UNIVERSITY OF CAPE COAST

STABILITY OF NONLINEAR VOLTERRA INTEGRODIFFERENTIAL EQUATIONS

AWO EYIDEY ASMAH

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BY

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DECLARATION

Candidate's Declaration

I hereby declare that this thesis is the result of my own original research and that no part of it has been presented for another degree in this university or elsewhere.

Candidate's Signature .... Date ....

Name: Awo Eyidey Asmah

Supervisor's Declaration

I hereby declare that the preparation and presentation of the thesis were supervised in accordance with the guidelines on supervision of thesis laid down by the University of Cape Coast.

Supervisor's Signature .... Date ....

Name: Prof. Ernest Yankson

ABSTRACT

In this thesis, sufficient conditions for the zero solution of nonlinear Volterra integrodifferential equations are established. The Lyapunov's direct method is the main mathematical technique used in the study. Thus, a Lyapunov functional is constructed. This Lyapunov functional is then used to derive sufficient conditions for the zero solution of nonlinear Volterra integrodifferential equations to be stable, uniformly stable and uniformly aymptotically

stable.

KEY WORDS

Integrodifferential Equations

Lyapunov Functional

Stability

Uniformly Asymptotically Stable

Uniformly Stable

Volterra Integrodifferential Equations

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Last but not least I am grateful to my lecturers, and all and sundry by whose support I have come this far. May God swell your blessings.

DEDICATION

To my devoted children, Micaiah and Michelle, and my beloved spouse, Rev. Michael Asmah.

TABLE OF CONTENTS

CHAPTER ONE

INTRODUCTION

The Volterra integrodifferential equation and the problem under investigation is introduced in this chapter. Additionally, the study's goals, importance, and organizational structure are all disclosed.

1.1 Background to the Study

Differential and integral equations have long been one of the most important tools employed in solving numerous problems in scientific fields (Jerri, 1999). However, application of the knowledge of stability properties of integrodifferential equations offer promising results and Volterra integrodifferential equations offer even more powerful results. Since the birth of these equations which investigate models about growth of population, Volterra integrodifferential equations have been employed by mathematicians to solve several complex problems in scientific fields, particularly, in biological and engineering fields (Hristova & Tunc, 2019, Aggarwal & Gupta, 2019). Understanding the concepts governing the theories of Volterra integrodifferential equations in its linear and nonlinear forms offers a great opportunity for solving several problems that arise in scientific fields (El Hajji, 2019). It is however not surprising how rapidly it has developed since the 1970s (Wazwaz, 2011).

Solutions to an ordinary differential equation is either one which is numerical or analytic. Most often than not, the complexities of some ordinary differential equations make it impossible to find an analytic solution. The research findings of Burton and Mahfoud (1983) and Wazwaz (2011), assert that, sometimes, efforts to find analytic solutions to some ordinary differential equations may not be successful.

Differential equations are used in several fields such as the amount of money in a savings bank, the orbit of a space ship, the description of radio waves, the size of a biological population, the voltage in an electric current, etc. To use differential equation(s) to understand a physical problem, mathematicians normally collect information about the physical problem of interest, create a model, usually differential equation(s) which describes the exactness of the physical problem and then solve this differential equation(s).

Stability theory involves how small changes affect physical systems involving time (Halanay & Rasvan, 2012). If these changes/disturbances are actually small and it is observed that the system stays closer to its state of equilibrium or even gets back to this state, then the system is said to be stable, otherwise the system is not stable. The desire to find solutions to problems that arise in real life leads mathematicians to formulate differential equations that depict physical problems. Surprisingly, what starts as a simple mathematical equation grows into practical physical models such as dynamics of fluids, heat flow, rate of growth of bacterial, growth rate of an economy, etc.

Several of these mathematical equations exist but an equation which comprises both an integral and the derivatives of a function which is not known qualifies to be an integrodifferential equation. In simple terms, a sole equation that includes the operation of differentiation and integration is an integrodifferential equation.

For his contributions to integral equations and mathematical biology, Italian mathematician and physicist Vito Volterra is well-known. His research on elasticity served as the basis for Volterra's integrodifferential equation theory. He discovered that the analysis of the matter's electromagnetic state at all prior instants, as well as the magnetic field for those substances at that specific instant, determines whether the matter is electrically or magnetically polarized. Using integrodifferential equations, these physical events are modelled (Paoloni & Simili, 2008).

Mathematical modelling demands that a real life problem is transformed into a mathematical equation. Volterra integrodifferential equations, a type of a mathematical equation which is under ordinary differential equations and named after its originator, Vito Volterra, has proven to be useful in different fields of study. Notably are fields of Biology, Engineering and Physics (Wazwaz, 2011).

Obtaining conditions that are actually sufficient to realize the attributes that makes Volterra integrodifferential equations stable in both their linear and nonlinear forms have captured the attention of several researchers over the years due to the vast opportunities it offer.

1.2 Statement of Problem

Recently, some mathematicians have shown interest in the study of the properties guaranteeing that the Volterra integrodifferential equation,

$$x'(t) = h(t)x(t) + \int_{t_0}^{t} c(at - s)x(s)ds, \tag{1.2.1}$$

where $a > 1$ is stable. Specifically, Islam and Raffoul (2005), studied the stability of equation 1.2.1 and its nonlinear perturbation of the form,

$$x'(t) = h(t)x(t) + \int_{t_0}^{t} c(at - s)x(s)ds + g(t, x(t)). \tag{1.2.2}$$

In the study, they obtained some conditions that ensured that the zero solution of equation 1.2.1 and equation 1.2.2 using a suitable Lyapunov's functional is stable, uniformly stable, uniformly asymptotically stable. However, the results obtained by Islam and Raffoul (2005), are for linear Volterra integrodifferential equation and does not hold for the nonlinear equation

$$x'(t) = h(t)f(x(t)) + \int_{t_0}^{t} c(at - s)f(x(s))ds. \tag{1.2.3}$$

It is therefore necessary that, additional research is conducted to obtain conditions that ensure that a nonlinear Volterra integrodifferential equation is uniformly asymptotically stable.

1.3 Purpose of the Study

The thesis' focus is to deduce conditions that are sufficient to guarantee that the solution (zero) of a nonlinear Volterra integrodifferential equation is stable, uniformly stable and uniformly asymptotically stable.

1.4 Research Objectives

The objectives of the study are to obtain sufficient conditions for the nonlinear Volterra integrodifferential equation

$$x'(t) = h(t)f(x(t)) + \int_{0}^{t} c(at - s)f(x(s))ds,$$

to be:

1. stable;
2. uniformly stable; and
3. uniformly asymptotically stable.

1.5 Significance of the Study

This research is of great importance because, it will offer conditions that are sufficient for nonlinear Volterra integrodifferential equations to be uniformly asymptotically stable which has not yet been investigated by researchers. Also, the results obtained in this research will not only add to existing literature but also contribute to the illumination of the vast opportunities Volterra integrodifferential equations offer.

1.6 Delimitations

The solution of a nonlinear Volterra integrodifferential equation that is zero was shown to be stable, uniformly stable and uniformly asymptotically stable.

1.7 Limitations

The study was restricted to using the direct Lyapunov approach while there are other ways to determine the stability characteristics of differential equation solutions. The study's conclusion also used a nonlinear Volterra integrodifferential equation rather than a linear one.

1.8 Organisation of the Research

The scope of the Volterra integrodifferential equation and its use in simulating physical processes was provided in Chapter One of the thesis. The problem statement, research objectives, as well as the study's organizational structure, are all included in this chapter.

A survey of pertinent and related literature on the stability characteristics of Volterra integrodifferential equations is given in Chapter Two of the research.

The method utilized to explore the stability properties of the Volterra integrodifferential equation was covered in Chapter Three of the study.

In Chapter Four, we present the findings of the thesis. Based on the study's goals, the conclusions were reached.

The summary of the findings of the study and its conclusions were covered in chapter Five.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

For the initial section of this chapter, what integral equations entail, in general sense, as well as some definition of terms used in this study will be considered. The latter section of the chapter will be dedicated to reviewing some literature on integrodifferential equations.

2.2 Integral Equations

An integral equation is any equation of the form

$$u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t)u(t)dt,$$

where $g(x)$ and $h(x)$ are the limits of the integral, and λ is a constant parameter, $K(x,t)$ is the kernel. Integral equations can be grouped under two headings based on the nature of the integral limits as well as the kernel, $K(x,t)$. An integral equation which is classified as a Fredholm integral equation has both upper and lower integral limits being constants whereas an integral equation is called a Volterra integral equation if it possesses at least a variable integral limit (Volterra, 1959).

2.3 Integrodifferential Equations

With respect to integrodifferential equations, both the integral and differential operators are seen together and they are found in many scientific fields of study, most importantly in the case where problems involving initial values or boundary values are converted into equations containing an integral. The Fredholm and Volterra integrodifferential equations are the major classifications.

The Fredholm integrodifferential equation is written as:

$$u^{(n)}(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt$$

with $u^{(n)}$ considered as the nth derivative of $u(x)$. An example of Fredholm integrodifferential equation is

$$u'(x) = 1 - \frac{1}{3}x + \int_0^1 xu(t)\,dt, \quad u(0) = 0.$$

A Volterra integrodifferential equation can be written as:

$$u^{(n)}(x) = f(x) + \lambda \int_0^x K(x,t)u(t)dt.$$

Consider any fixed $t \geq 0$, and let

$B(t) = \{\phi : [0,t] \to \mathbb{R},\ \phi$ is continuous and bounded in the supremum norm$\}$.

Then for each $\phi \in B(t_0),\ t_0 \geq 0$, there is a unique solution $x(t) = x(t, t_0, \phi)$ of equation 1.2.3 defined on the interval $[t_0, \gamma)$ with $x(s) = \phi$ for $0 \leq s \leq t_0$. For $\phi \in B(t_0)$, the supremum norm of ϕ is given by $||\phi|| = \sup\{|\phi(t)|: 0 \leq t \leq t_0\}$. If the solution remains bounded, then $\gamma = \infty$.

