UNIVERSITY OF CAPE COAST

# SOLUTION OF INVERSE EIGENVALUE PROBLEM FOR SINGULAR SYMMETRIC AND HERMITIAN MATRICES OF RANKS FIVE AND SIX

MICHAEL KUMORDZI

2023

UNIVERSITY OF CAPE COAST

# SOLUTION OF INVERSE EIGENVALUE PROBLEM FOR SINGULAR SYMMETRIC AND HERMITIAN MATRICES OF RANKS FIVE AND SIX

BY

MICHAEL KUMORDZI

Thesis submitted to the Department of Mathematics of the School of Physical Sciences, College of Agriculture and Natural Sciences, University of Cape Coast, in partial fulfillment of the requirements for the award of Master of Philosophy degree in Mathematics

DECEMBER 2023

# DECLARATION

## Candidate's Declaration

I hereby declare that this thesis is the result of my own original research and that no part of it has been presented for another degree in this university or elsewhere.

Candidate's Signature............................................ Date..............................

Name: Michael Kumordzi

## Supervisor's Declaration

I hereby declare that the preparation and presentation of the thesis were supervised in accordance with the guidelines on supervision of thesis laid down by the University of Cape Coast.

Supervisor's Signature ................................. Date..........................

Name: Prof. (Mrs.) Natalia Mensah

# ABSTRACT

In this work, the inverse eigenvalue problem is studied in the context of singular symmetric and Hermitian matrices, with a particular emphasis on ranks five and six. We looked into ways to solve singular symmetric and Hermitian matrices' Inverse Eigenvalue Problem (IEP). We devised a method to reconstruct such matrices from their eigenvalues, based on a solvability lemma. Through innovative methodologies, we aim to provide effective solutions for determining the original matrices from their eigenvalues, shedding light on challenges posed by singularity and higher rank. In the case of $n \times n$ matrix, the number of independent matrix elements would reduced.

# KEY WORDS

Hermitian matrices

Rank

Skew symmetric matrices

Symmetric matrices

Trace

Transpose

ACKNOWLEDGEMENTS

I am most grateful to God almighty for his grace, protection, and inspiration for this study.

I would like to thank Prof. (Mrs.) Natalia Mensah, my supervisor, for her kind guidance, counsel, and support throughout this study.

I owe an obligation of gratitude to all of the mathematics department lecturers and friends who have helped make this effort possible. Once more, I want to thank all of my coursemates for their assistance with the study.

# DEDICATION

To family and friends

# TABLE OF CONTENTS

LIST OF TABLES

Page

# LIST OF ABBREVIATIONS

| | |
|---|---|
| IEP | Inverse Eigenvalue Problem |
| NIEP | Non-negative Inverse Eigenvalue Problem |
| LSIEP | Least Square Inverse Eigenvalue Problem |
| PIEP | Parameterized Inverse Eigenvalue Problem |
| AIEP | Addictive Inverse Eigenvalue Problem |
| MIEP | Multiplicative Inverse Eigenvalue Problem |
| PDIEP | Partial Described Inverse Eigenvalue Problem |
| MVIEP | Multivariant Inverse Eigenvalue Problem |
| SIEP | Structured Inverse Eigenvalue Problem |

# CHAPTER ONE

# INTRODUCTION

In matrix analysis most specifically symmetric and Hermitian matrices, they are seen in science, engineering and physics in particular that inverse problems are among the most significant mathematical challenges. Several mathematicians and researchers contributed to the development of the inverse eigenvalue problem (Fallat & Hogben, 2007). The inverse eigenvalue problem can be found in several disciplines, including control theory, signal processing, and system identification. It has numerous real-world applications in the fields of engineering and science where it is necessary to comprehend or create systems based on defining characteristics.

## Background to the study

The inverse problem is a concept that arises in various fields of science, engineering, and mathematics. In essence, it involves the challenge of determining the input or cause of a given observed output or result. In other words, instead of starting with known inputs and predicting the outcomes, the inverse problem involves working backward to deduce the inputs from the observed outcomes. An inverse problem involves estimating the value of parameters defining the system under study for the results of real observation. An inverse problem uses the effect to calculate the causes. This type of problem is often encountered in situations where direct measurements of the cause or input are difficult or impossible to obtain, but the resulting outcomes or measurements are readily available. Solving the inverse problem requires making assumptions, using models, or employing computational techniques to approximate the unknown inputs that could have produced the given outputs.

Here are a few examples of inverse problems in different domains:

1. In medical imaging, ( X-ray, MRI, CT scans), the goal is to reconstruct the internal structures of the body from the observed measurements. This involves

solving the inverse problem to obtain an accurate representation of the distribution of tissues or substances within the body. The properties of subsurface geological structures (e.g., oil reservoirs, fault lines) from seismic wave measurements recorded on the Earth's surface.

2. Signal Processing is another application, such as audio or speech processing, the goal might be to retrieve the original signal (input) from the recorded signal (output) that has undergone various distortions, noise, or filtering.

3. In statistical modeling and machine learning, parameter estimation can be considered an inverse problem. Solving inverse problems can be challenging due to the potential non-uniqueness and noise in the observed data.

An inverse eigenvalue problem concerns the rebuilding of a coefficient structured matrix from prescribed spectral data (Chu & Golub, 2002).
According to Dehghan Niri, Shahzadeh Fazeli, and Heydari (2020), an inverse eigenvalue problem concerns the reconstruction of a matrix with a special structure from prescribed spectral data. Limiting the number of different options that are normally feasible requires that in addition to the spectrum requirement, the matrix produced maintains a required structure if a solution is not found. Addressing the inverse eigenvalue problem often involves constructing matrices that satisfy the given eigenvalue constraints and have connections to optimization and spectral theory. Rothblum (2006) explains that an inverse eigenvalue problem can also be used in mathematical modeling and parameter identification. The inverse eigenvalue problem is a mathematical problem where the goal is to find a matrix that has given eigenvalues and possibly other spectral properties (Rothblum, 2006). Specifically, given a set of eigenvalues and, optionally, some additional constraints, the task is to determine a matrix that possesses those eigenvalues. Solving the inverse eigenvalue problem for nonsingular symmetric matrices has occupied most research efforts (Oduro, 2014; Deakin & Luke, 1992; Chu & Golub, 2002; Chu, 1998). The non-negative eigenvalue problem of symmetric matrices remained the focus of several schol-

ars (Marijuán, Pisonero, & Soto, 2007; Soto & Rojo, 2006; Egleston, Lenker, & Narayan, 2004). The issue of singular symmetric matrices with varying size and order has been practically tackled by (Baah, 2012 and Aidoo, Gyamfi, Ackora-Prah, & Oduro, 2013), provided that linear dependence links are given.
Wu (2011) has also made clear statements regarding the prerequisites and solvability of the non-negative aspect of the inverse eigenvalue problem of symmetric matrices. The inverse eigenvalue problem is an important and challenging area of research with applications in various fields, including control theory, systems identification, and structural mechanics. Solving this problem is not always straightforward, and it heavily depends on the specific constraints imposed and the properties of the given eigenvalues.
Some variations of the inverse eigenvalue problem include.

1. Completing a given set of eigenvalues that is given a partial set of eigenvalues, the task is to find a matrix with those eigenvalues.

2. The symmetric inverse eigenvalue problem is the version of the problem that restricts the search to symmetric matrices .

Solving the inverse eigenvalue problem requires advanced mathematical techniques, and there may not always be a unique solution. Depending on the given conditions and constraints, there might be multiple matrices that satisfy the requirements. Researchers in this field employ various algorithms, optimization methods, and numerical techniques to tackle the problem efficiently. Due to the complexity of the problem and the lack of a unique solution in many cases, the inverse eigenvalue problem remains an active area of research in applied mathematics and engineering.

Iterative methods, sometimes referred to as iteration methods, are numerical approaches that are used to solve mathematical problems, usually when finding exact answers is difficult or computationally expensive. These techniques are iterative because they begin with a first guess and keep refining it until they arrive at a workable answer .

Typical iteration techniques include the following

1. Jacobi Method is an iteration approach used to determine the structure of linear equations. It updates each component of the solution vector based on the components of the previous iteration.

2. The Newton-Raphson Method is a technique for finding the roots of real-valued functions.

3. The Gauss-Seidel Method is another method to solve systems of linear equations, but it updates each component of the solution vector using the most recent values available in the current iteration.

4. Successive Over-Relaxation (SOR) is an extension of the Gauss-Seidel method that incorporates relaxation factors to improve convergence speed.

5. Conjugate Gradient Method systems of linear equations, often arising from problems in optimization and finite element analysis.

6. Richardson Iteration is a basic iterative method used to solve linear systems, based on a simple matrix transformation.

7. Krylov Subspace Methods is a family of iterative methods that seek approximate solutions within a subspace generated by the matrix and initial guesses.

A concept from linear algebra related to square matrices is called an eigenvalue. Eigenvalues are special scalars dealing with linear systems of equations (Marcus & Minc, 1988). We can use eigenvalue and eigenvector in physics and engineering applications. Each eigenvalue has a matching "eigenvector" that is coupled with it. Eigenvectors are a unique collection of vectors linked to a linear system of equations (Marcus & Minc, 1988). In the context of classification and matrix diagonalization, along with equilibrium and vibration inquiry, eigenvalues and eigenvectors are also utilized. According to Voyevodin (1983),

$$BX = \lambda X, \tag{1}$$

when scalars $\lambda$, its matching eigenvector,$X$ are present, $\lambda$ is referred to as the eigenvalue of $B$. Therefore $X \in \mathbf{R} \neq 0$.

Let $B$ be $n \times n$ square matrix

$$\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ b_{21} & b_{22} & \cdots & b_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ b_{k1} & b_{k2} & \cdots & b_{kk} \end{bmatrix} \tag{2}$$

the matching eigenvectors fulfill when the eigenvalues are $\lambda$,

$$\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ b_{21} & b_{22} & \cdots & b_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ b_{k1} & b_{k2} & \cdots & b_{kk} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix} = \lambda \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix} \tag{3}$$

it is the homogeneous method's counterpart $BY = 0$, that is

$$\begin{bmatrix} b_{11} - \lambda & b_{12} & \cdots & b_{1k} \\ b_{21} & b_{22} - \lambda & \cdots & b_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ b_{k1} & b_{k2} & \cdots & b_{kk} - \lambda \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{4}$$

There are different ways to write equation (4):

$$(B - \lambda I)Y = 0 \text{ (Voyevodin, 1983)}$$

The identity matrix is represented by $I$ in the equation above. When linear equations have nontrivial solutions and the determinant vanishes, then the given solution is $det(B - \lambda I)$.

A matrix is a collection of numbers arranged into fixed numbers of rows and columns. A matrix is a rectangular array of numbers, symbols, or expressions arranged in rows and columns. Matrices are an essential mathematical concept used in various fields, including linear algebra, computer graphics, physics, engineering, and many other disciplines.

Below are some forms of matrices;

A matrix with a single row is referred to as a row vector, while a matrix with a single column is referred to as a column vector.

An identity matrix is a special square matrix in which all diagonal elements are 1 and all other elements are 0.

One unique feature of a square is that it is identical to its transpose when it is a symmetric matrix. Moreover, if were to reflect a symmetric matrix over its main diagonal, the matrix would remain unchanged. That is $A^T = A$ (Strang, 2012).

A square matrix where the transpose of the matrix equals its negative is called a skew-symmetric matrix, $Q^T = -Q$ (Bronshtein & Semendyayev, 2013).

The transpose of a matrix is obtained by interchanging its rows and columns. Herstein and Winter (1988), the transpose of a matrix is obtained by interchanging the rows with the columns of a given matrix.

Example, if

$$Q = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ then } Q^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Connell (2002), transpose is a function to $R_{mn}$. $Q^T$,$R_{nm}$ is the matrix whereby the (i j) term is the (j i) term of Q. So, row i (column j) of Q generates column j (row i) of $Q^T$. The transpose of a matrix involves flipping its rows and columns, effectively turning its rows into columns and columns into rows. Keep in mind that not every matrix has a transpose. Only rectangular and square matrices, undertake transpose. Matrices of the same dimensions matrices can added or subtracted. Matrix multiplication is a bit more complex. For two matrices $X(m \times p)$ and $Y(p \times n)$, the resulting matrix $Z(m \times n)$ is obtained by taking the dot product of the rows of $X$ and the columns of $Y$. Matrices have numerous applications in various fields, such as solving systems of linear equations, representing transformations in computer graphics, and analyzing networks and data sets. They provide a powerful tool-set for solving complex problems and are a fundamental building block in linear algebra and related mathematical disciplines.

A symmetric matrix is a square matrix that is the same as its transpose. $Q = Q^T$ (Lipovetsky, 2013). In a symmetric matrix $Q$, for instance, the element at the $ith$ row and $jth$ column is the same for all $i$ and $j$ as it is at the $jth$ row and $ith$ column (Draper & Smith, 1998). Mathematically it can be represented as $Q = Q^T$. A symmetric matrix is always diagonalizable, meaning it can be expressed as a product of three matrices: $Q = PDP^{-1}$. Symmetric matrices are applied in linear algebra, physics, computer science, and statistics among others. Their symmetry simplifies calculations and makes them valuable in solving certain types of problems efficiently. If a matrix is symmetric then $Q^n$ is also symmetric where n is an integer. Also, if Q is symmetric then $Q^{-1}$ is also symmetric.

For singular symmetric matrices special techniques and considerations are required. Therefore combining both conditions, a singular symmetric matrix that is symmetric ($A^T = A$) and singular ($det(A) = O$). If the determinant is $0$, the matrix cannot be inverted. The presence of zero eigenvalues in matrix $A$ possesses additional difficulties in finding a suitable matrix $B$. If matrix $A$ has zero eigenvalues, it means the traditional inverse eigenvalue problem may not apply directly. So you need to use an altered algorithm to deal with it. In many different branches of mathematics and applications, such as linear algebra, optimization, and physics, singular symmetric matrices are present. Singular symmetric matrices can also be applied in engineering and various science fields.

A Non-singular symmetric matrix is a square matrix that has an inverse and is also symmetric. Due to their well-defined features and ability to frequently represent significant transformations or relationships, such matrices are of special importance in several branches of mathematics, physics, and engineering.

The determinant of a square matrix is a scalar value that can be computed from the elements of the matrix. It is a fundamental idea in linear algebra with several geometric and algebraic applications. The determinant is commonly written as $det(B)$ for a square matrix $B$ of size $n \times n$.

**Theorem 1.1.** For real symmetric characteristics, the eigenvalue and polynomial $C_A(x)$ have real roots.

**Theorem 1.2.** Eigenvectors with independent eigenvalues for actual symmetric matrices are diagonal (Weiss, 2019).

Hermitian matrices are the complex extension of real symmetric matrices. It can be easily proved that when two Hermitian matrices add up the results will be Hermitian. A square matrix of the form $Q = [q_{ij}]_{n\times n}$, where $Q^T = [q_{ji}]_{n\times n}$ is the conjugate transpose of $Q$, is the definition of a Hermitian matrix. This means that for every $q_{ij} \in Q$, there is also $q_{ji} \in Q^T$. Example: Matrices $Q$ and $P$ shows example of complex and real Hermitian matrices respectively;

$$Q = \begin{bmatrix} 2 & 1+5i & 3 \\ 1-5i & 4 & 2i \\ 3 & -2i & 7 \end{bmatrix}, \qquad P = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 6 & 5 \\ 4 & 5 & 8 \end{bmatrix}$$

**Theorem 1.3.** The sum of the components of the major diagonal or( the diagonal from the top left to the lower right), defines the trace of a $n \times n$ square matrix

$A$.

$$tr(A) = \sum_{i=1}^{n} \lambda\alpha_{ij}\text{,i=j}$$

(Sasikumar, Karthikeyan, Suganthi, & Madheswaran, 2009).

