_z}{I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \frac{1}{uwb_s b_z} \int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] \Delta_z \Delta_s \cos\frac{Z_s}{\hbar} \cos\frac{Z_z}{\hbar}$$
$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right) \frac{\nabla_s T}{T} \quad \text{(G.152)}$$

$$V_{10} = -\frac{1}{(\pi\hbar)^2} \frac{u^2 \Delta_s^2 b_s^2}{\hbar^2} \frac{n_o uwb_s b_z}{I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \frac{1}{uwb_s b_z} \int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] \Delta_z \Delta_s \cos\frac{Z_s}{\hbar} \cos\frac{Z_z}{\hbar}$$
$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right) \frac{\nabla_s T}{T} \quad \text{(G.153)}$$

$$V_{10} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z\Delta_s\cos\frac{Z_s}{\hbar}\cos\frac{Z_z}{\hbar}$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.154)$$

$$V_{10} = -\frac{\tau u^2\Delta_s^2 b_s^2\Delta_s\Delta_z n_o}{4\hbar^2 k_B T}\sum_{n=-\infty} J_n^2(\chi)\frac{\nabla_s T}{T} \quad (G.155)$$

$$V_{10} = -\frac{\tau u^2\Delta_s b_s^2\Delta_s^*\Delta_z n_o k}{\hbar^2}\sum_{n=-\infty} J_n^2(\chi) \quad (G.156)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.157)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.158)$$

Making use of the identity:

$$\cos^2\frac{Z_z}{\hbar} = \frac{1}{2}\left(1+\cos\frac{Z_z}{\hbar}\right) \qquad \sin^2\frac{Z_s}{\hbar} = \frac{1}{2}\left(1-\cos\frac{Z_s}{\hbar}\right)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$

$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right) \frac{\nabla_s T}{T} \quad (G.159)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2} \frac{u^2\Delta_s^2 b_s^2}{\hbar^2} \frac{n_o uwb_s b_z}{I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \frac{1}{uwb_s b_z} \int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] \Delta_z^2 \cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right) \frac{\nabla_s T}{T} \quad (G.160)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2} \frac{u^2\Delta_s^2 b_s^2}{\hbar^2} \frac{n_o uwb_s b_z}{I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \frac{1}{uwb_s b_z} \int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] \Delta_z^2 \cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right) \frac{\nabla_s T}{T} \quad (G.161)$$

Evaluating the recurrence relation:

$$1 + \frac{I_2(\Delta_z^*)}{I_o(\Delta_z^*)} = 1 + \frac{I_o(\Delta_z^*) - \frac{2}{\Delta_z^*} I_1(\Delta_z^*)}{I_o(\Delta_z^*)} = 2 - \frac{2}{\Delta_z^*} \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} = 2\left(1 - \frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2} \frac{u^2\Delta_s^2 b_s^2}{\hbar^2} \frac{n_o uwb_s b_z}{I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \frac{1}{uwb_s b_z} \int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] \Delta_z^2 \cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right) \frac{\nabla_s T}{T} \quad (G.162)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2} \frac{u^2\Delta_s^2 b_s^2}{\hbar^2} \frac{n_o uwb_s b_z}{I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \frac{1}{uwb_s b_z} \int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] \Delta_z^2 \cos^2\frac{Z_z}{\hbar}$$

$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right) \frac{\nabla_s T}{T} \quad \text{(G.163)}$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2} \frac{u^2\Delta_s^2 b_s^2}{\hbar^2} \frac{n_o uwb_s b_z}{I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \frac{1}{uwb_s b_z} \int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] \Delta_z^2 \cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right) \frac{\nabla_s T}{T} \quad \text{(G.164)}$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2} \frac{u^2\Delta_s^2 b_s^2}{\hbar^2} \frac{n_o uwb_s b_z}{I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \frac{1}{uwb_s b_z} \int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] \Delta_z^2 \cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right) \frac{\nabla_s T}{T} \quad \text{(G.165)}$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2} \frac{u^2\Delta_s^2 b_s^2}{\hbar^2} \frac{n_o uwb_s b_z}{I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \frac{1}{uwb_s b_z} \int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] \Delta_z^2 \cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right) \frac{\nabla_s T}{T} \quad \text{(G.166)}$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2} \frac{u^2\Delta_s^2 b_s^2}{\hbar^2} \frac{n_o uwb_s b_z}{I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \frac{1}{uwb_s b_z} \int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] \Delta_z^2 \cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right) \frac{\nabla_s T}{T} \quad \text{(G.167)}$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.168)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.169)$$

Summing the up $V_1....V_{12}$ yields:

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.170)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.171)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$

$$\times \sin^2\frac{Z_s}{\hbar}\cos\left(\frac{uebs_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T}T \quad (G.172)$$

where in the presence of a weak field, $\sigma_s(\vec{E})$ is given as:

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.173)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.174)$$

Now, $\vec{S}^* = \vec{S}_1^* + \vec{S}_2^*$ which yields:

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.175)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$

$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad \text{(G.176)}$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad \text{(G.177)}$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad \text{(G.178)}$$

where $\vec{E}_{sn}^* = \vec{E}_n + \nabla_s\mu/e$. Following the same procedure above:

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad \text{(G.179)}$$

The axial and circumferential components of the thermal current density are respectively given as:

$$\vec{q}_z = \vec{Z}^* + \vec{S}^*\sin\theta_h \qquad \vec{q}_c = \vec{S}^*\cos\theta_h$$

The axial current density yields:

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$

$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad \text{(G.180)}$$

and the circumferential component gives:

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad \text{(G.181)}$$

But $\vec{E}_s = \vec{E}_z \sin\theta_h$ and $\nabla_s T = \nabla_z T \sin\theta_h$. Hence:

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad \text{(G.182)}$$

Multiplying through by $k_B T$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad \text{(G.183)}$$

The circumferential current density is given as

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$

$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.184)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.185)$$

Multiplying and dividing by $k_B T$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.186)$$

Let κ_{ec} and κ_{ez} be the circumferential and axial components of the electron thermal conductivity respectively

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.187)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.188)$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right.$$
$$-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_z\sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\left.\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right]$$
$$\times\left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right.$$
$$\left.-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right\} \quad (G.100)$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right.$$
$$-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$
\begin{aligned}
&-\Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]\\
&\times\left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right.\\
&\left.-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right\} \qquad (G.101)
\end{aligned}
$$

$$
\begin{aligned}
S_1^* &= \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\\
&\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\left[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right.\\
&-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&-\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&-\Delta_z\sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\left.\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right]\\
&\times\left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right.\\
&\left.-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right\} \qquad (G.102)
\end{aligned}
$$

$$
\begin{aligned}
S_1^* &= \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\\
&\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\left[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right.\\
&-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]
\end{aligned}
$$

$$
\begin{aligned}
&-\Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&-\Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&-\Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]\\
&\times\Bigg\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\Bigg\}
\end{aligned}
\tag{G.103}
$$

$$
\begin{aligned}
S_1^* = &\frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\\
&\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\Bigg[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&-\Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&-\Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&-\Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&-\Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]\\
&\times\Bigg\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\Bigg\}
\end{aligned}
\tag{G.104}
$$

$$
\begin{aligned}
S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\
\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\left[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right. \\
-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
-\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
-\Delta_z\sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
\left.\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right] \\
\times\left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right. \\
\left.-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right\} \qquad (G.105)
\end{aligned}
$$

$$
\begin{aligned}
S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\
\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\left[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right. \\
-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
-\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]
\end{aligned}
$$

$$- \Delta_z \sin \frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right.$$
$$\left. - \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \quad (G.106)$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right.$$
$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right.$$
$$\left. - \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \quad (G.107)$$

$$\left[1 - \frac{I_2(\Delta_s^*)}{I_o(\Delta_s^*)}\right] = \left(1 - \frac{I_o(\Delta_s^*) - \frac{2}{\Delta_s^*}I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right)$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right.$$

$$
\begin{aligned}
&-\Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&-\Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&-\Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&-\Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]\\
&\times\left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right.\\
&\left.-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right\}
\end{aligned}
\tag{G.108}
$$

$$
V_4=-\frac{\tau u^2\Delta_s b_s^2 n_o}{\hbar^2}(\varepsilon_o-\mu)\sum_{n=-\infty}J_n^2(\chi)
\tag{G.109}
$$

$$
\begin{aligned}
S_1^* &= \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\\
&\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\left[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right.\\
&-\Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&-\Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&-\Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&-\Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)
\end{aligned}
$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times\left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right.$$
$$\left. -\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right\} \quad (G.110)$$

$$\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\cos^2\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$

For weak electric field

$$= \frac{\tau}{2} + \frac{1}{2}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+4(e\vec{E}_o b_s/\hbar + n\omega\hbar)^2\tau^2}\right] =$$

$$V_5 = +\frac{\tau u^2\Delta_s^2 b_s^2\Delta_s n_o(\varepsilon_o-\mu)}{8(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\left(1+\sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$[I_1(\Delta_s^*) - I_3(\Delta_s^*)I_o(\Delta_z^*)] \quad (G.111)$$

$$V_5 = +\frac{\tau u^2\Delta_s^2 b_s^2\Delta_s n_o(\varepsilon_o-\mu)}{8(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\left(1+\sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$[I_1(\Delta_s^*) - I_3(\Delta_s^*)I_o(\Delta_z^*)] \quad (G.112)$$

$$V_5 = +\frac{\tau u^2\Delta_s^2 b_s^2\Delta_s n_o(\varepsilon_o-\mu)}{8(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\left(1+\sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$[I_1(\Delta_s^*) - I_3(\Delta_s^*)I_o(\Delta_z^*)] \quad (G.113)$$

$$V_5 = +\frac{\tau u^2\Delta_s^2 b_s^2\Delta_s n_o(\varepsilon_o-\mu)}{8(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\left(1+\sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$[I_1(\Delta_s^*) - I_3(\Delta_s^*)I_o(\Delta_z^*)] \quad (G.114)$$

$$V_5 = +\frac{\tau u^2\Delta_s^2 b_s^2\Delta_s n_o(\varepsilon_o-\mu)}{8(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\left(1+\sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$[I_1(\Delta_s^*) - I_3(\Delta_s^*)I_o(\Delta_z^*)] \quad (G.115)$$

$$V_5 = +\frac{\tau u^2\Delta_s^2 b_s^2\Delta_s n_o(\varepsilon_o-\mu)}{8(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\left(1+\sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\mathrm{I}_1(\Delta_s^*) - I_3(\Delta_s^*)I_o(\Delta_z^*)$$
(G.116)

$$V_5 = +\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s n_o(\varepsilon_o - \mu)}{8(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T} \left[I_1(\Delta_s^*) - I_3(\Delta_s^*)I_o(\Delta_z^*)\right] \quad (G.117)$$

$$V_5 = +\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s n_o(\varepsilon_o - \mu)}{8(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T} \left[I_1(\Delta_s^*) - I_3(\Delta_s^*)I_o(\Delta_z^*)\right] \quad (G.118)$$

Evaluating the identity:

$$\cos^2\frac{Z_s}{\hbar}\sin^2\frac{Z_s}{\hbar} = \frac{1}{4}\sin^2\frac{2Z_s}{\hbar} =$$

$$V_6 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s^2 n_o}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T} \times \int_0^\pi dZ_z \exp\left[\Delta_z^*\cos\frac{Z_z}{\hbar}\right]\int_0^\pi dZ_s \exp\left[\Delta_s^*\cos\frac{Z_s}{\hbar}\right]\frac{1}{8}\left(1 - \cos\frac{4Z_s}{\hbar}\right) \quad (G.119)$$

$$V_6 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s^2 n_o}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T} \times \int_0^\pi dZ_z \exp\left[\Delta_z^*\cos\frac{Z_z}{\hbar}\right]\int_0^\pi dZ_s \exp\left[\Delta_s^*\cos\frac{Z_s}{\hbar}\right]\frac{1}{8}\left(1 - \cos\frac{4Z_s}{\hbar}\right) \quad (G.120)$$

$$V_6 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s^2 n_o}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T} \times \int_0^\pi dZ_z \exp\left[\Delta_z^*\cos\frac{Z_z}{\hbar}\right]\int_0^\pi dZ_s \exp\left[\Delta_s^*\cos\frac{Z_s}{\hbar}\right]\frac{1}{8}\left(1 - \cos\frac{4Z_s}{\hbar}\right) \quad (G.121)$$

$$V_6 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s^2 n_o}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T} \times \int_0^\pi dZ_z \exp\left[\Delta_z^*\cos\frac{Z_z}{\hbar}\right]\int_0^\pi dZ_s \exp\left[\Delta_s^*\cos\frac{Z_s}{\hbar}\right]\frac{1}{8}\left(1 - \cos\frac{4Z_s}{\hbar}\right) \quad (G.122)$$

