UNIVERSITY OF CAPE COAST

SUFFICIENT CONDITIONS FOR THE STABILITY OF A SYSTEM OF NON-LINEAR VOLTERRA DIFFERENCE EQUATIONS WITH VARIABLE DELAY

KWEKU ABEEKU MENSAH

2022

UNIVERSITY OF CAPE COAST

SUFFICIENT CONDITIONS FOR THE STABILITY OF A SYSTEM OF NON-LINEAR VOLTERRA DIFFERENCE EQUATIONS WITH VARIABLE DELAY

BY

KWEKU ABEEKU MENSAH

Thesis submitted to the Department of Mathematics of the School of Physical Sciences, College of Agriculture and Natural Sciences, University of Cape Coast, in partial fulfilment of the requirements for the award of Master of Philosophy degree in Mathematics

DECEMBER, 2022

DECLARATION

Candidate's Declaration

I hereby declare that this thesis is the result of my own original research and that no part of it has been presented for another degree in this university or elsewhere.

Candidate's Signature .... Date ....

Name: Kweku Abeeku Mensah

Supervisor's Declaration

I hereby declare that the preparation and presentation of the thesis were supervised in accordance with the guidelines on supervision of thesis laid down by the University of Cape Coast.

Principal Supervisor's Signature .... Date ....

Name: Prof. Ernest Yankson

ABSTRACT

This research seeks to obtain sufficient conditions for the zero solution of a system of non-linear Volterra difference equations with variable delay to be stable. The Lyapunov's direct method is employed in the research to establish the sufficient conditions. In the process a suitable Lyapunov function is constructed which is then used to obtain inequalities that relate the solution of the system of difference equations.

The obtained inequalities were then utilized to obtain results for which the system of non-linear Volterra difference equations has a stable zero solution.

To illustrate the power of the obtained sufficient conditions, an example was constructed.

KEY WORDS

Difference equation

Lyapunov function

Non-linear difference equation

Stability

Variable delay

Volterra difference equation

ACKNOWLEDGEMENTS

Thanks be to God for his wisdom, protection, guidance and traveling mercies on me throughout for making this thesis a reality.

Professor Ernest Yankson, (my supervisor) of the Department of Mathematics, University of Cape Coast deserves the deepest appreciation for his encouragement, open-door policy approach to students supervision, understanding, passion, pieces of advice, to mention only a few.

Special thanks goes to Mrs. Esther Mensah (my wife), Kweku Adom Mensah (my son), Maame Araba Nyamebekyere and Efuwa Nhyira Mensah (my daughters) for their support and cooperation physically and spiritually from beginning to the end of this work.

I appreciate the assistance from Mary Caroline Kakraba (ucc cash office, Cape Coast), Adwoa Akyaamaa (Assin Foso), my church ministers and members of Wesley Methodist Cathedral, Assin Foso, Ghana, Mr. Anthony Quansah (Akotokyir, Cape Coast), and also encouragement and advice from teachers of Obiri Yeboah Senior High School, Assin Foso, Ghana. Thanks to all my friends who in one way or another supported me for the success of this piece of research work. God bless you all.

DEDICATION

To my family

TABLE OF CONTENTS

CHAPTER ONE

INTRODUCTION

1.1 Overview

In this chapter, the background of the study, some identified areas of application of the difference equations, stability, the problem statement and the objective of the research will be briefly looked at.

1.2 Background to the Study

Almost everything happening in this physical world can be modelled in mathematical relationship (equation), and Thomas Malthus in 1798 happened to be one of those who made earliest attempts to model the human population growth by mathematical means.

It is most often so fascinating studying the behaviour of most of the real-life system or phenomenon, be it sociological, physical, or economics in expressions of mathematical terms, that are referred to as mathematical models. The formulated model is usually made up of time-difference of one or more of the variables, and so the mathematical expression (model) contains differences with time making the entire model a difference equation or a system of difference equations, and mathematicians challenge themselves to solving them.

Sultana (2015), explained that quite a number of parameters are permitted to assume any suitable amount on given intervals of real line in an attempt to representing real-life problems by using mathematical concepts. The parameters are referred to as continuous parameters/variables. Difficult computations are encounted because the parameters sometimes turn out to be discontinuous for some real-world problems, and an example is an investment with compound interest. In this case, it is necessary and appropriate to choose discrete variables. The parameters are then evaluated at discrete times. The values would be the same over each period of time and then switch to another period as time progresses, that is, the time interval becomes ± 1 to the next. In this case, each of the parameters under consideration is measured at one time only, at each interval of time, and a finite result is attained for the measurements that are in-between any two consecutive periods of time. The result being that, either their own initial values or other values can represent these parameters and the outcome relation of this operation is termed a recurrence formula, now called a difference equation.

Difference equations are said to have numerous uses in all facets of field of study except for few areas. The applicable areas are in electrical networks, stochastic time series, genetics, probabilities, economics, number theory, sociology, psychology, and combinatorics (see Sultana (2015)).

According to Elaydi (2005), difference equations describe populations or objects that develop discretely in which time (or independent variable) is a subset of the set of integers.

The studies on difference equations have received significant improvement and development particularly in recent times due to the advancements in computerization, and this is due to the fact that approximated difference equations are used to develop the needed formulation in computer-assisted programs to solve problems (see Sultana (2015)).

There are times that, it becomes very difficult when solving certain types of differential equations. Under these circumstaces, a numerical scheme becomes necessary to be used in approximating the solution of the differential equation. An attempt to use numerical approach leads to the construction of a related difference equation that is more compliant with computations. Therefore, difference equations as described by scholars is the discrete analogue representation for differential equations.

Solving mathmatical problems usually involves equations that lead to computing the value of a function repeatedly using a set of values given, and resulting equations are said to be difference equations. The equations frequently occur in distinct structures in all aspects of mathematics in theory and uses in dynamical systems, electrical circuit analysis, computing, statistics, biology, economics, and other related fields.

The study of dynamical systems (systems that vary with time flow, or set of time-dependent variables) is so fashionable in modern day mathematics and a lot has been done by scholars to show that indeed mathematics has real-world uses. Dynamical systems are classified into discrete-time or continuous-time systems. For the purposes of this thesis we will be concerned with only the discrete-time dynamical systems. That is, systems in which all the variables are defined over discrete range of values of time. Some systems have one or more of their parameters inherently discrete and appropriate difference equations are used to model such parameters due to their discrete character. De Moivre and his associates Euler, Lagrange, Laplace, and others in the eighteenth century developed the fundamental concept of linear difference equations.

One of the famous Italian mathematicians who also doubled as the great physicist known as Vito Volterra made tremendous contributions to the development of difference equations which cannot be overemphasized particularly in the discovery and development of mathematical biology.

According to Raffoul (2018), Volterra difference equations give a more pragmatic and practical model for comprehensive scope of occurrences in engineering and natural sciences.

1.3 Some Areas of Application of Difference Equations

In this section, the discussions will be on how difference equations have been employed to solve problems in other areas of study including sciences, economics, business, logistics, engineering and many more.

For instance, according to Elaydi (2005) difference equations were used to model how a drug is administered to a patient once every five hours at a health facility; let $G(i)$ represent the quantity of drug in the patient's blood system of a patient at the ith interval of time. Certain amount d of the drug is eliminated by the body over each interval of time. It is assumed that G_0 represents the amount administered, and the quantity of drug that is contained in patient's blood system at time $(i+1)$ is equal to the amount at time i minus the fraction d that has been eliminated from the body, plus the new dosage G_0, then

$$G(i+1) = (1-d)G(i) + G_0.$$

The term Amortization refers to the sequential repayment of a loan over a period of time. Each instalment constitutes part payment of interest and part payment of the outstanding principal; If $x(i)$ represents the outstanding principal after the ith payment $h(i)$. Let us assume that interest is charged compoundedly at the rate n per payment period. The model formulated here is based on the fact that the outstanding principal $q(i+1)$ after the $(i+1)st$ payment is equal to the outstanding principal $x(i)$ after the ith payment plus the interest $nx(i)$ incurred during the $(i+1)st$ period minus the ith payment $h(i)$. Hence

$$q(i+1) = x(i) + nx(i) - h(i)$$

represents the modelled difference equation (see Elaydi (2005)).

Electronic transmission of information in signal system can be modelled with difference equations; Let z_1 and z_2 be two signals that require exactly η_1 and η_2 units of time respectively, for transmission. Let $T(\eta)$ be the number of possible message sequences of duration η, a signal of duration time η either ends with an z_1 signal or with an z_2 signal. If the message ends with z_1, the last signal must start at $\eta - \eta_1$ (since z_1 takes η_1 units of time). Hence there are $M(\eta - \eta_1)$ possible messages to which the last z_1 may be appended. Hence there are $T(\eta - \eta_1)$ messages of duration η that end with z_1. In the same vein, one may conclude that there are $T(\eta - \eta_2)$ messages of duration η that end with z_2. Eventually, the total number of messages $x(\eta)$ of duration η may be given by

$$T(\eta) = T(\eta - \eta_1) + T(\eta - \eta_2).$$

If $\eta_1 \geq \eta_2$, then the above equation may be written in the familiar form of an $\eta_1 th$-order equation

$$T(\eta + \eta_1)T(\eta + \eta_1\eta_2) - T(\eta) = 0.$$

On the other hand, if $\eta_1 \leq \eta_2$, then we obtain the $\eta_2 th$-order equation given by

$$T(\eta + \eta_2) - T(\eta + \eta_2 - \eta_1)T(\eta) = 0$$

is obtained.

In inventory analysis, define $S(\eta)$ to be the number of units of consumer goods produced for sale in period η, and $R(\eta)$ be the number of units of consumer goods produced for inventories in period η. Assume that there is a constant noninduced net investment B_0 in each period. Then the total income $A(\eta)$ produced in time η is given by

$$A(\eta) = R(\eta) + S(\eta) + B_0.$$

A system or an experiment is said to stable if its behaviour is such that a little variation in the initial information leads to a little variation for future time. In other words, if ψ being a solution for a difference equation turns out to be stable, then it means that all other solutions which have initial information near to ψ will continue to stay close to ψ in the fullness of time.

1.4 Statement of the Problem

Records show that a good number of research work have been done on the qualitative belaviour of nonlinear Volterra difference equations or systems without variable delay. Elaydi et al. (1999), Medina (2001) and Eid et al. (2015) obtained stability results for discrete Volterra systems with finite delay. The results obtained however by Elaydi et al. (1999), Medina (2001) and Eid et al. (2015) do not hold for the discrete Volterra system of equations with variable delay

$$x(i+1) = Mx(i) + \sum_{j=i-\gamma(i)}^{i-1} F(i,j)g(x(j)). \tag{1.1}$$

where M is a $\tau \times \tau$ constant matrix, $F(i, j)$ is a $\tau \times \tau$ matrix of functions, $g : \mathbb{R}^\tau \to \mathbb{R}^\tau$, and $\gamma : \mathbb{Z} \to \mathbb{R}^+$. Equation (1.1) is referred to as a non-linear system of Volterra difference equation with variable delay. Against this background, there is the need to study the non-linear system of Volterra difference equation with variable delay.

1.5 Research Objectives

The objectives of this thesis are:

i) to construct a suitable Lyapunov function for determining the stability of the zero solution of (1.1); and

ii) to obtain sufficient conditions that will ensure that the zero solution of equation (1.1) is stable.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

Difference equations and Lyapunov functions in general will be discussed in this Chapter. Some of the earlier works on difference equations, Lyapunov functions, Volterra equations, stability, and non-linear equations will also be looked at in this chapter. A review of some relevant literature on difference equations will be considered.