An $n \times n$ matrix $Q$ has as its determinant $(Q - \lambda I)$. To obtain a polynomial or characteristics equation, set the determinant to zero.

$$\Delta(\lambda) = \begin{bmatrix} q_{11} - \lambda & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} - \lambda & \cdots & q_{21} \\ \vdots & \vdots & & \vdots \\ q_{n1} & q_{n2} & \cdots & q_{nn} - \lambda \end{bmatrix}$$

The polynomial in $\lambda$ that makes up the secular determinant as stated as follows

$$\Delta\lambda = (-\lambda)^n - p_1(-\lambda)^{n-1} + p_2(-\lambda)^{n-2} + p_3(-\lambda)^{n-3} + \cdots + p_{n-1}(-\lambda) + p_n = 0$$

**Theorem 1.4.** All eigenvalues of Hermitian matrices are real.

**Theorem 1.5.** Hermitian Matrices possess perpendicular.

**Theorem 1.6.** A symmetric matrix is produced when two symmetric matrices are added together. Proof: Assume $S = A + B$ and $A, B \in Symm(n)$. Concerning the inequality trace

$$tr(A) + tr(B) = tr(A + B)$$

$$\sum_{i=1}^{n} \lambda\mu\alpha_i(Q) + \sum_{j=1}^{n} \lambda\mu\beta_j(B) = \sum_{k=1}^{n} \delta\mu_k(S)$$

(Simovici, 2012 ; Bronshtein & Semendyayev, 2013 ; Zhang & Golub, 2001).

**Theorem 1.7.** The inverse is distinct. In other words, if $X$ has inverses $Y$ and $Z$, then $Y = Z$. This is proven. Let us assume $XY = ZX = I$. In contrast, $Y = IY = (ZX)Y = Z(XY) = ZI = Z$. Moreover, if $X$ is non-singular, then so is $X^{-1}$. $(X^{-1})^{-1} = X$.

**Theorem 1.8.** If $Q$ and $M$ are similar $n \times n$ matrices, then they have the same eigenvalues.

Proof: An invertible matrix $P$ exists such that $Q = P^{-1}MP$ since $M$ and $Q$ are similar. Based on determinant qualities, it can be inferred that

$$\begin{aligned}|\lambda I - Q| &= |\lambda I - P^{-1}MP| \\ &= |P^{-1}\lambda IP - P^{-1}MP| \\ &= |P^{-1}(\lambda I - M)P| \\ &= |P^{-1}||\lambda I - M||P| \\ &= |P^{-1}||P||\lambda I - M| \\ &= |P^{-1}P||\lambda I - M| \\ &= |\lambda I - M|\end{aligned}$$

**Theorem 1.9.** Let $S$ be a symmetric matrix such that $S = S^T$ and let $A$ be a non-symmetric matrix such that $-A = A^T$. Then $tr(SA) = tr(AS)$ (Soto, 2016).

Rank is the dimension of the vector space spanned by the rows (or columns) of the matrix. For any matrix $A$, the rank-nullity theorem states that the sum of the rank of $A$ and the nullity of $A$ (the dimension of the null space) is equal to the number of columns of $A$. Mathematically, $rank(A) + nullity(A) = columns(A)$. The rank of a matrix is useful in various applications, including solving systems of linear equations, finding the inverse of a matrix, and understanding the number of independent equations in a system. There are several methods, according to Johnston (2021) to compute the rank of a matrix, we use row reduction techniques (eg., Gaussian elimination).

Rank and Solutions of Linear Systems:

The rank of a coefficient matrix in a system of linear equations is related to the existence and uniqueness of solutions. Otherwise, there may be infinitely many solutions or no solutions at all. The rank of matrices is essential in linear algebra and is applicable in various fields, such as engineering, physics, computer science, and data analysis. The matrix is stable and unique at the higher ranks. As a result, the likelihood that the inverse eigenvalue problem's solution is unique increases when the provided matrix has full rank. The problem becomes more ill-posed and less unique if the matrix has a lower rank, which occurs when some rows or columns are linearly dependent.

For a system to be stable, all eigenvalues must have negative real portions for continuous-time systems or must be located inside the complex plane's unite circle for discrete-time systems.

## Statement of the Problem

The current research in the area of solving inverse eigenvalue problem for singular symmetric matrices is up to rank four. The research aims to address the Inverse Eigenvalue and Problem (IEP) specifically for singular symmetric and Hermitian matrices with ranks five and six. The problem involves determining a suitable set of matrix entries that satisfy the given eigenvalue constraints, focusing on the unique challenges posed by matrices of these ranks. The goal is to provide a solution algorithm that effectively reconstructs singular symmetric and Hermitian matrices of ranks five and six based on specified eigenvalue conditions. The proposed research seek to provide solutions and insights into this special domain. This research will improves the understanding of matrices properties and also lead to numerical stability and applicable in quantum mechanics and engineering domain.

## Purpose of the study

Finding a solution to the inverse eigenvalue problem for singular symmetric and Hermitian matrices of ranks five and six is the goal of this research. The research will maintain a particular specific structure that satisfies a specific spectral property. The two main concerns that arise are theoretical ones related to solvability and practical ones related to computation for inverse eigenvalue problems. This research focuses on the solvability of the inverse eigenvalue problem.

**Theorem 1.10.** A condition that must be met for a matrix with the principal diagonal $a_1, a_2, \cdots, a_n$ and eigenvalues, $\lambda_1, \lambda_2, \cdots \lambda_n$ is

$$\sum_{n-i}^{n} a_i = \sum_{i=1}^{n} \lambda_i$$

The next stage in the investigation is unquestionably to establish, as indicated by Mirsky (1964), the relationship between large diagonal elements and the one values of the general matrix. This correlation was found independently by (Sing, 1976 and Thompson, 1977).

**Theorem 1.11.** (Sing- Thompson theorem) Each of the two vectors $d, s \in \mathbf{R}^n$ has entries organized in the following orders: $s_1 \geq s_2 \geq \cdots \geq s_n$ and $|d_1| \geq |d_2| \geq \cdots \geq |d_n|$, respectively. Then, a real matrix $A \in \mathbf{R}^{n \times n}$ exists, with main diagonal entries (d) having a potentially distinct order and a single symmetric value (s).

$$\sum_{i=1}^{n} |d_i| \leq \sum_{i=1}^{n} s_i$$

for all $i = 1, 2, \cdots n$ $(\sum_{i=1}^{n-1} |d_i|) - |d_n| \leq (\sum_{i=1}^{n-1} s_i) - s_n$

The primary focus has instead been on developing a method for numerically constructing a matrix beforehand where the given spectral data are possible

## Research Objectives

Aidoo et al. (2013) design strategy of constructing singular symmetric matrices displaying up to rank four using eigenvalues. Therefore, the objectives are:

1. Construct singular symmetric matrices of ranks five and six using the eigenvalues.

2. Generate singular Hermitian matrices of ranks five and six using the eigenvalues.

### Research Questions

The research questions is based on the objective stated.

1. How can singular symmetric matrices of specific ranks, particularly ranks five and six be systematically constructed?

2. What are the procedures of generating a singular Hermitian matrix of ranks five and six?

### Significance of the Study

As far as the researcher is aware, research on solving the inverse eigenvalue problem for singular symmetric matrices is currently conducted at a rank four level. Consequently, increasing the scope of the inverse eigenvalue problem search for singular symmetric matrices of ranks five and six will improve knowledge in academia. The research find it applications in quantum mechanics and engineering domain.

### Delimitations

The work may have taken into consideration the inverse eigenvalue problems for non-singular symmetric matrices and skew-symmetric matrices of rank five and six. This paper, however, focuses on the inverse eigenvalue problem for ranks five and six singular symmetric and Hermitian matrices.

### Limitations

The work on the inverse eigenvalue has been restricted to rank five and six singular symmetric and Hermitian matrices since only three matrices were examined.

### Definition of Terms

Hermitian matrices are the complex extension of real symmetric matrices.

Rank is the dimension of a vector space spanned by the rows and columns of the matrix.

Skew-symmetric matrix, also known as an anti-symmetric matrix, is a square matrix where the transpose of the matrix is equal to its negative (Riondato, García-Soriano, & Bonchi, 2017).

Symmetric matrix is a type of square that has a special property: it is equal to its transpose. Moreover, if were to reflect a symmetric matrix over its main diagonal, the matrix would remain unchanged. That is $A^T = A$.

Trace is the total of all the elements on the matrix's major diagonal.

Transpose of a matrix is an operation that switches its rows with its columns.

## Organization of the Study

We have previously talked about the first chapter. The subsequent chapters are explicated as follows as well: In Chapter Two, some relevant material on the IEP of Hermitian and singular symmetric matrices is reviewed. The research approach and techniques are presented in Chapter Three. The primary findings of the study are presented in Chapter Four. In the study, a summary, findings, conclusions, and recommendations are found in Chapter Five.

# CHAPTER TWO

# LITERATURE REVIEW

## Introduction

The IEP of singular symmetric matrices specified ranks up to four were discussed by the author. Acknowledging the paucity of literature on the topic, the researcher chose to consider the many variations of the symmetric matrix inverse eigenvalue problem (Kandić, Reljin, et al., 2008). The researcher decided to take into account the numerous ways that the eigenvalue problem came to be solvable. We provide an overview of findings for general inverse eigenvalue problems as well as nonnegative eigenvalue problems in our work. Researchers continued to explore different approaches, numerical algorithms, and techniques to address this problem. It offers insights into the properties of specific matrix structures and can lead to the development of novel mathematical techniques and algorithms for solving complex problems in real-world scenarios. It is noted that the theoretical framework of eigenvalues is a powerful tool in linear algebra that enable us to understand and analyze complex system and structures (Aidoo et al., 2013). Circumstances under which Chu (1998), demonstrate how a collection of inverse eigenvalue problems are recognized and categorized following their properties.

## Least Square Inverse Eigenvalue Problem (LSIEP)

The goal of LSIEP is to identify a matrix whose eigenvalue difference from a specific set of target eigenvalues as minimal as possible. This implies that there exist situations where an approximation, best in the least squares sense, might be adequate. We go over how to get the least squares solution in this part. All the topics discussed at this point can be naturally extended to the least squares formulation. We therefore need to make clear two definitions for a least squares approximation, depending on whether the requirement is to be imposed explicitly. Reducing the difference between the eigenvalues is an appropriate way to do this (LSIEP) (Chu, 1998).

Given a set of scalars $\lambda_1, \lambda_2, \cdots, \lambda_n \in F$, $(m \leq n)$, find a matrix $X \in N$ of indices with $1 \leq \sigma_1 < \cdots \leq \sigma_n$.

$$F(X, \sigma) = \frac{1}{2} \sum_{i=1}^{m} \lambda_{\sigma_i}(X) - \lambda_i)^2 \quad (5)$$

where the matrix X's eigenvalues, $\lambda(X), i = 1, \cdots, n$, are reduced. Note that the cardinally of the required set of eigenvalues is m, which could be less than n. As such, for any fixed point, LSIEP is always associated with a computational problem. By measuring and reducing the difference between the matrices the least square approximation can also be expressed in this manner. Determine the set

$$F(X) := \frac{1}{2}||X - P(X)||^2 \quad (6)$$

that reduces the function from a set of $P$ that fulfills a specific spectral restriction and a set $N$ that defines a structure restriction.

**Parameterized Inverse Eigenvalue Problem(PIEP)**

In both linear algebra and control theory, there is a mathematical issue known as the (PIEP). The objective of this task is to locate a matrix whose eigenvalues, subject to some extra constraints imposed by parameters, match a given set of desired eigenvalues, and maybe eigenvectors as well. Determine the value of the parameter $r$ such that $1, \cdots n = Q(r)$ given a family of matrices $Q(r) \in X$. Remember that $m$ might not have as many parameters in $r$ as $n$ does. The formal definition of the family of matrices $Q(r)$ in terms of $r$ determines how the PIEP appears and is solved. Its format typically appears in factor analysis and discrete modeling. One element common to all PIEP versions is the usage of the parameter $r$ as a "control" that provides a particular, preset solution to the underlying problem; examples illustrating various characteristics are given in the next section.

**Addictive Inverse Eigenvalue Problem (AIEP)**

The AIEP involves finding a matrix whose eigenvalues are the negatives of the eigenvalues of a given matrix. Conditions for the existence and uniqueness of solutions to AIEP are frequently explored in this field of study. Moreover, the set $N$ can be used pretty widely aside from this. Therefore, we can put a specific structural restriction on the solution matrix $X$ using the set $N$. Structure on $N$ can occasionally develop naturally because of the engineer's design constraints. On the other hand, Chu, Diele, and Ragni (2005) $F = \mathbf{R}$ and $M = H(n)$, $N = D_R(n)$ can be used to represent AIEP (Downing Jr & Householder, 1956). In this regard, the separation of the boundaries valuation challenge with the $M$ Jacobi matrix. As an illustration

$$\lambda u(x) = U"(x) + p(x)u(x) \tag{7}$$
$$0 = u(\pi) = u(0).$$

the eigenvalue problem in a tridiagonal configuration is automatically brought about the fundamental variation equation with homogeneous correspond

$$h = \frac{\pi}{n+1}$$

$$\left( \frac{1}{h^2} \begin{bmatrix} 2 & -1 & 0 & & & \\ -1 & 2 & -1 & & & \\ 0 & -1 & 2 & & \cdots & 0 \\ \vdots & & & \ddots & & \\ 0 & & & & 2 & -1 \\ 0 & & & & -1 & 2 \end{bmatrix} + X \right) u = \lambda u$$

In this case, $p(x)$'s separation is reflected by the diagonal matrix $X$. In addition to this, an AIEP can be considered the discrete analog of the recognized inverse Strum-Liouville problem, in which it is essential to determine the potential $p(x)$ to guarantee that the system has the intended spectrum (Zhao, Hu, & Zhang, 2011). There is an association between the collaboration problem as stated by (Friedland, Nocedal, & Overton, 1987; Friedland, 1977) and the issue of schooling. In the first one, given an actual symmetric matrix $B$ with zero diagonal figures, create a diagonal matrix $D$ such that the total of $B + D$ contains as numerous zero eigenvalues as possible. To ensure that $B - D$ and $B - D$ stay positive semidefinite while $D$'s record is lowered, in the second case, a positive diagonal matrix $D$ must be created given a real symmetric positive definite matrix $B$.