The recurrence relation for I_4 is evaluated as:

$$I_4(\Delta_s^*) = I_2(\Delta_s^*) - \frac{6}{\Delta_s^*}I_3(\Delta_s^*) = I_o(\Delta_s^*) - \frac{2I_1(\Delta_s^*)}{\Delta_s^*} - \frac{6}{\Delta_s^*}\left(I_1(\Delta_s^*) - \frac{4I_o(\Delta_s^*)}{\Delta_s^*} - \frac{8I_1(\Delta_s^*)}{\Delta_s^{*2}}\right)$$
$$= I_o(\Delta_s^*) - \frac{8I_1(\Delta_s^*)}{\Delta_s^*} + \frac{24I_o(\Delta_s^*)}{\Delta_s^{*2}} - \frac{48I_1(\Delta_s^*)}{\Delta_s^{*3}}$$

Thus,

$$1-\frac{I_4(\Delta_s^*)}{I_o(\Delta_s^*)} = 1-1+\frac{8I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)} - \frac{24I_o(\Delta_s^*)}{\Delta_s^{*2} I_o(\Delta_s^*)} + \frac{48I_1(\Delta_s^*)}{\Delta_s^{*3} I_o(\Delta_s^*)}$$
$$= \frac{8}{\Delta_s^*}\left(\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} - \frac{3}{\Delta_s^*} + \frac{6I_1(\Delta_s^*)}{\Delta_s^{*2} I_o(\Delta_s^*)}\right)$$

Substituting the relation obtained into G.122 yields:

$$V_6 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s^2 n_o}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1+\sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^\pi dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right]\int_0^\pi dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right]\frac{1}{8}\left(1-\cos\frac{4Z_s}{\hbar}\right) \quad (G.123)$$

$$V_6 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s^2 n_o}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1+\sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^\pi dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right]\int_0^\pi dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right]\frac{1}{8}\left(1-\cos\frac{4Z_s}{\hbar}\right) \quad (G.124)$$

$$V_6 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s^2 n_o}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1+\sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^\pi dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right]\int_0^\pi dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right]\frac{1}{8}\left(1-\cos\frac{4Z_s}{\hbar}\right) \quad (G.125)$$

$$V_6 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s^2 n_o}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1+\sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^\pi dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right]\int_0^\pi dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right]\frac{1}{8}\left(1-\cos\frac{4Z_s}{\hbar}\right) \quad (G.126)$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1+\sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^\pi dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right]\cos\frac{Z_z}{\hbar}\int_0^\pi dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right]\frac{1}{4}\left(\cos\frac{Z_s}{\hbar}-\cos\frac{3Z_s}{\hbar}\right) \quad (G.127)$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1+\sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$

$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.128}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.129}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.130}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.131}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.132}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.133}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$

$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.134}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.135}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.136}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.137}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.138}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.139}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$

$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.140}$$

$$\begin{aligned}
\int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right) \\
\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right) \\
= \frac{1}{2}\left[\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt' + \frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\right. \\
\left. + \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt' - \frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\right] \\
= \frac{1}{2}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+[(e\vec{E}_o b_s/\hbar + n\omega\hbar) + (e\vec{E}_o b_s/\hbar + n\omega\hbar)]^2\tau^2}\right. \\
\left. + \frac{\tau}{1+[(e\vec{E}_o b_s/\hbar + n\omega\hbar) - (e\vec{E}_o b_s/\hbar + n\omega\hbar)]^2\tau^2}\right] \\
= \frac{1}{2}\sum_{n=-\infty} J_n^2(\chi)\left\{2\tau\left[1 - 0\left[\left(\frac{e\vec{E}_o b_s}{\hbar} + n\omega\hbar\right) + \left(\frac{e\vec{E}_o b_s}{\hbar} + n\omega\hbar\right)\right]^2\right.\right. \\
\left.\left. - 0\left[\left(\frac{e\vec{E}_o b_s}{\hbar} + n\omega\hbar\right) - \left(\frac{e\vec{E}_o b_s}{\hbar} + n\omega\hbar\right)\right]^2\right]\right\} \\
= \sum_{n=-\infty} J_n^2(\chi)\tau
\end{aligned}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.141}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.142}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$

$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.143}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.144}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.145}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.146}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.147}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.148}$$

$$V_{10} = -\frac{1}{(\pi\hbar)^2}\frac{u^2 \Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] \Delta_z \Delta_s \cos\frac{Z_s}{\hbar} \cos\frac{Z_z}{\hbar}$$

$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.149)$$

$$V_{10} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right]\Delta_z\Delta_s \cos\frac{Z_s}{\hbar}\cos\frac{Z_z}{\hbar}$$
$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.150)$$

$$V_{10} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right]\Delta_z\Delta_s \cos\frac{Z_s}{\hbar}\cos\frac{Z_z}{\hbar}$$
$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.151)$$

$$V_{10} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right]\Delta_z\Delta_s \cos\frac{Z_s}{\hbar}\cos\frac{Z_z}{\hbar}$$
$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.152)$$

$$V_{10} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right]\Delta_z\Delta_s \cos\frac{Z_s}{\hbar}\cos\frac{Z_z}{\hbar}$$
$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.153)$$

$$V_{10} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z\Delta_s\cos\frac{Z_s}{\hbar}\cos\frac{Z_z}{\hbar}$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.154)$$

$$V_{10} = -\frac{\tau u^2\Delta_s^2 b_s^2\Delta_s\Delta_z n_o}{4\hbar^2 k_B T}\sum_{n=-\infty} J_n^2(\chi)\frac{\nabla_s T}{T} \quad (G.155)$$
$$V_{10} = -\frac{\tau u^2\Delta_s b_s^2\Delta_s^*\Delta_z n_o k}{\hbar^2}\sum_{n=-\infty} J_n^2(\chi) \quad (G.156)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.157)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.158)$$

Making use of the identity:

$$\cos^2\frac{Z_z}{\hbar} = \frac{1}{2}\left(1+\cos\frac{Z_z}{\hbar}\right) \qquad \sin^2\frac{Z_s}{\hbar} = \frac{1}{2}\left(1-\cos\frac{Z_s}{\hbar}\right)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$

$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.159)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.160)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.161)$$

Evaluating the recurrence relation:

$$1+\frac{I_2(\Delta_z^*)}{I_o(\Delta_z^*)} = 1+\frac{I_o(\Delta_z^*)-\frac{2}{\Delta_z^*}I_1(\Delta_z^*)}{I_o(\Delta_z^*)} = 2-\frac{2}{\Delta_z^*}\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} = 2\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.162)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$

$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.163)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.164)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.165)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.166)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.167)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.168)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.169)$$

Summing the up $V_1 \ldots V_{12}$ yields:

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.170)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.171)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$

$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T}T \quad (G.172)$$

where in the presence of a weak field, $\sigma_s(\vec{E})$ is given as:

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right]\Delta_z^2 \cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.173)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right]\Delta_z^2 \cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.174)$$

Now, $\vec{S}^* = \vec{S}_1^* + \vec{S}_2^*$ which yields:

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right]\Delta_z^2 \cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.175)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right]\Delta_z^2 \cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$

$$\times\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad \text{(G.176)}$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad \text{(G.177)}$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad \text{(G.178)}$$

where $\vec{E}_{sn}^* = \vec{E}_n + \nabla_s\mu/e$. Following the same procedure above:

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad \text{(G.179)}$$

The axial and circumferential components of the thermal current density are respectively given as:

$$\vec{q}_z = \vec{Z}^* + \vec{S}^*\sin\theta_h \qquad \vec{q}_c = \vec{S}^*\cos\theta_h$$

The axial current density yields:

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$

$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.180)$$

and the circumferential component gives:

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right]\Delta_z^2 \cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.181)$$

But $\vec{E}_s = \vec{E}_z \sin\theta_h$ and $\nabla_s T = \nabla_z T \sin\theta_h$. Hence:

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right]\Delta_z^2 \cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.182)$$

Multiplying through by $k_B T$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right]\Delta_z^2 \cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.183)$$

The circumferential current density is given as

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right]\Delta_z^2 \cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2 \frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$

$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.184)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.185)$$

Multiplying and dividing by $k_B T$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.186)$$

Let κ_{ec} and κ_{ez} be the circumferential and axial components of the electron thermal conductivity respectively

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.187)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.188)$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_z\sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times\Bigg\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\Bigg\} \quad (G.189)$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$
\begin{aligned}
&- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right) \\
&\times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg] \\
&\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)\right. \\
&\left. - \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)\right\} \quad \text{(G.190)}
\end{aligned}
$$

$$
\begin{aligned}
S_1^* = &\frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\
&\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right. \\
&- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right) \\
&\times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right) \\
&\times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right) \\
&\times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right) \\
&\times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg] \\
&\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)\right. \\
&\left. - \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)\right\} \quad \text{(G.191)}
\end{aligned}
$$

$$
\begin{aligned}
S_1^* = &\frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\
&\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right. \\
&- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right) \\
&\times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]
\end{aligned}
$$

$$
\begin{aligned}
& - \Delta_s \sin \frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
& \times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
& - \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
& \times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
& - \Delta_z \sin \frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
& \times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg] \\
& \times \Bigg\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
& - \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\Bigg\} \qquad (G.192)
\end{aligned}
$$

$$
\begin{aligned}
S_1^* = {} & \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\
& \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right. \\
& - \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
& \times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
& - \Delta_s \sin \frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
& \times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
& - \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
& \times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
& - \Delta_z \sin \frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
& \left.\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right] \\
& \times \Bigg\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
& - \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\Bigg\} \qquad (G.193)
\end{aligned}
$$

$$
\begin{aligned}
S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\
\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
-\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
-\Delta_z\sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg] \\
\times\left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right. \\
\left.-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right\} \qquad \text{(G.194)}
\end{aligned}
$$

$$
\begin{aligned}
S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\
\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
-\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]
\end{aligned}
$$

$$- \Delta_z \sin \frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times \left\{\sin \frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right.$$
$$\left. - \cos \frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \quad (G.195)$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar}\right]\right.$$
$$- \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_s \sin \frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \sin \frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times \left\{\sin \frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right.$$
$$\left. - \cos \frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \quad (G.196)$$

$$\left[1 - \frac{I_2(\Delta_s^*)}{I_o(\Delta_s^*)}\right] = \left(1 - \frac{I_o(\Delta_s^*) - \frac{2}{\Delta_s^*}I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right)$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar}\right]\right.$$

$$-\Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times\left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right.$$
$$\left.-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right\} \quad \text{(G.197)}$$

$$V_4 = -\frac{\tau u^2\Delta_s b_s^2 n_o}{\hbar^2}(\varepsilon_o-\mu)\sum_{n=-\infty} J_n^2(\chi) \quad \text{(G.198)}$$

$$S_1^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\left[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right.$$
$$-\Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$

$$\times \left\{ \sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \quad (G.199)$$

$$\int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt \cos^2\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right) =$$

For weak electric field

$$= \frac{\tau}{2} + \frac{1}{2}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1+4(e\vec{E}_o b_s/\hbar + n\omega\hbar)^2\tau^2}\right] =$$

$$V_5 = +\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s n_o(\varepsilon_o - \mu)}{8(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T} [I_1(\Delta_s^*) - I_3(\Delta_s^*) I_o(\Delta_z^*)] \quad (G.200)$$

$$V_5 = +\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s n_o(\varepsilon_o - \mu)}{8(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T} [I_1(\Delta_s^*) - I_3(\Delta_s^*) I_o(\Delta_z^*)] \quad (G.201)$$

$$V_5 = +\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s n_o(\varepsilon_o - \mu)}{8(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T} [I_1(\Delta_s^*) - I_3(\Delta_s^*) I_o(\Delta_z^*)] \quad (G.202)$$

$$V_5 = +\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s n_o(\varepsilon_o - \mu)}{8(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T} [I_1(\Delta_s^*) - I_3(\Delta_s^*) I_o(\Delta_z^*)] \quad (G.203)$$

$$V_5 = +\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s n_o(\varepsilon_o - \mu)}{8(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T} [I_1(\Delta_s^*) - I_3(\Delta_s^*) I_o(\Delta_z^*)] \quad (G.204)$$

$$V_5 = +\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s n_o (\varepsilon_o - \mu)}{8(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \frac{\nabla_s T}{T}$$
$$[I_1(\Delta_s^*) - I_3(\Delta_s^*) I_o(\Delta_z^*)] \quad \text{(G.205)}$$

$$V_5 = +\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s n_o (\varepsilon_o - \mu)}{8(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \frac{\nabla_s T}{T}$$
$$[I_1(\Delta_s^*) - I_3(\Delta_s^*) I_o(\Delta_z^*)] \quad \text{(G.206)}$$

$$V_5 = +\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s n_o (\varepsilon_o - \mu)}{8(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \frac{\nabla_s T}{T}$$
$$[I_1(\Delta_s^*) - I_3(\Delta_s^*) I_o(\Delta_z^*)] \quad \text{(G.207)}$$