Definition 2.1[Stability]

The zero solution of equation 1.2.3 is said to be stable if for each $\varepsilon > 0$ and each $t_0 \geq 0$, there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that $[\phi \in B(t_0), \| \phi \| < \delta, t \geq t_0]$ imply $|x(t, t_0, \phi)| < \varepsilon$.

Definition 2.2[Uniform stability]

The zero solution of equation 1.2.3 is said to be uniformly stable if it is stable and δ is independent of t_0.

Definition 2.3[Uniform Asymptotic stability]

The zero solution of equation 1.2.3 is said to be uniformly asymptotically stable if it is uniformly stable and there is a $\gamma_1 > 0$ and for each $\epsilon > 0$ there exists a $T > 0$ such that $[t_0 \geq 0, \phi \in B(t_0), \| \phi \| < \gamma_1, t \geq T + t_0]$ imply $|x(t, t_0, \phi)| < \varepsilon$.

2.4 Review of Related Literature

The later part of the year 1900 saw an increase in the research into the qualitative analysis of differential equations and became rather popular in the 1940s. The early research ever known in connection with qualitative analysis of differential equations is that which is found in the work of Poincare and Magini (1899). Researchers have undoubtedly gained interest in its enormous opportunities it offer in understanding the behaviour of solutions of differential equations and that is seen in the numerous researches conducted in this field since it first came to light. Some of these reasearch works can be put in piece in the monographs by(Agarwal, et al., 2005; Burton, 2006; Coddington & Levinson, 1955; Hahn, et al., 1963; Halanay, 1966; Krasovskill, 1963).

In 1892, Lyapunov constructed functions which were later called Lyapunov functionals in studying stability problems, existence and boundedness of periodic solutions of differential equations (Lyapunov, 1892).

The Lyapunov's direct method is the most used tool by researchers in obtaining stability properties of differential equations. Several research works have been done on the stability properties of the convolution and nonconvolution forms of the Volterra integrodifferential equation.

Burton & Mahfould (1983) considered the stability criteria for the system of integrodifferential equations of the form

$$x' = A(t)x + \int_0^t c(t,s)x(s)ds. \tag{2.3.1}$$

In their study, the researchers investigated relations between stability properties of solutions of equation 2.3.1 and established necessary and sufficient conditions for stability and boundedness of solutions of equation 2.3.1 in their perturbed forms. The researchers as well constructed several Lyapunov functionals from which they obtained necessary and sufficient conditions for stability of the solution of equation 2.3.1.

However, Eloe & Islam (1995) studied the stability properties of the zero solution of the linear Volterra integrodifferential system,

$$x' = A(t)x(t) + \int_0^t B(t,s)x(s)ds. \tag{2.3.2}$$

Although the authors did not employ the use of the Lyapunov method, they were able to show that the zero solution of equation 2.3.2 is;

1. uniformly stable if the resolvent is integrable in some sense.
2. uniformly asymptotically stable if and only if the resolvent is integrable and in addition some conditions in terms of the resolvent and the kernel is satisfied.

Knyazhishche & Shcheglov (1998) studied the scalar equation,

$$\dot{x} = a(t)x(t) + b(t)x(t - r(t)), \quad t > 0. \tag{2.3.3}$$

In their research, the authors obtained a new definition of the positive definiteness of the Lyapunov functional involved in investigating stability and asymptotic stability. The authors used this new definition to prove

Lyapunov type theorems from which the results were applied to equation 2.3.3 where $r(t)$ and $b(t)$ may be bounded.

Burton & Somolinos (1999) also studied the stability properties of the scalar equation.

$$x' = -h(t)x - b(t)x^3 + \int_0^t c(at-s)x(s)ds. \tag{2.3.4}$$

The authors in their work considered the case where $a > 1$ in equation 2.3.4 which is a Volterra integrodifferential equation. The researchers used Lyapunov's direct method to obtain conditions that guarantee the stability properties of equation 2.3.4 for the case where $a > 1$.

Inspired by the equation,

$$x'(t) = Ax(t) + \int_0^t c(at-s)x(s)ds \tag{2.3.5}$$

and its nonlinear perturbations, Zhang (2000) constructed a new Lyapunov functional for linear Volterra integrodifferential equations. The author proved some general stability theorems for the functional differential equations with infinite delay and weakened the usual requirement for positive definiteness of Lyapunov functionals used in stability theory.

Islam & Raffoul (2005) studied the stability properties of the scalar linear Volterra integrodifferential equation

$$x'(t) = h(t)x(t) + \int_0^t c(at-s)x(s)ds, \tag{2.3.6}$$

and its perturbed form,

$$x'(t) = h(t)x(t) + \int_0^t c(at-s)x(s)ds + g(t, x(t)), \tag{2.3.7}$$

using the Lyapunov's direct method.

The authors studied only the case where $a > 1$ and pointed out that for $0 < a < 1$ no meaningful stability results were obtained. The authors employed Lyapunov's direct method in their study to obtain stability results of the zero solution of equation 2.3.6. The authors established that the zero solution of equation 2.3.6 is uniformly asymptotically stable without requiring $\lambda(t) \in L^1[0, \infty)$ contrary to what exists in literature.

2.5 Chapter Summary

This chapter, reviewed relevant and related literature on stability properties by Lyapunov direct method on Volterra integrodifferential equations. This was drawn from the research findings of other authors as published in journals and scholarly articles.

CHAPTER THREE

METHODOLOGY

3.1 Introduction

In this chapter, the Lyapunov's method which is the major tool that is used in obtaining the uniform asymptotic stability properties of equation 1.2.3 is discussed.

3.2 Lyapunov's Method

Stability analysis has been of keen interest to engineers as well as mathematicians. There are several tools employed in the mathematical field to either qualitatively or analytically obtain the stability properties of a system of differential equations. Some of these tools are fixed point theorems, Routh Hurwitz method and the Lyapunov's method. The Lyapunov's method is widely known for its major advantage that, stability and boundedness in the large can be achieved ignoring any prior knowledge of solutions.

In 1892, Lyapunov obtained conditions for stability using two methods. One of the methods known as the First method requires knowledge of the solution of the equation under consideration. The First method is limited to some relevant cases to some extent.

Lyapunov's second method(Direct method), however, does not require the knowledge of the solutions themselves to determine the stability as well as boundedness behaviour of solutions of linear and nonlinear systems of ODEs.

3.3 Lyapunov Functional/Functions

The use of Lyapunov's method entails the construction of the function usually denoted by $V(t,x)$ which is a scalar function as well as its derivatives and possesses some properties.

Definition 3.1[Lyapunov Function]

Suppose the zero vector is included in the domain $Q \subset R^n$. Then a function $V(x)$ defined by $V : Q \rightarrow [0, \infty)$ is called a Lyapunov function if it satisfies the following conditions:

1. $V(0) = 0$;
2. $V(x)$ is positive definite;
3. $V(x)$ has continuous first-order partial derivatives.

Definition 3.2[Locally positive definite functions]

A continuous function $V : D \rightarrow \mathbb{R}$ for $D \subset \mathbb{R}^n \times \mathbb{R}^+$ is a positive locally positive definite function if for some $\epsilon > 0$ and some continuous, strictly increasing function $\alpha : \mathbb{R}^+ \rightarrow \mathbb{R}$, $V(0, t) = 0$ and $V(x, t) \geq \alpha(||x||) \quad$ for all $x \in B_\epsilon$, for all $t \geq 0$.

Definition 3.3[Positive Definite Functions]

A continuous function $V : D \rightarrow \mathbb{R}$ for $D \subset \mathbb{R}^n \times \mathbb{R}^+$ is a positive definite function if it satisfies definition 3.2 and, additionally $\alpha(p) \rightarrow \infty$ as $p \rightarrow \infty$.

There is no general way of finding Lyapunov functions for nonlinear system. Faced with a specific systems, one has to use experience, intuition and physical insights to search for an appropriate Lyapunov function. The Lyapunov's direct method was used to obtain sufficient conditions for the zero solution of the Volterra integrodifferential equation to be stable. This requires that the following are done:

- construct a suitable Lyapunov functional. This must satisfy certain properties;

- compute the derivative of the Lyapunov functional along the solution of the Volterra integrodifferential equation; and
- simplify the resulting expression to obtain the desired results.

Lemma 3.4 [Leibnitz Rule, Wazwaz (2011)]

Let the function $f(x, \iota)$ be continuous as well as its partial derivative in a domain $[a, b] \times [t_0, t_1]$ and let

$$F(x) = \int_{g(x)}^{h(x)} f(x, t)dt, \tag{3.2.1}$$

then there is the existence of the differential of the integral in equation3.2.1 and is stated as

$$F'(x) = \frac{dF}{dx} = f(x, h(x))\frac{dh(x)}{dx} - f(x, g(x))\frac{dg(x)}{dx} + \int_{g(x)}^{h(x)} \frac{\partial f(x, t)}{\partial x}dt.$$

If $g(x) = a$ and $h(x) = b$ given that a and b are actually constants, then the Leibnitz rule shrinks to

$$F'(x) = \frac{dF}{dx} = \int_a^b \frac{\partial f(x, t)}{\partial x}dt.$$

This indicates that interchanging the differential sign and the integral sign gives

$$\frac{d}{dx}\int_a^b \mathsf{e}^{xt}dt = \int_a^b t\mathsf{e}^{xt}dt$$

Lemma 3.5[Cauchy–Schwarz Inequality]

Let $f, g : [a, b] \to \mathbb{R}$ be two Lebesque measurable functions on $[a, b]$ such that f^2, g^2 are Lebesque on $[a, b]$; then fg is integrable on $[a, b]$, then the following inequality is obtained.

$$\left(\int_a^b f(x)g(x)dx\right)^2 \leq \int_a^b g^2(x)dx \int_a^b f^2(x)dx$$

3.4 Chapter Summary

This chapter discussed Lyapunov's direct method, which is the major tool used in this research. The construction process was elaborated as well the definition of some other tools that was employed in the study.

CHAPTER FOUR

RESULTS AND DISCUSSION

4.1 Introduction

In this chapter, sufficient conditions for the zero solution of nonlinear Volterra integrodifferential equations are obtained with the use of Lyapunov functionals.