2.2 Difference Equations

Elaydi (2005), explained that difference equations customarily outline the growth of definite occurrences as time goes by. He sited that, choosing a definite population that has discrete generations, and as a result, x at $(i+1)st$, the generation is $x(i+1)$. Moreover, the population size defines a function of the generation $x(i)$ at ith. Consequently, this relation can simply be expressed in the difference equation as given by

$$x(i+1) = \tau(x(i)).$$

Agarwal (1992), stated that the importance of difference equations has recently been enhanced by the discretization methods applied to differential equations when seeking their numerical solution. The theory of non-linear and linear Volterra difference equations provides significant mathematical models for such field of study as science, business, sociology, economics and also engineering for several world-life phenomena. By this, it is very necessary for researchers to provide insightful analysis on the qualitative behaviour of both non-linear and linear Volterra difference equations without necessarily figuring out for their solutions.

2.2.1 Linear Difference Equations The following linear difference equation is of first order

$$x(i+1) - m(i)x(i) = \gamma(i). \tag{2.1}$$

This is because it contains the values of x at i and $i+1$ only, just as it is in the first order difference operator

$$\Delta x(i) = x(i+1) - x(i).$$

If $m(i) = 1$ for all i, then equation (2.1) above is as simple as

$$\Delta x(i) = \gamma(i),$$

and the corresponding solution is

$$x(i) = \sum \gamma(i) + F(i),$$

where $\Delta F(i) = 0$. For convenience, let us assume that a discrete set, $i = k, k+1, k+2, ,$ defines the domain and also a function $m(i)$ with $m(i) \neq 0 \; \forall \; i$ exists. So the following relation

$$\theta(i+1) = m(i)\theta(i) \tag{2.2}$$

defines the linear homogeneous first-order difference equation (See Kelley & Peterson (2001))

Thus, the following relation is obtained

$$\begin{aligned} \theta(k+2) &= m(k+1)\theta(k+1) \\ &= m(k+1)m(k)\theta(k), \\ &\vdots \\ \theta(k+i) &= \theta(k) \textstyle\prod_{j=0}^{i-1} M(k+j). \end{aligned}$$

In a more convenient form the solution is represented as

$$\theta(i) = \theta(k) \prod_{j=k}^{i-1} M(j), (i = k, k+1, ...)$$

where $\prod_{j=k}^{i-1} \equiv 1$ and, for $i \geq k+1$, for which the product is defined over $k, k+1, ..., i-1$.

Now, the difference equation

$$z(i+1) - m(i)z(i) = \gamma(i), \tag{2.3}$$

which is linear and nonhomogeneous is considered.

Applying a technique similar to the variation of parameters (reduction of order) used in differential equations, equation (2.3) can then be solved. Let $z(i) = v(i)\rho(i)$, and that $v(i)$ be the solution of the nonhomogeneous difference equation (2.3), that is, any form of nontrivial solution of (2.3) and $\rho(i)$ is found as

$$\theta(i+1)\rho(i+1) - m(i)\theta(i)\rho(i) = \gamma(i). \tag{2.4}$$

Using equation (2.2) in equation (2.4) gives

$$m(i)\theta(i)\rho(i+1) - m(i)\theta(i)\rho(i) = \gamma(i).$$

which gives

$$m(i)\theta(i)\Big[\rho(i+1) - \rho(i)\Big] = \gamma(i).$$

Thus,

$$m(i)\theta(i)\Delta\rho(i) = \gamma(i).$$

This implies that

$$\theta(i+1)\Delta\rho(i) = \gamma(i)$$

It follows that

$$\Delta\rho(i) = \sum \frac{\gamma(i)}{E\theta(i)},$$

and so

$$\rho(i) = \sum \frac{\gamma(i)}{E\theta(i)} + \Omega,$$

where Ω is any arbitrary constant.

Thus,

$$\begin{aligned} z(i) &= \theta(i)\Big[\sum \frac{\gamma(i)}{E\theta(i)} + \Omega\Big] \\ &= \theta(k)\prod_{j=k}^{i-1} M(j)\Big[\sum \frac{\gamma(i)}{E\theta(i)} + \Omega\Big]. \end{aligned}$$

2.2.2 The Difference Operator According to Kelley & Peterson (2001), the difference operator is one of the very basic properties which are very crucial with regard to the study of difference equations. It is the same as the differential operator which acts as a pivot to the study of the differential calculus.

Definition 2.1 Let us assume that $\chi(i)$ is a sequence consisting of real numbers or complex. The difference operator Δ as described above is defined by

$$\Delta\chi(i) = \chi(i+1) - \chi(i),$$

where $i \in \mathbb{N}_0$, and $\mathbb{N}_0 = \{0, 1, 2, ...\}$.

It is clearly noticed that higher order differences can be obtained via the usual iterations of the fundamental difference operator.

Thus, the next order difference (i. e. the 2nd order) can be iteratively obtained as follows:

$$\begin{aligned}\Delta^2\chi(i) &= \Delta(\Delta\chi(i)) \\ &= \Delta(\chi(i+1) - \chi(i)) \\ &= (\chi(i+2) - \chi(i+1)) - (\chi(i+1) - \chi(i)) \\ &= (\chi(i+2) - 2\chi(i+1) + \chi(i)\end{aligned}$$

Definition 2.2 For any $i \in \mathbb{N}$,

$$\Delta^i h(\gamma) = \sum_{\chi=0}^{i} (-1)^\chi \begin{pmatrix} i \\ \chi \end{pmatrix} h(\gamma + i - \chi)$$

The shift and the identity operators which are also useful in difference calculus are defined as follows:

Definition 2.3 The Shift operator E is defined by

$$E\chi(i) = h(i+1).$$

Definition 2.4 The Identity operator I is defined by

$$I\chi(i) = \chi(i)$$

It is noted that the composition of I and E is the same as multiplication of numbers. Clearly, $\Delta = E - I$, therefore, the Definition 2.2 is verified as the binomial theorem:

$$\begin{aligned}\Delta^n x(i) &= (E-I)^n x(i) \\ &= \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} (-I)^k E^{n-k} x(i) \\ &= \sum_{k=0}^{n} (-I)^k \begin{pmatrix} n \\ k \end{pmatrix} x(i+n-k).\end{aligned}$$

It follows that,

$$E^n x(i) = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \Delta^{n-k} x(i).$$

The fundamental properties of Δ are clearly stated in the following theorem.

Theorem 2.1

For any $\theta, \rho \in \mathbb{N}$ and any $i \in \mathbb{R}$:

(a) $\Delta^{\theta}(\Delta^{\rho}\chi(i)) = \Delta^{\theta+\rho}\chi(i)$.

(b) $\Delta(x(i) + \chi(i)) = \Delta x(i) + \Delta\chi(i)$.

(c) $\Delta(i\chi(i)) = i\Delta\chi(i)$.

(d) $\Delta(x(i)\chi(i)) = x(i)\Delta\chi(i) + E\chi(i)\Delta x(i)$.

(e) $\Delta\left(\frac{x(i)}{\chi(i)}\right) = \frac{\chi(i)\Delta x(i) x(i)\Delta\chi(i)}{\chi(i)E\chi(i)}$.

Proof. The definitions 2.1 - 2.4 can be used to prove parts (a), (b) and (c) of Theorem 2.1. The following calculation verifies part (d) of Theorem 2.1. Thus,

$$\begin{aligned} \Delta(x(i)\chi(i)) &= x(i+1)\chi(i+1)x(i)\chi(i) \\ &= x(i+1)\chi(i+1)x(i)\chi(i+1) + x(i)\chi(i+1)x(i)\chi(i) \\ &= \chi(i+1)(x(i+1)x(i)) + x(i)(\chi(i+1)\chi(i)) \\ &= x(i)\Delta\chi(i) + E\chi(i)\Delta x(i). \end{aligned}$$

In a similar marner, part (e) of Theorem 2.1 can also be proven.

2.2.3 Summation: Kelley & Peterson (2001) further explained that it is important to introduce an antidifference operator which is termed as the right inverse operator, here, it is sometimes called the "indefinite sum" in order to make effective application of the difference operator.

Definition 2.5 The Indefinite Sum which is also known as Antidifference of $\chi(i)$, represented by $\sum \chi(i)$, is defined as any given function such that

$$\Delta\left(\sum \chi(i)\right) = \chi(i)$$

for all i in the domain of χ.

As it is in integration of differential calculus, summation in difference calculus demands what we call a summation constant, though might not be constant all the time.

Theorem 2.2

Let $u(i)$ be an indefinite sum of $\chi(i)$, then every indefinite sum of $\chi(i)$ can be expressed as

$$\sum \chi(i) = u(i) + G(i),$$

believing that $G(i)$, χ and $\Delta G(i) = 0$ have same domain.

Let χ have the real numbers set domain, then

$$\Delta G(i) = 0,$$

which clearly indicates that

$$G(i + 1) = G(i),$$

and that means G is a one-periodic function.

Definition 2.6 Let $\chi(i)$ have its domain set to be the form $\{\beta, \beta + 1, \beta + 2, ...\}$, and β is any real number, and also $u(i)$ being an indefinite sum of $\chi(i)$, then every indefinite sum of $\chi(i)$ will have the form

$$\sum \chi(i) = u(i) + K,$$

where K is an arbitrary constant.

For the sake of convenience, the following convention is applied

$$\sum_{\theta=i}^{j} \chi(\theta) = 0 \tag{2.5}$$

whenever $i > j$. Note that if m is fixed and $n \geq m$,

$$\Delta_i\Big(\sum_{\theta=m}^{i-1}\chi(\theta)\Big) = \chi(i), \tag{2.6}$$

and also if r is fixed and $q \geq p$,

$$\Delta_i\Big(\sum_{\theta=p}^{q}\chi(\theta)\Big) = -\chi(i). \tag{2.7}$$

Some of the general properties of the indefinite sum can be obtained from Theorem 2.1.

Theorem 2.3

Given an arbitrary constant c

(a) $\sum(x(i) + \chi(i)) = \sum x(i) + \sum \chi(i)$.

(b) $\sum(cx(i)) = c\sum x(i)$.

(c) $\sum(x(i)\Delta\chi(i)) = x(i)\chi(i) - \sum E\chi(i)\Delta x(i)$.

(d) $\sum(Ex(i)\Delta\chi(i)) = x(i)\chi(i) - \sum \chi(i)\Delta x(i)$.

Remark 2.1. Parts (c) and (d) of Theorem 2.3 are referred to as "summation by parts" relations.

Proof. Clearly, inductions from Theorem 2.1 lead directly to parts (a) and (b). From part (d) of Theorem 2.1, it follows that

$$\Delta(x(i)\chi(i)) = x(i)\Delta\chi(i) + E\chi(i)\Delta x(i).$$

Also from Theorem 2.2, it follows that

$$\Delta(x(i)\Delta\chi(i) + E\chi(i)\Delta x(i)) = x(i)\chi(i) + F(i).$$

and so (c) follows from manupulations of (a) and (c) is also manupulated to yield (d). The proof is complete.

The summation by parts relations in difference calculus could also be utilized to calculate some types of indefinite sums and this appears to same as the formula for integration by parts which is used to calculate integrals in differential calculus. Also, these formulas appear to be essential fundamental tools for difference equations analysis. Now, given $r < q$, from Definition (2.3) the following can be deduced

$$\sum y_q = \sum_{\tau=r}^{q-1} y_\tau + K \tag{2.8}$$

for certain constant K and, alternatively, for $q \leq p$,

$$\sum y_q = -\sum_{\tau=q}^{p} y_\tau + K \tag{2.9}$$

for some constant K. Indefinite sums can now be related to definite sums from the above two equations.

The theorem below carries a practical formula for calculating definite sums, which is analogous to the fundamental theorem of differential calculus.

Theorem 2.4 (The Fundamental Theorem of difference Calculus)
Let $j(k)$ be an indefinite sum of $\eta(k)$, then for $z < k$

$$\sum_{\gamma=z}^{i-1} \eta(\gamma) = \Big[j(\gamma)\Big]_z^k = j(k) - j(z).$$

The next theorem provides a version of the summation by parts formula for definite sums.