Evaluating the identity:

$$\cos^2 \frac{Z_s}{\hbar} \sin^2 \frac{Z_s}{\hbar} = \frac{1}{4} \sin^2 \frac{2Z_s}{\hbar} =$$

$$V_6 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s^2 n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \frac{\nabla_s T}{T}$$
$$\times \int_0^\pi dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \int_0^\pi dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{8}\left(1 - \cos\frac{4Z_s}{\hbar}\right) \quad \text{(G.208)}$$

$$V_6 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s^2 n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \frac{\nabla_s T}{T}$$
$$\times \int_0^\pi dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \int_0^\pi dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{8}\left(1 - \cos\frac{4Z_s}{\hbar}\right) \quad \text{(G.209)}$$

$$V_6 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s^2 n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \frac{\nabla_s T}{T}$$
$$\times \int_0^\pi dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \int_0^\pi dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{8}\left(1 - \cos\frac{4Z_s}{\hbar}\right) \quad \text{(G.210)}$$

$$V_6 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s^2 n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \frac{\nabla_s T}{T}$$
$$\times \int_0^\pi dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \int_0^\pi dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{8}\left(1 - \cos\frac{4Z_s}{\hbar}\right) \quad \text{(G.211)}$$

The recurrence relation for I_4 is evaluated as:

$$I_4(\Delta_s^*) = I_2(\Delta_s^*) - \frac{6}{\Delta_s^*} I_3(\Delta_s^*) = I_o(\Delta_s^*) - \frac{2I_1(\Delta_s^*)}{\Delta_s^*} - \frac{6}{\Delta_s^*}\left(I_1(\Delta_s^*) - \frac{4I_o(\Delta_s^*)}{\Delta_s^*} - \frac{8I_1(\Delta_s^*)}{\Delta_s^{*2}}\right)$$
$$= I_o(\Delta_s^*) - \frac{8I_1(\Delta_s^*)}{\Delta_s^*} + \frac{24I_o(\Delta_s^*)}{\Delta_s^{*2}} - \frac{48I_1(\Delta_s^*)}{\Delta_s^{*3}}$$

Thus,

$$1 - \frac{I_4(\Delta_s^*)}{I_o(\Delta_s^*)} = 1 - 1 + \frac{8I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)} - \frac{24I_o(\Delta_s^*)}{\Delta_s^{*2} I_o(\Delta_s^*)} + \frac{48I_1(\Delta_s^*)}{\Delta_s^{*3} I_o(\Delta_s^*)}$$
$$= \frac{8}{\Delta_s^*}\left(\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} - \frac{3}{\Delta_s^*} + \frac{6I_1(\Delta_s^*)}{\Delta_s^{*2} I_o(\Delta_s^*)}\right)$$

Substituting the relation obtained into G.122 yields:

$$V_6 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s^2 n_o}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^\pi dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \int_0^\pi dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{8}\left(1 - \cos\frac{4Z_s}{\hbar}\right) \quad \text{(G.212)}$$

$$V_6 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s^2 n_o}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^\pi dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \int_0^\pi dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{8}\left(1 - \cos\frac{4Z_s}{\hbar}\right) \quad \text{(G.213)}$$

$$V_6 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s^2 n_o}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^\pi dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \int_0^\pi dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{8}\left(1 - \cos\frac{4Z_s}{\hbar}\right) \quad \text{(G.214)}$$

$$V_6 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s^2 n_o}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^\pi dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \int_0^\pi dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{8}\left(1 - \cos\frac{4Z_s}{\hbar}\right) \quad \text{(G.215)}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2\hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T}\left(1 + \sum_{n=-\infty} J_n^2(\chi)\right)\frac{\nabla_s T}{T}$$
$$\times \int_0^\pi dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^\pi dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right)$$
(G.216)

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \frac{\nabla_s T}{T}$$
$$\times \int_0^\pi dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^\pi dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.217}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \frac{\nabla_s T}{T}$$
$$\times \int_0^\pi dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^\pi dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.218}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \frac{\nabla_s T}{T}$$
$$\times \int_0^\pi dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^\pi dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.219}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \frac{\nabla_s T}{T}$$
$$\times \int_0^\pi dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^\pi dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.220}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \frac{\nabla_s T}{T}$$
$$\times \int_0^\pi dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^\pi dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.221}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \frac{\nabla_s T}{T}$$
$$\times \int_0^\pi dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^\pi dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.222}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \frac{\nabla_s T}{T}$$
$$\times \int_0^\pi dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^\pi dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.223}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \frac{\nabla_s T}{T}$$
$$\times \int_0^\pi dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^\pi dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.224}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \frac{\nabla_s T}{T}$$
$$\times \int_0^\pi dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^\pi dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.225}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \frac{\nabla_s T}{T}$$
$$\times \int_0^\pi dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^\pi dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.226}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \frac{\nabla_s T}{T}$$
$$\times \int_0^\pi dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^\pi dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.227}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \frac{\nabla_s T}{T}$$
$$\times \int_0^\pi dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^\pi dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.228}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \frac{\nabla_s T}{T}$$
$$\times \int_0^\pi dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^\pi dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.229}$$

$$\int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^t [\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^t [\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$= \frac{1}{2}\left[\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^t [\vec{E}_o + \vec{E}\cos\omega t']dt' + \frac{web_z}{\hbar}\int_{t-t'}^t [\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\right.$$
$$\left. + \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^t [\vec{E}_o + \vec{E}\cos\omega t']dt' - \frac{web_z}{\hbar}\int_{t-t'}^t [\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\right]$$
$$= \frac{1}{2}\sum_{n=-\infty} J_n^2(\chi)\left[\frac{\tau}{1 + [(e\vec{E}_o b_s/\hbar + n\omega\hbar) + (e\vec{E}_o b_s/\hbar + n\omega\hbar)]^2\tau^2}\right.$$
$$\left. + \frac{\tau}{1 + [(e\vec{E}_o b_s/\hbar + n\omega\hbar) - (e\vec{E}_o b_s/\hbar + n\omega\hbar)]^2\tau^2}\right]$$
$$= \frac{1}{2}\sum_{n=-\infty} J_n^2(\chi)\left\{2\tau\left[1 - 0\left[\left(\frac{e\vec{E}_o b_s}{\hbar} + n\omega\hbar\right) + \left(\frac{e\vec{E}_o b_s}{\hbar} + n\omega\hbar\right)\right]^2\right.\right.$$
$$\left.\left. - 0\left[\left(\frac{e\vec{E}_o b_s}{\hbar} + n\omega\hbar\right) - \left(\frac{e\vec{E}_o b_s}{\hbar} + n\omega\hbar\right)\right]^2\right]\right\}$$
$$= \sum_{n=-\infty} J_n^2(\chi)\tau$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \frac{\nabla_s T}{T}$$
$$\times \int_0^\pi dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^\pi dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.230}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \frac{\nabla_s T}{T}$$
$$\times \int_0^\pi dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^\pi dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.231}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.232}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.233}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.234}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.235}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.236}$$

$$V_7 = -\frac{\tau u^2 \Delta_s^2 b_s^2 \Delta_s \Delta_z n_o}{2(\pi\hbar)^2 \hbar^2 I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \left(1 + \sum_{n=-\infty} J_n^2(\chi)\right) \frac{\nabla_s T}{T}$$
$$\times \int_0^{\pi} dZ_z \exp\left[\Delta_z^* \cos\frac{Z_z}{\hbar}\right] \cos\frac{Z_z}{\hbar} \int_0^{\pi} dZ_s \exp\left[\Delta_s^* \cos\frac{Z_s}{\hbar}\right] \frac{1}{4}\left(\cos\frac{Z_s}{\hbar} - \cos\frac{3Z_s}{\hbar}\right) \tag{G.237}$$

$$V_{10} = -\frac{1}{(\pi\hbar)^2} \frac{u^2 \Delta_s^2 b_s^2}{\hbar^2} \frac{n_o u w b_s b_z}{I_o(\Delta_z^*) I_o(\Delta_s^*) k_B T} \frac{1}{u w b_s b_z} \int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right) dt$$

$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] \Delta_z \Delta_s \cos\frac{Z_s}{\hbar} \cos\frac{Z_z}{\hbar}$$
$$\times \sin^2\frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.238)$$

$$V_{10} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] \Delta_z \Delta_s \cos\frac{Z_s}{\hbar} \cos\frac{Z_z}{\hbar}$$
$$\times \sin^2\frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.239)$$

$$V_{10} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] \Delta_z \Delta_s \cos\frac{Z_s}{\hbar} \cos\frac{Z_z}{\hbar}$$
$$\times \sin^2\frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.240)$$

$$V_{10} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] \Delta_z \Delta_s \cos\frac{Z_s}{\hbar} \cos\frac{Z_z}{\hbar}$$
$$\times \sin^2\frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.241)$$

$$V_{10} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] \Delta_z \Delta_s \cos\frac{Z_s}{\hbar} \cos\frac{Z_z}{\hbar}$$
$$\times \sin^2\frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$

$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad \text{(G.242)}$$

$$V_{10} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z\Delta_s\cos\frac{Z_s}{\hbar}\cos\frac{Z_z}{\hbar}$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad \text{(G.243)}$$

$$V_{10} = -\frac{\tau u^2\Delta_s^2 b_s^2\Delta_s\Delta_z n_o}{4\hbar^2 k_B T}\sum_{n=-\infty} J_n^2(\chi)\frac{\nabla_s T}{T} \quad \text{(G.244)}$$

$$V_{10} = -\frac{\tau u^2\Delta_s b_s^2\Delta_s^*\Delta_z n_o k}{\hbar^2}\sum_{n=-\infty} J_n^2(\chi) \quad \text{(G.245)}$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad \text{(G.246)}$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad \text{(G.247)}$$

Making use of the identity:

$$\cos^2\frac{Z_z}{\hbar} = \frac{1}{2}\left(1+\cos\frac{Z_z}{\hbar}\right) \qquad \sin^2\frac{Z_s}{\hbar} = \frac{1}{2}\left(1-\cos\frac{Z_s}{\hbar}\right)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$

$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] \Delta_z^2 \cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2\frac{Z_s}{\hbar} \cos\left(\frac{uebs_s}{\hbar}\int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.248)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] \Delta_z^2 \cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2\frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.249)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] \Delta_z^2 \cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2\frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.250)$$

Evaluating the recurrence relation:

$$1 + \frac{I_2(\Delta_z^*)}{I_o(\Delta_z^*)} = 1 + \frac{I_o(\Delta_z^*) - \frac{2}{\Delta_z^*}I_1(\Delta_z^*)}{I_o(\Delta_z^*)} = 2 - \frac{2}{\Delta_z^*}\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} = 2\left(1 - \frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] \Delta_z^2 \cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2\frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.251)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$

$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] \Delta_z^2 \cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2\frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.252)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] \Delta_z^2 \cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2\frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.253)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] \Delta_z^2 \cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2\frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.254)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] \Delta_z^2 \cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2\frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.255)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] \Delta_z^2 \cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2\frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$

$$\times\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \tag{G.256}$$

$$\begin{aligned}V_{11} = &-\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\\ &\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}\\ &\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\\ &\times\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T}\end{aligned} \tag{G.257}$$

$$\begin{aligned}V_{11} = &-\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\\ &\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}\\ &\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\\ &\times\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T}\end{aligned} \tag{G.258}$$

Summing the up $V_1....V_{12}$ yields:

$$\begin{aligned}V_{11} = &-\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\\ &\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}\\ &\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\\ &\times\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T}\end{aligned} \tag{G.259}$$

$$\begin{aligned}V_{11} = &-\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\\ &\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}\\ &\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\\ &\times\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T}\end{aligned} \tag{G.260}$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T}T \quad (G.261)$$

where in the presence of a weak field, $\sigma_s(\vec{E})$ is given as:

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.262)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.263)$$

Now, $\vec{S}^* = \vec{S}_1^* + \vec{S}_2^*$ which yields:

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.264)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$

$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] \Delta_z^2 \cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2\frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad \text{(G.265)}$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] \Delta_z^2 \cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2\frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad \text{(G.266)}$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] \Delta_z^2 \cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2\frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad \text{(G.267)}$$

where $\vec{E}_{sn}^* = \vec{E}_n + \nabla_s \mu / e$. Following the same procedure above:

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_B T}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] \Delta_z^2 \cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2\frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad \text{(G.268)}$$

The axial and circumferential components of the thermal current density are respectively given as:

$$\vec{q}_z = \vec{Z}^* + \vec{S}^* \sin\theta_h \qquad \vec{q}_c = \vec{S}^* \cos\theta_h$$

The axial current density yields:

$$
\begin{aligned}
V_{11} = & -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\
& \times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar} \\
& \times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right) \\
& \times\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \qquad \text{(G.269)}
\end{aligned}
$$

and the circumferential component gives:

$$
\begin{aligned}
V_{11} = & -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\
& \times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar} \\
& \times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right) \\
& \times\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \qquad \text{(G.270)}
\end{aligned}
$$

But $\vec{E}_s = \vec{E}_z\sin\theta_h$ and $\nabla_s T = \nabla_z T\sin\theta_h$. Hence:

$$
\begin{aligned}
V_{11} = & -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\
& \times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar} \\
& \times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right) \\
& \times\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \qquad \text{(G.271)}
\end{aligned}
$$

Multiplying through by k_BT

$$
\begin{aligned}
V_{11} = & -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\
& \times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar} \\
& \times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right) \\
& \times\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \qquad \text{(G.272)}
\end{aligned}
$$