4.2 Preliminary Results

The scalar nonlinear Volterra integrodifferential equation considered is

$$x'(t) = h(t)f(x(t)) + \int_0^t c(at-s)f(x(s))ds, \tag{4.1.1}$$

where $a > 1$ is a constant. Additionally, $h(t)$ is continuous for all $t \geq 0$ and $c : R \longrightarrow R$ is continuous, $f : R \longrightarrow R$ is continuous and $f(0) = 0$.

In this thesis we assume that

$$f(x) = xf_1(x).$$

In Lemma 4.1, the Lyapunov function that will be used to obtain results for the stability, uniform stability and uniform asymptotic stability of the zero solution of equation 4.1.1 is proposed.

Lemma 4.1.

Let $a > 1$ and p be a positive constant. If $f(0) = 0$ then the functional defined by

$$V(t,x) = \frac{1}{2}\Big(x(t) + \frac{1}{a}\int_0^t G_\alpha(at-s)f(x(s))ds\Big)^2$$

$$+ \; p\int_0^t \int_{at-s}^\infty \mid G_\alpha(u) \mid du f^2(x(s))ds \tag{4.1.2}$$

is a Lyapunov functional.

Proof.

To verify that Eq. (4.1.2) is a Lyapunov function, consider

$$\begin{aligned} V(t,0) &= \frac{1}{2}\Big(0+\frac{1}{a}\int_0^t G_\alpha(at-s)f(0)ds\Big)^2 \\ &\quad + p\int_0^t\int_{at-s}^{\infty} \mid G_\alpha(u)\mid du f^2(0)ds \\ &= 0. \end{aligned}$$

Now, it is clear from Eq. (4.1.2) that $V(t,x) > 0$ for all x, except $x = 0$. Thus, $V(t,x)$ is positive definite.

Finally,

$$\frac{\partial V}{\partial x} = x(t) + \frac{1}{a}\int_0^t G_\alpha(at-s)f(x(s))ds,$$

which is continuous. Therefore, $V(t)$ defined by Eq. (4.1.2) is a Lyapunov functional. This completes the proof.

4.3 Main Results

In this section, sufficient conditions for the zero solution of Eq. (4.1.1) to be uniformly asymptotically stable are obtained.

For $\alpha < 0$, let

$$G_\alpha(t) = \int_t^\infty C(u)e^{\alpha u}due^{-\alpha t} \tag{4.2.1}$$

Assuming $G_\alpha(t)$ exists and $G_\alpha(t) \in L^1[0,\infty)$, define $V(t)$ by Eq. (4.1.2), where p is a positive constant to be determined.

In the next Lemma, the derivative of $V(t)$ with respect to t along solutions of Eq. (4.1.2) is computed.

Let

$$a_1(t) = \frac{h(t)f_1(x(t))}{a} + \frac{1}{a^2}G_\alpha(at-t)f_1(x(t)), \tag{4.2.2}$$

and

$$a_2(t) = \frac{h(t)}{a} + \frac{1}{a^2}G_\alpha(at-t). \tag{4.2.3}$$

Lemma 4.2 If $V(t)$ is given by equation Eq. (4.1.2), then for some positive constant L,

$$\begin{aligned} V'(t) \leq & \Big(aa_1(t) + \frac{(a_1(t)-\alpha)^2}{2L^2} + p\int_{at-t}^{\infty}|G_\alpha(u)|du f_1^2(x(t))\Big)x^2(t) \\ & + \Big[\Big(\frac{L^2}{2} + \frac{|\alpha|}{a}\Big)\int_{at-t}^{\infty}|G_\alpha(u)|du - ap\Big] \\ & \times \int_0^t |G_\alpha(at-s)|f_1^2(x(t))x^2(t)ds. \end{aligned} \tag{4.2.4}$$

Proof.

Let $x(t) = x(t, t_o, \phi)$ be a solution of Eq. (4.1.1)and define $V(t)$ by Eq. (4.1.2). Then along the solutions of Eq. (4.1.2)

$$
\begin{aligned}
V'(t) &= \Big(x(t) + \frac{1}{a}\int_0^t G_\alpha(at-s)f(x(s))ds\Big) \\
&\quad \times \Big(x'(t) + \frac{1}{a}\frac{d}{dt}\int_0^t G_\alpha(at-s)f(x(s))ds\Big) \\
&\quad + \frac{d}{dt}\Big(p\int_0^t \int_{at-s}^{\infty} \mid G_\alpha(u) \mid du f^2(x(s))ds\Big) \\
&= \Big(x(t) + \frac{1}{a}\int_0^t G_\alpha(at-s)f(x(s))ds\Big) \\
&\quad \times \Big(h(t)f(x(t)) + \int_0^t c(at-s)f(x(s))ds \\
&\quad + \frac{1}{a}\frac{d}{dt}\int_0^t G_\alpha(at-s)f(x(s))ds\Big) \\
&\quad + \frac{d}{dt}\Big(p\int_0^t \int_{at-s}^{\infty} \mid G_\alpha(u) \mid du f^2(x(s))ds\Big) \qquad (4.2.5)
\end{aligned}
$$

Now, by Leibnitz rule,

$$
\begin{aligned}
\frac{1}{a}\frac{d}{dt}\int_0^t G_\alpha(at-s)f(x(s))ds &= \frac{1}{a}\Big[G_\alpha(at-t)f(x(t))\ (1) \\
&\quad - G_\alpha(at)f(x(0))\ (0) \\
&\quad + \int_0^t \frac{\partial}{\partial t}G_\alpha(at-s)f(x(s))ds\Big] \\
&= \frac{1}{a}\Big[G_\alpha(at-t)f(x(t)) \\
&\quad + \int_0^t \frac{\partial}{\partial t}G_\alpha(at-s)f(x(s))ds\Big] \qquad (4.2.6)
\end{aligned}
$$

From Eq. (4.2.1)

$$G_\alpha(at-s) = \int_{at-s}^{\infty} c(u)e^{\alpha u}due^{-\alpha(at-s)}$$

Hence,

$$
\begin{aligned}
\frac{\partial}{\partial t} G_\alpha(at-s) &= \frac{\partial}{\partial t}\int_{at-s}^{\infty} c(u)e^{\alpha u}du e^{-\alpha(at-s)} \\
&= 0 - c(at-s)e^{\alpha(at-s)}e^{-\alpha(at-s)} \cdot a \\
&\quad + \int_{at-s}^{\infty} c(u)e^{\alpha u}du[-\alpha a e^{-\alpha(at-s)}] \\
&= -c(at-s)e^{\alpha(at-s)}e^{-\alpha(at-s)} \cdot a \\
&\quad - \alpha a \int_{at-s}^{\infty} c(u)e^{\alpha u}due^{-\alpha(at-s)} \\
&= -c(at-s)a \\
&\quad - \alpha a \int_{at-s}^{\infty} c(u)e^{\alpha u}due^{-\alpha(at-s)} \\
&= -c(at-s)a - \alpha a G_\alpha(at-s) \qquad (4.2.7)
\end{aligned}
$$

Substituting Eq. (4.2.7) into Eq. (4.2.6) gives

$$
\begin{aligned}
\frac{1}{a}\frac{d}{dt}\int_0^t G_\alpha(at-s)f(x(s))ds = \frac{1}{a}\Big[G_\alpha(at-t)f(x(t)) \ + \int_0^t \Big[-c(at-s)a \\
- \ \alpha a G_\alpha(at-s)\Big]f(x(s))ds\Big] \\
= \ \frac{1}{a}G_\alpha(at-t)f(x(t)) \\
- \ \frac{a}{a}\int_0^t c(at-s)f(x(s))ds \\
- \ \frac{\alpha a}{a}\int_0^t G_\alpha(at-s)f(x(s))ds \\
= \ \frac{1}{a}G_\alpha(at-t)f(x(t)) \\
- \int_0^t c(at-s)f(x(s))ds \\
- \ \alpha\int_0^t G_\alpha(at-s)f(x(s))ds
\end{aligned} \tag{4.2.8}
$$

Also,

$$\frac{d}{dt}\Big(p\int_0^t\int_{at-s}^{\infty} \mid G_\alpha(u)\mid du f^2(x(s))ds\Big)$$

$$= \quad p\int_0^t \frac{d}{dt}\int_{at-s}^{\infty} \mid G_\alpha(u)\mid du f^2(x(s))ds$$

$$= \quad p\int_{at-s}^{\infty} \mid G_\alpha(u)\mid du f^2(x(t))ds$$

$$+\, p\int_0^t \lim_{b\longrightarrow\infty}\frac{d}{dt}\int_{at-s}^{b} \mid G_\alpha(u)\mid du$$

$$\times\, f^2(x(s))ds$$

$$= \quad p\int_{at-s}^{\infty} \mid G_\alpha(u)\mid du f^2(x(t))ds$$

$$+\, p\int_0^t\Big[- \mid G_\alpha(at-s)\mid a f^2(x(s))ds\Big]$$

$$= \quad p\int_{at-s}^{\infty} \mid G_\alpha(u)\mid du f^2(x(t))ds$$

$$-\; ap\int_0^t \mid G_\alpha(at-s)\mid f^2(x(s))ds \qquad (4.2.9)$$

Substituting Eq. (4.2.8) and Eq. (4.2.9) into Eq. (4.2.5), gives

$$\begin{aligned}
V'(t) &= \left(x(t) + \frac{1}{a}\int_0^t G_\alpha(at-s)f(x(s))ds\right) \\
&\quad \times \left(h(t)f(x(t)) + \int_0^t c(at-s)f(x(s))ds\right. \\
&\quad + \frac{1}{a}G_\alpha(at-s)f(x(t)) \\
&\quad - \int_0^t c(at-s)f(x(s))ds \\
&\quad \left. - \alpha\int_0^t G_\alpha(at-s)f(x(s))ds\right) \\
&\quad + p\int_{at-t}^{\infty} \mid G_\alpha(u) \mid du f^2(x(t)) \\
&\quad - ap\int_0^t \mid G_\alpha(at-s) \mid du f^2(x(s))ds \\
&= \left(x(t) + \frac{1}{a}\int_0^t G_\alpha(at-s)f(x(s))ds\right) \\
&\quad \times \left(h(t)f(x(t)) + \frac{1}{a}G_\alpha(at-s)f(x(t))\right. \\
&\quad \left. - \alpha\int_0^t G_\alpha(at-s)f(x(s))ds\right) \\
&\quad + p\int_{at-t}^{\infty} \mid G_\alpha(u) \mid du f^2(x(t)) \\
&\quad - ap\int_0^t \mid G_\alpha(at-s) \mid du f^2(x(s))ds
\end{aligned}$$