Theorem 2.5 *If $r < k$, then*

$$\sum_{\tau=r}^{i-1} a_\tau \Delta b_\tau = \Big[a_\tau b_\tau\Big]_r^k = \sum_{\tau=r}^{i-1} \Big(\Delta a_\tau\Big) b_{\tau+1}.$$

Proof. Let $x_k = a_k$ and $h_k = b_k$, then by part (c) of Theorem 2.3, it follows that

$$\sum a_k \Delta b_k = a_k b_k - \sum (\Delta a_k) b_{k+1}.$$

By Theorem 2.4,

$$\sum_{\tau=r}^{i-1} a_\tau \Delta b_\tau = \Big[a_\tau b_\tau\Big]_r^k - \sum_{\tau=r}^{i-1} \Big(\Delta a_\tau\Big) b_{\tau+1} + K.$$

Substituting $k = r+1$ in the above equation, we have

$$\sum a_r \Delta b_r = a_{r+1} b_{r+1} - (\Delta a_r) b_{r+1} + K.$$

implying that $K = -a_r b_r$, and that completes the proof.

Remark 2.2 An equivalent form of Theorem 2.5 is Abel's summation formula

$$\sum_{\tau=z}^{i-1} a_\tau b_\tau = b_k \sum_{\tau=z}^{i-1} a_\tau - \sum_{\tau=z}^{i-1} \Big(\sum_{q=z}^{\tau} a_q\Big) \Delta b_\tau, (k > z)$$

and the alternative form is

$$\sum_{\tau=k}^{\eta} a_\tau b_\tau = b_{i-1} \sum_{\tau=i}^{\eta} a_\tau + \sum_{\tau=k}^{\eta} \Big(\sum_{q=\tau}^{\eta} a_q\Big) \Delta b_{\tau-1}, (\eta > k).$$

2.3 Stability

It is prudent to study the behaviour of a transition of say, the discrete generation of a certain population at initial time i_0 and $(i_0 + 1)$. This brings about the study of stability theory for difference equations.

Definition 2.7 (Stability)

A solution $\psi(i)$ of a difference equation is stable if for $\alpha > 0 \; \exists \; \beta > 0$, $\beta = \beta(\alpha, i_0)$ such that $|x_0 - \psi(i_0)| \leqslant \beta$ implies that $|x(i, i_0, x_0) - \psi(i)| \leqslant \alpha$ for $i \in [i_0, \infty) \cap \mathbb{Z}$.

Definition 2.8 (Uniformly Stability)

ψ is said to be uniformly stable provided for $\alpha > 0 \; \exists \; \beta > 0$, $\beta(\alpha)$ such that $|x_0 - \psi(i_0)| \leqslant \beta$ implies that $|x(i, i_0, x_0) - \psi(i)| \leqslant \alpha$, for $i > i_0$

Definition 2.9 (Asymptotically Stability)

ψ is said to be asymptotically stable provided it is stable and in addition to that $\exists \; \gamma(i) > 0$ such that $|x_0 - \psi(i_0)| \leqslant \gamma(i)$ implies that

$\lim_{i\to\infty}|x(i,i_0,x_0)-\psi(i)|=0$

Definition 2.10 (Uniformly Asymptotically Stability)

ψ is said to uniformly asymptotically stable provided it is uniformly stable and in addition to that $\exists\, r>0$ such that $|x_0-\phi(i_0)|\leqslant\gamma$ implies that $\lim_{i\to\infty}|x(i,i_0,x_0)-\phi(i)|=0$

2.4 Fredholm and Volterra Difference Equations

Any equation which is of the form

$$x(i+1)=\sum_{j=a}^{b}\tau(x(j)),$$

where a and b are constants, is called a Fredholm difference equation. Fredholm difference equations are characterized by fixed/constant limits of summation. On the other hand, any equation which is represented by the form

$$x(i+1)=\sum_{j=a}^{i-1}\tau(x(j)),$$

where a and i are the limits of the summation is called a Volterra difference equation.

The following are some types of delay Volterra difference equations:

(a) A linear Volterra difference equation with finite delay γ is given by

$$x(i+1)=Mx(i)+\sum_{j=i-\gamma}^{i-1}F(i,j)x(j).$$

(b) A linear Volterra difference equation with variable delay $\gamma(i)$ is given by

$$x(i+1)=Mx(i)+\sum_{j=i-\gamma(i)}^{i-1}F(i,j)x(j).$$

(c) A nonlinear Volterra difference equation with finite delay γ is given by

$$x(i+1) = Mx(i) + \sum_{j=i-\gamma}^{i-1} F(i,j)g(x(j)).$$

(d) A nonlinear Volterra difference equation with variable delay $\gamma(i)$ is given as

$$x(i+1) = Mx(i) + \sum_{j=i-\gamma(i)}^{i-1} F(i,j)g(x(j)),$$

where $g(x(i))$ is a non-linear function of $x(i)$ such as $g^3(x)$, $cos(g(x))$ and $e^{g(x)}$.

2.5 Review of Related Literature

Recently, a lot of attention has been devoted to the investigation of difference equations. For instance, Burton and Mahfoud in (1983) investigated an integro-differential system of equations of the form

$$x' = G(k)x + \int_0^i F(i,j)x(j)ds$$

where G and F are $k \times k$ matrices. A few varieties of stability were identified and defined, and the results were established indicating when one type of stability is equivalent to another type. Different kinds of Lyapunov functions were constructed and as a result conditions which are sufficiently enough and necessary for the above system to be stable were obtained. Finally, many other results regarding the qualitative behaviour of solutions of the system were found.

Kolmanovskii et al. (1998), investigated stability problem of some Volterra difference equations and established stability conditions formulated in terms of the characteristics of equations.

Elaydi et al. (1999), utilized the idea of total stability to establish results on the asymptotic behaviour of the solutions of discrete Volterra systems. The asymptotic equivalence of bounded solutions of the following Volterra difference systems which has an infinite delay was the target:

$$\rho(i+1) = \sum_{j=-\infty}^{i} M(i-j)W(i), \quad i_0 \geq 0$$

and

$$\theta(i+1) = \sum_{j=-\infty}^{i} M(i-j) + D(i,j)\theta(j); \quad i_0 \geq 0.$$

Medina (2001), obtained results for boundednees and stability properties of some classes of discrete Volterra equations. In the work, the main tool was the use of a representation formula which allowed the solution of discrete Volterra equations to be expressed in terms of the resolvent matrix of the corresponding system of Volterra difference equations.

Medina (2001), Győri and Horváth (2008), Migda and Morchalo (2013), studied the asymptotic properties of discrete Volterra systems, and also Volterra difference equations.

Song et al. (2004), used fixed point theory to investigate nonlinear Volterra difference equations that are perturbed versions of linear equations. Under perturbation, sufficient conditions were obtained to ensure that the stability properties of linear Volterra difference equations were preserved. The existence of asymptotically periodic solutions of perturbed Volterra difference equations was also proved.

Also, Song & Baker (2004), utilized the fundamental and resolvent matrices, and under appropriate assumptions, to odtain stability results from the linear case.

$$(K(n, j, x(n)) = B(n, j)x(j))$$

extracted from the discrete Volterra equation

$$x(n) = f(n) + \sum_{j=0}^{n}(K(n, j, x(n)), \quad (n \geq 0)$$

Several necessary and sufficient conditions for stability were obtained for solutions of the linear equation by considering the equations in various choices of Banach space.

Raffoul (2006), studied the stability of the zero solution of delay difference equations and the existence of unique periodic solution by utilizing fixed point theory. The interest was mainly in the qualitative analysis of the completely delay difference equation

$$\Delta x(i) = -a(i)x(i - \tau).$$

Győri & Horvath (2008), analysed the asymptotic behaviour of solutions of linear Volterra difference equations. Some sufficient conditions were presented under which the solutions to a general linear equation converged to limits, which were given by a limit formula. This result was then utilized to establish the exact asymptotic representation of the solutions of a class of convolution scalar difference equations, which have real characteristic roots.

Yankson (2009), utilized fixed point theory to analyse the stability of the zero solution of difference equations with variable delays. In particular the author considered the scalar delay equation

$$\Delta x(n) = -a(n)x(n - \tau(n))$$

and its generalization

$$\Delta x(n) = -\sum_{j=1}^{N} a_j(n)x(n - \tau_j(n)).$$

Adivar et al. (2013), invetigated the existence of periodic and asymptotically periodic solutions of a system of nonlinear Volterra difference equations with infinite delay. Fixed point theory was utilized to establish conditions that guarantee the existence of such periodic solutions given by

$$\begin{cases} \Delta x_n = h_n x_n + \sum_{i=-\infty}^{n} a_{n,i} f(y_i) \\ \\ \Delta y_n = p_n y_n + \sum_{i=-\infty}^{n} b_{n,i} g(x_i) \end{cases}$$

where f and g are real valued and continuous functions, and $a_{n,i}$, $b_{n,i}$, h_n, and p_n are real sequences.

Sultana (2015), studied various types of discrete Volterra equations with distinct orders. The convergence rate of solutions of scalar linear Volterra sum-difference equations with delay were addressed. The existence of bounded solutions on an unbounded domain of more general nonlinear Volterra sum-difference equations using the Schaefer fixed point theorem and the Lyapunov's direct method were also discussed (See Sultana (2015)).

Eid et al. (2015), utilized Lyapunov functions and obtained sufficient conditions necessary for the zero solution of the discrete Volterra system of the form

$$x(i+1) = Nx(i) + \sum_{j=i-\gamma}^{i-1} F(i,j)g(x(j)).$$

to be stable.

2.6 Chapter Summary

In this chapter, the general structure and concept of difference equations were identified. Fredholm equations were stated and related them to Volterra discrete equations. Some relevant theorems and lemmas regarding Lyapunov method of solving stability of difference equations were proved. Some relevant literature on various areas of this research were also looked at in this chapter.

CHAPTER THREE

METHODOLOGY

3.1 Introduction

In this chapter, the development and the application of Lyapunov function will be discussed. The stability property of difference equations will also be considered in relation to use of Lyapunov function.

3.2 Lyapunov Function

Lyapunov (1892), delth with stability by two distinct methods; these are the first and second (direct) methods. The first method pre-supposes an explicit solution known and this is applicble to some restricted but important cases. As against this, the second method, which is also called the direct method, is of great generality and power and, above all does not require the knowledge of the solutions themselves.

Lyapunov proposed a fundamental method for studying the problem of stability by constructing functions known as Lyapunov functions. This function is often represented as $V(i, x)$ defined in some region or the whole state phase that contains the unperturbed solution $x = 0$ for all $i > 0$ and which together with its difference $\Delta V(i, x)$ satisfy some sign definiteness.

In this thesis, the stability properties of non-linear difference equations shall be investigated by stating the conditions for stability. constructing Lyapunov functions. The constructed suitable Lyapunov function shall be used to discuss stability properties of the solution of the non-linear system of difference equations.

Throughout this thesis, the goal is to costruct suitable Lyapunov function and use it to obtain the results that guarantee the stability of the non-linear discrete Volterra system considered in (1.1)

The application of the Lyapunov method lies in constructing a scalar function (say V) and its differences such that they possess certain properties. When these properties of V and ΔV are shown, the stability behaviour of the system is established.

Definition 3.1 (For Difference Equations)

A function W defines a Lyapuvov function on a subset $\Gamma \in$R^k provided that

i) $W(0) = 0$, and $W(x) > 0$, for $x \neq 0$ and

ii) $\Delta W(x) \leqslant 0$, whenever x and $f(x)$ belong to the set Γ.

The function W is said to be a strict Lyapuvov function on a subset Γ of $\mathbb{R}^k$ if $\Delta W(x) < 0$.

The considered Lyapunov function must satisfy the following conditions;

- $V(i, x) = 0$ at $x = 0$, meaning V must be zero at $x = 0$;
- $V(i, x) \geq 0$ except $x = 0$, meaning V must be positive definite;
- $\Delta V(x) \leqslant 0$.

It is noteworthy that our constructed Lyapunov function V will need the existence of a positive definite matrix that will depend on the coefficient matrix M.

In this thesis, the stability properties of non-linear difference equations shall be looked into by constructing Lyapunov functions. The constructed suitable Lyapunov function shall be used to discuss stability properties of the solution of the non-linear system of difference equations.