**Multiplicative Inverse Eigenvalue Problem (MIEP)**

An $n \times n$ square symmetric matrix $B$ is per-multiplied by a vector $Y$ which includes the parameter $d = \{d_1, d_2, \cdots, d_n\}$ so that $YB = \sum_{i=1}^{n} d_i B$ where $d_i \in Y$, (Oliveira, 1972). Some rows become linear combination of other rows after the multiplication. Some authors have treated the solvability of the MIEP, for example (Hadeler, 1969; Oliveira, 1972; Shapiro, 1983). For practical applications, numerous mathematical approaches have also been developed.

$$\text{If } Y = \begin{bmatrix} 1 & 5 \\ 2 & 1 \end{bmatrix} \text{ and } P = \begin{bmatrix} e & f \\ 0 & 1 \end{bmatrix},$$

consequently

$$YP = \begin{bmatrix} 1 & 5 \\ 2 & 1 \end{bmatrix} P = \begin{bmatrix} e & f \\ 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} e + 2f & 5e + f \\ 2 & 1 \end{bmatrix}$$

Consequently, the structure meets the requirements by the eigenvalues of the matrices.

$$\left\| \begin{matrix} \lambda_1\lambda_2 & = & 1e \\ \lambda_1 + \lambda_2 & = & 1 + e + 2f \end{matrix} \right\|$$

We show that there is certainly a set $(e, f)$ of real numbers that solve MIEP given any $\lambda \in \mathbf{R}^2$. In this instance, the outcome is distinct. From a mathematical perspective, compound analysis and linear equations are related to the multiplicative inverse eigenvalue problem. The product of a square matrix $P$ and a matrix $Q$ is the identity matrix $I$ since matrix $Q$ is located. Numerous disciplines, including control theory, system analysis, and numerical analysis, all have uses for this issue. Finding the multiplicative inverse of a matrix, for instance, is essential in control theory for resolving specific control problems and doing stability analysis. Dumond and Baddour (2016), it can be difficult to solve the multiplicative inverse eigenvalue problem, and it frequently calls for complex mathematical procedures and methods. It's crucial to remember that not every matrix has a multiplicative inverse. Invertible or non-singular matrices are matrices with multiplicative inverses. If and only if a matrix's determinant is nonzero, the matrix is invertible. The multiplicative inverse eigenvalue problem is a key issue in linear algebra that has applications in many fields of science and mathematics.

**Partial Described Inverse Eigenvalue Problem (PDIEP)**

There are times when partial eigenvalues and eigenvectors are available during the rebuilding of a structure, rather than the entire range. With limited information available, the PDIEP entails identifying a subset of a matrix's eigenvalues and matching eigenvectors. Numerous applications, including control theory and fundamental dynamics, encounter this problem. To solve this issue approaches such as system detection and methods of optimization are frequently utilized, which involve building the missing eigenvalue and eigenvectors from the given data (Hald, 1976). The process of answering the partial inverse eigenvalue problem usually entails expressing the issue analytically and creating techniques or procedures to identify a matrix on the given subset of elements that meets the provided constraints of (PDIEP) (Hald, 1983). Given vectors $v^1, \cdots, v^k. \subset F^n$ scalars $\lambda_1, \cdots, \lambda_n \subset F$ where $1 \leq k < n$, determine a matrix $X \in N$ which is $Xv^1 = \lambda_i v^1$ for $i = 1, \cdots, k$. For example,

$$M\frac{d^2}{dt^2} + C\frac{d}{dt}v + kv = 0. \tag{8}$$

That happens in many instances in which $M$ is a positive integer and $M$, $C$, and $K$ all symmetric. When variables are separated, the system inevitably results in the quadratic -matrix issue.

Consider now the state feedback pressuring mechanism of the following provides:

$$u(t) = b(f^T \frac{d}{dt}v(t)) + g^T v(t) \tag{9}$$

is applied to the system, assuming vectors $b, f, g \in R^n$ are constant. The $\lambda$ matrix problem with pencil is the resultant the closed-loop system.

$$Q\lambda = M\lambda^2 + (c - bf^T)\lambda + (k - bg^T). \tag{10}$$

This feedback control $u(t)$ aims to shift the unstable or unsatisfactory eigenvalues or substantially induce motion occurrences in the structure while keeping the positive eigenvalues. This concept leads to the partial pole assignment challenge that follows. The result of applying the previously described feedback control $u(t)$ to a dynamical system is a pole assignment. A different kind of PDIEP that is similar to the PDIEP has the form provided with the corresponding eigenvalues, $M$, $C$, and $K$ along with their corresponding eigenvalues $(\lambda_1, \cdots \lambda_n)$ of the quadratic pencil

$$P(\lambda) = M\lambda^2 + C\lambda + K \tag{11}$$

Given a predetermined vector $b \in R^n$ and m complex numbers, such as $u_1, \cdots, u_m$, $m \leq n$, find $f, g \in \mathbf{C}^n$ that equals the closed loop pencil's spectrum.

$$Q(\lambda) = M\lambda^2 + (c - bf^T)\lambda + (k - bg^T) \tag{12}$$

has a spectrum $u_1 \cdots, u_x, \lambda_{a+1}, \cdots, \lambda_{2y}$ (Chu, 1998).

**Theorem 2.1.** Divide the matrices representing the eigenvalue and eigenvector of into $P = [P_1, P_2]$ as $\lambda = diag(\Lambda_1, \Lambda_2 \cdots \Lambda_n)$ where $X_2 \in \mathbf{C}^{n\times(2x-y)}$, $P_1 \in C^x \times y$. $\Lambda_1 \in Dc(x)$, and $\Lambda_2 \in Dc(2yx$. Define $\beta = [\beta_1, \cdots, \beta_x]^y\, T \in C^x$ by

$$\beta_j = \frac{1}{b^T x_j}\frac{\mu_j - \lambda}{\lambda_j} \prod_{i=1\neq j} \frac{\mu_i - \lambda_j}{\lambda_i - \lambda_j}$$

Afterward the vector pair

$$f := MX_{1\Lambda_1\beta}$$

$$g := -KX_{1\beta}$$

**Multivariate Inverse Eigenvalue Problem (MVIEP)**

It involves constructing a matrix with specified eigenvalues and eigenvectors. A mathematical issue in control theory and linear algebra is the multivariate inverse eigenvalue problem. Sharma and Sen (2018), the objective of MVIEP is to locate a matrix with a predetermined set of eigenvalues given a set of matrices. Solving the multivariate inverse eigenvalue problem often involves optimization techniques and algebraic methods to generate matrices that satisfy the prescribed eigenvalue and eigenvector conditions. Formally, the multivariate inverse eigenvalue problem seeks a matrix $X$ whose eigenvalues are precisely $(\lambda_1, \lambda_2, \cdots, \lambda_k)$ given a set of matrices $A_1, A_2, \cdots, A_k$ and a set of target eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_k$. Other restrictions may be placed on X as well. In a multivariate eigenvalue problem, actual integers $\lambda_1, \cdots, \lambda_m$ and actual variables $x \in R^n$ must be found in MVIEP to solve problems. Real scalars and a real vector must be located so that equations. $Bx = \Lambda x$

In the case when $B \in S(n)$ is a block-divided positive definite matrix, $i = 1, \cdots, m =$,$1 = ||x_i||$ , are satisfied

$$A = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1m} \\ B_{21} & B_{22} & \cdots & B_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ B_{m1} & B_{m2} & \cdots & B_{mm} \end{bmatrix}.$$

There is a lateral the matrix $\lambda$.

$$\Lambda = diag\{\lambda_1 I^{n_1}, \cdots, \lambda_m I^{n_m}\}$$

Blocks are created by partitioning $x \in R^n$ into identity matrices of size $n_i$ and $I^{n_i}.x = x_1^T, \cdots, x_m^T$ (Chu, 1998 ; Efron & Morris, 1976) using $x_i \in R^{n_i}$. In a trivial sense, the $m = 1$ single variate situation is just a standard symmetric eigenvalue issue.

Multivariate Canonical analysis in statistics causes the overall issue. There are several uses for the multivariate inverse eigenvalue problem in many different disciplines, such as system identification, structural dynamics, and control theory. The challenge of solving multivariate inverse eigenvalue problems is difficult, and there may not always be a single solution. Researchers attack this issue and seek approximations of answers using various numerical approaches and optimization algorithms.

### Structured Inverse Eigenvalue Problem (SIEP)

The structured inverse eigenvalue problem is a mathematical problem connected to control theory and the science. Calculate a matrix with a certain structure, often known as a structured matrix, whose eigenvalues match a set of target eigenvalues. The following is an example of a generic SIEP: To obtain $\Delta(X) = \{\lambda_1, \cdots, \lambda_n\}$ (SIEP), find $X \in \mathbf{N}$ that is composed of specifically structure matrices. It is important to keep in mind that the generating and solvability of the problem are prerequisites for the solution of $n \times n$ symmetric matrices (Chu & Golub, 2002).

**Inverse Eigenvalue Problem for Jacobi Matrices**

The inverse eigenvalue problem for Jacobi matrices involves determining the matrix entries when the eigenvalue are given. It's a challenging problem with applications in various fields, including signal processing and quantum mechanics. Various methods such as moment methods and iteration algorithms are used to tackle this problem. Asymmetric tridiagonal matrix is an accepted norm Jacobi matrix. We demonstrate how to create a matrix of this kind with two sets of eigenvalues that meet overlapping conditions, (Boley & Golub, 1987). Due to its creation by the removal of the initial row and final column of the primary diagonal matrix, the lower principal subordinate matrix has order $(n-1)\times(n-1)$ (Bronshtein & Semendyayev, 2013). The IEP of the Jacobi and Periodic Jacobi matrices has attracted a lot of scientific concern. Consider, for instance (Andrea & Berry, 1992 ; Xu & Jiang, 2006). One technique for creating symmetric Jacobi matrices is as follows. Considering a real symmetric tridiagonal matrix of the following kind denoted as Jacobi matrix $Q$.

$$Q = \begin{pmatrix} a_1 & b_1 & \cdots & 0 \\ b_1 & a_2 & \cdots & 0 \\ 0 & \cdots & \cdots & b_{n-1} \\ 0 & \cdots & b_{n-1} & a_n \end{pmatrix}$$

with $b_i > 0$

$$q_{1i}^2 = \frac{\prod_j^{n-1} = i(\mu_i - \lambda_j)}{\prod_{j=1}^{n} j \neq 1(\lambda_j - \lambda_i)}$$

All the eigenvalues are normalized to have norm 1, $a_{11}$ is obtained from the relation

$$\sum_1^n \lambda_i - \sum_1^{n-1} \mu_i.$$

The IEP for the periodic Jacobi matrix is then covered. Inverse scattering theory problems typically involve this inverse problem. Real entries can be found in the tridiagonal periodic Jacobi matrix. Consequently, the corresponding orthonormal eigenvectors and eigenvalues are real. There are at most $2^{n-m-1}$ solutions for the periodic Jacobi matrix, where $m$ is the number of common eigenvalues between the leading primary sub-matrix and the main matrix. There is no unique solution for the periodic Jacobi matrix. Throughout our discussions, we refer to $J_n$ and $J_{n-1}$ as the principal sub-matrix and the main matrix, respectively (Xu & Jiang, 2006).

Bronshtein and Semendyayev (2013), a real symmetric matrix of the following type is called a periodic Jacobi matrix:

$J_n$

$$= \begin{pmatrix} a_1 & b_1 & \cdots & \cdots & b_n \\ b_1 & a_2 & b_2 & \cdots & 0 \\ 0 & b_2 & a_3 & \cdots & b_{n-1} \\ 0 & \cdots & \cdots & a_{n-1} & b_{n-1} \\ b_n & 0 & \cdots & b_{n-1} & a_n \end{pmatrix}$$

We form the following matrices from the matrix above:

$$J^+ = \begin{pmatrix} a_1 & (b^+)^t \\ b^+ & K \end{pmatrix}$$

and

$$J^+ = \begin{pmatrix} a_1 & (b^-)^t \\ b^- & K \end{pmatrix}$$

where $K$ is a Jacobi matrix given by $J_{n-1}$.

The eigenvalues meet the specified linking characteristic and are unique. Using the connection, we calculate the matrix $A$'s row eigenvectors (Baah, 2012 ; Aidoo et al., 2013). Solving the inverse eigenvalue problem is a challenging task and can be ill-posed in some situations, implying that significant changes in the solution might result from minor adjustments to the input data. Researchers and mathematicians continue to explore different approaches, numerical algorithms, and techniques to address this problem. It offers insights into the properties of specific matrix structures and can lead to the development of novel mathematical techniques and algorithms for solving complex problems in real-world scenarios. Finding a solution to the inverse eigenvalue problem of singular symmetric matrices has received a lot of attention from research ( Fiedler, 1974; Marijuán et al., 2007; Soto & Rojo, 2006; Egleston et al., 2004; Wu, 2011; Aidoo et al., 2013 ). Inverse eigenvalue problems of singular symmetric matrices are the main focus of this research, and before we go into depth about our work, we need first to describe the various approaches. In addition to the spectra requirement, it is usually important for the matrix regenerated to keep certain features. To reduce the amount of potentially endlessly many solutions that are usually possible if a solution does not exist. A specific case of the IEP, which Kolmogorov initially posed in 1937, is the nonnegative inverse eigenvalue problem. In the nonnegative inverse eigenvalue problem, provided the collection of complex numbers that comprise the indeterminate matrix's range, it is necessary to determine whether any entrywise nonnegative matrices exist. Numerous approaches have been used to resolve the extensions and associated conditions of the principal submatrix of $J$. Numerous studies on the eigenvalues of nonnegative matrices have been published since 1949. If the integers $(\lambda_1, \geq \lambda_2, \geq \cdots, \geq, \lambda_n)$ are real numbers such that $\lambda_1 + \sum_{i\lambda_i \leq 0} \lambda \geq 0$ as reported by Suleimanova [S] and demonstrated by Perfect [5],

Radwan (1996) argue that the spectrum of a $n \times n$ symmetric non-negative matrix is equal to $n$ actual numbers, and the spectra of a $n \times n$ normal non-negative matrix is equal to $n$ complex numbers , some new sufficient requirements are discovered.

Soto and Rojo (2006) presented several simple necessary conditions and demonstrated the creation of a realizing matrix.

**Theorem 2.2.** $\Lambda = \{\lambda_1\lambda_2\cdots\lambda_n\}$ be a list of real numbers satisfying $\lambda \geq \cdots \geq \lambda_p \geq 0 \geq \lambda_n . Let S_j = \lambda_j + _{n-j+1}, j = 2, 3, \cdots \frac{n}{2}$.
$\frac{n+1}{2} = min\{\lambda_{\frac{n+1}{2}}, 0\}$ If $\lambda_1 \geq \lambda_n - \sum_{s_j<0} S_j$ Then $A \in \delta\mathbf{R}$ ( by a non-negative matrix).

Regarding this theorem, they established.

$$T(A) = \lambda_1 + \lambda_n + \sum_{S_j} < 0$$

Recall that (2) corresponds to $T(A) \geq 0$ and that if $\Lambda = \{\lambda_1\lambda_2, \cdots, \lambda_n\}$ satisfies the sufficient condition (2) then,

$$A^l = \{-\lambda - \sum_{s_j<0} S_j, \lambda_2, \cdots \lambda_n\}$$

is a list that can be achieved, and the number $-\lambda_n - \sum_{j<0} S_j$ is the minimum value that $\lambda_i$ may take order that $\Lambda$ must meet the realizability criteria specified by the theorem utilized, which is that $\Lambda$ must be realizable. Suppose that $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ is partitioned as $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_n$
Then according to the theorem, they have for each sub-list $\Lambda = \{\lambda_{k1}, \lambda_{k2}\cdots, \lambda_{kp}k\}$ of the partition, $k = 1, 2, \cdots, n$ the number.

$$T(\Lambda_k) = T_k = \lambda_{k_1} + \lambda_{k_{pk}} + \sum_{S_kj}.$$

In their research, there is an extending of Perfect's result, they give a new reliability criteria for the real Non-negative inverse eigenvalue problem, which contain Soto's realization criteria.