$$
\begin{aligned}
= \;& x(t)\Big(h(t)f(x(t)) + \frac{1}{a}G_\alpha(at-t)f(x(t)) \\
& - \alpha\int_0^t G_\alpha(at-s)f(x(s))ds\Big) \\
& + \frac{1}{a}\int_0^t G_\alpha(at-s)f(x(s))ds \\
& \times \Big(h(t)f(x(t)) + \frac{1}{a}G_\alpha(at-t)f(x(t)) \\
& - \alpha\int_0^t G_\alpha(at-s)f(x(s))ds\Big) \\
& + p\int_{at-t}^{\infty} \mid G_\alpha(u) \mid du f^2(x(t)) \\
& - ap\int_0^t \mid G_\alpha(at-s) \mid f^2(x(s))ds \\
= \;& h(t)f(x(t))x(t) + \frac{1}{a}G_\alpha(at-t)f(x(t))x(t) \\
& - \alpha x(t)\int_0^t G_\alpha(at-s)f(x(s))ds \\
& + h(t)f(x(t))\frac{1}{a}\int_0^t G_\alpha(at-s)f(x(s))ds \\
& + \frac{1}{a^2}\int_0^t G_\alpha(at-s)f(x(s))dsG_\alpha(at-t)f(x(t)) \\
& - \frac{\alpha}{a}\Big(\int_0^t G_\alpha(at-s)f(x(s))ds\Big)^2 + p\int_{at-t}^{\infty} \mid G_\alpha(u) \mid du f^2(x(t)) \\
& - ap\int_0^t \mid G_\alpha(at-s) \mid f^2(x(s))ds
\end{aligned}
$$

$$
\begin{aligned}
= \; & h(t)f(x(t))x(t) \; + \; \frac{1}{a}G_\alpha(at-t)f(x(t))x(t) \\
& + h(t)f(x(t))\frac{1}{a}\int_0^t G_\alpha(at-s)f(x(s))ds \\
& + \frac{1}{a^2}\int_0^t G_\alpha(at-s)f(x(s))dsG_\alpha(at-t)f(x(t)) \\
& - \alpha x(t)\int_0^t G_\alpha(at-s)f(x(s))ds \; - \; \frac{\alpha}{a}\Big(\int_0^t G_\alpha(at-s)f(x(s))ds\Big)^2 \\
& + p\int_{at-t}^\infty \mid G_\alpha(u) \mid du f^2(x(t)) \; - \; ap\int_0^t \mid G_\alpha(at-s) \mid f^2(x(s))ds
\end{aligned}
$$

$$
\begin{aligned}
= \; & h(t)f(x(t))x(t) \; + \; \frac{1}{a}G_\alpha(at-t)f(x(t))x(t) \\
& + \; \Big[\frac{1}{a}h(t)f(x(t)) + \frac{1}{a^2}G_\alpha(at-t)f(x(t)) - \alpha x(t)\Big] \\
& \cdot \int_0^t G_\alpha(at-s)f(x(s))ds \\
& - \; \frac{\alpha}{a}\Big(\int_0^t G_\alpha(at-s)f(x(s))ds\Big)^2 \\
& + \; p\int_{at-t}^\infty \mid G_\alpha(u) \mid du f^2(x(t)) \\
& - \; ap\int_0^t \mid G_\alpha(at-s) \mid f^2(x(s))ds \qquad (4.2.10)
\end{aligned}
$$

$$
\begin{aligned}
= \; & h(t)f_1(x(t))x^2(t) \; + \; \frac{1}{a}G_\alpha(at-t)f_1(x(t))x^2(t) \\
& + \Big[\frac{1}{a}h(t)f_1(x(t))x(t) \; + \; \frac{1}{a^2}G_\alpha(at-t)f_1(x(t))x(t) - \; \alpha x(t)\Big] \\
& \times \int_0^t G_\alpha(at-s)f(x(s))ds \; - \; \frac{\alpha}{a}\Big(\int_0^t G_\alpha(at-s)f(x(s))ds\Big)^2 \\
& + p\int_{at-t}^{\infty} \mid G_\alpha(u) \mid du f_1^2(x(t))x^2(t) \\
& - ap\int_0^t \mid G_\alpha(at-s) \mid f^2(x(s))ds \\
= \; & h(t)f_1(x(t))x^2(t) \\
& + \frac{1}{a}G_\alpha(at-t)f_1(x(t))x^2(t) \\
& + \Big[\frac{1}{a}h(t)f_1(x(t))x(t) + \frac{1}{a^2}G_\alpha(at-t)f_1(x(t))x(t) - \alpha x(t)\Big] \\
& \cdot \int_0^t G_\alpha(at-s)f(x(s))ds \\
& - \frac{\alpha}{a}\Big(\int_0^t G_\alpha(at-s)f(x(s))ds\Big)^2 \\
& + p\int_{at-t}^{\infty} \mid G_\alpha(u) \mid du f_1^2(x(t))x^2(t) \\
& - ap\int_0^t \mid G_\alpha(at-s) \mid f^2(x(s))ds
\end{aligned}
$$

$$= h(t)f_1(x(t))x^2(t) \; + \; \frac{1}{a}G_\alpha(at-t)f_1(x(t))x^2(t)$$

$$+ \; \Big[a_1(t)-\alpha\Big]x(t)\int_0^t G_\alpha(at-s)f(x(s))ds$$

$$- \; \frac{\alpha}{a}\Big(\int_0^t G_\alpha(at-s)f(x(s))ds\Big)^2$$

$$+ \; p\int_{at-t}^{\infty} \mid G_\alpha(u)\mid du f_1^2(x(t))x^2(t)$$

$$- \; ap\int_0^t \mid G_\alpha(at-s)\mid f^2(x(s))ds \tag{4.2.11}$$

For any real number y and z and any nonzero constant k, one has $2yz \leq (y^2/k^2) + k^2z^2$. Using this inequality gives for the positive constant L

$$(a_1(t)-\alpha)x(t)\int_0^t G_\alpha(at-s)f(x(s))ds \;\leq\; \frac{(a_1(t)-\alpha)^2x^2(t)}{2L^2}$$

$$+ \; \frac{L^2}{2}\Big(\int_0^t G_\alpha(at-s)f(x(s))ds\Big)^2 \tag{4.2.12}$$

Also by the Cauchy-Schwarz inequality,

$$\Big(\int_0^t G_\alpha(at-s)f(x(s))ds\Big)^2 \;=\; \Big(\int_0^t \sqrt{\mid G_\alpha(at-s)\mid}$$

$$\cdot\sqrt{\mid G_\alpha(at-s)\mid} f(x(s))ds\Big)^2$$

$$\leq \Big(\int_0^t \Big(\sqrt{\mid G_\alpha(at-s)\mid}\Big)^2 ds\Big)$$

$$\times\Big(\int_0^t \Big(\sqrt{\mid G_\alpha(at-s)\mid} f(x(s))\Big)^2 ds\Big)$$

$$= \int_0^t |G_\alpha(at-s)|ds \int_0^t |G_\alpha(at-s)| f^2(x(s))ds.$$

Therefore inequality (4.2.12) becomes

$$(a_1(t)-\alpha)x(t)\int_0^t G_\alpha(at-s)f(x(s))ds$$

$$\leq \frac{(a_1(t)-\alpha)^2 x^2(t)}{2L^2}$$

$$+ \frac{L^2}{2}\int_0^t |G_\alpha(at-s)|ds$$

$$\times \int_0^t |G_\alpha(at-s)| f^2(x(s))ds \tag{4.2.13}$$

Similarly, by the Cauchy Schwarz inequality for integrals,

$$-\frac{\alpha}{a}\Big(\int_0^t G_\alpha(at-s)f(x(s))ds\Big)^2$$

$$\leq \frac{|\alpha|}{a}\int_0^t |G_\alpha(at-s)|ds \int_0^t |G_\alpha(at-s)| f^2(x(s))ds. \tag{4.2.14}$$

Substituting Eq. (4.2.13) and Eq. (4.2.14) into Eq. (4.2.11) yields,

$$\begin{aligned}
V'(t) \quad \leq \quad & h(t)f_1(x(t))x^2(t) \\
& + \frac{1}{a}G_\alpha(at-t)f_1(x(t))x^2(t) \\
& + \frac{(a_1(t)-\alpha)^2x^2(t)}{2L^2} \\
& + \frac{L^2}{2}\int_0^t |G_\alpha(at-s)|ds \int_0^t |G_\alpha(at-s)|f^2(x(s))ds \\
& + \frac{|\alpha|}{a}\int_0^t |G_\alpha(at-s)|ds \int_0^t |G_\alpha(at-s)|f^2(x(s))ds \\
& + p\int_{at-t}^{\infty} \mid G_\alpha(u) \mid du f_1^2(x(t))x^2(t) \\
& - ap\int_0^t \mid G_\alpha(at-s) \mid f^2(x(s))ds \\
= \quad & h(t)f_1(x(t))x^2(t) \ + \ \frac{1}{a}G_\alpha(at-t)f_1(x(t))x^2(t) \\
& + \frac{(a_1(t)-\alpha)^2x^2(t)}{2L^2} \ + \ p\int_{at-t}^{\infty} \mid G_\alpha(u) \mid du f_1^2(x(t))x^2(t) \\
& + \frac{L^2}{2}\int_0^t |G_\alpha(at-s)|ds \int_0^t |G_\alpha(at-s)|f^2(x(s))ds \\
& + \frac{|\alpha|}{a}\int_0^t |G_\alpha(at-s)|ds \int_0^t |G_\alpha(at-s)|f^2(x(s))ds \\
& - ap\int_0^t \mid G_\alpha(at-s) \mid f^2(x(s))ds
\end{aligned}$$

$$
\begin{aligned}
=&\Big(h(t)f_1(x(t)) \ + \ \frac{1}{a}G_\alpha(at-t)f_1(x(t)) \\
&+ \ \frac{(a_1(t)-\alpha)^2}{2L^2} \ + \ p\int_{at-t}^{\infty}|G_\alpha(u)|du f_1^2(x(t))\Big)x^2(t) \\
&+ \ \Big(\frac{L^2}{2}\int_0^t|G_\alpha(at-s)|ds \\
&+ \ \frac{|\alpha|}{a}\int_0^t|G_\alpha(at-s)|ds - ap\Big) \\
&\times \int_0^t|G_\alpha(at-s)|f^2(x(s))ds
\end{aligned} \tag{4.2.15}
$$