In this thesis, the ultimate goal is to establish stability results for the zero solution of the non-linear system of Volterra difference equations with variable delay. A Lyapunov function is utilized to obtain sufficient conditions that can be used to achieve stability for the system of Volterra difference equations under consideration. Consider the following system of non-linear Volterra difference equations

$$x(i+1) = Mx(i) + \sum_{j=i-\gamma(i)}^{i-1} F(i,j)g(x(j)), \tag{3.1}$$

where γ is a function of time, i, M is a $k \times k$ constant matrix and $F(i,j)$ is an $k \times k$ matrix of functions that are defined on $-\omega \leq i \leq j < \infty$, where $\gamma(i) \leq \omega$, and $i, j \in [-\omega, \infty]$.

A Lyapunov function denoted by $V(i,x) = V(i)$ is constructed and used to show that along the solutions of equation (3.1), $\Delta V(i) \leq 0$. In the process, equation (3.1) is rewritten in order to obtain a suitable Lyapunov function so that ΔV can easily be calculated along the solutions of (3.1).

Let x_i be a function mapping an interval $[-\omega, 0] \cap \mathbb{Z}$ into $\mathbb{R}^k$. Therefore, $x(i) \equiv x(i, i_0, \psi)$ is a solution of (3.1) , provided $x(i)$ satisfies (3.1) for $i \leq i_0$ and $x_{i_0} = x(i_0 + j) = \psi(j), j \in [-\omega, 0] \cap \mathbb{Z}$.
All through this thesis it is to be understood that the argument of a function is i unless otherwise it is stated.

3.3 Chapter Summary

This chapter discusses the methodology used in this thesis. The form and concepts of the Lyapunov function were discussed. The conditions that justify a function to be Lyapunov as well as the conditions for stability were also dicussed in this chapter.

CHAPTER FOUR

RESULTS AND DISCUSSION

4.1 Introduction

In this chapter, the main results of the thesis are obtained. The Lyapunov's direct method is used to obtain inequalities that guarantee the stability of the zero solution of the Volterra difference equations with variable delay.

4.2 Preliminary Results

Let

$$B(i,j) := \sum_{\pi=i-j}^{\gamma} F(\pi+j,j), \tag{4.1}$$

where $0 < \gamma \leq r(i-1)$, for all $i \in Z^{+}$.

Also, let $x \in \mathbb{R}^k$ and $W = (w)_{ij}$ be a $k \times k$ matrix and define the norms $\mid x \mid$ to be the Euclidean norm so that

$$\mid W \mid = \max_{1\leq j\leq k} \sum_{i=1}^{k} \mid w_{ij} \mid .$$

The norm of a sequence function $\psi : [-\omega, \infty) \cap \mathbb{Z} \to \mathbb{R}^k$ is also denoted by

$$\parallel \psi \parallel = \sup_{-\omega\leq j\leq\infty} \mid \psi(j) \mid .$$

It is assumed that there exists a positive definite symmetric and constant $k \times k$ matrix G such that for positive constants μ_1,

$$M^T GN + N^T G = -\mu_1 I. \tag{4.2}$$

Also assumed that for some positive constant μ_2,

$$x^T\Big(M^T GB(i+1,i) + GB(i+1,i)\Big)g(x) \leq -\mu_2 \mid x \mid^2, \text{if } x \neq 0, \tag{4.3}$$

and for some positive constant η,

$$| g(x) | \leq \eta | x | . \tag{4.4}$$

Conditions (4.3) and (4.4) implies that $g(0) = 0$ and that $x = 0$ is a solution for system (3.1).

In order to construct a suitable Lyapunov function, equation (3.1) is rewritten in an equivalent form as given in Lemma 4.1.

Lemma 4.1 If $B(i, j)$ is defined by (4.1), then equation (3.1) is equivalent to the equation

$$\begin{aligned} \Delta x(i) = Nx(i) &+ B(i+1, i)g(x(i)) \\ &- \Delta_i \sum_{j=i-r(i-1)-1}^{i-1} B(i, j)g(x(j)), \end{aligned} \tag{4.5}$$

where N is a matrix given by $N = M - I$, and I is the identity $k \times k$ matrix.

Proof. Computing the difference of the summation term in equation (4.5) gives,

$$\begin{aligned} \Delta_i \sum_{j=i-r(i-1)-1}^{i-1} B(i, j)g(x(j)) = & \sum_{j=i-r(i)}^{k} B(i+1, i)g(x(j)) \\ & - \sum_{j=i-r(i-1)-1}^{i-1} B(i, j)g(x(j)) \\ = B(i & + 1, i)g(x(i)) \\ & + \sum_{j=i-r(i)}^{i-1} B(i+1, i)g(x(j)) \\ & - \sum_{j=i-r(i)}^{i-1} B(i, j)g(x(j)) - B(i, i-r(i-1)-1) \\ & \times g(x(i-r(i-1)-1)). \end{aligned} \tag{4.6}$$

It follows from equations (4.1) and (2.5) that

$$B(i, i-r(i-1)-1) = \sum_{\pi=r(i-1)+1}^{\gamma} F(i, i-r(i-1)-1) = 0.$$

Thus, equation (4.6) becomes,

$$\begin{aligned}\Delta_i \sum_{j=i-r(i-1)-1}^{i-1} B(i,j)g(x(j)) &= B(i+1,i)g((x(i)) \\ &+ \sum_{j=i-r(i)}^{i-1} B(i+1,i)g(x(j)) \\ &- \sum_{j=i-r(i)}^{i-1} B(i,j)g(x(j)) \\ = B(i+1,i)g((x(i)) &+ \sum_{j=i-r(i)}^{i-1} \Big[B(i+1,i) - B(i,j)\Big] g(x(j)).\end{aligned}$$

Thus,

$$\begin{aligned}\Delta_i \sum_{j=i-r(i-1)-1}^{i-1} B(i,j)g(x(j)) = & \quad B(i+1,i)g(x(i)) \\ & + \sum_{j=i-r(i)}^{i-1} \Delta B(i,j)g(x(j)) \end{aligned} \tag{4.7}$$

Now, from equation (4.1),

$$\begin{aligned}\Delta B(i,j) &= \Delta \sum_{\pi=i-j}^{\gamma} F(\pi+j,j) \\ &= \sum_{\pi=i-j+1}^{\gamma} F(\pi+j,j) - \sum_{\pi=i-j}^{\gamma} F(\pi+j,j) \\ &= \sum_{\pi=i-j+1}^{\gamma} F(\pi+j,j) - \sum_{\pi=i-j+1}^{\gamma} F(\pi+j,j) - F(i,j) \\ &= -F(i,j).\end{aligned}$$

Thus,

$$\Delta B(i,j) = -F(i,j). \tag{4.8}$$

Substituting (4.8) into (4.7) gives,

$$\Delta_i \sum_{j=i-r(i-1)-1}^{i-1} B(i,j)g(x(j)) = B(i+1,i)g(x(i)) - \sum_{j=i-r(i)}^{i-1} F(i,j)g(x(j)). \quad (4.9)$$

Using equation (4.9) in equation (4.5) gives

$$x(i+1) - x(i) = Mx(i) - x(i) + B(i+1,i)g(x(i)) - \left[B(i+1,i)g(x(i)) - \sum_{j=i-r(i)}^{i-1} F(i,j)g(x(j))\right]$$

Thus,

$$x(i+1) = Mx(i) + \sum_{j=i-r(i)}^{i-1} F(i,j)g(x(j)). \quad (4.10)$$

This completes the proof.

At this point, there is the need to construct a Lyapunov function. In Lemma 4.2, a Lyapunov function is prosposed.

Lemm 4.2 Let β be a constant such that $\beta > 0$. Then the function defined by

$$V(i,x) = \Big(x(i) + \sum_{j=i-r(i-1)-1}^{i-1} B(i,j)g(x(j))\Big)^T G \times \Big(x(i) + \sum_{j=i-r(i-1)-1}^{i-1} B(i,j)g(x(j))\Big) + \beta \sum_{j=-\omega}^{-1} \sum_{z=i+j}^{i-1} \mid B(i,z) \mid\mid g(x(z)) \mid^2 . \quad (4.11)$$

is a Lyapunov function.

Proof. To verify that equation (4.11) is a Lyapunov function, first, let $x = 0$, then

$$
\begin{aligned}
V(i,0) = \Big(0 \;+ & \sum_{j=i-r(i-1)-1}^{i-1} B(i,j)g(0)\Big)^T G \\
& \times \Big(0 + \sum_{j=i-r(i-1)-1}^{i-1} B(i,j)g(0)\Big) \\
& + \beta \sum_{j=-\omega}^{-1} \sum_{z=i+j}^{i-1} \mid B(i,z) \mid\mid g(0) \mid^2 \\
= 0 &
\end{aligned}
$$

Thus $V(i,0) = 0$ for $x = 0$.

From equation (4.11), $V(i,x) > 0$ for all x except for $x = 0$. Thus, $V(i,x)$ is positive definite.

Finally, under appropriate inequalities it is established in Theorem 4.1 that $\Delta V \leq -\psi \mid x \mid^2$ for $\psi > 0$ implying that $\Delta V \leq 0$. Therefore, equation (4.11) is a Lyapunov function. This completes the proof.

It must be noted that, if G is a positive definite symmetric matrix, then there is a positive constant Ω such that

$$\Omega \mid x \mid^2 \leq x^T G x, \text{ for all } x. \tag{4.12}$$

Lemma 4.3 Let $G(i)$ and $x(i)$ to be two sequences, then

$$\Delta G(i)x(i) = G(i+1)\Delta x(i)) + \Delta G(i)x(i).$$

Proof. If $G(i)$ and $x(i)$ are two sequences, then

$$\Delta[G(i)x(i)] = G(i+1)x(i+1) - G(i)x(i), \tag{4.13}$$

But

$$\Delta G(i) = G(i+1) - G(i),$$

and so

$$G(i) = G(i+1) - \Delta G(i).$$

Also,

$$\Delta x(i) = x(i+1) - x(i)$$

implies that

$$x(i+1) = \Delta x(i) + x(i)$$

Thus equation

$$\begin{aligned}\Delta[G(i)x(i)] = \; & G(i+1)[\Delta x(i) + x(i)] - [G(i+1) - \Delta\tau]x(i) \\ = \; & G(i+1)\Delta x(i) + G(i+1)x(i) - G(i+1)x(i) \\ & +\Delta G(i)x(i)\end{aligned}$$

implying that,

$$\Delta[G(i)x(i)] = G(i+1)\Delta x(i)) + \Delta G(i)x(i)$$

4.3 Main Results

In the next Theorem it is shown that $\Delta V \leq 0$.

Theorem 4.1 *Let* (4.2)- (4.4) *hold, and suppose there are constants* $\beta > 0$ *and* $\psi > 0$ *such that*

$$\begin{aligned}-\mu_1 - \mu_2 + \beta\omega\eta^2 B(i+1,i) \; & + \; \Big(2\eta \mid B^T(i+1,i) \mid + \mid N^T G \mid \Big) \\ & \times \sum_{j=i-r(i)}^{i-1} \mid B(i,j) \mid \\ & \leq \; -\psi, \qquad (4.14)\end{aligned}$$

$$-\beta + 2\eta \mid B^T(i+1,i)G \mid + \mid N^T G \mid \leqslant 0, \tag{4.15}$$

$$\mid GN \mid + \eta \mid GB(i+1,i) \mid \leq 0 \tag{4.16}$$

and

$$\Delta \mid B(i,z) \mid \leq 0 \tag{4.17}$$

then,

$$\Delta V(i) \leq -\psi \mid (x(i) \mid^2 .$$

Proof.