Marijuán et al. (2007) established the fact to determine the necessary condition and sufficient condition for a list of real numbers, $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ to be the spectrum of an entrywise non-negative matrix. The study demonstrates how to compare these conditions or sufficient conditions for the spectrum of an entrywise non-negative matrix. According to Johnston (2021), the entrywise non-negative matrix $A = (a_{ij})^n_{ij} = 1$ is said to have constant row sums if all its rows sum up to the same constant say, $\lambda$ i.e. $\sum_{j=1}^{n} a_{ij} = \lambda i = 1, 2, \cdots, n$. Suleimanova 1949 cited by Marijuán et al. (2007), demonstrate or figuring out the prerequisites for a list of real numbers entry-wise non-negative matrix $\Lambda = \{\lambda_0, \lambda_1, \cdots, \lambda_n\}$

Wu (2011), established the fact that for three unsolved non-negative inverse eigenvalues problem (NIEP) problems that have lain unsolved for up to 70 years, they provide solvability requirements. It will provide practical methods for determining whether an NIEP is solvable. The researcher establishes his factors based on theorems to find unique solutions to the problem. According to Fielderas cited by Wu (2011) concerning the first systematic treatment of eigenvalue of symmetric matrices. Additionally, the researcher demonstrated how Boley and Golub (1987) can use the concept of symbolic dynamics to describe the circumstances in which a particular set is a part of the spectrum of a primitive matrix or non-negative matrix. They also talk about the basic requirement of the solution inverse eigenvalue problem (IEP) and non-negative inverse eigenvalue problem (NIEP). The researcher based his argument on the

**Theorem 2.3.** For a given list of complex numbers $\lambda_1, \lambda_2, \cdots, \lambda_n$, if it has closed property under complex conjugation, then the sufficient condition that has at least one non-negative matrix $A$ with the spectrum, $\{\lambda_1, \lambda_2 \cdots, \lambda_n\}$.

**Theorem 2.4.** For a given list of complex numbers $\Lambda = \{\lambda_1, \lambda_2 \cdots \lambda_n\}$, there must be at least one real matrix A with spectrum $\Lambda$ if it has closed property under complex conjugation, meaning that $\Lambda$ includes an element and its complex conjugation (Aidoo et al., 2013).

Proof. The polynomial is constructed via Aidoo et al. (2013) the closed property of $\Lambda$ under complex conjugation:

$$F(x) = \prod_{i=1}^{n} (x - \lambda_1).$$

Aidoo et al. (2013), the above equation can be expressed as: by multiplying out, merging related terms, and simplifying:

$F(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ with the zeros in the polynomial (1) are $\lambda_1, \lambda_2, \cdots, \lambda_n$ and the real numbers are $a_1, a_2, \cdots, a_n$. Thus, the matrix A can be formed as using $a_1, a_2, \cdots a_n$.

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & & \\ \vdots & \vdots & \vdots & & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 \end{bmatrix}$$

**Theorem 2.5.** For rank one with a $n \times n$ singular symmetric matrix, the inverse eigenvalue problem is solvable given the spectrum and the row multipliers $k_i, i = 1, 2, \cdots n-1$. For the proof of the theorem, see (Baah, 2012)

**Theorem 2.6.** For a $n \times n$ singular Hermitian matrix of rank one, the inverse eigenvalue problem is solvable given the spectrum and the row multipliers $k_i = 1, 2, \cdots, n-1$. For the proof of the theorem, see (Baah, 2012)

**Theorem 2.7.** For ranks two, three, and four with a $n \times n$ singular symmetric matrix, the inverse eigenvalue problem is solvable given the spectrum and the row multipliers $k_i, i = 1, 2, \cdots n-1$. For the proof of the theorem, see (Aidoo et al., 2013).

Aidoo et al. (2013) investigate the conditions under which symmetric matrices become singular as well as how this affects the matrix's internal structure. Determine the conditions in which $\Lambda$ will form the spectrum of a dense $n \times n$ singular symmetric matrix, given a list of real numbers $\Lambda = \{\lambda_1, \lambda_2, \cdots \lambda_n\}$. For a given list $\Lambda$ and dependence parameters, an algorithm to compute the matrix's members is derived based on a solvability lemma. To demonstrate the findings of their research, computations are done for $n \leq 5$ and $r \leq 4$. Their research is also based on the following theorems below.

**Theorem 2.8.** Let $\Lambda = \{\lambda, \cdots, \lambda\}$be a set of real numbers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p \geq 0 > \lambda_{p+1} \geq \cdots \geq \lambda_n$. If there exists a partition $\Lambda = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_s$

with $\lambda_{k_1} \geq \lambda_{k_2} \geq \lambda_{k_pk}, \lambda_{k_1} \geq 0 = \lambda_1$ k=1,2,$\cdots, s$

$\Lambda = \{\lambda_{k_1}, \lambda_{k_2} \cdots \lambda_{k_pk}\}$

$S_{k_j} = \lambda_{k_j} + \lambda k_{p^k - j+1}$, j=2,,$\cdots, [\frac{k_{pk}}{2}]$, and for $k_{pk}$ odd

$\frac{S_{k_{pk}+1}}{2} = min\{\frac{\lambda_{pk}+1}{2}0\}$

$T_k = \lambda_{k_1} + \lambda_{k_{pk}} + \sum_{S_{kj}<o} S_{kj}$

k=1,2,$\cdots s$

and

$L = max\{-\lambda_{1_{p1}} - \sum_{s_{s_{1_j}}<0} : max\{\lambda k_1\}\} \; 2 \leq k \leq s$

satisfying

$\lambda \geq L - \sum_{T_{k<0k=0}}^{s} T_k$

Following that, $\Lambda$ can be realized by a non-negative matrix with constant row sum. Theorem $\lambda_1\lambda_2\lambda_3$ has been demonstrated to be sufficient for the existence of a $n \times n$ symmetric non-negative matrix with real spectrum.

In their research, they take a $5 \times 5$ singularly symmetric matrices of rank 4 into consideration. According to Aidoo et al. (2013), the following quartic equation in $a_{11}$, where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the nonzero members of the spectrum, is derived using the general polynomial equation.

Which yield the result

$$\begin{bmatrix} 1.3 & 2.7 & 8 & -4 & 5 \\ 2.7 & 5.4 & 16 & -8 & 10 \\ 8 & 16 & 4 & 6 & 14 \\ -4 & -8 & 6 & 7 & 17 \\ 5 & 10 & 14 & 17 & 12 \end{bmatrix}$$

**Chapter Summary**

Several types of inverse eigenvalue problems have been reviewed and explained in this chapter. The matrix's internal structure solution provided by (Baah, 2012; Aidoo et al., 2013). Aidoo et al. (2013)) uses an approach of generating singular symmetric matrices from the eigenvalues.

# CHAPTER THREE

# METHODOLOGY

## Introduction

This chapter contains a detailed description of how to take given size and order of matrices to generate singular symmetric and Hermitian matrices of rank. This chapter covers the research methodology that was applied to this investigation. The technique used by Aidoo et al. (2013) and Baah (2012) has been altered to generate n$\times$ n singular symmetric matrices of ranks five and six depending on the solvability lemma. Subsequently, a supplement is developed to construct singular Hermitian matrices of ranks five and six. Initially, the approach for constructing singular symmetric and Hermitian matrices is explained followed by an improved approach for solving the same problem.

## Procedure for Constructing Singular Symmetric and Hermitian Matrices

The procedure is to determine a particular size and rank of a square matrix. This research uses $2 \times 2$, $3 \times 3$, and $4 \times 4$ as the foundation for this procedure.

$$B=\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

Making matrix $B$ simultaneously singular and symmetric is our objective. A symmetric matrix has the same value as its transpose $B = B^T$

$$\text{If } B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

then

$$B^T = \begin{pmatrix} b_{11} & b_{21} \\ b_{21} & b_{22} \end{pmatrix}$$

this indicates $b_{12} = b_{21}$.

The singularity $det(B) = 0$ which implies

$$det(B) = b_{11}b_{22} - b_{12}b_{21} = 0$$
$$\implies b_{11}b_{22} = b_{12}b_{21}$$

Given that we expressed matrix $B$ in terms of $b_{11}, b_{12}, b_{21}, b_{22}$, we can see that the row's linear dependence is given by $b_{12} = b_{21} = kb_{11}$ where $k$ is a parameter, and the column and row becomes scalar multiples of one another.

$$\implies b_{11}b_{22} = kb_{11}kb_{11}$$
$$b_{22} = k^2b_{11}$$

Hence for $2 \times 2$ singular symmetric matrix, we have

$$B = \begin{pmatrix} b_{11} & kb_{11} \\ kb_{11} & k^2b_{11} \end{pmatrix} \quad (13)$$

$$\implies b_{11} \begin{pmatrix} 1 & k \\ k & k^2 \end{pmatrix}$$

Since the number of scalars given by equation (3.0), $n - r$, for $B_{(2,1)}$ with rank $(r) = 1$, the value of $n - r = 2 - 1 = 1$

Assume $\Lambda = \{\lambda_1, \lambda_2\}$. Given that $B_{2,1}$ is singular of rank one, $\lambda_2 = 0$. We have gotten:

$$Tr(B_{(2,1)}) = \lambda = b_{11}(1 + k^2).$$

Consequently

$$b_{11} = \frac{\lambda}{1 + k^2}$$

Thus:

$$B_{(2,1)} = \frac{\lambda}{1 + k^2} \begin{pmatrix} 1 & k \\ k & k^2 \end{pmatrix}$$

Since $b_{12} = b_{21}$

Consequently, for given scalars, $k$, $B_{(2,1)}$ has created and provided $\lambda$ (Aidoo et al., 2013). The $2 \times 2$ singular symmetric matrix of rank one is stated clearly below. For example, if $\lambda = 5$ and $k = 2$, then

$$b_{2,1} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

$$\implies scalar = k$$

The following methods are defined by the research and produce (n$\times n$) singular symmetric matrices with the same size and rank as given by equation (13).

Step1: Take into account the dimensions of the symmetric square matrix's constituent pieces.

Accordingly,

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} = B^T = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{n1} \\ b_{21} & b_{22} & \cdots & b_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ b_{1n} & b_{2n} & \cdots & b_{nn} \end{bmatrix}$$

The research also takes into consideration $3 \times 3$ singular symmetric matrix.

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

The research presume that the row dependence connections shown below are the cause of the singular:

$$R_{i+1} = k_i R_i.$$

Additionally, for rank 1 singular symmetric matrices with a specified row multiplier, there is only one solution to the inverse eigenvalue problem. In other words, $n - r = 3 - 1 = 2$, thus get the number of scalars next

$$b_{12} = b_{21} = k_1 b_{11}$$

$$b_{13} = b_{31} = k_1 k_2 b_{11}$$

$$b_{13} = b_{32} = k_1^2 k_2 b_{11}.$$

$$b_{22} = k_1 b_{12}$$

$$b_{23} = k_1 b_{13}$$

$$b_{31} = k_2 b_{11}$$
$$b_{32} = k_2 b_{12}$$
$$b_{33} = k_2 b_{13}$$
$$\implies b_{22} = k_1 b_{12} = k_1 b_{21} = k_1^2 b_{11}$$
$$b_{23} = k_1 b_{13} = k_1 b_{31} = k_1 k_2 b_{11}$$
$$b_{33} = k_2 b_{13} = k_2 b_{31} = k_1^2 k_2^2 b_{11}$$

Now $B_{(3,1)}$ is the form:

$$B_{(3,1)} = \begin{bmatrix} b_{11} & k_1 b_{11} & k_1 k_2 b_{11} \\ k_1 b_{11} & k_1^2 b_{11} & k_1^2 k_2 \\ k_1 k_2 b_{11} & k_1^2 k_2 b_{11} & k_1^2 k_2^2 \end{bmatrix} = b_{11} \begin{bmatrix} 1 & k_1 & k_1 k_2 \\ k_1 & k_1^2 & k_1^2 k_2 \\ k_1 k_2 & k_1^2 k_2 & k_1^2 k_2^2 \end{bmatrix} \tag{14}$$

We also look at when $n = 4$ and $r = 1$.

$$B_{(4,1)} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{32} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix}$$

By using the same process to determine that there is $n - r = 4 - 1 = 3$ scalars.

Thus:

$$b_{12} = b_{21} = k_1 b_{11}$$

$$b_{13} = b_{31} = k_2 b_{11}$$

$$b_{22} = k_1 b_{12}$$

$$b_{23} = k_1 b_{13}$$

$$b_{24} = k_1 b_{14}$$

$$b_{31} = k_2 b_{11}$$

$$b_{32} = k_2 b_{12}$$

$$b_{33} = k_2 b_{13}$$

$$b_{34} = k_2 b_{14}$$

$$b_{41} = k_3 b_{11}$$

$$b_{42} = k_3 b_{12}$$

$$b_{43} = k_3 b_{13}$$

$$b_{44} = k_3 b_{14}$$

$$\begin{aligned}
\implies b_{22} &= k_1 b_{12} = k_1 b_{21} = k_1^2 b_{11} \\
b_{23} &= k_1 b_{13} = k_1 b_{31} = k_1 k_2 b_{11} \\
b_{24} &= k_1 b_{14} = k_1 b_{41} = k_1 k_3 b_{11} \\
b_{33} &= k_2 b_{13} = k_2 b_{31} = k_2^2 b_{11} \\
b_{34} &= k_2 b_{14} = k_2 b_{41} = k_2 k_3 b_{11} \\
b_{44} &= k_3 b_{14} = k_3 b_{41} = k_3^2 b_{11}
\end{aligned}$$

Thus

$$B_{(4,1)}=\begin{bmatrix} 1 & k_1 & k_2 & k_3 \\ k_1 & k_1^2 & k_1^2k_2 & k_1^2k_2k_3 \\ k_1k_2 & k_1^2k_2 & k_1^2k_2^2 & k_1^2k_2^2k_3 \\ k_1k_2k_3 & k_1^2k_2k_3 & k_1^2k_2^2k_3 & k_1^2k_2^2k_3^2 \end{bmatrix} \tag{15}$$

$i = 1, 2, 3, 4, \cdots, n-1$ where $R_i$ is a non-zero scalar. The matrix's non-zero eigenvalue, $\lambda$ can be used to generate the following:

If a matrix of rank one has the following row dependence relations, $R_{i+1} = k_i Ri$,

$$B_{(n,1)}=b_{11}\begin{bmatrix} b_{11} & k_1b_{11} & k_1k_2b_{11} & \cdots & k_{n-1} \\ k_1b_{11} & k_1^2b_{11} & k_1^2k_2b_{11} & \cdots & k_{n-1} \\ k_1k_2a_{11} & k_1^2k_2b_{11} & k_1^2k_2^2b_{11} & \cdots & k_{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ k_1k_2b_{11}\cdots k_{i-1} & k_1^2k_2a_{11} & k_1^2k_2^2k_3b_{11}\cdots k_{i-1} & \cdots & k^2n-1 \end{bmatrix}$$

Step 2: Calculate the scalars multiplication number, $n - r$

Step 3: Using equation $(13)$, we employ the following formulas based on the number of scalars:

Number

of

scalars

$$\begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ N \end{matrix} \quad b_{11}\begin{bmatrix} & & & 1 & & & & & \\ & & 1 & & k_1 & & & & \\ & 1 & & k_1 & & k_1k_2 & & & \\ 1 & & k_1 & & k_1k_2 & & k_1k_2k_3 & & \\ \vdots & & & & & & & & \\ 1 & k_1 & k_1k_2 & k_1k_2k_3 & \cdots & & k_2k_3k_4 & \cdots & k_n \end{bmatrix} \tag{16}$$

elements in the matrix's first row that correspond to the requested entry's first, second, third and so on. The first row's elements in the corresponding rank of the symmetric matrix are replaced by the data elements depending on the number of scalars. The table below provides information:

**Table 1: The List of Elements that should be substituted for Symmetric Elements depends on the Matrix's Size, Rank and Number of Scalars**

| Size of matrices | Rank(r) | Scalars | Symmetric elements |
|---|---|---|---|
| $2 \times 2$ | 1 | 1 | $b_{11}k_1b_{11}$ |
| $3 \times 3$ | 1 | 2 | $b_{11}k_1b_{11}k_1k_2b_{11}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $n \times n$ | 1 | n-1 | $b_{11}k_1b_{11}k_1k_2b_{11}\cdots k_{n-1}b_{11}$ |
| $3 \times 3$ | 2 | 1 | $b_{11}k_1b_{11b_{13}}$ |
| $4 \times 4$ | 2 | 2 | $b_{11}k_1b_{11}k_1k_2b_{14}$ |
| $4 \times 4$ | 2 | 2 | $b_{11}k_1b_{11}b_{13}b_{14}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $n \times n$ | $r > 1$ | n-r | $b_{11}k_1b_{11},\cdots k_{n-r}b_{11}\cdots a_{1n}$ |

Step 4a. Make use of the structure's row reliance.

$$R_2 = K_1R_1$$
$$R_3 = K_2R_2$$
$$R_4 = K_3R_3$$
$$R_5 = K_4R_4$$
$$\vdots$$
$$R_n = K_{n-1}R_{n-1}$$

to produce the next third and so on, up to the designated column or row number. Once more, the number of scalars is the only factor that determines the next stated number of rows.