Now using the substitution $u = at - s$ in the integral $\int_0^t \mid G_\alpha(at-s) \mid ds$ gives

$$
\begin{aligned}
\int_0^t \mid G_\alpha(at-s) \mid ds &= -\int_{at}^{at-t} \mid G_\alpha(u) \mid du \\
&= \int_{at-t}^{at} \mid G_\alpha(u) \mid du \\
&\leq \int_{at-t}^{\infty} \mid G_\alpha(u) \mid du
\end{aligned} \tag{4.2.16}
$$

Thus, using Eq. (4.2.16) in Eq. (4.2.15) gives

$$
\begin{aligned}
V'(t) \leq & \Big(h(t)f_1(x(t)) + \frac{1}{a}G_\alpha(at-t)f_1(x(t)) \\
& + \frac{(a_1(t)-\alpha)^2}{2L^2} + p\int_{at-t}^{\infty}|G_\alpha(u)|duf_1^2(x(t))\Big)x^2(t) \\
& + \Big(\frac{L^2}{2}\int_{at-t}^{at}|G_\alpha(u)|du + \frac{|\alpha|}{a}\int_{at-t}^{at}|G_\alpha(u)|du - ap\Big) \\
& \times \int_0^t |G_\alpha(at-s)|f^2(x(s))ds \\
= & \Big(a\Big[\frac{h(t)f_1(x(t))}{a} + \frac{1}{a^2}G_\alpha(at-t)f_1(x(t))\Big] \\
& + \frac{(a_1(t)-\alpha)^2}{2L^2} + p\int_{at-t}^{\infty}|G_\alpha(u)|duf_1^2(x(t)\Big)x^2(t) \\
& + \Big[\Big(\frac{L^2}{2} + \frac{|\alpha|}{a}\Big)\int_{at-t}^{\infty}|G_\alpha(u)|du - ap\Big] \\
& \times \int_0^t |G_\alpha(at-s)|f^2(x(s))ds \\
= & \Big(aa_1(t) + \frac{(a_1(t)-\alpha)^2}{2L^2} + p\int_{at-t}^{\infty}|G_\alpha(u)|duf_1^2(x(t)\Big)x^2(t) \\
& + \Big[\Big(\frac{L^2}{2} + \frac{|\alpha|}{a}\Big)\int_{at-t}^{\infty}|G_\alpha(u)|du - ap\Big] \\
& \times \int_0^t |G_\alpha(at-s)|f^2(x(s))ds
\end{aligned}
$$

This completes the proof.

In Theorem 4.1, stability result for the zero solution of Eq. (4.1.1) is stated.

Theorem 4.1 Let $G_\alpha(u) \in L^1[0,\infty)$ with $(1/a)\int_0^\infty \mid G_\alpha(u) \mid du < 1$ and $1 \leq f_1(x)$. Suppose

$$aa_2(t) + \frac{(a_2(t)-\alpha)^2}{2} + a_3 Q \leq -\beta, \tag{4.2.17}$$

for $\beta \geq 0$, and $\mid f_1(x) \mid \leq \lambda \mid x \mid$, for $\lambda > 0$, where

$$Q = \left(\frac{1}{\sqrt{a}}\int_{at-t}^{\infty} \mid G_\alpha(u) \mid du\right)^2$$

and

$$a_3 = \left(\frac{L^2}{2} + \frac{\mid \alpha \mid}{a}\right). \tag{4.2.18}$$

Then the zero solution of Eq. (4.1.1) is stable.

Proof. From Eq. (4.1.2),

$$\begin{aligned}
V(t) &= \frac{1}{2}\left(x(t) + \frac{1}{a}\int_0^t G_\alpha(at-s)f(x(s))ds\right)^2 \\
&\quad + P\int_0^t\int_{at-s}^{\infty} \mid G_\alpha(u) \mid du f^2(x(s))ds \\
&= \frac{1}{2}\Big[x^2(t) + \frac{2x(t)}{a}\int_0^t G_\alpha(at-s)f(x(s))d \\
&\quad + \left(\frac{1}{a}\int_0^t G_\alpha(at-s)f(x(s))ds\right)^2\Big] \\
&\quad + P\int_0^t\int_{at-s}^{\infty} \mid G_\alpha(u) \mid du f^2(x(s))ds
\end{aligned}$$

$$V(t) = \frac{x^2(t)}{2} + \frac{1}{2a^2}\Big(\int_0^t G_\alpha(at-s)f(x(s))ds\Big)^2$$

$$+ \frac{x(t)}{a}\int_0^t G_\alpha(at-s)f(x(s))ds$$

$$+ P\int_0^t \int_{at-s}^{\infty} \mid G_\alpha(u) \mid du f^2(x(s))ds \tag{4.2.19}$$

By the Cauchy-Schwarz inequality,

$$\frac{1}{2a^2}\Big(\int_0^t G_\alpha(at-s)f(x(s))ds\Big)^2$$

$$\leq \frac{1}{2a^2}\int_0^t \mid G_\alpha(at-s) \mid ds$$

$$\times \int_0^t \mid G_\alpha(at-s) \mid f^2(x(s))ds \tag{4.2.20}$$

and

$$\frac{x(t)}{a}\int_0^t G_\alpha(at-s)f(x(s))ds$$

$$= x(t) \times \frac{1}{a}\Big(\int_0^t G_\alpha(at-s)f(x(s))ds\Big)$$

$$\leq \frac{x^2(t)}{2} + \frac{1}{2a^2}\Big(\int_0^t \mid G_\alpha(at-s) \mid f(x(s))ds\Big)^2$$

$$= \frac{x^2(t)}{2} + \frac{1}{2a^2}\int_0^t \mid G_\alpha(at-s) \mid ds$$

$$\times \int_0^t \mid G_\alpha(at-s) \mid f^2(x(s))ds \qquad (4.2.21)$$

Using Eq. (4.2.20) and Eq. (4.2.21) in Eq. (4.2.19) gives

$$\begin{aligned}
V(t) \leq\ & \frac{x^2(t)}{2} + \frac{1}{2a^2}\int_0^t \mid G_\alpha(at-s) \mid ds \int_0^t \mid G_\alpha(at-s) \mid f^2(x(s))ds \\
& + \frac{x^2(t)}{2} + \frac{1}{2a^2}\int_0^t \mid G_\alpha(at-s) \mid ds \int_0^t \mid G_\alpha(at-s) \mid f^2(x(s))ds \\
& + P\int_0^t \int_{at-s}^\infty \mid G_\alpha(u) \mid du f^2(x(s))ds \\
=\ & \frac{x^2(t)}{2} + \frac{x^2(t)}{2} \\
& + \frac{1}{2a^2}\int_0^t \mid G_\alpha(at-s) \mid ds \int_0^t \mid G_\alpha(at-s) \mid f^2(x(s))ds \\
& + \frac{1}{2a^2}\int_0^t \mid G_\alpha(at-s) \mid ds \int_0^t \mid G_\alpha(at-s) \mid f^2(x(s))ds \\
& + P\int_0^t \int_{at-s}^\infty \mid G_\alpha(u) \mid du f^2(x(s))ds \\
=\ & x^2(t) + \frac{1}{a^2}\int_0^t \mid G_\alpha(at-s) \mid ds \int_0^t \mid G_\alpha(at-s) \mid f^2(x(s))ds \\
& + P\int_0^t \int_{at-s}^\infty \mid G_\alpha(u) \mid du f^2(x(s))ds
\end{aligned}$$

$$\leq x^2(t) + \frac{1}{a^2}\int_0^\infty \mid G_\alpha(u) \mid du \int_0^t \mid G_\alpha(at-s) \mid f^2(x(s))ds$$

$$+ P\int_0^t \int_{at-s}^\infty \mid G_\alpha(u) \mid du f^2(x(s))ds. \qquad (4.2.22)$$

Using the fact that $\frac{1}{a}\int_0^\infty \mid G_\alpha(u) \mid du < 1$ in Eq. (4.2.22) gives

$$V(t) \leq x^2(t) + \frac{1}{a}\int_0^t \mid G_\alpha(at-s) \mid f^2(x(s))ds$$

$$+ P\int_0^t \int_{at-s}^\infty \mid G_\alpha(u) \mid du f^2(x(s))ds.$$

$$= x^2(t) + \frac{1}{a}\int_0^t \mid G_\alpha(at-s) \mid f_1^2(x(s))x^2(s)ds$$

$$+ P\int_0^t \int_{at-s}^\infty \mid G_\alpha(u) \mid du f_1^2(x(s))x^2(s)ds \qquad (4.2.23)$$