Let $V(i) = V(i,x)$ be the Lyapunov function defined by

$$\begin{aligned} V(i) = \Big(x(i) \quad + & \sum_{j=i-r(i-1)-1}^{i-1} B(i,j)g(x(j))\Big)^T G \\ & \times \Big(x(i) + \sum_{j=i-r(i-1)-1}^{i-1} B(i,j)g(x(j))\Big) \\ & + \beta \sum_{s=-\omega}^{-1} \sum_{z=i+j}^{i-1} \mid B(i,z) \mid\mid g(x(z)) \mid^2 . \end{aligned} \tag{4.18}$$

Thus, taking the difference along the solutions of equation (3.1) gives

$$\begin{aligned} \Delta V(i) = \Delta\Big[\Big(x(i) + & \sum_{j=i-r(i-1)-1}^{i-1} B(i,j)g(x(j))\Big)^T G \\ & \times \Big(x(i) + \sum_{j=i-r(i-1)-1}^{i-1} B(i,j)g(x(j))\Big) \\ & + \beta \sum_{z=-\omega}^{-1} \sum_{z=i+j}^{i-1} \mid B(i,z) \mid\mid g(x(z)) \mid^2 \Big]. \end{aligned} \tag{4.19}$$

Applying Lemma 4.3, the following is obtained

$$\begin{aligned}
\Delta V(i) = \Big(x(i+1) &+ \sum_{j=i-r(i)}^{t} B(i+1,i)g(x(j))\Big)^T G \\
&\times \Delta\Big(x(i) + \sum_{j=i-r(i-1)-1}^{i-1} B(i,j)g(x(j))\Big) \\
&+ \Delta\Big(x(i) + \sum_{j=i-r(i-1)-1}^{i-1} B(i,j)g(x(j))\Big)^T G \\
&\times \Big(x(i) + \sum_{j=i-r(i-1)-1}^{i-1} B(i,j)g(x(j))\Big) \\
&+ \beta \sum_{j=-\omega}^{-1} \Big[\sum_{z=i+j+1}^{i} \mid B(i+1,z) \mid\mid g(x(z)) \mid^2 \\
&- \sum_{z=i+j}^{i-1} \mid B(i,z) \mid\mid g(x(z)) \mid^2 \Big]. \qquad (4.20)
\end{aligned}$$

But

$$\beta \sum_{h=-\omega}^{-1} \Big[\sum_{z=i+j+1}^{i} \mid B(i+1,z) \mid\mid g(x(z)) \mid^2 - \sum_{z=i+j}^{i-1} \mid B(i,z) \mid\mid g(x(z)) \mid^2 \Big]$$

$$= \beta \sum_{j=-\omega}^{-1} \Big[\mid B(i+1,i) \mid\mid g(x(i)) \mid^2 + \sum_{z=i+j+1}^{i-1} \mid B(i+1,z) \mid\mid g(x(z)) \mid^2$$

$$- \mid B(i,i+j) \mid\mid g(x(i+j)) \mid^2 - \sum_{z=i+j+1}^{i-1} \mid B(i,z) \mid\mid g(x(z)) \mid^2 \Big]$$

$$= \beta \sum_{j=-\omega}^{-1} \Big[\mid B(i+1,i) \mid\mid g(x(i)) \mid^2 - \mid B(i,i+j) \mid\mid g(x(i+j)) \mid^2$$

$$+ \sum_{z=i+j+1}^{i-1} \Delta \mid B(i,z) \mid\mid g(x(z)) \mid^2 \Big]. \qquad (4.21)$$

In view of condition (4.17),

$$\sum_{z=i+j+1}^{i-1} \Delta \mid B(i,z) \mid\mid g(x(z)) \mid^2 \leq 0$$

Thus, (4.21) becomes

$$\beta \sum_{j=-\omega}^{-1} \Big[\sum_{z=i+j+1}^{i} \mid B(i+1,z) \mid\mid g(x(z)) \mid^2 - \sum_{z=i+j}^{i-1} \mid B(i,z) \mid\mid g(x(z)) \mid^2 \Big]$$

$$\leq \beta \sum_{j=-\omega}^{-1} \Big[\mid B(i+1,i) \mid\mid g(x(i)) \mid^2 - \mid B(i,i+j) \mid\mid g(x(i+j)) \mid^2 \Big]$$

$$= \beta \sum_{j=-\omega}^{-1} \mid B(i+1,i) \mid\mid g(x(i)) \mid^2 - \beta \sum_{j=-\omega}^{-1} \mid B(i,i+j) \mid\mid g(x(i+j)) \mid^2$$

$$\begin{aligned} = \beta\Big((-1) &- (-\omega)+1\Big) \mid B(i+1,i) \mid\mid g(x(i)) \mid^2 \\ &- \beta \sum_{j=-\omega}^{-1} \mid B(i,i+j) \mid\mid g(x(i+j)) \mid^2 \end{aligned}$$

$$\begin{aligned} = \beta\omega \mid B(i &+ 1,i) \mid\mid g(x(i)) \mid^2 \\ &- \beta \sum_{j=-\omega}^{-1} \mid B(i,i+j) \mid\mid g(x(i+j)) \mid^2 . \end{aligned} \tag{4.22}$$

Substituting (4.20) into (4.19) gives

$$\begin{aligned}\Delta V(i) = \Big(x(i) &+ \sum_{j=i-r(i)}^{t} B(i,j)g(x(j))\Big)^T G \\ &\times \Delta\Big(x(i) + \sum_{j=i-r(i-1)-1}^{i-1} B(i,j)g(x(j))\Big) \\ &+ \Delta\Big(x(i) + \sum_{j=i-r(i-1)-1}^{i-1} B(i,j)g(x(j))\Big)^T G \\ &\times \Big(x(i) + \sum_{j=i-r(i-1)-1}^{i-1} B(i,j)g(x(j))\Big) \\ &+ \beta\omega \mid B(i+1,i) \mid\mid g(x(i)) \mid^2 \\ &- \beta \sum_{j=-\omega}^{-1} \mid B(i,i+j) \mid\mid g(x(i+j)) \mid^2 . \end{aligned} \tag{4.23}$$

But, reference to Lemma 4.1 indicates that

$$\begin{aligned}\Delta x(i) = Nx(i) + B(i &+ 1,i)g(x(i)) - \Delta_i \sum_{j=i-r(i-1)-1}^{i-1} B(i,j)g(x(j)) \\ = (M-I)x(i) &+ B(i+1,i)g(x(i)) \\ &- \Delta_i \sum_{j=i-r(i-1)-1}^{i-1} B(i,j)g(x(j)).\end{aligned}$$

Thus,

$$\begin{aligned}x(i+1) - x(i) = Mx(i) &- x(i) + B(i+1,i)g(x(i)) \\ &- \sum_{j=i-r(i)}^{i} B(i+1,i)g(x(j)) \\ &+ \sum_{j=i-r(i-1)-1}^{i-1} B(i,j)g(x(j)).\end{aligned}$$

This implies that

$$\begin{aligned}x(i+1) = Mx(i) &+ B(i+1,i)g(x(i)) - \sum_{j=i-r(i)}^{i} B(i+1,i)g(x(j)) \\ &+ \sum_{j=i-r(i-1)-1}^{i-1} B(i,j)g(x(j))\end{aligned}$$

Thus,

$$\begin{aligned}
x(i+1) + \sum_{j=i-r(i)}^{i} B(i+1,i)g(x(j)) = \quad & Mx(i) + B(i+1,i)g(x(i)) \\
& + \sum_{j=i-r(i-1)-1}^{i-1} B(i,j)g(x(j)) \\
= Mx(i) + B(i \quad & + \quad 1, i)g(x(i)) + \sum_{j=i-r(i)}^{i-1} B(i,j) \\
\times g(x(j)) \quad & + \quad B(i, i-r(i-1)-1) \\
& \times \quad g(x(i-r(i-1)-1)).
\end{aligned}$$

Since $\gamma \leq r(i-1)$,

$$\begin{aligned}
B(i, i-r(i-1)-1) &= \sum_{w=r(i-1)+1}^{\gamma} F(i, i-r(i-1)-1) \\
&= 0.
\end{aligned}$$

This gives

$$\begin{aligned}
x(i+1) + \sum_{j=i-r(i)}^{i} B(i \quad & + \quad 1, i)g(x(j)) \\
= Mx(i) \quad & + \quad B(i+1,i)g(x(i)) \\
& + \quad \sum_{j=i-r(i)}^{i-1} B(i,j)g(x(j)). \qquad (4.24)
\end{aligned}$$

Again, from Lemma 4.1, the following can be obtained

$$\Delta x(i) + \Delta_i \sum_{j=i-r(i-1)-1}^{i-1} B(i,j)g(x(j)) = Nx(i) + B(i+1,i)g(x(i))$$

and so

$$\begin{aligned}
\Delta\Big(x(i) + \sum_{j=i-r(i-1)-1}^{i-1} & B(i,j)g(x(j))\Big) \\
&= Nx(i) + B(i+1,i)g(x(i)). \qquad (4.25)
\end{aligned}$$

Substituting (4.24) and (4.23) into (4.21) gives

$$
\begin{aligned}
\Delta V(i) = \Big(Mx(i) &+ B(i+1,i)g(x(i)) + \sum_{j=i-r(i)}^{i-1} B(i,j)g(x(j))\Big)^T G \\
&\times \Big(Nx(i) + B(i+1,i)g(x(i))\Big) \\
&+ \Big(Nx(i) + B(i+1,i)g(x(i))\Big)^T G \\
&\times \Big(x(i) + \sum_{j=i-r(i-1)-1}^{i-1} B(i,j)g(x(j))\Big) \\
&+ \beta\omega \mid B(i+1,i) \mid\mid g(x(i)) \mid^2 \\
&- \beta \sum_{j=-\omega}^{-1} \mid B(i,i+j) \mid\mid g(x(i+j)) \mid^2 \\
= \Big[x^T M^T &+ g^T(x(i))B^T(i+1,i) + \Big(\sum_{j=i-r(i)}^{i-1} B(i,j)g((x(j))\Big)^T\Big] \\
&\times \Big[GNx(i) + GB(i+1,i)g(x(i))\Big] \\
&+ \Big[x^T N^T + g^T(x(i))B^T(i+1,i)\Big] \\
&\times \Big[Gx(i) + G \sum_{j=i-r(i-1)-1}^{i-1} B(i,j)g(x(j))\Big] \\
&+ \beta\omega \mid B(i+1,i) \mid\mid g(x(i)) \mid^2 \\
&- \beta \sum_{j=-\omega}^{-1} \mid B(i,i+j) \mid\mid g(x(i+j)) \mid^2 \\
= x^T M^T GNx(i) &+ x^T M^T GB(i+1,i)g(x(i)) \\
&+ g^T(x(i))B^T(i+1,i)GNx(i) \\
&+ g^T(x(i))B^T(i+1,i)GB(i+1,i)g(x(i)) \\
&+ \Big(\sum_{j=i-r(i)}^{i-1} B(i,j)g(x(j))\Big)^T GNx(i) \\
&+ \Big(\sum_{j=i-r(i)}^{i-1} B(i,j)g(x(j))\Big)^T GB(i+1,i)g(x(i)) \\
&+ x^T N^T Gx(i) + x^T N^T G \sum_{j=i-r(i-1)-1}^{i-1} B(i,j)g(x(j)) \\
&+ g^T(x(i))B^T(i+1,i)Gx(i) \\
&+ g^T(x(i))B^T(i+1,i)G \sum_{j=i-r(i-1)-1}^{i-1} B(i,j)g(x(j))
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \beta\omega \mid B(i+1,i) \mid\mid g(x(i)) \mid^2 \\
&- \quad \beta \sum_{j=-\omega}^{-1} \mid B(i,i+j) \mid\mid g(x(i+j)) \mid^2 .
\end{aligned}
$$