The circumferential current density is given as

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.273)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.274)$$

Multiplying and dividing by k_BT

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.275)$$

Let κ_{ec} and κ_{ez} be the circumferential and axial components of the electron thermal conductivity respectively

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\times\int_0^{\pi}dZ_z\int_0^{\pi}dZ_s\exp\left[\Delta_s\cos\frac{Z_s}{\hbar}+\Delta_z\cos\frac{Z_z}{\hbar}\right]\Delta_z^2\cos^2\frac{Z_z}{\hbar}$$
$$\times\sin^2\frac{Z_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)$$
$$\times\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad (G.276)$$

$$V_{11} = -\frac{1}{(\pi\hbar)^2}\frac{u^2\Delta_s^2 b_s^2}{\hbar^2}\frac{n_o uwb_s b_z}{I_o(\Delta_z^*)I_o(\Delta_s^*)k_BT}\frac{1}{uwb_s b_z}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$

$$\times \int_0^{\pi} dZ_z \int_0^{\pi} dZ_s \exp\left[\Delta_s \cos\frac{Z_s}{\hbar} + \Delta_z \cos\frac{Z_z}{\hbar}\right] \Delta_z^2 \cos^2\frac{Z_z}{\hbar}$$
$$\times \sin^2\frac{Z_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right)$$
$$\times \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t} [\vec{E}_o + \vec{E}\cos\omega t']dt'\right)\frac{\nabla_s T}{T} \quad \text{(G.277)}$$

APPENDIX H

ENTROPY AND ENERGY DISSIPATED

Because the FSWCNT is such a small device, the physical mechanisms that operate within it are completely elastomeric.

$$\vec{J}\cdot\vec{E} = \sigma E^2 = j^2/\sigma \tag{H.1}$$

This is Joule's law. The evolution of heat results in an increase in the entropy of the FSWCNT. When an amount of heat $dQ = \vec{J}\cdot\vec{E}dV$ is evolved, the entropy of the volume element dV increases by dQ/T. The rate of change of the total entropy of the material is therefore given as:

$$\frac{ds}{dt} = \int(\vec{J}\cdot\vec{E})dV \tag{H.2}$$

Since the entropy increases, its derivative must be positive. Putting $\vec{J} = \sigma\vec{E}$, shows that the conductivity is also positive. The symmetry of the kinetic coefficients gives a relation between the coefficient β and the coefficient α in

$$\vec{J} = \sigma(\vec{E} - \alpha\nabla T) \tag{H.3}$$

To derive this, the rate of change of the total entropy of the FSWCNT is calculated. The amount of heat evolved per unit time and volume is $-\nabla\cdot\mathbf{q}$. Hence;

$$\frac{ds}{dt} = -\int\frac{\nabla\cdot\mathbf{q}}{T}dV \tag{H.4}$$

Using the equation $div\vec{J} = 0$, yields;

$$\frac{\nabla\cdot\mathbf{q}}{T} = \frac{1}{T}\{div(\mathbf{q}-\phi\vec{J}) + div\phi\vec{J}\} = \frac{1}{T}div(\mathbf{q}-\phi\vec{J}) - \frac{\vec{J}\cdot\vec{E}}{T} \tag{H.5}$$

Integrating the first term by parts gives:

$$\frac{ds}{dt} = \int\frac{\vec{J}\cdot\vec{E}}{T}dV - \int\frac{(\mathbf{q}-\phi\vec{J})\cdot\nabla T}{T^2}dV \tag{H.6}$$

Considering $\vec{J}$ and $\mathbf{q}$, this quantity contains an amount $\phi\vec{J}$ resulting from the fact that each charged particle carries with it an energy $e\phi$. The difference $\mathbf{q}-\phi\vec{J}$, however, does not depend on the potential and can be written as a linear function of the gradients $\nabla\phi$ and ∇T, similarly to (H.3) for the current density. Presently this can be written as:

$$\vec{J} = \sigma T\frac{\vec{E}}{T} - \sigma\alpha\nabla T^2\frac{\nabla T}{T^2} \tag{H.7}$$

$$\mathbf{q}-\phi\vec{J} = \beta T\frac{\vec{E}}{T} - \gamma T^2\frac{\nabla T}{T^2} \tag{H.8}$$

the coefficients $\sigma\alpha T^2$ and βT must be equal. Thus $\beta = \sigma\alpha T$, so that $\mathbf{q}-\phi\vec{J} = \sigma\alpha T\vec{E} - \gamma\nabla T$. Finally, expressing $\vec{E}$ in terms of $\vec{J}$ and ∇T by (H.3), we have the result

$$\mathbf{q} = (\phi + \alpha T)\vec{J} - \kappa \nabla T \tag{H.9}$$

where $\kappa = \gamma - T\alpha^2\sigma$ is simply the ordinary thermal conductivity, which gives the heat flux in the absence of an electric current. It should be pointed out that the condition that ds/dt should be positive places no new restriction on the thermoelectric coefficients. Substituting (H.3) and (H.9) in (H.6) gives:

$$\frac{ds}{dt} = \int \left(\frac{J^2}{\sigma T} + \frac{\kappa(\nabla T)^2}{T^2} \right) dV > 0 \tag{H.10}$$

whence the coefficients of the thermal and electrical conductivity must be positive. The rate of change of entropy along the axial direction is obtained as:

$$\frac{ds_{zz}}{dt} = \int \left(\frac{J_{zz}^2}{\sigma_{zz} T} + \frac{\kappa_{zz}(\nabla T)^2}{T^2} \right) dV > 0 \tag{H.11}$$

Similarly, i the circumferential direction

$$\frac{ds_{cz}}{dt} = \int \left(\frac{J_{cz}^2}{\sigma_{cz} T} + \frac{\kappa_{cz}(\nabla T)^2}{T^2} \right) dV > 0 \tag{H.12}$$

Let us consider the amount of heat $-div\mathbf{q}$ evolved per unit time and volume in the conductor. Taking the divergence of (H.6), we have

$$Q = -div\mathbf{q} = \nabla(\kappa \nabla T) + \vec{J} \cdot \vec{E} + \vec{J} \cdot \nabla(\alpha T) \tag{H.13}$$

or, substituting (H.3),

$$Q = \nabla(\kappa \nabla T) + \frac{J^2}{\sigma} - T\vec{J} \cdot \nabla \alpha \tag{H.14}$$

The first term on the right is the thermal conduction, and the second term is the Joule heat. The term of interest here is the third, which gives the thermoelectric effects. Thus, the workone done by the electric field on the axial component of the current density is given as:

$$Q_{zz} = \nabla(\kappa_{zz} \nabla T) + \frac{J_{zz}^2}{\sigma_{zz}} - T\vec{J}_{zz} \cdot \nabla \alpha_{zz} \tag{H.15}$$

$$Q_{zz} = \vec{\nabla} \left(-\frac{k_B^2 T}{e^2} \left\{ \sigma_z(\vec{E}) \left[\left(\frac{\varepsilon_o - \mu}{k_B T} \right)^2 \sum_{n=-\infty}^{\infty} J_n^2(\chi) \right. \right. \right.$$
$$- \frac{\Delta_z^*}{2} \left(\frac{\varepsilon_o - \mu}{k_B T} \right) \left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)} - \frac{2}{\Delta_z^*} \right) \left(1 + 3 \sum_{n=-\infty}^{\infty} J_n^2(\chi) \right)$$
$$- 2\Delta_s^* \left(\frac{\varepsilon_o - \mu}{k_B T} \right) \frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)} \sum_{n=-\infty}^{\infty} J_n^2(\chi)$$
$$+ \frac{\Delta_z^{*2}}{2} \left(1 - \frac{3I_o(\Delta_z^*)}{\Delta_z^* I_1(\Delta_z^*)} + \frac{6}{\Delta_z^{*2}} \right) \left(1 + \sum_{n=-\infty}^{\infty} J_n^2(\chi) \right)$$

$$
\begin{aligned}
&+ \frac{\Delta_s^*\Delta_z^*}{2}\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)\\
&+ \Delta_s^{*2}\left(1-\frac{I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)}\right)\sum_{n=-\infty}^{\infty}J_n^2(\chi)\Bigg]\\
&+ \sigma_s(\vec{E})\sin^2\theta_h\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right.\\
&- \frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)\\
&- 2\Delta_z^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty}^{\infty}J_n^2(\chi)\\
&+ \frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)\\
&+ \frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)\\
&+ \Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right)\sum_{n=-\infty}^{\infty}J_n^2(\chi)\Bigg]\Bigg\}\vec{\nabla}T\Bigg)+\frac{j_{zz}^2}{\sigma_{zz}}\\
&+T\vec{J}_{zz}\cdot\vec{\nabla}\left(\frac{\sigma_z(\vec{E})}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_z^*\frac{I_o(\Delta_z^*)}{I_1(\Delta_z)}+2-\Delta_s^*\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right]\right.\\
&+\left.\frac{\sigma_s(\vec{E})\sin^2\theta_h}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z)}{I_o(\Delta_z^*)}\right]\right)
\end{aligned}
\tag{H.16}
$$

$$
\begin{aligned}
Q_{zz}=\vec{\nabla}\Bigg(-\frac{k_B^2T}{e^2}\Bigg\{&\sigma_z(\vec{E})\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right.\\
&- \frac{\Delta_z^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)\\
&- 2\Delta_s^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\sum_{n=-\infty}^{\infty}J_n^2(\chi)\\
&+ \frac{\Delta_z^{*2}}{2}\left(1-\frac{3I_o(\Delta_z^*)}{\Delta_z^* I_1(\Delta_z^*)}+\frac{6}{\Delta_z^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)\\
&+ \frac{\Delta_s^*\Delta_z^*}{2}\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)\\
&+ \Delta_s^{*2}\left(1-\frac{I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)}\right)\sum_{n=-\infty}^{\infty}J_n^2(\chi)\Bigg]\\
&+ \sigma_s(\vec{E})\sin^2\theta_h\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right.
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)\\
&- 2\Delta_z^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty}^{\infty}J_n^2(\chi)\\
&+ \frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^*I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)\\
&+ \frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)\\
&+ \Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^*I_o(\Delta_z^*)}\right)\sum_{n=-\infty}^{\infty}J_n^2(\chi)\Bigg]\Bigg\}\vec{\nabla}T\Bigg)+\frac{j_{zz}^2}{\sigma_{zz}}\\
&+T\vec{J}_{zz}\cdot\vec{\nabla}\Bigg(\frac{\sigma_z(\vec{E})}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_z^*\frac{I_o(\Delta_z^*)}{I_1(\Delta_z)}+2-\Delta_s^*\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right]\\
&+\frac{\sigma_s(\vec{E})\sin^2\theta_h}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z)}{I_o(\Delta_z^*)}\right]\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
Q_{zz}=\vec{\nabla}\Bigg(&-\frac{k_B^2T}{e^2}\Bigg\{\sigma_z(\vec{E})\Bigg[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty}^{\infty}J_n^2(\chi)\\
&- \frac{\Delta_z^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)\\
&- 2\Delta_s^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\sum_{n=-\infty}^{\infty}J_n^2(\chi)\\
&+ \frac{\Delta_z^{*2}}{2}\left(1-\frac{3I_o(\Delta_z^*)}{\Delta_z^*I_1(\Delta_z^*)}+\frac{6}{\Delta_z^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)\\
&+ \frac{\Delta_s^*\Delta_z^*}{2}\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)\\
&+ \Delta_s^{*2}\left(1-\frac{I_1(\Delta_s^*)}{\Delta_s^*I_o(\Delta_s^*)}\right)\sum_{n=-\infty}^{\infty}J_n^2(\chi)\Bigg]\\
&+ \sigma_s(\vec{E})\sin^2\theta_h\Bigg[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty}^{\infty}J_n^2(\chi)\\
&- \frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)\\
&- 2\Delta_z^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty}^{\infty}J_n^2(\chi)\\
&+ \frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^*I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)
\end{aligned}
$$

$$+ \frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$

$$+ \Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right)\sum_{n=-\infty}^{\infty}J_n^2(\chi)\Bigg]\Bigg\}\vec{\nabla}T\Bigg)+\frac{j_{zz}^2}{\sigma_{zz}}$$

$$+T\vec{J}_{zz}\cdot\vec{\nabla}\left(\frac{\sigma_z(\vec{E})}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_z^*\frac{I_o(\Delta_z^*)}{I_1(\Delta_z)}+2-\Delta_s^*\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right]\right.$$

$$\left.+\frac{\sigma_s(\vec{E})\sin^2\theta_h}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z)}{I_o(\Delta_z^*)}\right]\right)$$

$$Q_{zz}=\vec{\nabla}\Bigg(-\frac{k_B^2T}{e^2}\Bigg\{\sigma_z(\vec{E})\Bigg[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty}^{\infty}J_n^2(\chi)$$

$$-\frac{\Delta_z^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$

$$-2\Delta_s^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\sum_{n=-\infty}^{\infty}J_n^2(\chi)$$