Now for $L = 1$ take $p = \frac{a_3}{a}\int_{at-t}^\infty \mid G_\alpha(u) \mid du$. Then, from inequality (4.2.4) and from the fact that $1 \leq f_1(x(t))$

$$V'(t) \leq \Big[aa_1(t) + \frac{(a_1(t)-\alpha)^2}{2} + \frac{a_3}{a}\int_{at-t}^\infty \mid G_\alpha(u) \mid du \int_{at-t}^\infty |G_\alpha(u)| du f_1^2(x)\Big]x^2(t)$$

$$+ \Big[\Big(\frac{1}{2} + \frac{\mid \alpha \mid}{a}\Big)\int_{at-t}^\infty \mid G_\alpha(u) \mid du - a\Big(\frac{a_3}{a}\int_{at-t}^\infty \mid G_\alpha(u) \mid du\Big)\Big]$$

$$\times \int_0^t \mid G_\alpha(at-s) \mid f^2(x(s))$$

$$= \Big[aa_1(t) + \frac{(a_1(t)-\alpha)^2}{2} + \frac{a_3}{a}\Big(\int_{at-t}^{\infty} \mid G_\alpha(u) \mid du\Big)^2 f_1^2(x)\Big]x^2(t)$$

$$+ \Big[\Big(\frac{1}{2} + \frac{\mid \alpha \mid}{a}\Big)\int_{at-t}^{\infty} \mid G_\alpha(u) \mid du - a_3 \int_{at-t}^{\infty} \mid G_\alpha(u) \mid du\Big]$$

$$\times \int_0^t \mid G_\alpha(at-s) \mid f^2(x(s))$$

$$= \Big[aa_1(t) + \frac{(a_1(t)-\alpha)^2}{2} + \frac{a_3}{a}\Big(\int_{at-t}^{\infty} \mid G_\alpha(u) \mid du\Big)^2 f_1^2(x)\Big]x^2(t)$$

$$+ \Big[\Big(\frac{1}{2} + \frac{\mid \alpha \mid}{a}\Big)\int_{at-t}^{\infty} \mid G_\alpha(u) \mid du - a_3 \int_{at-t}^{\infty} \mid G_\alpha(u) \mid du\Big]$$

$$\times \int_0^t \mid G_\alpha(at-s) \mid f^2(x(s))$$

$$= \Big[aa_1(t) + \frac{(a_1(t)-\alpha)^2}{2} + a_3 \times \frac{1}{a}\Big(\int_{at-t}^{\infty} \mid G_\alpha(u) \mid du\Big)^2 f_1^2(x)\Big]x^2(t)$$

$$+ \Big[\Big(\frac{1}{2} + \frac{\mid \alpha \mid}{a}\Big)\int_{at-t}^{\infty} \mid G_\alpha(u) \mid du - a_3 \int_{at-t}^{\infty} \mid G_\alpha(u) \mid du\Big]$$

$$\times \int_0^t \mid G_\alpha(at-s) \mid f^2(x(s))ds$$

$$
\begin{aligned}
V'(t) &= \Big[aa_1(t) + \frac{(a_1(t)-\alpha)^2}{2} + a_3 Q f_1^2(x(t))\Big]x^2(t) \\
&\quad + \Big[\Big(\frac{1}{2} + \frac{\mid \alpha \mid}{a}\Big)\int_{at-t}^{\infty} \mid G_\alpha(u) \mid du - \Big(\frac{1}{2} + \frac{\mid \alpha \mid}{a}\Big)\int_{at-t}^{\infty} \mid G_\alpha(u) \mid du\Big] \\
&\quad \times \int_0^t \mid G_\alpha(at-s) \mid f^2(x(s)) \\
&= \Big[aa_1(t) + \frac{(a_1(t)-\alpha)^2}{2} + a_3 Q f_1^2(x(t))\Big]x^2(t) \\
&\leq \Big[aa_2(t) + \frac{(a_2(t)-\alpha)^2}{2} + a_3 Q\Big] f_1^2(x(t))x^2(t) \\
&\leq -\beta f_1^2(x(t))x^2(t) \\
&\leq -\beta x^2(t). \qquad (4.2.24)
\end{aligned}
$$

Let $J = (1/a)\int_0^\infty \mid G_\alpha(u) \mid du$. Given an $\varepsilon > 0$ and a fixed $t_0 \geq 0$, choose $\delta > 0$ with $0 < \delta < \varepsilon$ such that

$$
\sqrt{2}\Big(1 + J + Japt_0\Big)^{1/2}\delta < \varepsilon(1-J) \qquad (4.2.25)
$$

Let $x(t) = x(t, t_0, \phi)$ be a solution of Eq. (4.1.1) with $\parallel \phi \parallel < \delta$. Then for $t \geq t_0$, using Eq. (4.2.23) and inequality (4.2.24) gives

$$
\frac{1}{2}\Big(x(t) + \frac{1}{a}\int_0^t G_\alpha(at-s)f(x(s))ds\Big)^2 \leq V(t) \leq V(t_0) \qquad (4.2.26)
$$

This implies that

$$\frac{1}{2}\Big(x(t)+\frac{1}{a}\int_0^t G_\alpha(at-s)f(x(s))ds\Big)^2$$

$$\leq \; x^2(t) \; + \; \frac{1}{a}\int_0^t \mid G_\alpha(at-s)\mid f_1^2(x(s))x^2(s)ds$$

$$+\,P\int_0^t\int_{at-s}^{\infty} \mid G_\alpha(u)\mid du f_1^2(x(s))x^2(s)ds$$

$$\leq \; x^2(t) \; + \; \frac{1}{a}\int_0^{\infty} \mid G_\alpha(at-s)\mid\mid f_1^2(x(s))\mid\mid x^2(s)\mid ds$$

$$+\,P\int_0^t\int_{at-s}^{\infty} \mid G_\alpha(u)du\mid\mid f_1^2(x(s))\mid\mid x^2(s)\mid ds.$$

$$\leq \; x^2(t) \; + \; \frac{1}{a}\int_0^{\infty} \mid G_\alpha(at-s)\mid\mid \lambda^2 x^4(s)\mid ds$$

$$+\,P\int_0^t\int_{at-s}^{\infty} \mid G_\alpha(u)\mid\mid x^4(s)\mid \lambda^2 ds$$

$$\leq \; x^2(t) \; + \; \frac{\lambda^2}{a}\int_0^{\infty} \mid G_\alpha(at-s)\mid\mid x^4(s)\mid ds$$

$$+\,\lambda^2 P\int_0^t\int_{at-s}^{\infty} \mid G_\alpha(u)du\mid\mid x^4(s)\mid ds$$

$$\leq \; x^2(t_0) \; + \; \frac{\lambda^2}{a}\int_0^{\infty} \mid G_\alpha(u)\mid x^4(s)ds$$

$$+\,\lambda^2 P\int_0^{t_0}\int_0^{\infty} \mid G_\alpha(u)\mid du x^4(s)ds.$$

The fact that $\| \phi \| < \delta$ implies that

$$\begin{aligned}
\frac{1}{2}\Big(x(t) + \frac{1}{a}\int_0^t G_\alpha(at-s)f(x(s))ds\Big)^2 &\leq \delta^2 + \frac{\lambda^2}{a}\int_0^\infty \mid G_\alpha(u) \mid \delta^4 ds \\
&+ \lambda^2 P \int_0^{t_0}\int_0^\infty \mid G_\alpha(u) \mid du\delta^4 ds \\
&\leq \Big(1 + \frac{\lambda^2}{a}\int_0^\infty \mid G_\alpha(u) \mid \delta^2 ds \\
&+ \lambda^2 P \int_0^{t_0}\int_0^\infty \mid G_\alpha(u) \mid du\delta^2 ds\Big)\delta^2 \\
&\leq (1 + \lambda^2\delta^2 J + \lambda^2 apJt_0\delta^2)\delta^2
\end{aligned}$$

This implies that

$$\begin{aligned}
x(t) + \frac{1}{a}\int_0^t G_\alpha(at-s)f(x(s))ds &\leq \sqrt{2\Big(1 + \lambda^2\delta^2 J + \lambda^2 apJt_0\delta^2\Big)\delta^2} \\
&\leq \sqrt{2}\Big(1 + \lambda^2\delta^2 J + \lambda^2 apJt_0\delta^2\Big)^{1/2}\delta
\end{aligned} \tag{4.2.27}$$

It is claimed that $\mid x(t) \mid < \varepsilon$ for all $t \geq t_0$. Note also that $\mid x(u) \mid < \delta < \varepsilon$ for all $0 \leq u \leq t_0$. If the claim is not true, let $t = t_*$ be the first t such that $\mid x(t_*) \mid = \varepsilon$ and $\mid x(s) \mid < \varepsilon$ for all $t_0 \leq s < t_*$. Then inequality (4.2.27) yields

$$\varepsilon(1-J) = \varepsilon\left(1 - \frac{1}{a}\int_0^\infty \mid G_\alpha(u) \mid du\right)$$

$$\leq \quad | \, x(t_*) + \frac{1}{a}\int_0^{t_*} G_\alpha(at_* - s)x(s)ds \, |$$

$$\leq \quad \sqrt{2}\Big(1 + J + Japt_0\Big)^{1/2}\delta,$$

which contradicts inequality (4.2.25) and completes the proof.

Theorem 4.2 Suppose the hypotheses of Theorem 4.1 hold and there is a positive constant R such that

$$\int_{(a-1)t}^{at}\int_v^\infty | \, G_\alpha(u) \, | \, dudv \leq R \tag{4.2.28}$$

for all $t > 0$ and a positive constant K such that $|f_1(x(t))| \leq K$. Then the zero solution of Equation (1) is uniformly stable.