It then follows that

$$
\begin{aligned}
\Delta V(i) = x^T(i) \quad & \times \quad \Big(M^TGN + N^TG\Big)x(i) + x^TM^TGB(i+1,i) \\
& \times \quad g(x(i)) + g^T(x(i))B^T(i+1,i)Gx(i) \\
& + \quad g^T(x(i))B^T(i+1,i)GNx(i) \\
& + \quad g^T(x(i))B^T(i+1,i)GB(i+1,i)g(x(i)) \\
& + \quad \Big(\sum_{j=i-r(i)}^{i-1} B(i,j)g(x(j))\Big)^T GNx(i) \\
& + \quad x^TN^TG \sum_{j=i-r(i-1)-1}^{i-1} B(i,j)g(x(j)) \\
& + \quad \Big(\sum_{j=i-r(i)}^{i-1} B(i,j)g(x(j))\Big)^T GB(i+1,i) \\
& \times \quad g(x(i)) + g^T(x(i))B^T(i+1,i)G \\
& \times \quad \sum_{j=i-r(i-1)-1}^{i-1} B(i,j)g(x(j)) \\
& + \quad \beta\omega \mid B(i+1,i) \mid\mid g(x(i)) \mid^2 \\
& - \quad \beta \sum_{j=-\omega}^{-1} \mid B(i,i+j) \mid\mid g(x(i+j)) \mid^2 . \qquad (4.26)
\end{aligned}
$$

It is observed that

$$
\Big[g^T(x(i))B^T(i+1,i)GMx(i)\Big]^T = x^TM^TGB(i+1,i)g(x(i)).
$$

Now, considering the sum of the second and third terms on the right hand side of (4.26) gives

$$
\begin{aligned}
x^TM^T \quad G \quad & B(i+1,i)g(x(i)) + g^T(x(i))B^T(i+1,i)Gx(i) \\
= \quad & x^TM^TGB(i+1,i)g(x(i)) + \Big[x^TGB(i+1,i)g(x(i))\Big]^T \\
= \quad & x^T\Big(M^TGB(i+1,i) + GB(i+1,i)\Big)g(x(i)). \qquad (4.27)
\end{aligned}
$$

Also, considering the sum of the fourth and fifth terms on the right hand side of equation (4.26) gives

$$\begin{aligned} & g^T(x(i))B^T(i + 1,i)GNx(i) + g^T(x(i))B^T(i+1,i)GB(i+1,i)g(x(i)) \\ \leq & \mid g^T(x(i))B^T(i + 1,i)GN \mid\mid x(i) \mid \\ & + \mid g^T(x(i))B^T(i+1,i)GB(i+1,i) \mid\mid g(x(i)) \mid \\ \leq & \mid g^T(x(i))B^T(i + 1,i)GN \mid\mid x(i) \mid \\ & + \eta \mid g^T(x(i))B^T(i+1,i)DB(i+1,i) \mid\mid (x(i) \mid \\ \leq & \mid g^T(x(i))B^T(i + 1,i)GN \mid \Big(\mid GN \mid + \eta \mid GB(i+1,i) \mid \Big) \mid (x(i) \mid \end{aligned} \quad (4.28)$$

Summing the sixth and seventh terms on the right hand side of equation (4.26) gives

$$\begin{aligned} & \Big(\sum_{j=i-r(i)}^{i-1} B(i,j)g(x(j)) \Big)^T GNx(i) + x^T N^T G \sum_{j=i-r(i-1)-1}^{i-1} B(i,j)g(x(j)) \\ = & \Big[x^T N^T G \sum_{j=i-r(i)}^{i-1} B(i,j)g(x(j)) \Big]^T + x^T N^T G \sum_{j=i-r(i)}^{i-1} B(i,j)g(x(j)) \\ = & 2x^T N^T G \sum_{j=i-r(i)}^{i-1} B(i,j)g(x(j)) \\ \leq & 2^T \mid\mid N^T G \mid \sum_{j=i-r(i)}^{i-1} \mid B(i,j) \mid\mid g(x(j)) \mid . \end{aligned} \quad (4.29)$$

Using the inequality $2ab \leq a^2 + b^2$, equation (4.28) becomes

$$\begin{aligned} & 2^T \mid\mid N^T G \mid \sum_{j=i-r(i)}^{i-1} \mid B(i,j) \mid\mid g(x(j)) \mid \\ \leq & \mid N^T G \mid \sum_{j=i-r(i)}^{i-1} \mid B(i,j) \mid \Big(^T (t) \mid^2 + \mid g(x(j)) \mid^2 \Big). \end{aligned} \quad (4.30)$$

Furthermore, a sum of the eighth and nineth terms on the right hand side of equation (4.26) gives

$$
\begin{aligned}
&\Big(\sum_{j=i-r(i)}^{i-1} B(i,j)g(x(j))\Big)^T GB(i+1,i)g(x(i)) + g^T(x(i)) \\
&\qquad\qquad \times B^T(i+1,i)G \sum_{j=i-r(i-1)-1}^{i-1} B(i,j)g(x(j)) \\
&= \Big[g^T(x(i))B^T(i+1,i)G \sum_{j=i-r(i)}^{i-1} B(i,j)g(x(j))\Big]^T \\
&\qquad + g^T(x(i))B^T(i+1,i)G \sum_{j=i-r(i-1)-1}^{i-1} B(i,j)g(x(j)) \\
&= 2g^T(x(i))B^T(i+1,i)G \sum_{j=i-r(i)}^{i-1} B(i,j)g(x(j)) \\
&\leq 2 \mid g^T(x(i)) \mid\mid B^T(i+1,i)G \mid \sum_{j=i-r(i)}^{i-1} \mid B(i,j) \mid\mid g(x(j)) \mid \\
&\leq 2\eta \mid (x(i)) \mid\mid B^T(i+1,i)G \mid \sum_{j=i-r(i)}^{i-1} \mid B(i,j) \mid\mid g(x(j)) \mid \\
&\leq 2\eta \mid B^T(i+1,i)G \mid \sum_{j=i-r(i)}^{i-1} \mid B(i,j) \mid \\
&\qquad \times \Big(\mid (x(i)) \mid^2 + \mid g(x(j)) \mid^2 \Big)
\end{aligned} \tag{4.31}
$$

Finally, the last term of equation (4.26) is simplified by letting $\pi = i + j$. Thus, the upper limit of the summation becomes $\pi = i - \omega$. Hence,

$$
\begin{aligned}
&\beta \sum_{j=-\omega}^{-1} \mid B(i,i + j) \mid\mid g(x(i+j)) \mid^2 \\
&\qquad = \beta \sum_{j=i-\omega}^{i-1} \mid B(i,\pi) \mid\mid g(x(\pi)) \mid^2 \\
&\qquad \geq \beta \sum_{\pi=i-r(i)}^{i-1} \mid B(i,\pi) \mid\mid g(x(\pi)) \mid^2 .
\end{aligned} \tag{4.32}
$$

Replacing π with j in the R.H.S of equation (4.32) gives

$$
\beta \sum_{\pi=i-r(i)}^{i-1} \mid B(i,\pi) \mid\mid g(x(\pi)) \mid^2 = \beta \sum_{j=i-r(i)}^{i-1} \mid B(i,j) \mid\mid g(x(j)) \mid^2 .
$$

Thus,

$$\beta \sum_{j=-\omega}^{-1} \mid B(i,i+j) \mid\mid g(x(i+j)) \mid^2 \geq \beta \sum_{j=i-r(i)}^{i-1} \mid B(i,j) \mid\mid g(x(j)) \mid^2 . \quad (4.33)$$

Substituting equations (4.27) - (4.32) into equation (4.26) gives

$$\begin{aligned}
\Delta V(i) \leq \quad \mid x^T(i) \mid \quad & \times \quad \Big(P^T GN + N^T G\Big) \mid x(i) \mid \\
& + \quad x^T \Big(M^T GB(i+1,i) + GB(i+1,i)\Big) \\
& \times \quad g(x(i)) + \mid g^T(x(i)) B^T(i+1,i) GN \mid \\
& \times \quad \Big(\mid GN \mid + \eta \mid GB(i+1,i) \mid \Big) \\
& \times \quad \mid (x(i) \mid + \mid N^T G \mid \sum_{j=i-r(i)}^{i-1} \mid B(i,j) \mid \\
& \times \quad \Big(^T(i) \mid^2 + \mid g(x(j)) \mid^2 \Big) \\
& + \quad 2\eta \mid B^T(i+1,i) G \mid \sum_{j=i-r(i)}^{i-1} \mid B(i,j) \mid \\
& \times \quad \Big(\mid x(i) \mid^2 + \mid g(x(j)) \mid^2 \Big) \\
& + \quad \beta\omega \mid B(i+1,i) \mid\mid g(x(i)) \mid^2 \\
& - \quad \beta \sum_{j=i-r(i)}^{i-1} \mid B(i,j) \mid\mid g(x(j)) \mid^2 . \qquad (4.34)
\end{aligned}$$

In view of equations (4.9), (4.10), (4.13), (4.15) and (4.16), inequality (4.32) becomes

$$\begin{aligned}
\Delta V(i) \leq \Big[& - \mu_1 I\Big] \mid x(i) \mid^2 + \Big[- \mu_2 \mid x(i) \mid^2 \Big] \\
& + \mid g^T(x(i)) B^T(i+1,i) G N \mid \Big[0\Big] \mid (x(i) \mid \\
& + \Big[\mid N^T G \mid \sum_{j=i-r(i)}^{i-1} \mid B(i,j) \mid \Big] \mid x(i) \mid^2 \\
& + \Big[\mid N^T G \mid \sum_{j=i-r(i)}^{i-1} \mid B(i,j) \mid \Big] \mid g(x(j)) \mid^2 \\
& + \Big[2\eta \mid B^T(i+1,i) G \mid \sum_{j=i-r(i)}^{i-1} \mid B(i,j) \mid \Big] \mid x(i) \mid^2 \\
& + \Big[2\eta \mid B^T(i+1,i) G \mid \sum_{j=i-r(i)}^{i-1} \mid B(i,j) \mid \Big] \mid g(x(j)) \mid^2 \\
& + \beta\omega \mid B(i+1,i) \mid\mid g(x(i)) \mid^2 - \beta \sum_{j=i-r(i)}^{i-1} \mid B(i,j) \mid\mid g(x(j)) \mid^2 \\
\leq \Big[& - \mu_1 - \mu_2 \Big] \mid x(i) \mid^2 + \mid N^T G \mid \sum_{j=i-r(i)}^{i-1} \mid B(i,j) \mid\mid x(i) \mid^2 \\
& + 2\eta \mid B^T(i+1,i) G \mid \sum_{j=i-r(i)}^{i-1} \mid B(i,j) \mid\mid x(i) \mid^2 \\
& + \beta\omega\eta^2 \mid B(i+1,i) \mid\mid x(i) \mid^2 \\
& + \mid N^T G \mid \sum_{j=i-r(i)}^{i-1} \mid B(i,j) \mid\mid g(x(j)) \mid^2 \\
& + 2\eta \mid B^T(i+1,i) G \mid \sum_{j=i-r(i)}^{i-1} \mid B(i,j) \mid\mid g(x(j)) \mid^2 \\
& - \beta \sum_{j=i-r(i)}^{i-1} \mid B(i,j) \mid\mid g(x(j)) \mid^2 .
\end{aligned}$$

Thus,

$$\begin{aligned}
\Delta V(i) \leq \Big[& - \mu_1 - \mu_2 + \beta\omega\eta^2 \mid B(i+1,i) \mid \\
& + 2\eta \mid B^T(i+1,i)G \mid \sum_{j=i-r(i)}^{i-1} \mid B(i,j) \mid \\
& + \mid N^T G \mid \sum_{j=i-r(i)}^{i-1} \mid B(i,j) \mid \Big] \mid x(i) \mid^2 \\
& + \Big[2\eta \mid B^T(i+1,i)G \mid \sum_{j=i-r(i)}^{i-1} \mid B(i,j) \mid \\
& + \mid N^T G \mid \sum_{j=i-r(i)}^{i-1} \mid B(i,j) \mid \Big] \mid g(x(j)) \mid^2 \\
& - \beta \sum_{j=i-r(i)}^{i-1} \mid B(i,j) \mid\mid g(x(j)) \mid^2 \\
\leq \Big[& - \mu_1 - \mu_2 + \beta\omega\eta^2 \mid B(i+1,i) \mid \\
& + \Big(2\eta \mid B^T(i+1,i)G \mid + \mid N^T G \mid \Big) \\
& \times \sum_{j=i-r(i)}^{i-1} \mid B(i,j) \mid \Big] \mid x(i) \mid^2 \\
& + \Big[-\beta + 2\eta \mid B^T(i+1,i)G \mid + \mid N^T G \mid \Big] \\
& \times \sum_{j=i-r(i)}^{i-1} \mid B(i,j) \mid\mid g(x(j)) \mid^2 \\
\leq - \ & \psi \ \mid x(i) \mid^2 + \Big[0 \Big] \sum_{j=i-r(i)}^{i-1} \mid B(i,j) \mid .
\end{aligned}$$

Therefore,

$$\Delta V(i) \leq -\psi \mid x(i) \mid^2 . \tag{4.35}$$

This completes the proof.