Step 4b. We use rank $(r)$ and dimension $(n)$ to generate a singular symmetric matrix.

Step 5. We use rank $(r)$ and dimension $(n)$ available to generate the singular Hermitian matrix.

**Altered Algorithm for Generating the Singular Symmetric and Hermitian Matrices**

We build the structure given the matrix's eigenvalues and certain parameters that create the distinction between the square matrix's size and the ranks five and six, or $n-5$ and $n-6$ respectively. Which results in $k_1, k_2, \cdots k_n - r$, we construct the structure.

We designed the real case

$$= diag(b_{11}[1 + k_1^2 + k_1^2 k_2^2 + k_1^2 k_2^2 k_3^2 + \cdots + k_1^2 \cdots k_{n-r}] + \sum_{i=m+2}^{2} b_{ij}, i = j)$$

For complex

$$= diag(b_{11}[1 + k_1^2 + k_1^2 k_2^2 + k_1^2 k_2^2 k_3^2 + \cdots + k_1^2 \cdots k_{n-r}] + \sum_{i=m+2}^{2} b_{ij}, i = j)$$

Next, the research generates the terms that come from the sum of the eigenvalues, which correspond to the elements in the main diagonal order.

$$\lambda_1 \lambda_i i = 2, \cdots, r \tag{17}$$

$$\lambda_2 \lambda_i i = 3, \cdots r \tag{18}$$

$$\lambda_3 \lambda_i, \cdots \lambda_i i = 4, \cdots r \tag{19}$$

$$\lambda_4 \lambda_i i = 5, \cdots, r \tag{20}$$

$$\lambda_5 \lambda_i i = 6, \cdots, r \tag{21}$$

$$\lambda_1 \lambda_2 \lambda_i, \cdots \lambda_{i-2} \lambda_{i-1} \lambda_i i = 3, \cdots, r \tag{22}$$

$$\lambda_1 \lambda_3 \lambda_i, \cdots \lambda_{i-2} \lambda_{i-1} \lambda_i i = 4, \cdots, r \tag{23}$$

$$\lambda_2 \lambda_3 \lambda_i, \cdots \lambda_{i-2} \lambda_{i-1} \lambda_i i = 4, \cdots, r \tag{24}$$

$$\lambda_1 \lambda_2 \lambda_3 \lambda_i, \cdots \lambda_1 \lambda_2 \lambda_3 \lambda_{i-1}, i = 5, \cdots, r \tag{25}$$

$$\lambda_1\lambda_2\lambda_4\lambda_i, \cdots \lambda_{i-2}\lambda_1\lambda_2\lambda_4 i = 5\cdots, r \tag{26}$$

$$\lambda_1\lambda_3\lambda_4\lambda_i, \cdots \lambda_{i-2}\lambda_1\lambda_2\lambda_4 i = 6\cdots, r \tag{27}$$

$$\lambda_1\lambda_2\lambda_3\lambda_4\lambda_i, \cdots \lambda_1\lambda_2\lambda_3\lambda_4\lambda_{i-2}\lambda_{i-1}\lambda_i i = 6, \cdots, r \tag{28}$$

Lastly, the polynomial of the matrix is found by adding the terms in the previous equation presented above , that is from (17) up to (28). This is addressed to find a relationship between the eigenvalues and diagonal matrix elements. This connection is then used to construct the structure with extra independent variables, resulting in singular symmetric matrices.

## Chapter Summary

This chapter outlined the steps of solving the IEP for singular symmetric and Hermitian matrices of ranks.

# CHAPTER FOUR

# RESULTS AND DISCUSSION

## Introduction

The generation of singular symmetric and Hermitian matrices is examined in this chapter. This important subject explores the characteristics and uses of symmetric matrices that meet singularity requirements. We seek to comprehend and produce these specialized matrices using a range of techniques and algorithms, illuminating their importance in a variety of mathematical and real-world contexts.

## Constructing Singular Symmetric Matrix

Using rank five as a starting point, let's address the research topic. In the beginning, we construct dimension, $n = 6$, and rank, $r = 5$ singular symmetric matrix. Assume

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & b_{26} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} & b_{36} \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} & b_{46} \\ b_{51} & b_{52} & b_{53} & b_{55} & b_{55} & b_{56} \\ b_{61} & b_{62} & b_{63} & b_{64} & b_{65} & b_{66} \end{bmatrix} = B^T = \begin{bmatrix} b_{11} & b_{21} & b_{31} & b_{41} & b_{51} & b_{61} \\ b_{12} & b_{22} & b_{32} & b_{42} & b_{52} & b_{62} \\ b_{13} & b_{23} & b_{33} & b_{43} & b_{53} & b_{63} \\ b_{14} & b_{24} & b_{34} & b_{44} & b_{54} & b_{64} \\ b_{15} & b_{25} & b_{35} & b_{45} & b_{55} & b_{65} \\ b_{16} & b_{26} & b_{36} & b_{46} & b_{56} & b_{66} \end{bmatrix}$$

Initially, we use row procedure to find a set of scalars.

$$R_2 = k_1 R_1$$
$$R_3 = k_2 R_2$$
$$\vdots$$
$$R_{n+1} = K_n R_n$$

Given $n = 6$ and $r = 5$

$$\implies n - r = 6 - 5 = 1$$

Next, we substitute $kb_{11}$ for $b_{12}$ the number of scalars equals one. After that

$$R_2 = KR_1$$

The entries made are symmetric within the key diagonal, but this concludes our row procedure. Therefore;

$$B_{(6,5)} = \begin{bmatrix} b_{11} & kb_{11} & b_{13} & b_{14} & b_{15} & b_{16} \\ kb_{11} & k^2b_{11} & kb_{13} & kb_{14} & kb_{15} & kb_{16} \\ b_{13} & kb_{13} & b_{33} & b_{34} & b_{35} & b_{36} \\ b_{14} & kb_{14} & b_{34} & b_{44} & b_{45} & b_{46} \\ b_{15} & kb_{15} & b_{35} & b_{45} & b_{55} & b_{56} \\ b_{16} & kb_{16} & b_{36} & b_{46} & b_{56} & b_{66} \end{bmatrix} \quad (29)$$

Secondly,

for $B_{(6,5)}$ in equation (29),

we have

$\lambda_1 \neq 0$,$\lambda_2 \neq 0$, $\lambda_3 \neq 0$,$\lambda_4 \neq 0$,$\lambda_4 \neq 0$ $\lambda_5 \neq 0$ and $\lambda_6 = 0$.

such that

$$tr(B) = \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 = b_{11}(1 + k^2) + b_{33} + b_{44} + b_{55} + b_{66}$$

However:

$$\lambda_1 = b_{11}[1 + k^2]$$
$$\lambda_2 = b_{33}$$
$$\lambda_3 = b_{44}$$
$$\lambda_4 = b_{55}$$
$$\lambda_5 = b_{66}$$

Next:

$$\lambda_1\lambda_2 = b_{11}[1 + k^2]b_{33}$$
$$\lambda_1\lambda_3 = b_{11}[1 + k^2]b_{44}$$
$$\lambda_1\lambda_4 = b_{11}[1 + k^2]b_{55}$$
$$\lambda_1\lambda_5 = b_{11}[1 + k^2]b_{66}$$
$$\lambda_2\lambda_3 = b_{33}b_{44}$$
$$\lambda_2\lambda_4 = b_{33}b_{55}$$
$$\lambda_2\lambda_5 = b_{33}b_{66}$$
$$\lambda_3\lambda_4 = b_{44}b_{55}$$
$$\lambda_3\lambda_5 = b_{44}b_{66}$$
$$\lambda_4\lambda_5 = b_{55}b_{66}$$
$$\lambda_1\lambda_2\lambda_3 = b_{11}[1 + k^2]b_{33}b_{44}$$
$$\lambda_1\lambda_2\lambda_4 = b_{11}[1 + k^2]b_{33}b_{55}$$
$$\lambda_1\lambda_2\lambda_5 = b_{11}[1 + k^2]b_{33}b_{66}$$
$$\lambda_1\lambda_3\lambda_4 = b_{11}[1 + k^2]b_{44}b_{55}$$
$$\lambda_1\lambda_3\lambda_5 = b_{11}[1 + k^2]b_{44}b_{66}$$
$$\lambda_1\lambda_4\lambda_5 = b_{11}[1 + k^2]b_{55}b_{66}$$
$$\lambda_2\lambda_3\lambda_4 = b_{33}b_{44}b_{55}$$
$$\lambda_2\lambda_3\lambda_5 = b_{33}b_{44}b_{66}$$
$$\lambda_3\lambda_4\lambda_5 = b_{44}b_{55}b_{66}$$
$$\lambda_1\lambda_2\lambda_3\lambda_4 = b_{11}[1 + k^2]b_{33}b_{44}b_{55}$$
$$\lambda_1\lambda_2\lambda_3\lambda_5 = b_{11}[1 + k^2]b_{33}b_{44}b_{66}$$
$$\lambda_1\lambda_2\lambda_4\lambda_5 = b_{11}[1 + k^2]b_{33}b_{55}b_{66}$$

$$\lambda_1\lambda_2\lambda_3\lambda_4 = b_{11}[1+k^2]b_{44}b_{55}b_{66}$$

$$\lambda_2\lambda_3\lambda_4\lambda_5 = b_{33}b_{44}b_{55}b_{66}$$

$$\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5 = b_{11}[1+k^2]b_{33}b_{44}b_{55}b_{66}$$

$$\implies b_{33}b_{44}b_{55}b_{66} = \frac{\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5}{b_{11}(1+k^2)}$$

Now

$\lambda_1\lambda_2\lambda_3\lambda_4 + \lambda_1\lambda_2\lambda_3\lambda_5 + \lambda_1\lambda_2\lambda_4\lambda_5 + \lambda_1\lambda_3\lambda_4\lambda_5 + \lambda_2\lambda_3\lambda_4\lambda_5 =$ $b_{11}[1+k^2]b_{33}b_{44}b_{55} + b_{11}[1+k^2]b_{33}b_{44}b_{66} + b_{11}[1+k^2]b_{33}b_{55}b_{66} + b_{11}[1+k^2]b_{44}b_{55}b_{66} + b_{33}b_{44}b_{55}b_{66}$

Which gives

$\lambda_1\lambda_2\lambda_3\lambda_4+\lambda_1\lambda_2\lambda_3\lambda_5+\lambda_1\lambda_2\lambda_4\lambda_5+\lambda_1\lambda_3\lambda_4\lambda_5+\lambda_2\lambda_3\lambda_4\lambda_5 = b_{11}[1+k^2][b_{33}b_{44}b_{55}+ b_{33}b_{44}b_{66} + b_{33}b_{55}b_{66} + b_{44}b_{55}b_{66}] + b_{33}b_{44}b_{55}b_{66}$

Also

$\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_2\lambda_5 + \lambda_1\lambda_3\lambda_4 + \lambda_1\lambda_3\lambda_5 + \lambda_1\lambda_4\lambda_5 + \lambda_2\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_5 + \lambda_3\lambda_4\lambda_5$

$= b_{11}[1+k^2][b_{33}b_{44} + b_{33}b_{55} + b_{33}b_{66} + b_{44}b_{55} + b_{44}b_{66} + b_{55}b_{66}] + b_{33}b_{44}b_{55} + b_{33}b_{44}b_{66} + b_{44}b_{55}b_{66}$

Furthermore,

$$\frac{\begin{array}{c} b_{11}[1+k^2][\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_2\lambda_5 \\ +\lambda_1\lambda_3\lambda_4 + \lambda_1\lambda_3\lambda_5 + \lambda_1\lambda_4\lambda_5 + \lambda_2\lambda_3\lambda_4 + \lambda_2 \\ \lambda_3\lambda_5 + \lambda_3\lambda_4\lambda_5] - [\lambda_1\lambda_2\lambda_3\lambda_4 \\ +\lambda_1\lambda_2\lambda_3\lambda_5 + \lambda_1\lambda_2\lambda_4\lambda_5 + \lambda_2\lambda_3\lambda_4\lambda_5] \\ +\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5 \end{array}}{b_{11}^2[1+k^2]^2} \tag{30}$$

$b_{11}[1+k^2][b_{33}+b_{44}+b_{55}+b_{66}]+b_{33}b_{44}+b_{33}b_{55}+b_{33}b_{66}+b_{44}b_{55}+b_{44}b_{66}+b_{55}b_{66}$

$= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_1\lambda_5 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_2\lambda_5 + \lambda_3\lambda_4 + \lambda_3\lambda_5 + \lambda_4\lambda_5$

Substituting into (30),

we have

$$\frac{\begin{matrix} b_{11}^2[1+k^2]^2[\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_4+\lambda_1\lambda_5+\lambda_2\lambda_3+\lambda_2\lambda_4 \\ +\lambda_2\lambda_5+\lambda_3\lambda_4+\lambda_3\lambda_5+\lambda_4\lambda_5]-b_{11}[1+k^2][\lambda_1\lambda_2 \\ \lambda_3+\lambda_1\lambda_2\lambda_4+\lambda_1\lambda_2\lambda_5+\lambda_1\lambda_3\lambda_4 \\ +\lambda_1\lambda_3\lambda_5+\lambda_1\lambda_4\lambda_5+\lambda_2\lambda_3\lambda_4+\lambda_2\lambda_3\lambda_5+\lambda_3\lambda_4\lambda_5] \\ +b_{11}[1+k^2][\lambda_1\lambda_2\lambda_3\lambda_4+\lambda_1\lambda_2\lambda_3\lambda_5+\lambda_1\lambda_2\lambda_4\lambda_5+\lambda_2\lambda_3\lambda_4\lambda_5] \\ -\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5 \end{matrix}}{b_{11}^3[1+k^2]^3} \quad (31)$$

$$\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5=b_{11}[1+k^2]+b_{33}+b_{44}+b_{55}+b_{66}$$