Proof. For any $t_0 \geq 0$ there is

$$\begin{aligned}\int_0^{t_0}\int_{at_0-s}^\infty | \, G_\alpha(u) \, | \, f_1^2(x(s))duds \;&=\; \int_{(a-1)t_0}^{at_0}\int_v^\infty | \, G_\alpha(u) \, | \, f_1^2(x(v))dudv \\ &\leq\; KR\end{aligned}$$

Given an $\varepsilon > 0$ choose $\delta > 0$ with $0 < \delta < \varepsilon$ such that

$$\sqrt{2}\Big(1 + J + pR\Big)^{1/2}\delta < \varepsilon(1 - J) \tag{4.2.29}$$

Let $x(t) = x(t, t_0, \phi)$ be a solution of equation 4.1.1 with $\| \, \phi \, \| < \delta$. Then for $t \geq t_0$, using Eq. (4.2.23) and inequality (4.2.24) gives

$$\frac{1}{2}\Big(x(t) + \frac{1}{a}\int_0^t G_\alpha(at - s)f(x(s))ds\Big)^2 \leq V(t) \leq V(t_0) \tag{4.2.30}$$

Thus,

$$\frac{1}{2}\Big(x(t)+\frac{1}{a}\int_0^t G_\alpha(at-s)f(x(s))ds\Big)^2$$

$$\leq \quad x^2(t) \; + \; \frac{1}{a}\int_0^t \mid G_\alpha(at-s) \mid f_1^2(x(s))x^2(s)ds$$

$$+ \; P\int_0^t \int_{at-s}^\infty \mid G_\alpha(u) \mid du f_1^2(x(s))x^2(s)ds \qquad (4.2.31)$$

For $t \geq t_0$

$$\int_0^t \int_{at-s}^\infty \mid G_\alpha(u) \mid du f_1^2(x(s))x^2(s)ds$$

$$= \int_0^{t_0} \int_{at_0-s}^\infty \mid G_\alpha(u) \mid du f_1^2(x(s))x^2(s)ds$$

$$+ \int_{t_0}^t \int_{at-s}^\infty \mid G_\alpha(u) \mid du f_1^2(x(s))x^2(s)ds$$

It implies that

$$\frac{1}{2}\Big(x(t)+\frac{1}{a}\int_0^t G_\alpha(at-s)f(x(s))ds\Big)^2$$

$$\leq \Big(x(t)^2+\frac{1}{a}\int_0^\infty \mid G_\alpha(u)\mid f_1^2(x(u))x^2(u)du$$

$$+p\int_0^{t_0}\int_{at_0-s}^\infty \mid G_\alpha(u)\mid du f_1^2(x(s))x^2(s)ds\Big)$$

$$\leq \Big(1+\frac{1}{a}\int_0^\infty \mid G_\alpha(u)\mid f_1^2(x(u))du$$

$$+p\int_0^{t_0}\int_{at_0-s}^\infty \mid G_\alpha(u)\mid f_1^2(x(s))duds\Big)x(t)^2$$

Using, $\mid x(t)\mid<\delta$ and the fact that

$$\frac{1}{2}\Big(x(t)+\frac{1}{a}\int_0^t G_\alpha(at-s)f(x(s))ds\Big)^2$$

$$\leq \Big(1+\frac{1}{a}\int_0^\infty \mid G_\alpha(u)\mid f_1^2(x(u))du$$

$$+p\int_0^{t_0}\int_{at_0-s}^\infty \mid G_\alpha(u)\mid f_1^2(x(s))duds\Big)\delta^2$$

$$< \Big(1+J+pKR\Big)\delta^2$$

Hence;

$$
\begin{aligned}
\frac{1}{2}\Big(x(t)+\frac{1}{a}\int_0^t G_\alpha(at-s)f(x(s))ds\Big)^2 &< \Big(1+J+pKR\Big)\delta^2 \\
\Big(x(t)+\frac{1}{a}\int_0^t G_\alpha(at-s)f(x(s))ds\Big)^2 &< 2\Big(1+J+pKR\Big)\delta^2 \\
x(t)+\frac{1}{a}\int_0^t G_\alpha(at-s)f(x(s))ds &< \sqrt{2\Big(1+J+pKR\Big)\delta^2} \\
&< \sqrt{2}\Big(1+J+pKR\Big)^{1/2}\delta
\end{aligned}
$$

It is claimed that $\mid x(t) \mid< \varepsilon$ for all $t \geq t_0$. Note also that $\mid x(u) \mid< \delta < \varepsilon$ for all $0 \leq u \leq t_0$. If the claim is not true, let $t = t_*$ be the first t such that $\mid x(t_*) \mid= \varepsilon$ and $\mid x(s) \mid< \varepsilon$ for all $t_0 \leq s < t_*$. Then from inequality (4.2.27),

$$
\begin{aligned}
\varepsilon(1-J) &= \varepsilon\Big(1-\frac{1}{a}\int_0^\infty \mid G_\alpha(u)\mid f_1^2(x(u))du\Big) \\
&< \mid x(t_*)+\frac{1}{a}\int_0^{t_*} G_\alpha(at_*-s)f_1^2(x(s))x(s)ds \mid \\
&< \sqrt{2}\Big(1+J+pKR\Big)^{1/2}\delta
\end{aligned}
$$

which contradicts inequality (4.2.27) and completes the proof.

Theorem 4.3 Suppose the hypothesis of Theorem 4.2 hold with $\beta > 0$, where β satisfies condition 4.2.17. If $\int_0^\infty \mid G_\alpha(u) \mid du \in L^1[0,\infty)$ then the zero solution of Eq. (4.1.1) is uniformly asymptotically stable.

Proof.

By Theorem 4.2 the zero solution is uniformly stable. So, for $\epsilon = 1$, find δ of uniform stability. Let $\gamma_1(t) > 0$ be given. Then $T > 0$ would be found such that $[t_0 \geq 0, \| \phi \| < \delta, t \geq t_0 + T]$ implies $| x(t, t_0, \phi) | < \gamma_1(t)$. Since $V' \leq 0$, if t_f is found such that $V(t_f) < \gamma^2$ for a given $\gamma > 0$, then

$$\frac{1}{2}\Big(x(t) + \frac{1}{a}\int_0^t G_\alpha(at - s) f(x(s)) ds\Big)^2 \leq V(t) \leq V(t_f) < \gamma^2 \qquad (4.2.32)$$

for all $t \geq t_f$. Then the lower bound on $V(t)$ is used to show that $| x(t, t_0, \phi) | < \gamma_1(t)$, (γ_1 is a function of t). Now T is found so that for any such solution there will be a $t_f \in [t_0, t_0 + T]$. Since $G_\alpha(u) \in L^1[0, \infty)$, there is a T_* such that

$$\int_{(a-1)T_*}^{\infty} | G_\alpha(u) | f(x(u)) du \leq \frac{a\gamma^2}{4}$$

This gives

$$\frac{1}{a}\int_{(a-1)T_*}^{\infty} | G_\alpha(u) | f(x(u)) du \leq \frac{\gamma^2}{4}. \qquad (4.2.33)$$

Also, from the hypotheses, there is a T_1 such that for all $T > T_1$,

$$\int_{(a-1)T}^{\infty}\int_v^{\infty} | G_\alpha(u) | du dv \leq \frac{a\gamma^2}{4p} \qquad (4.2.34)$$

Thus for $t \geq T$, there is

$$\int_T^t \int_{at-s}^{\infty} \mid G_\alpha(u) \mid duds = \int_{at-t}^{at-T} \int_v^{\infty} \mid G_\alpha(u) \mid dudv$$

$$\leq \int_{(a-1)t}^{\infty} \int_v^{\infty} \mid G_\alpha(u) \mid dudv$$

$$\leq \frac{a\gamma^2}{4p}. \tag{4.2.35}$$

Fix a $T_2 > T_1$. For all $t > T_2$, there is

$$\int_0^t \int_{at-s}^{\infty} \mid G_\alpha(u) \mid duf^2(x(s))ds = \int_0^{T_2} \int_{at-s}^{\infty} \mid G_\alpha(u) \mid duf^2(x(s))ds$$

$$+ \int_{T_2}^t \int_{at-s}^{\infty} \mid G_\alpha(u) \mid duf^2(x(s))ds. \tag{4.2.36}$$

Also, since $G_\alpha(u) \in L^1$, pick a $T_3 > T_2$, such that for $t > T_3$,

$$T_2 \int_{at-T_2}^{\infty} \mid G_\alpha(v) \mid dv = \int_{t-T_3}^{\infty} \mid G_\alpha(v) \mid dv$$

$$< \frac{a\gamma^2}{4p}$$

Thus,

$$\int_{at-T_2}^{\infty} \mid G_\alpha(v) \mid dv < \frac{\gamma^2}{4pT_2}. \tag{4.2.37}$$

Using Eq. (4.2.37), the first integral on the right hand side of Eq. (4.2.36) satisfies

$$\int_0^{T_2}\int_{at-s}^{\infty} \mid G_\alpha(u) \mid du f^2(x(s))ds \leq T_2 \int_{at-T_2}^{\infty} \mid G_\alpha(u) \mid du < \frac{\gamma^2}{4p}$$

Also, using Eq. (4.2.35), the second integral on the right side of Eq. (4.2.36) becomes

$$\begin{aligned} \int_{T_2}^{t}\int_{at-s}^{\infty} \mid G_\alpha(u) \mid du f^2(x(s))ds &\leq \int_{T_2}^{t}\int_{at-s}^{\infty} \mid G_\alpha(u) \mid duds \\ &\leq \frac{\gamma^2}{4p}. \end{aligned}$$

Thus, Eq. (4.2.36), implies that

$$\int_0^{t}\int_{at-s}^{\infty} \mid G_\alpha(u) \mid du f^2(x(s))ds \leq \frac{\gamma^2}{4p}.$$

Hence,

$$p\int_0^{t}\int_{at-s}^{\infty} \mid G_\alpha(u) \mid du f^2(x(s))ds \leq \frac{\gamma^2}{4} \tag{4.2.38}$$

for $t > T_3$. Next it is claimed that $x(t) \in L^1$. To see this , let $t \geq t_0 \geq 0$. Then integrating inequality (4.2.24) from t_0 to t, yields;

$$\int_{t_0}^{t} \frac{dV(s)}{ds} ds \leq \int_{t_0}^{t} -\beta x^2(s)ds$$

$$\int_{t_0}^{t} V(s) \leq -\int_{t_0}^{t} \beta x^2(s)ds$$

$$V(t) - V(t_0) \leq -\int_{t_0}^{t} \beta x^2(s)ds.$$

This implies that

$$\int_{t_0}^{t} \beta x^2(s)ds \leq V(t_0) - V(t).$$

Therefore

$$\int_{t_0}^{t} \beta x^2(s)ds \leq V(t_0) - V(t) \leq V(t_0) < (1 + J + pKR)$$

Let $T_4 = (1 + J + pKR)/\beta(\gamma/2)^2$. Now it is claimed that every interval of length T_4 contains a τ such that $\mid x(\tau) \mid < \gamma/2$. If the claim is not true, then $\mid x(t) \mid < \gamma/2$ for $t \in [t_1, t_1 + T_4]$ for some $t_1 \geq t_0$. Inequality (4.2.24), implies that

$$V(t) - V(t_0) \leq -\int_{t_0}^{t} \beta x^2(s)ds$$

Thus,

$$V(t) \leq V(t_0) - \int_{t_0}^{t} \beta x^2(s)ds$$

This implies that

$$V(t_1 + T_4) \leq V(t_1) - \int_{t_1}^{t_1+T_4} \beta x^2(s) ds$$

$$\leq V(t_1) - \int_{t_1}^{t_1+T_4} \beta \Big(\frac{\gamma}{2}\Big)^2 ds$$

$$\leq V(t_1) - \beta \Big(\frac{\gamma}{2}\Big)^2 T_4$$

$$= V(t_0) - (1 + J + pKR)$$

$$< 0,$$

which contradicts $V(t) \geq 0$ for all $t \geq 0$.