Theorem 4.2 *Suppose the hypotheses of Lemmas 4.1 and 4.2 hold and*

$$1-\eta\sum_{j=i-r(i-1)-1}^{i-1}\mid B(i,j)\mid>0, \tag{4.36}$$

then the zero solution of equation (4.30) *is stable.*

Proof.

Let

$$\delta<\frac{k\epsilon}{L}\Big(1-\eta\sum_{j=i-r(i-1)-1}^{i-1}\mid B(i,j)\mid\Big)$$

and define

$$L^2=\mid G\mid\Big(1+\eta\sum_{j=i_0-r(i_0)}^{i_0-1}\mid B(i_0,j)\mid\Big)^2 + \eta^2\beta\sum_{j=-\omega}^{-1}\sum_{z=i_0+j}^{i_0-1}\mid B(i_0,j)\mid. \tag{4.37}$$

By (4.18),

$$\begin{aligned}V(i)&=\Big(x(i)+\sum_{j=i-r(i-1)-1}^{i-1}B(i,j)g(x(j))\Big)^TG\\&\quad\times\Big(x(i)+\sum_{j=i-r(i-1)-1}^{i-1}B(i,j)g(x(j))\Big)\\&\quad+\beta\sum_{j=-\omega}^{-1}\sum_{z=i+j}^{i-1}\mid B(i,z)\mid\mid g(x(z))\mid^2\\&\le\mid G\mid\Big(\mid x(i)\mid+\sum_{j=i-r(i-1)-1}^{i-1}\mid B(i,j)g(x(j))\mid\Big)^2\\&\quad+\beta\sum_{j=-\omega}^{-1}\sum_{z=i+j}^{i-1}\mid B(i,z)\mid\mid g(x(z))\mid^2\\&\le\mid G\mid\Big(\mid x(i)\mid+\sum_{j=i-r(i-1)-1}^{i-1}\mid B(i,j)\mid\mid g(x(j))\mid\Big)^2\\&\quad+\beta\sum_{j=-\omega}^{-1}\sum_{z=i+j}^{i-1}\mid B(i,z)\mid\mid g(x(z))\mid^2.\end{aligned}$$

Therefore,

$$V(i_0, \phi) \leq \mid G \mid \Big(\mid \phi(i_0) \mid + \eta \sum_{j=i_0-r(i_0-1)-1}^{i_0-1} \mid B(i_0, j) \mid\mid g(\phi(j)) \mid \Big)^2 + \eta^2 \beta \sum_{j=-\omega}^{-1} \sum_{z=i_0+j}^{i_0-1} \mid B(i_0, j) \mid\mid g(\phi(j)) \mid^2$$

In view of inequality (4.33), the function V is decreasing, and that,

$$\Delta V(i) \leq 0. \tag{4.38}$$

implying that

$$V(i+1) \leq V(i). \tag{4.39}$$

and that for $i_0 \geq 0$ it follows that $V(i, x) \leq V(i_0, \phi)$. Thus,

$$\begin{aligned} V(i, x) &\leq \mid \phi(i_0) \mid^2 \Big[\mid G \mid \Big(1 + \eta \sum_{j=i_0-r(i_0)}^{i_0-1} \mid B(i_0, j) \mid \Big)^2 + \eta^2 \beta \sum_{j=-\omega}^{-1} \sum_{z=i_0+j}^{i-1} \mid B(i_0, z) \mid \Big] \\ &\leq \delta^2 \Big[\mid G \mid \Big(1 + \eta \sum_{j=i_0-r(i_0)}^{i_0-1} \mid B(i_0, j) \mid \Big)^2 + \eta^2 \beta \sum_{j=-\omega}^{-1} \sum_{z=i_0+j}^{i-1} \mid B(i_0, z) \mid \Big]. \end{aligned}$$

Thus,

$$V(i, x) \leq \delta^2 L^2. \tag{4.40}$$

It follows from (4.17) that

$$V(i, x) \geq \Big(x(i) + \sum_{j=i-r(i-1)-1}^{i-1} B(i, j) g(x(j)) \Big)^T G \times \Big(x(i) + \sum_{j=i-r(i-1)-1}^{i-1} B(i, j) g(x(j)) \Big)$$

Thus, using the fact that $x^T G x \geq K \mid x \mid^2$ for all x, and $(m+n)^2 \geq (m-n)^2$,

$$V(i,x) \;\geq\; K\Big| \, x(i) + \sum_{j=i-r(i-1)-1}^{i-1} B(i,j)g(x(j))\Big|^{2}.$$

and also

$$V(i,x) \geq K^2\Big(\mid x(i) \mid - \sum_{j=i-r(i-1)-1}^{i-1} \mid B(i,j) \mid\mid g(x(j)) \mid \Big)^2. \qquad (4.41)$$

Combining the two inequalities (4.38) and (4.39) gives

$$\begin{aligned}
\delta^2 L^2 \;&\geq\; K^2\Big(\mid x(i) \mid - \sum_{j=i-r(i-1)-1}^{i-1} \mid B(i,j) \mid\mid g(x(j)) \mid \Big)^2 \\
\delta L \;&\geq\; K\Big(\mid x(i) \mid - \sum_{j=i-r(i-1)-1}^{i-1} \mid B(i,j) \mid\mid g(x(j)) \mid \Big) \\
&\geq\; \mid x(i) \mid - \sum_{j=i-r(i-1)-1}^{i-1} \mid B(i,j) \mid\mid g(x(j)) \mid \\
&\geq\; \mid x(i) \mid - \eta \sum_{j=i-r(i-1)-1}^{i-1} \mid B(i,j) \mid\mid x(j) \mid .
\end{aligned}$$

Thus,

$$\frac{\delta L}{K} \geq \mid x(i) \mid \Big(1 - \eta \sum_{j=i-r(i-1)-1}^{i-1} \mid B(i,j) \mid \Big).$$

Therefore,

$$\mid x(i) \mid \;\; \leq \epsilon.$$

This completes the proof.

4.4 Example

In this section, the results obtained in the previous section are applied to a constructed example, and the details are as follows:

Let

$$M = \begin{pmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{5} \end{pmatrix} \quad \text{and} \quad F(i,j) = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix},$$

such that

$$\begin{aligned} \begin{pmatrix} x_1(n+1) \\ x_2(n+1) \end{pmatrix} &= \begin{pmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{5} \end{pmatrix} \begin{pmatrix} x_1(n) \\ x_2(n) \end{pmatrix} \\ &+ \sum_{j=i-r(i)}^{i-1} \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} -\frac{16\mu_2}{5\mu_1\gamma} x_1 \\ -\frac{16\mu_2}{5\mu_1\gamma} x_2 \end{pmatrix}. \end{aligned} \tag{4.42}$$

It follows from equation (4.1),

$$\begin{aligned} B(i,j) &= \sum_{\pi=i-j}^{\gamma} F(\pi+j,j), \\ &= \Big[\gamma-(i-j)+1\Big] F(i,j) \\ &= \Big[\gamma-i+j+1\Big] \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{4}(\gamma-i+j+1) & 0 \\ 0 & \frac{1}{4}(\gamma-i+j+1) \end{pmatrix}, \end{aligned} \tag{4.43}$$

and so

$$
\begin{aligned}
B(i+1,i) &= B^T(i+1,i)\\
&= \begin{pmatrix} \frac{1}{4}(\gamma - i - 1 + i + 1) & 0 \\ 0 & \frac{1}{4}(\gamma - i - 1 + i + 1) \end{pmatrix}\\
&= \begin{pmatrix} \frac{1}{4}\gamma & 0 \\ 0 & \frac{1}{4}\gamma \end{pmatrix}.
\end{aligned}
$$

Also, N can be evaluated by using the condition below

$$
\begin{aligned}
N = N^T &= M - I\\
&= \begin{pmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{5} \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\\
&= \begin{pmatrix} -\frac{4}{5} & 0 \\ 0 & -\frac{4}{5} \end{pmatrix}.
\end{aligned}
$$

The following condition also holds to evaluate G

$$M^T G N + N^T G = -\mu_1 I.$$

Thus,

$$\begin{pmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{5} \end{pmatrix} G \begin{pmatrix} -\frac{4}{5} & 0 \\ 0 & -\frac{4}{5} \end{pmatrix} + \begin{pmatrix} -\frac{4}{5} & 0 \\ 0 & -\frac{4}{5} \end{pmatrix} G = -\mu_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This implies that

$$\begin{pmatrix} -\frac{4}{25} & 0 \\ 0 & -\frac{4}{25} \end{pmatrix} G + \begin{pmatrix} -\frac{4}{5} & 0 \\ 0 & -\frac{4}{5} \end{pmatrix} G = \begin{pmatrix} -\mu_1 & 0 \\ 0 & -\mu_1 \end{pmatrix}.$$

which gives

$$\begin{pmatrix} -\frac{24}{25} & 0 \\ 0 & -\frac{24}{25} \end{pmatrix} G = \begin{pmatrix} -\mu_1 & 0 \\ 0 & -\mu_1 \end{pmatrix}$$

.

Thus,

$$\begin{aligned} G &= \begin{pmatrix} -\mu_1 & 0 \\ 0 & -\mu_1 \end{pmatrix} \left(\frac{24}{25}\right)^{-2} \begin{pmatrix} -\frac{24}{25} & 0 \\ 0 & -\frac{24}{25} \end{pmatrix} \\ &= \left(\frac{24}{25}\right)^{-2} \begin{pmatrix} \frac{24}{25}\mu_1 & 0 \\ 0 & \frac{24}{25}\mu_1 \end{pmatrix} \\ &= \left(\frac{25}{24}\right)^{2} \begin{pmatrix} \frac{24}{25}\mu_1 & 0 \\ 0 & \frac{24}{25}\mu_1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{25}{24}\mu_1 & 0 \\ 0 & \frac{25}{24}\mu_1 \end{pmatrix}. \end{aligned}$$

It is noted that the inequality

$$\Big| g(x) \Big| \leq \eta \mid x \mid .$$

can be used to estimate η. That is,

$$\left[(-\frac{16\mu_2}{5\mu_1\gamma}x_1)^2 + (-\frac{16\mu_2}{5\mu_1\gamma}x_2)^2\right]^{\frac{1}{2}} \leq \eta\left[x_1^2 + x_2^2\right]^{\frac{1}{2}}$$

implying that

$$\frac{16\mu_2}{5\mu_1\gamma}\left[x_1^2 + x_2^2\right]^{\frac{1}{2}} \leq \eta\left[x_1^2 + x_2^2\right]^{\frac{1}{2}}$$

and so

$$\eta \geq \frac{16\mu_2}{5\mu_1\gamma}.$$

It follows from equation (4.43) that,

$$\begin{aligned}
\Big|B(i,j)\Big| &\leq \frac{1}{4}\Big|\Big(\gamma - i + j + 1\Big)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\Big| \\
&\leq \frac{1}{4}\Big|\Big(\gamma - i + j + 1\Big)\Big|\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix} \\
&\leq \frac{1}{4}\Big|\Big(\gamma - i + j + 1\Big)\Big| \\
&\leq \frac{\gamma}{4}, \qquad \text{for} \quad \mathrm{j} \in \Big[i-\gamma, i-1\Big].
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{j=i-r(i-1)-1}^{i-1} \mid B(i,j) \mid &= \frac{\gamma}{4}\Big[i - 1 - \Big(i - r(i-1) - 1\Big) - 1\Big] \\
&\leq \frac{\gamma}{4}\Big(r(i-1)+1\Big) \\
&\leq \frac{\gamma}{4}\Big(\gamma+1\Big), \quad \gamma \leq r\Big(i-1\Big).
\end{aligned}$$