$$+\frac{\Delta_z^{*2}}{2}\left(1-\frac{3I_o(\Delta_z^*)}{\Delta_z^*I_1(\Delta_z^*)}+\frac{6}{\Delta_z^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$

$$+\frac{\Delta_s^*\Delta_z^*}{2}\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$

$$+\Delta_s^{*2}\left(1-\frac{I_1(\Delta_s^*)}{\Delta_s^*I_o(\Delta_s^*)}\right)\sum_{n=-\infty}^{\infty}J_n^2(\chi)\Bigg]$$

$$+\sigma_s(\vec{E})\sin^2\theta_h\Bigg[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty}^{\infty}J_n^2(\chi)$$

$$-\frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$

$$-2\Delta_z^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty}^{\infty}J_n^2(\chi)$$

$$+\frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^*I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$

$$+\frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$

$$+\Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^*I_o(\Delta_z^*)}\right)\sum_{n=-\infty}^{\infty}J_n^2(\chi)\Bigg]\Bigg\}\vec{\nabla}T\Bigg)+\frac{j_{zz}^2}{\sigma_{zz}}$$

$$+T\vec{J}_{zz}\cdot\vec{\nabla}\left(\frac{\sigma_z(\vec{E})}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_z^*\frac{I_o(\Delta_z^*)}{I_1(\Delta_z)}+2-\Delta_s^*\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right]\right.$$

$$+\frac{\sigma_s(\vec{E})\sin^2\theta_h}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z)}{I_o(\Delta_z^*)}\right]\Bigg)$$

$$Q_{zz}=\vec{\nabla}\Bigg(-\frac{k_B^2T}{e^2}\Bigg\{\sigma_z(\vec{E})\Bigg[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty}^{\infty}J_n^2(\chi)$$
$$-\frac{\Delta_z^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$
$$-2\Delta_s^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\sum_{n=-\infty}^{\infty}J_n^2(\chi)$$
$$+\frac{\Delta_z^{*2}}{2}\left(1-\frac{3I_o(\Delta_z^*)}{\Delta_z^*I_1(\Delta_z^*)}+\frac{6}{\Delta_z^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$
$$+\frac{\Delta_s^*\Delta_z^*}{2}\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$
$$+\Delta_s^{*2}\left(1-\frac{I_1(\Delta_s^*)}{\Delta_s^*I_o(\Delta_s^*)}\right)\sum_{n=-\infty}^{\infty}J_n^2(\chi)\Bigg]$$
$$+\sigma_s(\vec{E})\sin^2\theta_h\Bigg[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty}^{\infty}J_n^2(\chi)$$
$$-\frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$
$$-2\Delta_z^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty}^{\infty}J_n^2(\chi)$$
$$+\frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^*I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$
$$+\frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$
$$+\Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^*I_o(\Delta_z^*)}\right)\sum_{n=-\infty}^{\infty}J_n^2(\chi)\Bigg]\Bigg\}\vec{\nabla}T\Bigg)+\frac{j_{zz}^2}{\sigma_{zz}}$$
$$+T\vec{J}_{zz}\cdot\vec{\nabla}\Bigg(\frac{\sigma_z(\vec{E})}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_z^*\frac{I_o(\Delta_z^*)}{I_1(\Delta_z)}+2-\Delta_s^*\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right]$$
$$+\frac{\sigma_s(\vec{E})\sin^2\theta_h}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z)}{I_o(\Delta_z^*)}\right]\Bigg)$$

$$Q_{zz}=\vec{\nabla}\Bigg(-\frac{k_B^2T}{e^2}\Bigg\{\sigma_z(\vec{E})\Bigg[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty}^{\infty}J_n^2(\chi)$$

$$-\frac{\Delta_z^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$
$$-2\Delta_s^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\sum_{n=-\infty}^{\infty}J_n^2(\chi)$$
$$+\frac{\Delta_z^{*2}}{2}\left(1-\frac{3I_o(\Delta_z^*)}{\Delta_z^*I_1(\Delta_z^*)}+\frac{6}{\Delta_z^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$
$$+\frac{\Delta_s^*\Delta_z^*}{2}\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$
$$+\Delta_s^{*2}\left(1-\frac{I_1(\Delta_s^*)}{\Delta_s^*I_o(\Delta_s^*)}\right)\sum_{n=-\infty}^{\infty}J_n^2(\chi)\Bigg]$$
$$+\sigma_s(\vec{E})\sin^2\theta_h\Bigg[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty}^{\infty}J_n^2(\chi)$$
$$-\frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$
$$-2\Delta_z^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty}^{\infty}J_n^2(\chi)$$
$$+\frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^*I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$
$$+\frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$
$$+\Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^*I_o(\Delta_z^*)}\right)\sum_{n=-\infty}^{\infty}J_n^2(\chi)\Bigg]\Bigg\}\vec{\nabla}T\Bigg)+\frac{j_{zz}^2}{\sigma_{zz}}$$
$$+T\vec{J}_{zz}\cdot\vec{\nabla}\Bigg(\frac{\sigma_z(\vec{E})}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_z^*\frac{I_o(\Delta_z^*)}{I_1(\Delta_z)}+2-\Delta_s^*\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right]$$
$$+\frac{\sigma_s(\vec{E})\sin^2\theta_h}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z)}{I_o(\Delta_z^*)}\right]\Bigg)$$

Similarly, the workdone in the circumferential direction is also given as:

$$Q_{cz}=\vec{\nabla}(\vec{\kappa}_{cz}\vec{\nabla}T)+\frac{J_{cz}^2}{\sigma_{cz}}-T\vec{J}_{cz}\cdot\vec{\nabla}\alpha_{cz} \tag{H.17}$$

$$Q_{cz}=\vec{\nabla}\Bigg(-\sigma_s(\vec{E})\frac{k_B^2T}{e^2}\sin\theta_h\cos\theta_h\Bigg\{\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty}^{\infty}J_n^2(\chi)$$
$$-\frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$

$$
\begin{aligned}
&- 2\Delta_z^* \left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty}^{\infty}J_n^2(\chi) \\
&+ \frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right) \\
&+ \frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right) \\
&+ \Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right)\sum_{n=-\infty}^{\infty}J_n^2(\chi)\Bigg\}\vec{\nabla}T\Bigg) + \frac{J_{cz}^2}{\sigma_{cz}} \\
&- T\vec{J}_{cz}\cdot\vec{\nabla}\left(-\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z)}{I_o(\Delta_z^*)}\right]\right)
\end{aligned}
\tag{H.18}
$$

$$
\begin{aligned}
Q_{cz} = \vec{\nabla}\Bigg(&-\sigma_s(\vec{E})\frac{k_B^2T}{e^2}\sin\theta_h\cos\theta_h\left\{\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right. \\
&- \frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right) \\
&- 2\Delta_z^* \left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty}^{\infty}J_n^2(\chi) \\
&+ \frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right) \\
&+ \frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right) \\
&+ \Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right)\sum_{n=-\infty}^{\infty}J_n^2(\chi)\Bigg\}\vec{\nabla}T\Bigg) + \frac{J_{cz}^2}{\sigma_{cz}} \\
&- T\vec{J}_{cz}\cdot\vec{\nabla}\left(-\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z)}{I_o(\Delta_z^*)}\right]\right)
\end{aligned}
\tag{H.19}
$$

$$
\begin{aligned}
Q_{cz} = \vec{\nabla}\Bigg(&-\sigma_s(\vec{E})\frac{k_B^2T}{e^2}\sin\theta_h\cos\theta_h\left\{\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right. \\
&- \frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right) \\
&- 2\Delta_z^* \left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty}^{\infty}J_n^2(\chi) \\
&+ \frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right) \\
&+ \frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)
\end{aligned}
$$

$$+ \Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right)\sum_{n=-\infty}^{\infty} J_n^2(\chi)\Bigg\}\vec{\nabla}T\Bigg) + \frac{J_{cz}^2}{\sigma_{cz}}$$

$$- T\vec{J}_{cz}\cdot\vec{\nabla}\left(-\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right) - \Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} + 2 - \Delta_z^*\frac{I_1(\Delta_z)}{I_o(\Delta_z^*)}\right]\right) \quad \text{(H.20)}$$

$$Q_{cz} = \vec{\nabla}\Bigg(-\sigma_s(\vec{E})\frac{k_B^2 T}{e^2}\sin\theta_h\cos\theta_h\Bigg\{\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty}^{\infty} J_n^2(\chi)$$

$$- \frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right)$$

$$- 2\Delta_z^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty}^{\infty} J_n^2(\chi)$$

$$+ \frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)} + \frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right)$$

$$+ \frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right)$$

$$+ \Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right)\sum_{n=-\infty}^{\infty} J_n^2(\chi)\Bigg\}\vec{\nabla}T\Bigg) + \frac{J_{cz}^2}{\sigma_{cz}}$$

$$- T\vec{J}_{cz}\cdot\vec{\nabla}\left(-\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right) - \Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} + 2 - \Delta_z^*\frac{I_1(\Delta_z)}{I_o(\Delta_z^*)}\right]\right) \quad \text{(H.21)}$$

$$Q_{cz} = \vec{\nabla}\Bigg(-\sigma_s(\vec{E})\frac{k_B^2 T}{e^2}\sin\theta_h\cos\theta_h\Bigg\{\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty}^{\infty} J_n^2(\chi)$$

$$- \frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right)$$

$$- 2\Delta_z^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty}^{\infty} J_n^2(\chi)$$

$$+ \frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)} + \frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right)$$

$$+ \frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right)$$

$$+ \Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right)\sum_{n=-\infty}^{\infty} J_n^2(\chi)\Bigg\}\vec{\nabla}T\Bigg) + \frac{J_{cz}^2}{\sigma_{cz}}$$

$$- T\vec{J}_{cz}\cdot\vec{\nabla}\left(-\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right) - \Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} + 2 - \Delta_z^*\frac{I_1(\Delta_z)}{I_o(\Delta_z^*)}\right]\right) \quad \text{(H.22)}$$

$$
\begin{aligned}
Q_{cz} = \vec{\nabla} \Bigg(& -\sigma_s(\vec{E}) \frac{k_B^2 T}{e^2} \sin\theta_h \cos\theta_h \Bigg\{ \left(\frac{\varepsilon_o - \mu}{k_B T} \right)^2 \sum_{n=-\infty}^{\infty} J_n^2(\chi) \\
& - \frac{\Delta_s^*}{2} \left(\frac{\varepsilon_o - \mu}{k_B T} \right) \left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*} \right) \left(1 + 3 \sum_{n=-\infty}^{\infty} J_n^2(\chi) \right) \\
& - 2\Delta_z^* \left(\frac{\varepsilon_o - \mu}{k_B T} \right) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \sum_{n=-\infty}^{\infty} J_n^2(\chi) \\
& + \frac{\Delta_s^{*2}}{2} \left(1 - \frac{3 I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)} + \frac{6}{\Delta_s^{*2}} \right) \left(1 + \sum_{n=-\infty}^{\infty} J_n^2(\chi) \right) \\
& + \frac{\Delta_z^* \Delta_s^*}{2} \left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*} \right) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \left(1 + 3 \sum_{n=-\infty}^{\infty} J_n^2(\chi) \right) \\
& + \Delta_z^{*2} \left(1 - \frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)} \right) \sum_{n=-\infty}^{\infty} J_n^2(\chi) \Bigg\} \vec{\nabla} T \Bigg) + \frac{J_{cz}^2}{\sigma_{cz}} \\
& - T \vec{J}_{cz} \cdot \vec{\nabla} \left(-\frac{k_B}{e} \left[\left(\frac{\varepsilon_o - \mu}{k_B T} \right) - \Delta_s^* \frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} + 2 - \Delta_z^* \frac{I_1(\Delta_z)}{I_o(\Delta_z^*)} \right] \right)
\end{aligned}
\tag{H.23}
$$

$$
\begin{aligned}
Q_{cz} = \vec{\nabla} \Bigg(& -\sigma_s(\vec{E}) \frac{k_B^2 T}{e^2} \sin\theta_h \cos\theta_h \Bigg\{ \left(\frac{\varepsilon_o - \mu}{k_B T} \right)^2 \sum_{n=-\infty}^{\infty} J_n^2(\chi) \\
& - \frac{\Delta_s^*}{2} \left(\frac{\varepsilon_o - \mu}{k_B T} \right) \left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*} \right) \left(1 + 3 \sum_{n=-\infty}^{\infty} J_n^2(\chi) \right) \\
& - 2\Delta_z^* \left(\frac{\varepsilon_o - \mu}{k_B T} \right) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \sum_{n=-\infty}^{\infty} J_n^2(\chi) \\
& + \frac{\Delta_s^{*2}}{2} \left(1 - \frac{3 I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)} + \frac{6}{\Delta_s^{*2}} \right) \left(1 + \sum_{n=-\infty}^{\infty} J_n^2(\chi) \right) \\
& + \frac{\Delta_z^* \Delta_s^*}{2} \left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*} \right) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \left(1 + 3 \sum_{n=-\infty}^{\infty} J_n^2(\chi) \right) \\
& + \Delta_z^{*2} \left(1 - \frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)} \right) \sum_{n=-\infty}^{\infty} J_n^2(\chi) \Bigg\} \vec{\nabla} T \Bigg) + \frac{J_{cz}^2}{\sigma_{cz}} \\
& - T \vec{J}_{cz} \cdot \vec{\nabla} \left(-\frac{k_B}{e} \left[\left(\frac{\varepsilon_o - \mu}{k_B T} \right) - \Delta_s^* \frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} + 2 - \Delta_z^* \frac{I_1(\Delta_z)}{I_o(\Delta_z^*)} \right] \right)
\end{aligned}
\tag{H.24}
$$