For $t > t_0 + T_* + T_3$, observe that both inequality (4.2.33) and inequality (4.2.38) hold. Moreover, there is a $t_f \in [t_0 + T_* + T_3, t_0 + T_* + T_3 + T_4]$ such that $x^2(t_f) < \gamma/4$ since this interval has length T_4. Consequently, by Eq. (4.1.2), for $t \geq t_f$,

$$\frac{1}{2}\Big(x(t) + \frac{1}{a}\int_0^t G_\alpha(at - s) f(x(s)) ds\Big)^2 \leq V(t) \leq V(t_f)$$

This implies that

$$
\begin{aligned}
&\frac{1}{2}\Big(x(t)+\frac{1}{a}\int_0^t G_\alpha(at-s)f(x(s))ds\Big)^2\\
=\;&\frac{1}{2}\Big(x(t_f)+\frac{1}{a}\int_0^{t_f} G_\alpha(at-s)f(x(s))ds\Big)^2\\
&+p\int_0^{t_f}\int_{at_f-s}^{\infty}\mid G_\alpha(u)\mid du f^2(x(s))ds\\
=\;&\frac{1}{2}\Big[x^2(t_f)+\frac{2x(t_f)}{a}\int_0^{t_f} G_\alpha(at_f-s)f(x(s))ds\\
&+\Big(\frac{1}{a}\int_0^{t_f} G_\alpha(at_f-s)f(x(s))ds\Big)^2\Big]\\
&+P\int_0^{t_f}\int_{at_f-s}^{\infty}\mid G_\alpha(u)\mid du f^2(x(s))ds\\
=\;&\frac{1}{2}\Big[x^2(t_f)+\Big(\frac{1}{a}\int_0^{t_f} G_\alpha(at_f-s)f(x(s))ds\Big)^2\\
&+\frac{2x(t_f)}{a}\int_0^{t_f} G_\alpha(at_f-s)f(x(s))ds\Big]\\
&+P\int_0^{t_f}\int_{at_f-s}^{\infty}\mid G_\alpha(u)\mid du f^2(x(s))ds\\
=\;&\frac{1}{2}\Big[x^2(t_f)+\frac{1}{a^2}\Big(\int_0^{t_f} G_\alpha(at_f-s)f(x(s))ds\Big)^2\\
&+\frac{2x(t_f)}{a}\int_0^{t_f} G_\alpha(at_f-s)f(x(s))ds\Big]\\
&+P\int_0^{t_f}\int_{at_f-s}^{\infty}\mid G_\alpha(u)\mid du f^2(x(s))ds
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{x^2(t_f)}{2} + \frac{1}{2a^2}\Big(\int_0^{t_f} G_\alpha(at_f - s)f(x(s))ds\Big)^2 \\
&\quad + \frac{x(t_f)}{a}\int_0^{t_f} G_\alpha(at_f - s)f(x(s))ds \\
&\quad + P\int_0^{t_f}\int_{at_f-s}^{\infty} \mid G_\alpha(u) \mid du f^2(x(s))ds \\
&\leq \frac{x^2(t_f)}{2} + \frac{1}{2a^2}\int_0^{t_f} \mid G_\alpha(at_f - s) \mid ds \int_0^{t_f} \mid G_\alpha(at_f - s) \mid f^2(x(s))ds \\
&\quad + \frac{x^2(t_f)}{2} + \frac{1}{2a^2}\int_0^{t_f} \mid G_\alpha(at_f - s) \mid ds \int_0^{t_f} \mid G_\alpha(at_f - s) \mid f^2(x(s))ds \\
&\quad + P\int_0^{t_f}\int_{at_f-s}^{\infty} \mid G_\alpha(u) \mid du f^2(x(s))ds \\
&= \frac{x^2(t_f)}{2} + \frac{x^2(t_f)}{2} \\
&\quad + \frac{1}{2a^2}\int_0^{t_f} \mid G_\alpha(at_f - s) \mid ds \int_0^{t_f} \mid G_\alpha(at_f - s) \mid f^2(x(s))ds \\
&\quad + \frac{1}{2a^2}\int_0^{t_f} \mid G_\alpha(at_f - s) \mid ds \int_0^{t_f} \mid G_\alpha(at_f - s) \mid f^2(x(s))ds \\
&\quad + P\int_0^{t_f}\int_{at_f-s}^{\infty} \mid G_\alpha(u) \mid du f^2(x(s))ds
\end{aligned}
$$

$$
\begin{aligned}
&= x^2(t_f) \\
&\quad + \frac{1}{a^2}\int_0^{t_f} | G_\alpha(at_f - s) | ds \int_0^{t_f} | G_\alpha(at_f - s) | f^2(x(s))ds \\
&\quad + P\int_0^{t_f}\int_{at_f-s}^{\infty} | G_\alpha(u) | du f^2(x(s))ds \\
&\leq x^2(t_f) \\
&\quad + \frac{1}{a^2}\int_0^{\infty} | G_\alpha(u) | ds \int_0^{t_f} | G_\alpha(at_f - s) | f^2(x(s))ds \\
&\quad + P\int_0^{t_f}\int_{at_f-s}^{\infty} | G_\alpha(u) | du f^2(x(s))ds \\
&\leq x^2(t_f) \\
&\quad + \frac{1}{a}\int_0^{t_f} | G_\alpha(at_f - s) | f^2(x(s))ds \\
&\quad + P\int_0^{t_f}\int_{at_f-s}^{\infty} | G_\alpha(u) | du f^2(x(s))ds \\
&\leq x^2(t_f) \\
&\quad + \frac{1}{a}\int_{(a-1)t_f}^{\infty} | G_\alpha(u) | f^2(x(u))du \\
&\quad + P\int_0^{t_f}\int_{at_f-s}^{\infty} | G_\alpha(u) | du f^2(x(s))ds \\
&< \frac{\gamma^2}{4} + \frac{\gamma^2}{4} + \frac{\gamma^2}{2} \\
&= \gamma^2
\end{aligned}
$$

Therefore, for $t \geq t_f$

$$\frac{1}{2}\Big(x(t) + \frac{1}{a}\int_0^t G_\alpha(at-s)f(x(s))ds\Big)^2 = \gamma^2$$

Thus,,

$$\Big(x(t) + \frac{1}{a}\int_0^t G_\alpha(at-s)f(x(s))ds\Big)^2 = 2\gamma^2$$

This implies that,

$$x(t) + \frac{1}{a}\int_0^t G_\alpha(at-s)f(x(s))ds = \sqrt{2\gamma^2}$$

which gives,

$$x(t) + \frac{1}{a}\int_0^t G_\alpha(at-s)f(x(s))ds = \sqrt{2}\gamma$$

Hence,

$$\mid x(t) + \frac{1}{a}\int_0^t G_\alpha(at-s)f(x(s))ds \mid < \sqrt{2}\gamma.$$

It follows from the above inequality that

$$\mid x(t) \mid - \left|\frac{1}{a}\int_0^t G_\alpha(at-s)f(x(s))ds\right| < \sqrt{2}\gamma \tag{4.2.39}$$

Since $t_f \geq T_*$, it follows from inequality (4.2.34) and inequality (4.2.39) that

$$\begin{aligned} \mid x(t) \mid &< \left|\frac{1}{a}\int_0^t G_\alpha(at-s)f(x(s))ds\right| + \sqrt{2}\gamma \\ &< \frac{1}{a}\int_0^t \mid G_\alpha(at-s) \mid f(x(s)) \mid ds + \sqrt{2}\gamma \end{aligned}$$

$$\leq \quad \frac{\gamma^2}{4} + \sqrt{2}\gamma$$

$$= \quad \gamma_1(t).$$

for $t \geq t_f$. This completes the proof.

4.4 Chapter Summary

In this chapter, results concerning stability, uniform stability, and uniform asymptotic stabilty of Eq. (4.2.1) were established. In the process, the Lyapunov function constructed was used to deduce inequalities regarding the solution of the nonlinear Volterra integrodifferential equation from which the stability, uniform stability, and uniform asymptotic stability were obtained. .

CHAPTER FIVE

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

5.1 Overview

This chapter provides the summary, conclusions and recommendations of the study. The summary explains briefly the research problem, objectives of the study, the method used and the results obtained. The conclusions entail the results of the study based on the set objectives. The recommendations suggest possible areas for further research.

5.2 Summary

The primary aim of this research was to obtain conditions that are sufficient for the zero solution of a scalar nonlinear Volterra integrodifferential equation to be stable, uniformly stable and uniformly asymptotically stable. In order to obtain these conditions, the Lyapunov's direct method was used and a Lyapunov functional was carefully constructed. The Lyapunov functional aided in the construction of some inequalities which were used to derive sufficient conditions for the stability of the zero solution of a scalar nonlinear Volterra integrodifferential equation to be stable, uniformly stable and uniformly asymptotically stable.

5.3 Conclusions

Sufficient conditions for the zero solution of nonlinear Volterra integrodifferential equations to be stable have been obtained.

Again, conditions that are sufficient for the zero solution of nonlinear Volterra integrodifferential equations to be uniformly stable have been obtained.

Lastly, conditions that are sufficient for the zero solution of nonlinear Volterra integrodifferential equations to be uniformly asymptotically stable have been established.

5.4 Recommendations

Future work can be done on Stability of nonlinear Volterra integrodifferential equations with numerical simulations.

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