This implies that

$$1 - \eta \sum_{j=i-r(i-1)-1}^{i-1} \mid B(i,j) \mid \quad > \quad 0,$$

which also implies that

$$1 - \eta\frac{\gamma}{4}(\gamma+1) \quad > \quad 0$$

Thus,

$$\eta < \frac{4}{\gamma(\gamma+1)}.$$

Therefore,

$$\frac{16\mu_2}{5\mu_1\gamma} \quad \leq \quad \eta \quad < \quad \frac{4}{\gamma(\gamma+1)}.$$

Now, using

$$K \mid x \mid^2 \quad \leq \quad x^T G x,$$

an estimation of K is found. That is,

$$K\left[\sqrt{\left(x_1^2 + x_2^2\right)}\,\right]^2 \quad \leq \quad \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} \frac{25}{24}\mu_1 & 0 \\ 0 & \frac{25}{24}\mu_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

And it follows that,

$$\begin{aligned} K\left(x_1^2 + x_2^2\right) \quad &\leq \quad \frac{25}{24}\mu_1 x_1^2 + \frac{25}{24}\mu_1 x_1^2 \\ &\leq \quad \frac{25}{24}\mu_1 \left[x_1^2 + x_1^2\right]. \end{aligned}$$

Thus,

$$K \quad = \quad \frac{25}{24}\mu_1.$$

To verify that

$$\begin{aligned} -\mu_1 - \mu_2 + \beta\omega\eta^2 \mid B(i+1,i) \mid \quad &+ \quad \left(2\eta \mid B^T(i+1,i)G \mid + \mid N^T G \mid\right) \\ &\times \quad \sum_{j=i-r(i)}^{i-1} \mid B(i,j) \mid \quad \leq \quad -\psi, \end{aligned}$$

consider

$$\begin{aligned} -\mu_1 - \mu_2 + \beta\omega\eta^2 \mid B(i+1,i) \mid \quad &+ \quad \left(2\eta \mid B^T(i+1,i)G \mid + \mid N^T G \mid\right) \\ &\times \quad \sum_{j=i-r(i)}^{i-1} \mid B(i,j) \mid. \end{aligned}$$

Substituting the estimates for $B(i+1,i)$, $B^T(i+1,i)$, G, η and N^T gives

$$\begin{aligned}
-\mu_1-\mu_2 &+ \beta\omega\eta^2\Big|\begin{pmatrix}\frac{1}{4}\gamma & 0\\ 0 & \frac{1}{4}\gamma\end{pmatrix}\Big|\\
&+ \Big[2\eta\Big|\begin{pmatrix}\frac{1}{4}\gamma & 0\\ 0 & \frac{1}{4}\gamma\end{pmatrix}\begin{pmatrix}\frac{25}{24}\mu_1 & 0\\ 0 & \frac{25}{24}\mu_1\end{pmatrix}\Big|\\
&+ \Big|\begin{pmatrix}-\frac{4}{5} & 0\\ 0 & -\frac{4}{5}\end{pmatrix}\begin{pmatrix}\frac{25}{24}\mu_1 & 0\\ 0 & \frac{25}{24}\mu_1\end{pmatrix}\Big|\Big]\times\frac{\gamma}{4}\Big(\gamma+1\Big)
\end{aligned}$$

$$\begin{aligned}
\le -\mu_1-\mu_2 &+ \beta\omega\eta^2\Big|\frac{1}{4}\gamma\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\Big|\\
&+ \Big[2\eta\Big|\frac{1}{4}\gamma\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\Big|\Big|\frac{25}{24}\mu_1\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\Big|\\
&+ \Big|-\frac{4}{5}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\Big|\Big|\frac{25}{24}\mu_1\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\Big|\Big]\times\frac{\gamma}{4}\Big(\gamma+1\Big).
\end{aligned}$$

Note that

$$\Big|\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\Big| = 1.$$

and so

$$-\mu_1-\mu_2+\frac{\beta\omega\eta^2\gamma}{4}+\Big(\frac{25\eta\gamma}{48}\mu_1+\frac{5}{6}\mu_1\Big)\times\frac{\gamma}{4}(\gamma+1)$$

$$\begin{aligned}
&= -\mu_1-\mu_2+\frac{\beta\omega\eta^2\gamma}{4}+\Big(\frac{25\eta\gamma+40}{48}\Big)\times\frac{\mu_1\gamma}{4}(\gamma+1)\\
&\le -\psi. \qquad (4.44)
\end{aligned}$$

To verify the condition

$$-\beta+2\eta\mid B^T(i+1,i)G\mid+\mid N^TG\mid \;\le\; 0,$$

In a similar manner, consider

$$
\begin{aligned}
-\beta+ \quad 2\eta \Big| & \begin{pmatrix} \frac{1}{4}\gamma & 0 \\ 0 & \frac{1}{4}\gamma \end{pmatrix} \begin{pmatrix} \frac{25}{24}\mu_1 & 0 \\ 0 & \frac{25}{24}\mu_1 \end{pmatrix} \Big| \\
+ \quad \Big| & \begin{pmatrix} -\frac{4}{5} & 0 \\ 0 & -\frac{4}{5} \end{pmatrix} \begin{pmatrix} \frac{25}{24}\mu_1 & 0 \\ 0 & \frac{25}{24}\mu_1 \end{pmatrix} \Big|
\end{aligned}
$$

$$
\begin{aligned}
\leq -\beta+ \quad 2\eta \Big| \frac{1}{4}\gamma & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Big| \Big| \frac{25}{24}\mu_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Big| \\
+ \quad \Big| -\frac{4}{5} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Big| \Big| \frac{25}{24}\mu_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Big|
\end{aligned}
$$

$$
\leq -\beta + 2\eta \times \frac{25\mu_1\gamma}{96} + \frac{5}{6}\mu_1
$$

$$
\begin{aligned}
\leq & \quad -\beta + \frac{25\eta\mu_1\gamma}{48} + \frac{5}{6}\mu_1 \\
\leq & \quad 0. \qquad (4.45)
\end{aligned}
$$

Now, to verify

$$
\mid GN \mid +\eta \mid GB(i+1, i) \mid \quad \leq \quad 0.
$$

consider

$$
\mid GN \mid +\eta \mid GB(i+1, i) \mid .
$$

By substtution,

$$
\Big| \begin{pmatrix} \frac{25}{24}\mu_1 & 0 \\ 0 & \frac{25}{24}\mu_1 \end{pmatrix} \begin{pmatrix} -\frac{4}{5} & 0 \\ 0 & -\frac{4}{5} \end{pmatrix} \Big| + \eta \Big| \begin{pmatrix} \frac{25}{24}\mu_1 & 0 \\ 0 & \frac{25}{24}\mu_1 \end{pmatrix} \begin{pmatrix} \frac{1}{4}\gamma & 0 \\ 0 & \frac{1}{4}\gamma \end{pmatrix} \Big|
$$

$$\leq \frac{5}{6}\mu_1 + \frac{25\eta\mu_1\gamma}{96}$$
$$\leq \quad 0.$$

Now inequalities (4.41) and (4.43) can be satisfied by the choice of appropriate μ_1, μ_2 and ω. Thus, it is shown that the zero solution of

$$x(i+1) = \begin{pmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{5} \end{pmatrix} x(i) + \sum_{j=i-r(i)}^{i-1} \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} -\frac{16\mu_2}{5\mu_1\gamma}x_1 \\ -\frac{16\mu_2}{5\mu_1\gamma}x_2 \end{pmatrix},$$

is stable by invoking Theorem 4.1.

4.5 Chapter Summary

In this chapter the stability concept was discussed. A suitable Lyapunov function which led to obtaining sufficient inequalities to achieve stability of the zero solution of the considered system of variable delay difference equation was constructed. Relevant lemmas, theorems to establish stability of the understudied difference equation werer proved. Further, an example was constructed to test the validity and power of the obtained results .

CHAPTER FIVE

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

5.1 Overview

In this chapter, summaries of results, conclusions, and a few recommendations were provided .

5.2 Summary

In this thesis, as stated in the objectives of the research, a suitable Lyapunov function was contructed to investigate the qualitative behaviour of the solution for a system of Volterra difference equations.
Along the line, the constructed Lyapunov function was used to prove that the zero solution for the system of non-linear Volterra equations was stable and sufficient conditions for the stability of the zero solution for the system of non-linear Volterra difference equations with variable delay wre established.
An example was constructed to acertain the power of the obtained results.

5.3 Conclusions

It is concluded that a suitable Lyapunov function for establishing the stability of a system of non-linear Volterra difference equations with variable delay has been constructed.
Also, sufficient conditions for the stability of the zero solution of a system of non-linear Volterra difference equations with variable delay have been obtained.
Finally, the work has been furnished with a suitable example and the strength of the obtained results has also been tested.

5.4 Recommendations

The Lyapunov function should be used to determine the stability of the zero solution of a system of non-linear Volterra difference equations.

REFERENCES

Adivar, M., Koyuncuoğlu, H. C., & Raffoul, Y. N. (2013). Periodic and asymptotically periodic solutions of systems of nonlinear difference equations with infinite delay. *Journal of Difference Equations and Applications, 19(12)* , 1927–1939.

Agarwal, R.P. (1992). *Difference equations and inequalities.* New York: Marcel Dekker.

Burton, T. A., & Mahfoud, W. E. (1983). Stability criteria for Volterra equations. *Transactions of the American Mathematical Society, 279(1),* 143–143.

Eid, G. M., Ghalayini, B., & Raffoul, Y. N. (2015). *Lyapunov functions and Stability in Nonlinear Finite Delay Volterra Discrete Systems10(1)*, 77-90.

Elaydi, S., Murakami, S., & Kamiyama, E. (1999). Asymptotic equivalence for difference equations with infinite delay. *Journal of Difference Equations and Applications, 5(1),* 1-23.

Elaydi, S. (2005). *An Introduction to Difference Equations. 3rd Edition,* Springer, Berlin.

Győri, I., & Horváth, L. (2008). Asymptotic Representation of the Solutions of Linear Volterra Difference Equations. *Advances in Difference Equations, 2008,* 1–23.

Kelley, W. G. & Peterson, A. C. (2001). *Difference equations: An introduction with applications.* (2nd Ed). Harcourt/Academic Press, San Diego, CA.

Kolmanovskii, V. B., Castellanos-Velasco, E., & Torres-Muñoz, J. A. (2003). A survey: stability and boundedness of Volterra difference equations. *Nonlinear Analysis: Theory, Methods & Applications, 53(7-8)* , 861–928.

Kolmanovskii, V. B., & Myshkis, A. D. (1998). Stability in the first approximation of some Volterra difference equations. *Journal of Difference Equations and Applications, 3(5-6)* , 401–410.

Medina, R. (2001). Asymptotic behavior of Volterra difference equations. *Computers & Mathematics with Applications, 41(5-6),* 679–687.

Migda, M., & Morchało, J. (2013). Asymptotic properties of solutions of difference equations with several delays and Volterra summation equations. *Applied Mathematics and Computation, 220,* 365–373.

Raffoul, Y. N. (2006). Stability and periodicity in discrete delay equations. *Journal of Mathematical Analysis and Applications, 324(2)*, 1356–1362.

Raffoul, Y. N. (2018). *Qualitative theory of Volterra difference equations (1st ed.)*. Springer Cham.

Song, Y., & Baker, C. T. H. (2004). Linearized stability analysis of discrete Volterra equations. *Journal of Mathematical Analysis and Applications, 294(1),* 310–333.

Sultana, N. (2015). Volterra difference equations. *Doctoral Dissertations.* 2396.

Yankson, E. (2009). Stability in discrete equations with variable delays. *Electronic Journal of Qualitative Theory of Differential Equations, 8,* 1–7.