Substituting (31), that is,

$$\frac{\begin{matrix} b_{11}^3[1+k^2]^3[\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_1\lambda_5+\lambda_2\lambda_3+\lambda_2\lambda_4 \\ +\lambda_2\lambda_5+\lambda_3\lambda_4+\lambda_3\lambda_5+\lambda_4\lambda_5]-b_{11}^2[1+k^2]^2[\lambda_1\lambda_2 \\ \lambda_3+\lambda_1\lambda_2\lambda_4+\lambda_1\lambda_2\lambda_5+\lambda_1\lambda_3\lambda_4 \\ +\lambda_1\lambda_3\lambda_5+\lambda_1\lambda_4\lambda_5+\lambda_2\lambda_3\lambda_4+\lambda_2\lambda_3\lambda_5+\lambda_3\lambda_4\lambda_5] \\ +b_{11}[1+k^2][\lambda_1\lambda_2\lambda_3\lambda_4+\lambda_1\lambda_2\lambda_3\lambda_5+\lambda_1\lambda_2\lambda_4\lambda_5+\lambda_2\lambda_3\lambda_4\lambda_5] \\ -\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5 \end{matrix}}{b_{11}^4[1+k^2]^4}$$

Leading to

$b_{11}^5[1+k^2]^5-[\lambda_1+\lambda_2+\lambda_3+\lambda_4\lambda_5]b_{11}[1+k^2]^4$

$+[\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_1\lambda_5+\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_2\lambda_5$

$+\lambda_3\lambda_4+\lambda_3\lambda_5+\lambda_4\lambda_5]b_{11}^3[1+k^2]^3$

$-[\lambda_1\lambda_2\lambda_3+\lambda_1\lambda_2\lambda_4+\lambda_1\lambda_2\lambda_5+\lambda_1\lambda_3\lambda_4+\lambda_1\lambda_3\lambda_5$

$+\lambda_1\lambda_4\lambda_5+\lambda_2\lambda_3\lambda_4+\lambda_2\lambda_3\lambda_5+\lambda_3\lambda_4\lambda_5]b_{11}^2[1+k^2]^2$

$+[\lambda_1\lambda_2\lambda_3\lambda_4+\lambda_1\lambda_2\lambda_3\lambda_5+\lambda_1\lambda_2\lambda_4\lambda_5+\lambda_1\lambda_3\lambda_4\lambda_5+\lambda_2\lambda_3\lambda_4\lambda_5]$

$b_{11}[1+k^2]-\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5=0$

The result of solving the quartic equation is

$b_{11}=\dfrac{\lambda_1}{1+k^2}, \lambda_2=b_{44}, \lambda_3=b_{55}, \lambda_4=b_{66}, \lambda_5=b_{77}$

Independent variables included:

$b_{13}, b_{14}, b_{15}, b_{16}, b_{34}, b_{35}, b_{36}, b_{45}, b_{46}, b_{56}, b_{57}.$

$tr(B)=\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5=b_{11}$

**Example 1.0**

Suppose $\lambda_1 = 5$, $\lambda_2 = 2$, $\lambda_3 = 3$, $\lambda_4 = 4$ and $\lambda_5 = 5$, $k = 2$ $b_{13} = 1$, $b_{14} = 2$, $b_{15} = 3$, $b_{16} = 4$, $b_{34} = 1$, $b_{35} = 2$, $b_{36} = 3$, $b_{45} = 4$, $b_{46} = 4$ and $b_{56} = -1$

It is necessary to construct a $6 \times 6$ singular symmetric matrix of rank five. With the knowledge at our disposal, we can now ascertain that

$$\implies b_{11} = \frac{5}{1 + 2^2} = 1$$

Hence

$$A_(6, 5) \begin{bmatrix} 1 & 2 & 1 & -2 & 3 & 4 \\ 2 & 4 & 2 & -4 & 6 & 8 \\ 1 & 2 & 2 & 1 & 2 & 3 \\ -2 & -4 & 1 & 3 & 4 & 5 \\ 3 & 6 & 2 & 4 & 4 & -1 \\ 4 & 8 & 3 & 5 & -1 & 5 \end{bmatrix}$$

Let us proceed to address the research topic with a $7 \times 7$ with rank five. First, we want to create a singular symmetric matrix with a rank of $r = 5$ and a dimension of $n = 7$. Assume

$$B=\begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} & b_{17} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & b_{26} & b_{27} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} & b_{36} & b_{37} \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} & b_{46} & b_{47} \\ b_{51} & b_{52} & b_{53} & b_{55} & b_{55} & b_{56} & b_{57} \\ b_{61} & b_{62} & b_{63} & b_{64} & b_{65} & b_{66} & b_{67} \\ b_{71} & b_{72} & b_{73} & b_{74} & b_{75} & b_{76} & b_{77} \end{bmatrix}$$

Afterwards,

$$B^T=\begin{bmatrix} b_{11} & b_{21} & b_{31} & b_{41} & b_{51} & b_{61} & b_{71} \\ b_{12} & b_{22} & b_{32} & b_{42} & b_{52} & b_{62} & b_{72} \\ b_{13} & b_{23} & b_{33} & b_{43} & b_{53} & b_{63} & b_{73} \\ b_{14} & b_{24} & b_{34} & b_{44} & b_{54} & b_{64} & b_{74} \\ b_{15} & b_{25} & b_{35} & b_{45} & b_{55} & b_{65} & b_{75} \\ b_{16} & b_{26} & b_{36} & b_{46} & b_{56} & b_{66} & b_{76} \\ b_{17} & b_{27} & b_{37} & b_{47} & b_{57} & b_{67} & b_{77} \end{bmatrix}$$

Row procedure is used to find the number of scalars.

$$R_2 = k_1 R_1$$
$$R_3 = k_2 R_2$$
$$\vdots$$
$$R_{n+1} = K_n R_n$$
$$\text{Given } n = 7 \text{ and } r = 5$$
$$\implies n - r = 7 - 5 = 2$$

Next, since the number of scalars equals 2, we should substitute $b_{12}$ with $kb_{11}$ $b_{13}$ with $k_1k_2b_{11}$. Then

$$R_2 = KR_1$$

$$R_3 = KR_2$$

It completes our row operation; however, the entries are symmetric concerning the primary diagonal. Additionally, we produce a rank five $7 \times 7$ singular symmetric matrix.

$$B_{(7,5)} = \begin{bmatrix} b_{11} & k_1b_{12} & k_1k_2b_{11} & b_{14} & b_{15} & b_{16} & b_{17} \\ k_1b_{11} & k_1^2b_{11} & k_1^2k_2b_{11} & k_1b_{14} & k_1b_{15} & k_1b_{16} & k_1b_{17} \\ k_1k_2b_{11} & k_1^2k_2b_{11} & k_1^2k_2^2b_{11} & k_1k_2b_{14} & k_1k_2b_{15} & k_1k_2b_{16} & k_1k_2b_{17} \\ b_{14} & k_1b_{14} & k_1k_2b_{14} & b_{44} & b_{45} & b_{46} & b_{47} \\ b_{15} & k_1b_{15} & k_1k_2b_{15} & b_{45} & b_{55} & b_{56} & b_{57} \\ b_{16} & k_1b_{16} & k_1k_2b_{16} & b_{46} & b_{56} & b_{66} & b_{67} \\ b_{17} & k_1b_{17} & k_1k_2b_{17} & b_{47} & b_{57} & b_{67} & b_{77} \end{bmatrix} \quad (32)$$

From (32), $\lambda_1 \neq 0$,$\lambda_2 \neq 0$,$\lambda_3 \neq 0$,$\lambda_4 \neq$,$0$ $\lambda_5 \neq 0$,$\lambda_6 = 0$, $\lambda_7 = 0$

$tr(B) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5$

Using the comparable method, we are able to

$$b_{11}^5[1 + k^2]^5 - [\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4\lambda_5]b_{11}[1 + k^2]^4$$
$$+ [\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_1\lambda_5 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_2\lambda_5$$
$$+ \lambda_3\lambda_4 + \lambda_3\lambda_5 + \lambda_4\lambda_5]b_{11}^3[1 + k^2]^3$$
$$- [\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_2\lambda_5 + \lambda_1\lambda_3\lambda_4 + \lambda_1\lambda_3\lambda_5$$
$$+ \lambda_1\lambda_4\lambda_5 + \lambda_2\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_5 + \lambda_3\lambda_4\lambda_5]b_{11}^2[1 + k^2]^2$$
$$+ [\lambda_1\lambda_2\lambda_3\lambda_4 + \lambda_1\lambda_2\lambda_3\lambda_5 + \lambda_1\lambda_2\lambda_4\lambda_5 + \lambda_1\lambda_3\lambda_4\lambda_5 + \lambda_2\lambda_3\lambda_4\lambda_5]$$
$$b_{11}[1 + k^2] - \lambda_1\lambda_2\lambda_3\lambda_4\lambda_5 = 0$$

The result of solving the quartic equation is

$$b_{11} = \frac{\lambda_1}{1 + k_1^2 + k_1^2k_2^2}, \ \lambda_2 = b_{44}, \ \lambda_3 = b_{55}, \ \lambda_4 = b_{66}, \ \lambda_5 = b_{77}$$

Independent variables included:

$b_{14}, b_{15}, b_{16}, b_{34}, b_{35}, b_{36}, b_{45}, b_{46}, b_{56}, b_{57}$ and $b_{67}$.

$tr(B) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = b_{11}[1 + k_1^2 + k_1^2 k_2^2] + b_{44} + b_{55} + b_{66} + b_{77}$

**Example 2.0**

Suppose,$\lambda_1 = 193.2$,$\lambda_2 = 3$,$\lambda_3 = 4$,$\lambda_4 = 6$,$\lambda_5 = 8$ $k_1 = 4$ $k_2 = 3$ $_{14} = 2$, $b_{15} = 3$, $b_{16} = 4$, $b_{17} = 2$, $b_{45} = 7$, $b_{46} = 3$, $b_{56} = 1$, $b_{57} = 4$ and $b_{67} = 6$

$$\implies b_{11} = \frac{193.2}{1 + 4^2 + 4^2(3^2)} = 1.2$$

We need to create a rank five, $7 \times 7$ singular symmetric matrix. Equipped with the given facts, we have

$$\begin{bmatrix} 1.2 & 4.8 & 14.4 & 2 & 3 & 4 & 2 \\ 4.8 & 19.2 & 57.6 & 8 & 12 & 16 & 8 \\ 14.4 & 57.6 & 172.8 & 24 & 36 & 48 & 24 \\ 2 & 8 & 24 & 3 & 7 & 3 & 5 \\ 3 & 12 & 36 & 7 & 4 & 1 & 4 \\ 4 & 16 & 48 & 3 & 1 & 6 & -6 \\ 2 & 8 & 24 & 5 & 4 & -6 & 8 \end{bmatrix}$$

The row procedure is used to determine the number of scalars.

$$R_2 = k_1 R_1$$
$$R_3 = k_2 R_2$$
$$\vdots$$
$$R_{n+1} = K_n R_n$$
Given $n = 7$ and $r = 6$
$$\implies n - r = 7 - 6 = 1$$

Next, since there is one scalar, we can replace $b_{12}$ with $kb_{11}$. Let us proceed to address the research topic using a $7 \times 7$ with rank six. The singular symmetric matrix with size $n = 7$ and rank $r = 6$ is initially generated. Assume

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} & b_{17} \\ b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & b_{26} & b_{27} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} & b_{36} & b_{37} \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} & b_{46} & b_{47} \\ b_{51} & b_{52} & b_{53} & b_{55} & b_{55} & b_{56} & b_{57} \\ b_{61} & b_{62} & b_{63} & b_{64} & b_{65} & b_{66} & b_{67} \\ b_{71} & b_{72} & b_{73} & b_{74} & b_{75} & b_{76} & b_{77} \end{bmatrix}$$

Then,

$$B^T = \begin{bmatrix} b_{11} & b_{21} & b_{31} & b_{41} & b_{51} & b_{61} & b_{71} \\ b_{12} & b_{22} & b_{32} & b_{42} & b_{52} & b_{62} & b_{72} \\ b_{13} & b_{23} & b_{33} & b_{43} & b_{53} & b_{63} & b_{73} \\ b_{14} & b_{24} & b_{34} & b_{44} & b_{54} & b_{64} & b_{74} \\ b_{15} & b_{25} & b_{35} & b_{45} & b_{55} & b_{65} & b_{75} \\ b_{16} & b_{26} & b_{36} & b_{46} & b_{56} & b_{66} & b_{76} \\ b_{17} & b_{27} & b_{37} & b_{47} & b_{57} & b_{67} & b_{77} \end{bmatrix}$$

A $7 \times 7$ singular symmetric matrix of rank six is also produced.

$B_{(7,6)}$

$$= \begin{bmatrix} b_{11} & kb_{11} & b_{13} & b_{14} & b_{15} & b_{16} & b_{17} \\ kb_{11} & k^2b_{11} & kb_{13} & kb_{14} & kb_{15} & kb_{16} & kb_{17} \\ b_{13} & kb_{13} & b_{33} & b_{34} & b_{35} & b_{36} & b_{37} \\ b_{14} & kb_{14} & b_{34} & b_{44} & b_{45} & b_{46} & b_{47} \\ b_{15} & kb_{15} & b_{35} & b_{45} & b_{55} & b_{56} & b_{57} \\ b_{16} & kb_{16} & b_{36} & b_{46} & b_{56} & b_{66} & b_{67} \\ b_{17} & kb_{17} & b_{37} & b_{47} & b_{57} & b_{67} & b_{77} \end{bmatrix} \quad (33)$$

From (33), we have

$\lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 \neq 0, \lambda_4 \neq, 0\ \lambda_5 \neq 0, \lambda_6 \neq 0, \lambda_7 = 0$

$tr(B_{(7,6)}) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = b_{11}[1 + k^2] + b_{33} + b_{44} + b_{55} + b_{66} + b_{77}$

By following the same method as earlier, we can get the quartic polynomial below;

$b_{11}^6[1 + k^2]^6 - [\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4\lambda_5 + \lambda_6]b_{11}^5[1 + k^2]^5$
$+[\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_1\lambda_5 + \lambda_1\lambda_6 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_2\lambda_5 + \lambda_2\lambda_6$
$+\lambda_3\lambda_4 + \lambda_3\lambda_5 + \lambda_3\lambda_6 + \lambda_4\lambda_5 + \lambda_4\lambda_6 + \lambda_5\lambda_6]b_{11}^4[1 + k^2]^4$
$-[\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_2\lambda_5 + \lambda_1\lambda_2\lambda_6 + \lambda_1\lambda_3\lambda_4 + \lambda_1\lambda_3\lambda_5 + \lambda_1\lambda_3\lambda_6$
$+\lambda_1\lambda_4\lambda_5 + \lambda_1\lambda_4\lambda_6 + \lambda_2\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_5 + \lambda_2\lambda_3\lambda_6 + \lambda_2\lambda_4\lambda_5 + \lambda_2\lambda_4\lambda_6 + \lambda_3\lambda_4\lambda_5$
$+\lambda_3\lambda_4\lambda_6 + \lambda_4\lambda_5\lambda_6]b_{11}^3[1 + k^2]^3$
$+[\lambda_1\lambda_2\lambda_3\lambda_4+\lambda_1\lambda_2\lambda_3\lambda_5+\lambda_1\lambda_2\lambda_3\lambda_6+\lambda_1\lambda_2\lambda_4\lambda_5+\lambda_1\lambda_2\lambda_4\lambda_6+\lambda_1\lambda_2\lambda_5\lambda_6 +\lambda_1\lambda_3\lambda_4\lambda_5$
$+\lambda_1\lambda_3\lambda_4\lambda_6 + \lambda_1\lambda_4\lambda_5\lambda_6 + \lambda_2\lambda_3\lambda_4\lambda_5 + \lambda_2\lambda_3\lambda_4\lambda_6 + \lambda_3\lambda_4\lambda_5\lambda_6]$
$b_{11}^2[1+k^2]^2-\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5+\lambda_1\lambda_2\lambda_3\lambda_4\lambda_6+\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6]b_{11}[1+k^2]+\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6$
$= 0$

The result of solving the quartic equation is

$b_{11} = \dfrac{\lambda_1}{1 + k^2}$, $\lambda_2 = b_{33}$, $\lambda_3 = b_{44}$, $\lambda_4 = b_{55}$ $\lambda_5 = b_{66}$ and $\lambda_6 = b_{77}$.