$$
\begin{aligned}
Q_{cz} = \vec{\nabla} \Bigg(& -\sigma_s(\vec{E}) \frac{k_B^2 T}{e^2} \sin\theta_h \cos\theta_h \Bigg\{ \left(\frac{\varepsilon_o - \mu}{k_B T} \right)^2 \sum_{n=-\infty}^{\infty} J_n^2(\chi) \\
& - \frac{\Delta_s^*}{2} \left(\frac{\varepsilon_o - \mu}{k_B T} \right) \left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*} \right) \left(1 + 3 \sum_{n=-\infty}^{\infty} J_n^2(\chi) \right) \\
& - 2\Delta_z^* \left(\frac{\varepsilon_o - \mu}{k_B T} \right) \frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)} \sum_{n=-\infty}^{\infty} J_n^2(\chi)
\end{aligned}
$$

$$+\frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$
$$+\frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$
$$+\Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right)\sum_{n=-\infty}^{\infty}J_n^2(\chi)\Bigg\}\vec{\nabla}T\Bigg)+\frac{J_{cz}^2}{\sigma_{cz}}$$
$$-T\vec{J}_{cz}\cdot\vec{\nabla}\left(-\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z)}{I_o(\Delta_z^*)}\right]\right) \quad \text{(H.25)}$$

$$Q_{zz}=\nabla(\kappa_{zz}\nabla T)+\frac{J_{zz}^2}{\sigma_{zz}}-T\vec{J}_{zz}\cdot\nabla\alpha_{zz} \quad \text{(H.26)}$$

$$Q_{zz}=\vec{\nabla}\Bigg(-\frac{k_B^2T}{e^2}\Bigg\{\sigma_z(\vec{E})\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right.$$
$$-\frac{\Delta_z^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$
$$-2\Delta_s^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\sum_{n=-\infty}^{\infty}J_n^2(\chi)$$
$$+\frac{\Delta_z^{*2}}{2}\left(1-\frac{3I_o(\Delta_z^*)}{\Delta_z^* I_1(\Delta_z^*)}+\frac{6}{\Delta_z^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$
$$+\frac{\Delta_s^*\Delta_z^*}{2}\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$
$$\left.+\Delta_s^{*2}\left(1-\frac{I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)}\right)\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right]$$
$$+\sigma_s(\vec{E})\sin^2\theta_h\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right.$$
$$-\frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$
$$-2\Delta_z^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty}^{\infty}J_n^2(\chi)$$
$$+\frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$
$$+\frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$
$$\left.+\Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right)\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right]\Bigg\}\vec{\nabla}T\Bigg)+\frac{j_{zz}^2}{\sigma_{zz}}$$

$$+T\vec{J}_{zz}\cdot\vec{\nabla}\left(\frac{\sigma_z(\vec{E})}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_z^*\frac{I_o(\Delta_z^*)}{I_1(\Delta_z)}+2-\Delta_s^*\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right]\right.$$
$$\left.+\frac{\sigma_s(\vec{E})\sin^2\theta_h}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z)}{I_o(\Delta_z^*)}\right]\right) \tag{H.27}$$

$$Q_{zz}=\vec{\nabla}\left(-\frac{k_B^2T}{e^2}\left\{\sigma_z(\vec{E})\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right.\right.\right.$$
$$-\frac{\Delta_z^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$
$$-2\Delta_s^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\sum_{n=-\infty}^{\infty}J_n^2(\chi)$$
$$+\frac{\Delta_z^{*2}}{2}\left(1-\frac{3I_o(\Delta_z^*)}{\Delta_z^*I_1(\Delta_z^*)}+\frac{6}{\Delta_z^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$
$$+\frac{\Delta_s^*\Delta_z^*}{2}\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$
$$\left.+\Delta_s^{*2}\left(1-\frac{I_1(\Delta_s^*)}{\Delta_s^*I_o(\Delta_s^*)}\right)\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right]$$
$$+\sigma_s(\vec{E})\sin^2\theta_h\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right.$$
$$-\frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$
$$-2\Delta_z^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty}^{\infty}J_n^2(\chi)$$
$$+\frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^*I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$
$$+\frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$
$$\left.\left.\left.+\Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^*I_o(\Delta_z^*)}\right)\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right]\right\}\vec{\nabla}T\right)+\frac{j_{zz}^2}{\sigma_{zz}}$$
$$+T\vec{J}_{zz}\cdot\vec{\nabla}\left(\frac{\sigma_z(\vec{E})}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_z^*\frac{I_o(\Delta_z^*)}{I_1(\Delta_z)}+2-\Delta_s^*\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right]\right.$$
$$\left.+\frac{\sigma_s(\vec{E})\sin^2\theta_h}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z)}{I_o(\Delta_z^*)}\right]\right)$$

$$Q_{zz} = \vec{\nabla}\left(-\frac{k_B^2 T}{e^2}\left\{\sigma_z(\vec{E})\left[\left(\frac{\varepsilon_o-\mu}{k_B T}\right)^2 \sum_{n=-\infty}^{\infty} J_n^2(\chi)\right.\right.\right.$$
$$-\frac{\Delta_z^*}{2}\left(\frac{\varepsilon_o-\mu}{k_B T}\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\left(1+3\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right)$$
$$-2\Delta_s^*\left(\frac{\varepsilon_o-\mu}{k_B T}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\sum_{n=-\infty}^{\infty} J_n^2(\chi)$$
$$+\frac{\Delta_z^{*2}}{2}\left(1-\frac{3I_o(\Delta_z^*)}{\Delta_z^* I_1(\Delta_z^*)}+\frac{6}{\Delta_z^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right)$$
$$+\frac{\Delta_s^*\Delta_z^*}{2}\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\left(1+3\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right)$$
$$\left.+\Delta_s^{*2}\left(1-\frac{I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)}\right)\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right]$$
$$+\sigma_s(\vec{E})\sin^2\theta_h\left[\left(\frac{\varepsilon_o-\mu}{k_B T}\right)^2 \sum_{n=-\infty}^{\infty} J_n^2(\chi)\right.$$
$$-\frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o-\mu}{k_B T}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right)$$
$$-2\Delta_z^*\left(\frac{\varepsilon_o-\mu}{k_B T}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty}^{\infty} J_n^2(\chi)$$
$$+\frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right)$$
$$+\frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right)$$
$$\left.\left.\left.+\Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right)\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right]\right\}\vec{\nabla}T\right)+\frac{j_{zz}^2}{\sigma_{zz}}$$
$$+T\vec{J}_{zz}\cdot\vec{\nabla}\left(\frac{\sigma_z(\vec{E})}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_B T}\right)-\Delta_z^*\frac{I_o(\Delta_z^*)}{I_1(\Delta_z)}+2-\Delta_s^*\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right]\right.$$
$$\left.+\frac{\sigma_s(\vec{E})\sin^2\theta_h}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_B T}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z)}{I_o(\Delta_z^*)}\right]\right)$$

$$Q_{zz} = \vec{\nabla}\left(-\frac{k_B^2 T}{e^2}\left\{\sigma_z(\vec{E})\left[\left(\frac{\varepsilon_o-\mu}{k_B T}\right)^2 \sum_{n=-\infty}^{\infty} J_n^2(\chi)\right.\right.\right.$$
$$-\frac{\Delta_z^*}{2}\left(\frac{\varepsilon_o-\mu}{k_B T}\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\left(1+3\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right)$$
$$-2\Delta_s^*\left(\frac{\varepsilon_o-\mu}{k_B T}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\sum_{n=-\infty}^{\infty} J_n^2(\chi)$$

$$
\begin{aligned}
&+ \frac{\Delta_z^{*2}}{2}\left(1-\frac{3I_o(\Delta_z^*)}{\Delta_z^* I_1(\Delta_z^*)}+\frac{6}{\Delta_z^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right) \\
&+ \frac{\Delta_s^*\Delta_z^*}{2}\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right) \\
&+ \Delta_s^{*2}\left(1-\frac{I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)}\right)\sum_{n=-\infty}^{\infty}J_n^2(\chi)\Bigg] \\
&+ \sigma_s(\vec{E})\sin^2\theta_h\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right. \\
&- \frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right) \\
&- 2\Delta_z^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty}^{\infty}J_n^2(\chi) \\
&+ \frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right) \\
&+ \frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right) \\
&+ \left.\Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right)\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right]\Bigg\}\vec{\nabla}T\Bigg)+\frac{j_{zz}^2}{\sigma_{zz}} \\
&+T\vec{J}_{zz}\cdot\vec{\nabla}\left(\frac{\sigma_z(\vec{E})}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_z^*\frac{I_o(\Delta_z^*)}{I_1(\Delta_z)}+2-\Delta_s^*\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right]\right. \\
&+\left.\frac{\sigma_s(\vec{E})\sin^2\theta_h}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z)}{I_o(\Delta_z^*)}\right]\right)
\end{aligned}
$$

$$
\begin{aligned}
Q_{zz} = \vec{\nabla}\Bigg(-\frac{k_B^2T}{e^2}\Bigg\{&\sigma_z(\vec{E})\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right. \\
&- \frac{\Delta_z^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right) \\
&- 2\Delta_s^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\sum_{n=-\infty}^{\infty}J_n^2(\chi) \\
&+ \frac{\Delta_z^{*2}}{2}\left(1-\frac{3I_o(\Delta_z^*)}{\Delta_z^* I_1(\Delta_z^*)}+\frac{6}{\Delta_z^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right) \\
&+ \frac{\Delta_s^*\Delta_z^*}{2}\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)}-\frac{2}{\Delta_z^*}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right) \\
&+ \left.\Delta_s^{*2}\left(1-\frac{I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)}\right)\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right]
\end{aligned}
$$

$$+ \sigma_s(\vec{E}) \sin^2\theta_h \Bigg[\left(\frac{\varepsilon_o - \mu}{k_B T}\right)^2 \sum_{n=-\infty}^{\infty} J_n^2(\chi)$$

$$- \frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o - \mu}{k_B T}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*}\right)\left(1 + 3\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right)$$

$$- 2\Delta_z^*\left(\frac{\varepsilon_o - \mu}{k_B T}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty}^{\infty} J_n^2(\chi)$$

$$+ \frac{\Delta_s^{*2}}{2}\left(1 - \frac{3I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)} + \frac{6}{\Delta_s^{*2}}\right)\left(1 + \sum_{n=-\infty}^{\infty} J_n^2(\chi)\right)$$

$$+ \frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1 + 3\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right)$$

$$+ \Delta_z^{*2}\left(1 - \frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right)\sum_{n=-\infty}^{\infty} J_n^2(\chi)\Bigg]\Bigg\}\vec{\nabla}T\Bigg) + \frac{j_{zz}^2}{\sigma_{zz}}$$

$$+ T\vec{J}_{zz}\cdot\vec{\nabla}\Bigg(\frac{\sigma_z(\vec{E})}{(\sigma_z(\vec{E}) + \sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o - \mu}{k_B T}\right) - \Delta_z^*\frac{I_o(\Delta_z^*)}{I_1(\Delta_z)} + 2 - \Delta_s^*\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right]$$

$$+ \frac{\sigma_s(\vec{E})\sin^2\theta_h}{(\sigma_z(\vec{E}) + \sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o - \mu}{k_B T}\right) - \Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} + 2 - \Delta_z^*\frac{I_1(\Delta_z)}{I_o(\Delta_z^*)}\right]\Bigg)$$

$$Q_{zz} = \vec{\nabla}\Bigg(-\frac{k_B^2 T}{e^2}\Bigg\{\sigma_z(\vec{E})\Bigg[\left(\frac{\varepsilon_o - \mu}{k_B T}\right)^2 \sum_{n=-\infty}^{\infty} J_n^2(\chi)$$

$$- \frac{\Delta_z^*}{2}\left(\frac{\varepsilon_o - \mu}{k_B T}\right)\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)} - \frac{2}{\Delta_z^*}\right)\left(1 + 3\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right)$$

$$- 2\Delta_s^*\left(\frac{\varepsilon_o - \mu}{k_B T}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\sum_{n=-\infty}^{\infty} J_n^2(\chi)$$

$$+ \frac{\Delta_z^{*2}}{2}\left(1 - \frac{3I_o(\Delta_z^*)}{\Delta_z^* I_1(\Delta_z^*)} + \frac{6}{\Delta_z^{*2}}\right)\left(1 + \sum_{n=-\infty}^{\infty} J_n^2(\chi)\right)$$

$$+ \frac{\Delta_s^*\Delta_z^*}{2}\left(\frac{I_o(\Delta_z^*)}{I_1(\Delta_z^*)} - \frac{2}{\Delta_z^*}\right)\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\left(1 + 3\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right)$$

$$+ \Delta_s^{*2}\left(1 - \frac{I_1(\Delta_s^*)}{\Delta_s^* I_o(\Delta_s^*)}\right)\sum_{n=-\infty}^{\infty} J_n^2(\chi)\Bigg]$$

$$+ \sigma_s(\vec{E}) \sin^2\theta_h \Bigg[\left(\frac{\varepsilon_o - \mu}{k_B T}\right)^2 \sum_{n=-\infty}^{\infty} J_n^2(\chi)$$

$$- \frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o - \mu}{k_B T}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)} - \frac{2}{\Delta_s^*}\right)\left(1 + 3\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right)$$

$$- 2\Delta_z^*\left(\frac{\varepsilon_o - \mu}{k_B T}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty}^{\infty} J_n^2(\chi)$$

$$+\frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right)$$
$$+\frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right)$$
$$+\Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right)\sum_{n=-\infty}^{\infty} J_n^2(\chi)\Bigg]\Bigg\}\vec{\nabla}T\Bigg)+\frac{j_{zz}^2}{\sigma_{zz}}$$
$$+T\vec{J}_{zz}\cdot\vec{\nabla}\Bigg(\frac{\sigma_z(\vec{E})}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_z^*\frac{I_o(\Delta_z^*)}{I_1(\Delta_z)}+2-\Delta_s^*\frac{I_1(\Delta_s^*)}{I_o(\Delta_s^*)}\right]$$
$$+\frac{\sigma_s(\vec{E})\sin^2\theta_h}{(\sigma_z(\vec{E})+\sigma_s(\vec{E})\sin^2\theta_h)}\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z)}{I_o(\Delta_z^*)}\right]\Bigg)$$

Similarly, the workdone in the circumferential direction is also given as:

$$Q_{cz}=\vec{\nabla}(\vec{\kappa}_{cz}\vec{\nabla}T)+\frac{J_{cz}^2}{\sigma_{cz}}-T\vec{J}_{cz}\cdot\vec{\nabla}\alpha_{cz} \tag{H.28}$$

$$Q_{cz}=\vec{\nabla}\Bigg(-\sigma_s(\vec{E})\frac{k_B^2T}{e^2}\sin\theta_h\cos\theta_h\Bigg\{\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty}^{\infty} J_n^2(\chi)$$
$$-\frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right)$$
$$-2\Delta_z^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty}^{\infty} J_n^2(\chi)$$
$$+\frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right)$$
$$+\frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right)$$
$$+\Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right)\sum_{n=-\infty}^{\infty} J_n^2(\chi)\Bigg\}\vec{\nabla}T\Bigg)+\frac{J_{cz}^2}{\sigma_{cz}}$$
$$-T\vec{J}_{cz}\cdot\vec{\nabla}\left(-\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z)}{I_o(\Delta_z^*)}\right]\right) \tag{H.29}$$

$$Q_{cz}=\vec{\nabla}\Bigg(-\sigma_s(\vec{E})\frac{k_B^2T}{e^2}\sin\theta_h\cos\theta_h\Bigg\{\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty}^{\infty} J_n^2(\chi)$$
$$-\frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty}^{\infty} J_n^2(\chi)\right)$$
$$-2\Delta_z^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty}^{\infty} J_n^2(\chi)$$

$$+\frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$
$$+\frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$
$$+\Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right)\sum_{n=-\infty}^{\infty}J_n^2(\chi)\Bigg\}\vec{\nabla}T\Bigg)+\frac{J_{cz}^2}{\sigma_{cz}}$$
$$-T\vec{J}_{cz}\cdot\vec{\nabla}\left(-\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z)}{I_o(\Delta_z^*)}\right]\right) \quad (H.30)$$

$$Q_{cz}=\vec{\nabla}\Bigg(-\sigma_s(\vec{E})\frac{k_B^2T}{e^2}\sin\theta_h\cos\theta_h\Bigg\{\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty}^{\infty}J_n^2(\chi)$$
$$-\frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$
$$-2\Delta_z^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty}^{\infty}J_n^2(\chi)$$
$$+\frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$
$$+\frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$
$$+\Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right)\sum_{n=-\infty}^{\infty}J_n^2(\chi)\Bigg\}\vec{\nabla}T\Bigg)+\frac{J_{cz}^2}{\sigma_{cz}}$$
$$-T\vec{J}_{cz}\cdot\vec{\nabla}\left(-\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z)}{I_o(\Delta_z^*)}\right]\right) \quad (H.31)$$

$$Q_{cz}=\vec{\nabla}\Bigg(-\sigma_s(\vec{E})\frac{k_B^2T}{e^2}\sin\theta_h\cos\theta_h\Bigg\{\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty}^{\infty}J_n^2(\chi)$$
$$-\frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$
$$-2\Delta_z^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty}^{\infty}J_n^2(\chi)$$
$$+\frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^* I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$
$$+\frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)$$
$$+\Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^* I_o(\Delta_z^*)}\right)\sum_{n=-\infty}^{\infty}J_n^2(\chi)\Bigg\}\vec{\nabla}T\Bigg)+\frac{J_{cz}^2}{\sigma_{cz}}$$

$$-T\vec{J}_{cz}\cdot\vec{\nabla}\left(-\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z)}{I_o(\Delta_z^*)}\right]\right) \tag{H.32}$$

$$\begin{aligned}Q_{cz}=\vec{\nabla}\Bigg(&-\sigma_s(\vec{E})\frac{k_B^2T}{e^2}\sin\theta_h\cos\theta_h\Bigg\{\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty}^{\infty}J_n^2(\chi)\\&-\frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)\\&-2\Delta_z^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty}^{\infty}J_n^2(\chi)\\&+\frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^*I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)\\&+\frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)\\&+\Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^*I_o(\Delta_z^*)}\right)\sum_{n=-\infty}^{\infty}J_n^2(\chi)\Bigg\}\vec{\nabla}T\Bigg)+\frac{J_{cz}^2}{\sigma_{cz}}\\&-T\vec{J}_{cz}\cdot\vec{\nabla}\left(-\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z)}{I_o(\Delta_z^*)}\right]\right)\end{aligned} \tag{H.33}$$

$$\begin{aligned}Q_{cz}=\vec{\nabla}\Bigg(&-\sigma_s(\vec{E})\frac{k_B^2T}{e^2}\sin\theta_h\cos\theta_h\Bigg\{\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty}^{\infty}J_n^2(\chi)\\&-\frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)\\&-2\Delta_z^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty}^{\infty}J_n^2(\chi)\\&+\frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^*I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)\\&+\frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)\\&+\Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^*I_o(\Delta_z^*)}\right)\sum_{n=-\infty}^{\infty}J_n^2(\chi)\Bigg\}\vec{\nabla}T\Bigg)+\frac{J_{cz}^2}{\sigma_{cz}}\\&-T\vec{J}_{cz}\cdot\vec{\nabla}\left(-\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z)}{I_o(\Delta_z^*)}\right]\right)\end{aligned} \tag{H.34}$$

$$Q_{cz}=\vec{\nabla}\Bigg(-\sigma_s(\vec{E})\frac{k_B^2T}{e^2}\sin\theta_h\cos\theta_h\Bigg\{\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty}^{\infty}J_n^2(\chi)$$

$$
\begin{aligned}
&- \frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)\\
&- 2\Delta_z^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty}^{\infty}J_n^2(\chi)\\
&+ \frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^*I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)\\
&+ \frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)\\
&+ \Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^*I_o(\Delta_z^*)}\right)\sum_{n=-\infty}^{\infty}J_n^2(\chi)\Bigg\}\vec{\nabla}T\Bigg)+\frac{J_{cz}^2}{\sigma_{cz}}\\
&- T\vec{J}_{cz}\cdot\vec{\nabla}\left(-\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z)}{I_o(\Delta_z^*)}\right]\right) \qquad \text{(H.35)}
\end{aligned}
$$

$$
\begin{aligned}
Q_{cz} = \vec{\nabla}\Bigg(&-\sigma_s(\vec{E})\frac{k_B^2T}{e^2}\sin\theta_h\cos\theta_h\Bigg\{\left(\frac{\varepsilon_o-\mu}{k_BT}\right)^2\sum_{n=-\infty}^{\infty}J_n^2(\chi)\\
&- \frac{\Delta_s^*}{2}\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)\\
&- 2\Delta_z^*\left(\frac{\varepsilon_o-\mu}{k_BT}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\sum_{n=-\infty}^{\infty}J_n^2(\chi)\\
&+ \frac{\Delta_s^{*2}}{2}\left(1-\frac{3I_o(\Delta_s^*)}{\Delta_s^*I_1(\Delta_s^*)}+\frac{6}{\Delta_s^{*2}}\right)\left(1+\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)\\
&+ \frac{\Delta_z^*\Delta_s^*}{2}\left(\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}-\frac{2}{\Delta_s^*}\right)\frac{I_1(\Delta_z^*)}{I_o(\Delta_z^*)}\left(1+3\sum_{n=-\infty}^{\infty}J_n^2(\chi)\right)\\
&+ \Delta_z^{*2}\left(1-\frac{I_1(\Delta_z^*)}{\Delta_z^*I_o(\Delta_z^*)}\right)\sum_{n=-\infty}^{\infty}J_n^2(\chi)\Bigg\}\vec{\nabla}T\Bigg)+\frac{J_{cz}^2}{\sigma_{cz}}\\
&- T\vec{J}_{cz}\cdot\vec{\nabla}\left(-\frac{k_B}{e}\left[\left(\frac{\varepsilon_o-\mu}{k_BT}\right)-\Delta_s^*\frac{I_o(\Delta_s^*)}{I_1(\Delta_s^*)}+2-\Delta_z^*\frac{I_1(\Delta_z)}{I_o(\Delta_z^*)}\right]\right) \qquad \text{(H.36)}
\end{aligned}
$$

APPENDIX I

PELTIER COEFFICIENT FOR FSWCNT

The analytical deductions to establish whether the FSWCNT obeys the Onsager relations is beyond the scope of this study. However, the Onsager relation is invoked to obtain the Peltier coeficient for FSWCNT for possible refridgeration applications. The electron current densities for the FSWCNT along the axial and circumferential directions are respectively, quoted as:

$$\vec{J}_c = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\left[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right.$$
$$-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_z\sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\left.\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right]$$
$$\times\left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right.$$
$$\left.-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right\}T \quad (I.1)$$

$$\vec{J}_z = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\left[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right.$$
$$-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$- \Delta_s \sin \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\times \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \cos \left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\times \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_z \sin \frac{w\vec{p}_z b_z}{\hbar} \sin \left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\times \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \Bigg]$$
$$\times \left\{ \sin \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \right.$$
$$\left. - \cos \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \right\} \quad \text{(I.2)}$$

In the compact form, this I.1 and I.2 takes the form;

$$\vec{J}_c = \sigma_{cz} \vec{E}_{zn}^* - \sigma_{cz} \alpha_{cz} \nabla_z T \quad \text{(I.3)}$$

$$\vec{J}_z = \sigma_{zz} \vec{E}_{zn}^* - \sigma_{zz} \alpha_{zz} \nabla_z T \quad \text{(I.4)}$$

Since the representation of **q** in terms of $\vec{E}_{zn}^*$ is not convenient when comparing theory with experiment, it becomes necessary to express **q** in termsof $\vec{J}$ and $\nabla_z T$. Hence, making the electric field the subject $\vec{E}_{zn}^*$ from the circumferential component in I.1 yields;

$$\vec{E}_{zn}^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2} \frac{u\Delta_s b_s}{\hbar} \frac{uwn_o b_s b_z}{2I_o(\Delta_s^*) I_o(\Delta_z^*)} \int_0^{-\infty} \exp\left(\frac{-t}{\tau} \right) dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu) \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \right.$$
$$- \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\times \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_s \sin \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\times \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$
$$- \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \cos \left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right)$$
$$\times \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right]$$

$$
\begin{aligned}
&- \Delta_z \sin \frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) \\
&\quad \times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg] \\
&\times \left\{ \sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right. \\
&\quad \left. - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \qquad (I.5)
\end{aligned}
$$

Substituting $\vec{E}_{zn}^*$ to $\mathbf{q}_c$ in G.186 yields;

$$
\begin{aligned}
\mathbf{q} =& \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\
&\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right. \\
&- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right) \\
&\quad \times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right) \\
&\quad \times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right) \\
&\quad \times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
&- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right) \\
&\quad \times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg] \\
&\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)\right. \\
&\quad \left. - \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)\right\} T \qquad (I.6)
\end{aligned}
$$

Simplifying yields:

$$
\begin{aligned}
\mathbf{q}_c ==& \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\
&\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right. \\
&- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s\cos\omega t']dt'\right)
\end{aligned}
$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right.$$
$$\left. - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} T \quad \text{(I.7)}$$