Independent variables include;

$b_{13}$,$b_{14}$ ,$b_{15}$,$b_{16}$, $b_{17}$,$b_{34}$,$b_{35}$,$b_{36}$ $b_{37}$,$b_{45}$,$b_{46}$, $b_{47}$,$b_{56}$, $b_{57}$ and $b_{67}$.

$tr(B) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = b_{11}[1 + k^2] + b_{33} + b_{44} + b_{55} + b_{66} + b_{77}$

**Example 3.0**

Assuming, $\lambda_1 = 20$, $\lambda_2 = 4$, $\lambda_3 = 5$, $\lambda_4 = 3$, $\lambda_5 = 1$, $\lambda_6 = 7$ , $k = 2$ $b_{13} = -1$, $b_{14} = 2$ ,$b_{15} = -3$, $b_{16} = 2$, $b_{17} = 7$, $b_{34} = -2$, $b_{35} = 4$, $b_{36} = 3$ $b_{37} = 4$, $b_{45} = 8$, $b_{46} = 1$, , $b_{47} = 5$, $b_{56} = 9$, $b_{57} = 3$ and $b_{67} = 1$

$$b_{11} = \frac{20}{1 + 2^2} = 4$$

We need to create a rank six, $7 \times 7$ singular symmetric matrix. Equipped with the given facts, we have

$$\begin{bmatrix} 4 & 8 & -1 & 2 & -3 & 6 & 7 \\ 8 & 16 & -2 & 4 & -6 & 12 & 14 \\ -1 & -2 & 4 & -2 & 4 & 3 & 4 \\ 2 & 4 & -2 & 5 & 8 & 1 & 5 \\ -3 & -6 & 4 & 8 & 3 & 9 & 3 \\ 6 & 12 & 3 & 1 & 9 & 1 & -1 \\ 7 & 14 & 4 & 5 & 3 & -1 & 7 \end{bmatrix}$$

Let us now use a $8 \times 8$ with rank six to address the research topic. initially, we constructed a seven-dimensional or $7 \times 7$ singular symmetric matrix with a rank of six.

To complete our row operation, however, the entries are symmetric concerning the primary diagonal. In addition, we produce a $8 \times 8$ singular.

$B_{(8,6)}$

$$\begin{bmatrix} b_{11} & k_1b_{12} & k_1k_2b_{11} & b_{14} & b_{15} & b_{16} & b_{17} & b_{18} \\ k_1b_{11} & k_1^2b_{11} & k_1^2k_2b_{11} & k_1b_{14} & k_1b_{15} & k_1b_{16} & k_1b_{17} & k_1b_{18} \\ k_1k_2b_{11} & k_1^2k_2b_{11} & k_1^2k_2^2b_{11} & k_1k_2b_{14} & k_1k_2b_{15} & k_1k_2b_{16} & k_1k_2b_{17} & k_1k_2b_{18} \\ b_{14} & k_1b_{14} & k_1k_2b_{14} & b_{44} & b_{45} & b_{46} & b_{47} & b_{48} \\ b_{15} & k_1b_{15} & k_1k_2b_{15} & b_{45} & b_{55} & b_{56} & b_{57} & b_{58} \\ b_{16} & k_1b_{16} & k_1k_2b_{16} & b_{46} & b_{56} & b_{66} & b_{67} & b_{68} \\ b_{17} & k_1b_{17} & k_1k_2b_{17} & b_{47} & b_{57} & b_{67} & b_{77} & b_{87} \\ b_{18} & k_1b_{18} & k_1k_2b_{18} & b_{48} & b_{58} & b_{68} & b_{78} & b_{88} \end{bmatrix} \tag{34}$$

From (34), we have

$\lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 \neq 0, \lambda_4 \neq, 0\ \lambda_5 \neq 0, \lambda_6 \neq 0,\ \lambda_7 = 0, \lambda_8 = 0$

$tr(B_{(7,6)}) = \lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5 = b_{11}[1+k_1^2+k_1^2k_2^2]+b_{44}+b_{55}+b_{66}+b_{77}+b_{88}$

By following the same method as earlier, we can get the quartic polynomial below;

$$\begin{aligned}
&b_{11}^6[1+k^2]^6 - [\lambda_1+\lambda_2+\lambda_3+\lambda_4\lambda_5+\lambda_6]b_{11}^5[1+k^2]^5 \\
&+[\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_1\lambda_5+\lambda_1\lambda_6+\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_2\lambda_5+\lambda_2\lambda_6 \\
&+\lambda_3\lambda_4+\lambda_3\lambda_5+\lambda_3\lambda_6+\lambda_4\lambda_5+\lambda_4\lambda_6+\lambda_5\lambda_6]b_{11}^4[1+k^2]^4 \\
&-[\lambda_1\lambda_2\lambda_3+\lambda_1\lambda_2\lambda_4+\lambda_1\lambda_2\lambda_5+\lambda_1\lambda_2\lambda_6+\lambda_1\lambda_3\lambda_4+\lambda_1\lambda_3\lambda_5+\lambda_1\lambda_3\lambda_6 \\
&+\lambda_1\lambda_4\lambda_5+\lambda_1\lambda_4\lambda_6+\lambda_2\lambda_3\lambda_4+\lambda_2\lambda_3\lambda_5+\lambda_2\lambda_3\lambda_6+\lambda_2\lambda_4\lambda_5+\lambda_2\lambda_4\lambda_6+\lambda_3\lambda_4\lambda_5 \\
&+\lambda_3\lambda_4\lambda_6+\lambda_4\lambda_5\lambda_6]b_{11}^3[1+k^2]^3 \\
&+[\lambda_1\lambda_2\lambda_3\lambda_4+\lambda_1\lambda_2\lambda_3\lambda_5+\lambda_1\lambda_2\lambda_3\lambda_6+\lambda_1\lambda_2\lambda_4\lambda_5+\lambda_1\lambda_2\lambda_4\lambda_6+\lambda_1\lambda_2\lambda_5\lambda_6+\lambda_1\lambda_3\lambda_4\lambda_5 \\
&+\lambda_1\lambda_3\lambda_4\lambda_6+\lambda_1\lambda_4\lambda_5\lambda_6+\lambda_2\lambda_3\lambda_4\lambda_5+\lambda_2\lambda_3\lambda_4\lambda_6+\lambda_3\lambda_4\lambda_5\lambda_6] \\
&b_{11}^2[1+k^2]^2-\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5+\lambda_1\lambda_2\lambda_3\lambda_4\lambda_6+\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6]b_{11}[1+k^2]+\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6 \\
&= 0
\end{aligned}$$

Using similar procedure, we arrived at:

$b_{11} = \dfrac{\lambda_1}{1+k_1^2+k_1^2k_2^2}$, $\lambda_2 = b_{44}$, $\lambda_3 = b_{55}$, $\lambda_4 = b_{66}$ $\lambda_5 = b_{77}$ and $\lambda_6 = b_{88}$.

The variables that are independent include;

$b_{14}$ ,$b_{15}$,$b_{16}$,$b_{17}$,$b_{18}$,$b_{45}$,$b_{46}$, $b_{47}$,$b_{48}$,$b_{56}$,$b_{57}$,$b_{58}$,$b_{67}$,$b_{68}$,$b_{78}$

**Example 4.0**

$\lambda_1 = 3$ , $\lambda_2 = 5$, $\lambda_3 = 1$, $\lambda_4 = 8$ , $\lambda_5 = 4$, $\lambda_6 = 2$, $k_1 = 4$, $k_2 = 3$
$b_{14} = 2$ ,$b_{15} = 1$,$b_{16} = 3$,$b_{17} = 4$,$b_{18} = 8$,$b_{45} = 7$,$b_{46} = 2$, $b_{47} = 3$, $b_{48} = 8$,
$b_{56} = 2$,$b_{57} = 3$,$b_{58} = 9$,$b_{67} = 1$,$b_{68} = 2$,$b_{78} = 1$

$$\implies b_{11} = \frac{3}{1+4^2+4^2(3)^2} = \frac{3}{161}$$

It is necessary to construct a rank six's $8 \times 8$ singular symmetric matrix. Considering the data provided, we have

$$\begin{bmatrix} \frac{3}{161} & \frac{12}{161} & \frac{36}{161} & 2 & 1 & 3 & 4 & 8 \\ \frac{12}{161} & \frac{48}{161} & \frac{144}{161} & 8 & 4 & 12 & 16 & 32 \\ \frac{36}{161} & \frac{144}{161} & \frac{432}{161} & 24 & 12 & 36 & 48 & 96 \\ 2 & 8 & 24 & 5 & 7 & 2 & 3 & 8 \\ 1 & 4 & 12 & 7 & 1 & 2 & 3 & 9 \\ 3 & 12 & 36 & 2 & 2 & 8 & 1 & 2 \\ 4 & 16 & 48 & 3 & 3 & 1 & 4 & 1 \\ 8 & 32 & 96 & 8 & 9 & 2 & 1 & 2 \end{bmatrix}$$

**Extension to Singular Hermitian Matrix**

Begin with $6 \times 6$ square matrix as the size of rank five. Consequently,

$$B_{(6,5)} = \begin{bmatrix} b_{11} & k^{-}b_{11} & b_{13}^{-} & b_{14}^{-} & b_{15}^{-} & b_{16}^{-} \\ kb_{11} & k^{2}b_{11} & kb_{13}^{-} & kb_{14}^{-} & kb_{15}^{-} & kb_{16}^{-} \\ b_{13} & k^{-}b_{13} & b_{33} & b_{34}^{-} & b_{35}^{-} & b_{36}^{-} \\ b_{14} & k^{-}b_{14} & b_{34} & b_{44} & b_{45}^{-} & b_{46}^{-} \\ b_{15} & k^{-}b_{15} & b_{35} & b_{45} & b_{55} & b_{56}^{-} \\ b_{16} & k^{-}b_{16} & b_{36} & b_{46} & b_{56} & b_{66} \end{bmatrix} \quad (35)$$

For (35), the eigenvalues will show the results that follow

$\lambda_1 \neq 0$,$\lambda_2 \neq 0$,$\lambda_3 \neq 0$,$\lambda_4 \neq 0$,$\lambda_5 \neq 0$, $\lambda_6 = 0$ such that $tr(B) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = b_{11}[1 + k^2] + b_{33} + b_{44} + b_{55} + b_{66}$

Then, using a comparable procedure, the polynomial properties will be

$b_{11}^5[1 + k^2]^5 - [\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4\lambda_5]b_{11}[1 + k^2]^4$
$+[\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_1\lambda_5 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_2\lambda_5$
$+\lambda_3\lambda_4 + \lambda_3\lambda_5 + \lambda_4\lambda_5]b_{11}^3[1 + k^2]^3$
-$[\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_2\lambda_5 + \lambda_1\lambda_3\lambda_4 + \lambda_1\lambda_3\lambda_5$
$\lambda_1\lambda_4\lambda_5 + \lambda_2\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_5 + \lambda_3\lambda_4\lambda_5]b_{11}^2[1 + k^2]^2$
$+[\lambda_1\lambda_2\lambda_3\lambda_4 + \lambda_1\lambda_2\lambda_3\lambda_5 + \lambda_1\lambda_2\lambda_4\lambda_5 + \lambda_1\lambda_3\lambda_4\lambda_5 + \lambda_2\lambda_3\lambda_4\lambda_5]$
$b_{11}[1 + k^2] - \lambda_1\lambda_2\lambda_3\lambda_4\lambda_5 = 0$

The result of solving the quartic equation is

$$b_{11} = \frac{\lambda_1}{1 + k^2}$$

$$\lambda_2 = b_{33}, \lambda_3 = b_{44}$$

$$\lambda_4 = b_{55}, \lambda_5 = b_{66}.$$

Additional independent factors include;

$b_{13}.b_{14}, b_{15}b_{16}, b_{34}, b_{35}, b_{36}, b_{45}, b_{46}, b_{56}$

**Example 5.0**

Suppose $\lambda_1 = 2$, $\lambda_2 = -1$, $\lambda_3 = 5$, $\lambda_4 = 3$, $\lambda_5 = 8$

$k = i$ $b_{13} = 2 - i$, $b_{14} = 4$, $b_{15} = 2 - 3i$, $b_{16} = 3$

$b_{34} = 1 + 3i$, $b_{35} = 6i$, $b_{36} = 2 - 5i$, $b_{45} = 2 - 7i$, $b_{46} = 1 + 4i$ $b_{56} = 6$

The solution of is

$$\implies \frac{2}{1 + 1^2} = 1$$

Therefore

$$A_(6, 5) = \begin{bmatrix} 1 & i & 2-i & 4 & 2+3i & 3 \\ -i & 1 & 1-2i & 4i & 3+2i & 3i \\ 2+i & 1+2i & -1 & 1+3i & -6i & 2-3i \\ 4 & -4i & 1-3i & 5 & 2-7i & 1+4i \\ 2-3i & 3-2i & 6i & 2+7i & 3 & 6 \\ 3 & -3i & 2+3i & 1-4i & 6 & 8 \end{bmatrix}$$

Now for rank, 6 and dimension, 7, will result:

$$B(7,6) = \begin{bmatrix} b_{11} & k^{-}b_{11} & b_{13}^{-} & b_{14}^{-} & b_{15} & b_{16}^{-} & b_{17}^{-} \\ kb_{11} & k^2b_{11} & kb_{13}^{-} & kb_{14}^{-} & kb_{15}^{-} & k^{-}b_{16} & kb_{17}^{-} \\ b_{13} & k^{-}b_{13} & b_{33} & b_{34}^{-} & b_{35}^{-} & b_{36}^{-} & b_{37}^{-} \\ b_{14} & k^{-}b_{14} & b_{34} & b_{44} & b_{45}^{-} & b_{46}^{-} & b_{47}^{-} \\ b_{15} & k^{-}b_{15} & b_{35} & b_{45} & b_{55} & b_{56}^{-} & b_{57}^{-} \\ b_{16} & k^{-}b_{16} & b_{36} & b_{46} & b_{56} & b_{66} & b_{67}^{-} \\ b_{17} & k^{-}b_{17} & b_{37} & b_{47} & b_{57} & b_{67} & b_{77} \end{bmatrix} \qquad (36)$$

This is what the eigenvalues for (36) will demonstrate

$\lambda_1 \neq 0$,$\lambda_2 \neq 0$,$\lambda_3 \neq 0$,$\lambda_4 \neq 0$, $\lambda_5 \neq 0$, $\lambda_6 neq 0$, $\lambda_7 = 0$ in such a manner that $tr(B) = \lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5+\lambda_6 = b_{11}[1+k^2]+b_{33}+b_{44}+b_{55}+b_{66}+b_{77}$