From I.2 $\vec{E}_{zn}^*$ is given as;

$$\vec{E}_{zn}^* = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right.$$
$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right.$$

$$- \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\Bigg\} \quad (I.8)$$

Substituting $\vec{E}^*_{zn}$ into $\mathbf{q}_z$ in G.183;

$$\mathbf{q}_z == \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\Bigg[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_z\sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times\Bigg\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\Bigg\} \quad (I.9)$$

Simplifying further yields:

$$\mathbf{q}_z = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\Bigg[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$- \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \sin \frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right.$$
$$\left. - \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \quad (I.10)$$

$$\mathbf{q}_z == \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right.$$
$$- \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_s \sin \frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \sin \frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right.$$
$$\left. - \cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \quad (I.11)$$

I.7 and I.11 are in the form of the Onsager relations quoted as:

$$\mathbf{q}_c = \Pi_{cz}\vec{J}_c - \kappa_{cz}\nabla_z T \quad (I.12)$$

$$\mathbf{q}_z = \Pi_{zz}\vec{J}_z - \kappa_{zz}\nabla_z T \quad (I.13)$$

where κ is the electron thermal conductivity when the carrier current density $\vec{J}$ is zero and Π is the Peltier coefficient is given as $\Pi = \alpha T$. As usual α is the thermopower. Comparing the equations, the circumferential component of the Peltier coefficient as:

$$\Pi_{cz} = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\left[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right.$$
$$-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_z\sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\left.\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right]$$
$$\times\left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right.$$
$$\left.-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right\} \quad (I.14)$$

and the axial component is obtained as;

$$\Pi_{zz} = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\left[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right.$$
$$-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$
\begin{aligned}
&-\Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&-\Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]\\
&\times\left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right.\\
&\left.-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right\} \qquad (I.15)
\end{aligned}
$$

Comparing I.12 and I.13, the thermal conductivity is obtained again as:

$$
\begin{aligned}
\kappa_{cz} &= \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\\
&\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\left[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right.\\
&-\Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&-\Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&-\Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&-\Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\left.\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right]\\
&\times\left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right.\\
&\left.-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right\} \qquad (I.16)
\end{aligned}
$$

$$
\kappa_{zz} = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt
$$

$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$

$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$-\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$

$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$

$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$-\Delta_z\sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$

$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$

$$\times\left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right.$$

$$\left.-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right\} \quad \text{(I.17)}$$

Transport of heat and charge is commonly estimated using the measured σ using the Wiedemann-Franz law $\kappa = T\sigma L$, where L is the Lorentz number. The Lorentz number along the circumferential direction L_{cz} is given as:

$$L_{cz} == \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$

$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$

$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$-\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$

$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$

$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$
\begin{aligned}
&- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]\\
&\times\Bigg\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\Bigg\} \qquad \text{(I.18)}
\end{aligned}
$$

and the axial Lorentz number is given as:

$$
\begin{aligned}
L_{zz} ==& \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\\
&\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\Bigg[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&-\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&-\Delta_z\sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&\times \exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]\\
&\times\Bigg\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\\
&-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\Bigg\} \qquad \text{(I.19)}
\end{aligned}
$$

$$
\begin{aligned}
L_{zz} =& \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt\\
&\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\Bigg[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\\
&-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)
\end{aligned}
$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right.$$
$$\left. - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \quad \text{(I.20)}$$

$$L_{zz} = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right.$$
$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times \left\{\sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right.$$

$$-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\Bigg\} \quad (I.21)$$

$$L_{zz}=\frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\left[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right.$$
$$-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_z\sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\left.\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right]$$
$$\times\Bigg\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\Bigg\} \quad (I.22)$$

$$L_{zz}=\frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\left[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right.$$
$$-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$
\begin{aligned}
& - \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \\
& \times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \\
& - \Delta_z \sin \frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \\
& \times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \Bigg] \\
& \times \left\{ \sin \frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \right. \\
& \left. - \cos \frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \right\} \quad \text{(I.23)}
\end{aligned}
$$

$$
\begin{aligned}
L_{zz} = & \frac{2\tau^{-1}}{(2\pi\hbar)^2} \frac{u\Delta_s b_s}{\hbar} \frac{uwn_o b_s b_z}{2I_o(\Delta_s^*) I_o(\Delta_z^*)} \int_0^{-\infty} \exp\left(\frac{-t}{\tau} \right) dt \\
& \int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \left[(\varepsilon_o - \mu) \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \right. \\
& - \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \\
& \times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \\
& - \Delta_s \sin \frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \\
& \times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \\
& - \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \\
& \times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \\
& - \Delta_z \sin \frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \\
& \left. \times \exp\left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \right] \\
& \times \left\{ \sin \frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \right. \\
& \left. - \cos \frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \right\} \quad \text{(I.24)}
\end{aligned}
$$

$$
L_{zz} = \frac{2\tau^{-1}}{(2\pi\hbar)^2} \frac{u\Delta_s b_s}{\hbar} \frac{uwn_o b_s b_z}{2I_o(\Delta_s^*) I_o(\Delta_z^*)} \int_0^{-\infty} \exp\left(\frac{-t}{\tau} \right) dt
$$

$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_z\sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times\left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right.$$
$$\left.-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right\} \quad \text{(I.25)}$$

$$L_{cz}=\frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$
$$\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$-\Delta_z\sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$

$$\times \left\{ \sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \quad (I.26)$$

and the axial Lorentz number is given as:

$$L_{zz} = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times \left\{ \sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right) - \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\right\} \quad (I.27)$$

$$L_{zz} = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$
\begin{aligned}
&- \Delta_s \sin \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \\
&\times \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \\
&- \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \cos \left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \\
&\times \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \\
&- \Delta_z \sin \frac{w\vec{p}_z b_z}{\hbar} \sin \left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \\
&\times \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \Bigg] \\
&\times \Bigg\{ \sin \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \\
&- \cos \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \Bigg\} \qquad (I.28)
\end{aligned}
$$

$$
\begin{aligned}
L_{zz} = &\frac{2\tau^{-1}}{(2\pi\hbar)^2} \frac{u\Delta_s b_s}{\hbar} \frac{uwn_o b_s b_z}{2I_o(\Delta_s^*) I_o(\Delta_z^*)} \int_0^{-\infty} \exp\left(\frac{-t}{\tau} \right) dt \\
&\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o - \mu) \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \\
&- \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \\
&\times \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \\
&- \Delta_s \sin \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \\
&\times \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \\
&- \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \cos \left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \\
&\times \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \\
&- \Delta_z \sin \frac{w\vec{p}_z b_z}{\hbar} \sin \left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \\
&\times \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \Bigg] \\
&\times \Bigg\{ \sin \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \\
&- \cos \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \Bigg\} \qquad (I.29)
\end{aligned}
$$

$$
\begin{aligned}
L_{zz} = & \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\
& \int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\left[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right. \\
& -\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
& \times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
& -\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
& \times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
& -\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
& \times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
& -\Delta_z\sin\frac{w\vec{p}_z b_z}{\hbar}\sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
& \left.\times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right] \\
& \times\left\{\sin\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right. \\
& \left.-\cos\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)\right\} \qquad (I.30)
\end{aligned}
$$

$$
\begin{aligned}
L_{zz} = & \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty}\exp\left(\frac{-t}{\tau}\right)dt \\
& \int_{-\pi/b_s}^{\pi/b_s}d\vec{p}_s\int_{-\pi/b_z}^{\pi/b_z}d\vec{p}_z\left[(\varepsilon_o-\mu)\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right]\right. \\
& -\Delta_s\cos\frac{u\vec{p}_s b_s}{\hbar}\cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
& \times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
& -\Delta_s\sin\frac{u\vec{p}_s b_s}{\hbar}\sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right) \\
& \times\exp\left[\Delta_s^*\cos\frac{u\vec{p}_s b_s}{\hbar}+\Delta_z^*\cos\frac{w\vec{p}_z b_z}{\hbar}\right] \\
& -\Delta_z\cos\frac{w\vec{p}_z b_z}{\hbar}\cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o+\vec{E}_s\cos\omega t']dt'\right)
\end{aligned}
$$

$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times \Bigg\{\sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$- \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\Bigg\} \quad (I.31)$$

$$L_{zz} = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_s \cos\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_s \sin\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \cos\frac{w\vec{p}_z b_z}{\hbar} \cos\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$
$$- \Delta_z \sin\frac{w\vec{p}_z b_z}{\hbar} \sin\left(\frac{web_z}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$\times \exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]\Bigg]$$
$$\times \Bigg\{\sin\frac{u\vec{p}_s b_s}{\hbar} \cos\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)$$
$$- \cos\frac{u\vec{p}_s b_s}{\hbar} \sin\left(\frac{ueb_s}{\hbar}\int_{t-t'}^{t}[\vec{E}_o + \vec{E}_s \cos\omega t']dt'\right)\Bigg\} \quad (I.32)$$

$$L_{zz} = \frac{2\tau^{-1}}{(2\pi\hbar)^2}\frac{u\Delta_s b_s}{\hbar}\frac{uwn_o b_s b_z}{2I_o(\Delta_s^*)I_o(\Delta_z^*)}\int_0^{-\infty} \exp\left(\frac{-t}{\tau}\right)dt$$
$$\int_{-\pi/b_s}^{\pi/b_s} d\vec{p}_s \int_{-\pi/b_z}^{\pi/b_z} d\vec{p}_z \Bigg[(\varepsilon_o - \mu)\exp\left[\Delta_s^* \cos\frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos\frac{w\vec{p}_z b_z}{\hbar}\right]$$

$$
\begin{aligned}
&- \Delta_s \cos \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \\
&\times \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \\
&- \Delta_s \sin \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \\
&\times \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \\
&- \Delta_z \cos \frac{w\vec{p}_z b_z}{\hbar} \cos \left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \\
&\times \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \\
&- \Delta_z \sin \frac{w\vec{p}_z b_z}{\hbar} \sin \left(\frac{web_z}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \\
&\times \exp \left[\Delta_s^* \cos \frac{u\vec{p}_s b_s}{\hbar} + \Delta_z^* \cos \frac{w\vec{p}_z b_z}{\hbar} \right] \Bigg] \\
&\times \Bigg\{ \sin \frac{u\vec{p}_s b_s}{\hbar} \cos \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \\
&- \cos \frac{u\vec{p}_s b_s}{\hbar} \sin \left(\frac{ueb_s}{\hbar} \int_{t-t'}^{t} [\vec{E}_o + \vec{E}_s \cos \omega t'] dt' \right) \Bigg\} \qquad \text{(I.33)}
\end{aligned}
$$

APPENDIX J

PUBLICATION(S)

- [1] Sekyi-Arthur, D., Mensah, S. Y., K.W. Adu, Mensah, N. G., Dompreh, K. A., and Edziah, R. Tunable power factor of fluorine-doped carbon nanotube. J. Appl. Phys. 128, 244301 (2020). doi: 10.1063/5.0031326

- [2] Sekyi-Arthur, D., Mensah, S.Y., Adu, K.W., Dompreh, K.A., Edziah, R.Mensah, N.G., and Jebuni-Adanu, C.(2020). Induced Hall-like current by Acoustic Phonons in Semiconductor Fluorinated Carbon Nanotube . World Journal of Condensed Matter Physics, 10, 71-87.https://doi.org/10.4236/wjcmp.2020. 102005

- [3] Sekyi-Arthur, D., Mensah, S.Y., Adu, K.W., Dompreh, K.A., Edziah, R. and Mensah, N.G. (2020). Acoustoelectric Effect in Fluorinated Carbon Nanotube in the Absence of External Electric Field. World Journal of Condensed Matter Physics, 10, 1-11.https://doi.org/10.4236/wjcmp.2020. 101001

- [4] Sekyi-Arthur, D., Mensah, S.Y., Adu, K.W., Dompreh, K.A., Edziah, R., Mensah, N.G., and Jebuni-Adanu, C. (2020). Semiconductor Fluorinated Carbon Nanotube as a Low Voltage Current Amplifier Acoustic Device. World Journal of Condensed Matter Physics, 10, 12-26.https://doi.org/ 10.4236/wjcmp.2020.101002

- [5] Sekyi-Arthur, D., S. Y. Mensah, N. G. Mensah, K. A. Dompreh, and R. Edziah. Absorption of Acoustic Phonons in Fluorinated Carbon Nanotube with Double Periodic Band. Phonons in Low Dimensional Structures (pp.129-142). InTech. http://dx.doi.org/10.5772/intechopen.78231 (2018)

- [6] Sekyi-Arthur, D., Mensah, S.Y., Adu, K.W., Dompreh, K.A., Edziah, R. and Mensah, N.G. (2021). Fluorine Doped Single Walled Carbon Nanotube for a SASER Device. World Journal of Condensed Matter Physics. ID: 4800486 (accepted)

- [7] Sekyi-Arthur,D.; Mensah, S.Y.; Dompreh, K.A.; Adu, K.W.;Edziah, R; Mensah, N.G. Giant Thermoelectric Figure of Merit in Fluorine Doped Single Walled Carbon Nanotube. Nanomaterials 2021, 1,0.https://doi.org/10.20944/preprints202106.0070.v1