Then, using a comparable procedure, the polynomial properties will be

$b_{11}^6[1+k^2]^6 - [\lambda_1+\lambda_2+\lambda_3+\lambda_4\lambda_5+\lambda_6]b_{11}^5[1+k^2]^5$
$+[\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_1\lambda_5+\lambda_1\lambda_6+\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_2\lambda_5+\lambda_2\lambda_6$
$+\lambda_3\lambda_4+\lambda_3\lambda_5+\lambda_3\lambda_6+\lambda_4\lambda_5+\lambda_4\lambda_6+\lambda_5\lambda_6]b_{11}^4[1+k^2]^4$
$-[\lambda_1\lambda_2\lambda_3+\lambda_1\lambda_2\lambda_4+\lambda_1\lambda_2\lambda_5+\lambda_1\lambda_2\lambda_6+\lambda_1\lambda_3\lambda_4+\lambda_1\lambda_3\lambda_5+\lambda_1\lambda_3\lambda_6$
$+\lambda_1\lambda_4\lambda_5+\lambda_1\lambda_4\lambda_6+\lambda_2\lambda_3\lambda_4+\lambda_2\lambda_3\lambda_5+\lambda_2\lambda_3\lambda_6+\lambda_2\lambda_4\lambda_5+\lambda_2\lambda_4\lambda_6+\lambda_3\lambda_4\lambda_5+$
$\lambda_3\lambda_4\lambda_6+\lambda_4\lambda_5\lambda_6]b_{11}^3[1+k^2]^3$
$+[\lambda_1\lambda_2\lambda_3\lambda_4+\lambda_1\lambda_2\lambda_3\lambda_5+\lambda_1\lambda_2\lambda_3\lambda_6+\lambda_1\lambda_2\lambda_4\lambda_5+\lambda_1\lambda_2\lambda_4\lambda_6+\lambda_1\lambda_2\lambda_5\lambda_6+$
$\lambda_1\lambda_3\lambda_4\lambda_5+\lambda_1\lambda_3\lambda_4\lambda_6+\lambda_1\lambda_4\lambda_5\lambda_6+\lambda_2\lambda_3\lambda_4\lambda_5+\lambda_2\lambda_3\lambda_4\lambda_6+\lambda_3\lambda_4\lambda_5\lambda_6]$
$b_{11}^2[1+k^2]^2 - \lambda_1\lambda_2\lambda_3\lambda_4\lambda_5+\lambda_1\lambda_2\lambda_3\lambda_4\lambda_6+\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6]b_{11}[1+k^2]+\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6 =$
$0$

The result of solving the quartic equation is

$$b_{11} = \frac{\lambda_1}{1+k^2}$$

$$\lambda_2 = b_{33}, \lambda_3 = b_{44}$$

$$\lambda_4 = b_{55},\ \lambda_5 = b_{66},\ \lambda_6 = b_{77}$$

Independent variables for addition consist of;

$b_{13}.b_{14}, b_{15}, b_{16}, b_{17}, b_{34}, b_{35}, b_{36}, b_{37}, b_{45}, b_{46}, b_{47}, b_{56}, b_{57}$ and $b_{67}$

**Example 6.0**

Assume $\lambda_1 = 5$, $\lambda_2 = -8$, $\lambda_3 = -3$,$\lambda_4 = 5$, $\lambda_5 = 2$,$\lambda_6 = 1$

$k = 2i$ $b_{13} = 1 - 2i$, $b_{14} = 2 + i$, $b_{15} = 3i$, $b_{16} = 4$,$b_{17} = 3$

$b_{34} = 7i$, $b_{35} = 5$, $b_{36} = 1 + 9i$, $b_{37} = 3 - 5i$ $b_{45} = 2 + 3i$, $b_{46} = 6 + 5i$,

$b_{47} = 7 - 4i$ $b_{56} = 8 - i$, $b_{57} = 3 + 8i$, $b_{67} = 1 + 4i$

With the rank, six and dimension, seven we generate complex square matrix leading to

$$\implies \frac{5}{1 + 2^2} = 1$$

Consequently

$$B_{(7,6)} = \begin{bmatrix} 1 & 2i & 1-2i & 2+i & -3i & 4 & 3 \\ -2i & 4 & 4+2i & -2+4i & 6 & 8i & 6i \\ 1+2i & 4-2i & 8 & 7i & 5 & 1+9i & 3-5i \\ 2-i & -2-4i & -7i & -3 & 2+3i & 6+5i & 7-4i \\ 3i & 6 & 5 & 2-3i & 7 & 8-i & 3+8i \\ 4 & -8i & 1-9i & 6-5i & 8+i & 5 & 1+4i \\ 3 & -6i & 3+5i & 7+4i & 3-8i & 1-4i & 1 \end{bmatrix}$$

The aforementioned process is then generalized for singular symmetric and Hermitian matrices by,

1. For a $n \times n$ square matrix of rank five, we generalize.

2. We extend this procedure to a rank six $n \times n$ matrix.

**Theorem 4.1.** The $n \times n$ singular symmetric matrix of rank five can have its inverse eigenvalue problem solved given the range of values and the number of variables $k_1 = 1, 2, 3, \cdots, n-5$.

It follows that given the rank of $\Lambda_5$ and the range $\Lambda_n = \{\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n\}$ for $i = 6, 7, 8, \cdots, n-1$ $\lambda_i = 0$ ,$\lambda_5 \neq 0$, $\lambda_4 \neq 0$,$\lambda_3 \neq 0$,$\lambda_2 \neq 0$ and $\lambda_1 \neq 0$

However, since $r = 5$, it follows that the number of scalars is $n-5$. Consequently, we swap out the column or row for

$$b_{11}\begin{bmatrix} 1 & k_1 & k_1k_2 & k_1k_2k_3 & \cdots & k_{n-5} \end{bmatrix}$$

then continue the row procedure until all of the variables are utilized to create the following rows. Ultimately, we acquire as the entries are symmetric about the major diagonal.

$$B_{(n,5)} = \begin{bmatrix} b_{11} & k_1b_{11} & k_1k_2b_{11}\cdots & k_1k_2\cdots & b_{1n} \\ k_1b_{11} & k_1^2b_{11} & k_1^2k_2a_{11}\cdots & k_1^2k_2^2\cdots & k_1b_{1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_1k_2\cdots k_{n-5}b_{11} & k_1^2k_2\cdots k_{n-5}b_{11} & k_1^2k_2^2\cdots k_{n-5}^2b_{11} & k_1k_2\cdots & b_{1n} \\ b_{1(n-2)} & k_1b_{1(n-2)} & k_1k_2b_{1(n-2)} & \cdots & b_{1(n-2)} \\ b_{1(n-1)} & k_1b_{1(n-1)} & k_1k_2b_{1(n-1)} & \cdots & b_{1(n-1)} \\ b_{1n} & k_1b_{1n} & k_1k_2b_{1n} & k_1k_2k_{n-5} & \cdots b_{nn} \end{bmatrix}$$

that is

$$tr(B_{(n,5)}) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 =$$
$$b_{11}\left[1+ \quad k_1^2+ \quad k_1^2k_2^2 + \cdots + \quad k_1^2k^2\times \quad \cdots \quad k_{n-5}^2\right]$$
$$+b_{(n-2)(n-2)} + b_{(n-1)(n-1)} + b_{nn}$$

It gives us the formula for quarts.

$$b_{11}^5[1 + k_1^2 + k_1^2k_2^2 + \cdots + k_1^2k_2^2 \times \cdots \times k_{n-5}^2]^5 - [\textstyle\sum_{i=1}^5 \lambda_i]b_{11}^4[1 + k_1^2 + k_1^2k_2^2 +$$
$$\cdots \times k_{n-5}^2]^4 + [\textstyle\sum_{i=1}^4 \lambda_1\lambda_{i+1} + \sum_{i=1}^3 \lambda_2\lambda_{i+2} + \sum_{i=1}^2 \lambda_3\lambda_{i+3}$$
$$+ \textstyle\sum_{i=1}^1 \lambda_4\lambda i + 4]\, b_{11}^2[1+k_1^2+k_1^2k_2^2+\cdots+\cdots k_1^2k_2^2\times\cdots\times k_{n-5}^2]^2 - [\sum_{i=1}^3 \lambda_1\lambda_2\lambda_{i+2}$$
$$+ \textstyle\sum_{i=1}^2 \lambda_1\lambda_3\lambda_{i+3} + \sum_{i=1}^2 \lambda_2\lambda_4\lambda_{i+4} + \sum_{i=1}^2 \lambda_2\lambda_3\lambda_{i+3} + \sum_{i=1}^1 \lambda_2\lambda_4\lambda_{i+4}$$
$$+ \textstyle\sum_{i=1}^2 \lambda_2\lambda_3\lambda_{i+3}$$
$$+ \textstyle\sum_{i=1}^1 \lambda_1\lambda_4\lambda i + 4]\, b_{11}[1 + k_1^2 + k_1^2k_2^2 + \cdots + \cdots + k_1^2k_2^2 \times \cdots \times k_{n-5}^2]$$
$$+[\textstyle\sum_{i=1}^2 \lambda_1\lambda_2\lambda_3\lambda_{i+3} + \sum_{i=1}^1 \lambda_1\lambda_2\lambda_4\lambda i + 2+$$
$$\textstyle\sum_{i=1}^1 \lambda_1\lambda_3\lambda_4\lambda_{i+1}] - ]\prod_{i=1}^5 \lambda_i] = 0$$

After resolving the quartic equation above, we get

$$b_{11} = \frac{\lambda_1}{1 + k_1^2 + k_1^2k_2^2 + \cdots + k_1^2k_2^2 \times \cdots \times k_{n-5}^2},$$
$$\lambda_2 = b_{(n-3)(n-3)}$$
$$\lambda_3 = b_{(n-2)(n-2)}$$
$$\lambda_4 = b_{(n-1)(n-1)}$$
$$\lambda_5 = b_{nn}$$

It is solved.

1. For $7 \times 7$ singular symmetric matrix, we obtain a single scalar, $k$, which accounts for the independent variables,

$$b_{13}\ ,b_{14},\ b_{15},\ b_{16},\ b_{17},\ b_{34},\ b_{35},\ b_{36},\ b_{37},\ b_{45}\ b_{46},\ b_{47},\ b_{56},\ b_{57} \text{ and } b_{67}.$$

Generally, we obtain $n-5$ scalars for a $n \times n$ singular symmetric matrix:

$$b_{(n-1)n},b_{1(n-2)n},b_{1(n-2)(n-1)},b_{1n},b_{1(n-1)} \text{ and } b_{1(n-2)}.$$

2. For a $n \times n$ singular symmetric matrix of rank six, the inverse eigenvalue problem can be solved given the range and the number of scalars

$$k_i, i = 1, 2, 3, \cdots, n-6.$$

Proof: Taking into account the range given the rank $\Lambda_6$ of
$\lambda_n = \{\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n\}$,
it can be deduced that for $i = 7, 8, 9, \cdots, n-1$ $\lambda_i = 0$ ,$\lambda_5 \neq 0$, $\lambda_4 \neq 0$ , $\lambda_3 \neq 0$, $\lambda_2 \neq 0$ and $\lambda_1 \neq 0$
However, since $r = 6$, it follows that the number of scalars is equal to $n$–6 (Bronshtein & Semendyayev, 2013). Consequently, we swap out the column or row for

$B_{(n,6)} =$

$$\begin{bmatrix} b_{11} & k_1b_{11} & k_1k_2b_{11}\cdots & k_1k_2\cdots & b_{1n} \\ k_1b_{11} & k_1^2b_{11} & k_1^2k_2b_{11}\cdots & k_1^2k_2^2\cdots & k_1b_{1n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_1k_2\cdots k_1k_{n-6}b_{11} & k_1^2k_2\cdots k_{n-6}b_{11} & k_1^2k_2^2\cdots k_{n-6}^2b_{11} & k_1k_2\cdots & b_{1n} \\ b_{1(n-2)} & k_1b_{1(n-2)} & k_1k_2b_{1(n-2)} & \cdots & b_{1(n-2)} \\ b_{1(n-1)} & k_1b_{1(n-1)} & k_1k_2b_{1(n-1)} & \cdots & b_{1(n-1)} \\ b_{1n} & k_1b_{1n} & k_1k_2b_{1n} & k_1k_2k_{n-6} & \cdots b_{nn} \end{bmatrix}$$

since

$$tr(B_{(n,6)}) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 =$$

$$b_{11}\left[1+ \quad k_1^2+ \quad k_1^2k_2^2+\cdots+ \quad k_1^2k^2\times \quad \cdots \quad k_{n-6}^2\right]$$

$$+b_{(n-2)(n-2)}+b_{(n-1)(n-1)}+b_{nn}$$

leading to quartic equation.

$b_{11}^6[1+k_1^2+k_1^2k_2^2+\cdots+k_1^2k_2^2\times\cdots\times k_{n-6}^2]^6-[\sum_{i=1}^6\lambda_i]b_{11}^5[1+k_1^2+k_1^2k_2^2+\cdots\times k_{n-6}^2]^5[\sum_{i=1}^4\lambda_1\lambda_{i+1}+\sum_{i=1}^3\lambda_2\lambda_{i+2}+\sum_{i=1}^2\lambda_3\lambda_{i+3}+\sum_{i=1}^1\lambda_4\lambda_{i+4}]b_{11}^2[1+k_1^2+k_1^2k_2^2+\cdots+\cdots k_1^2k_2^2\times\cdots\times k_{n-6}^2]^2-[\sum_{i=1}^3\lambda_1\lambda_2\lambda_{i+2}+\sum_{i=1}^2\lambda_1\lambda_3\lambda_{i+3}+\sum_{i=1}^2\lambda_2\lambda_4\lambda_{i+4}+\sum_{i=1}^2\lambda_2\lambda_3\lambda_{i+3}+\sum_{i=1}^1\lambda_2\lambda_4\lambda_{i+4}+\sum_{i=1}^2\lambda_2\lambda_3\lambda_{i+3}$
$+\sum_{i=1}^1\lambda_1\lambda_4\lambda i+4]b_{11}[1+k_1^2+k_1^2k_2^2+\cdots+\cdots+k_1^2k_2^2\times\cdots\times k_{n-6}^2]+[\sum_{i=1}^5\lambda_1\lambda_2\lambda_3\lambda_{i+3}+\sum_{i=1}^4\lambda_1\lambda_2\lambda_4\lambda_{i+2}+\sum_{i=1}^3\lambda_1\lambda_2\lambda_5\lambda_{i+3}+\sum^2\lambda_2\lambda_6\lambda_{i+4}+\sum_{i=1}^1\lambda_1\lambda_3$

$\lambda_4\lambda_{i+1}]b_{11}^2[1+k_1^2+k_1^2k_2^2+\cdots\times k_{n-6}^2]^2]\prod_{i=1}^6\lambda_i]=0$

We arrived at after resolving the quartic equation above.

$$b_{11}=\frac{\lambda_1}{1+k_1^2+k_1^2k_2^2+\cdots+k_1^2k_2^2\times\cdots\times k_{n-5}^2},$$
$$\lambda_2=b_{(n-4)(n-4)}$$
$$\lambda_3=b_{(n-3)(n-3)}$$
$$\lambda_4=b_{(n-2)(n-2)}$$
$$\lambda_5=b_{(n-1)(n-1)}$$
$$\lambda_6=b_{nn}$$

It is solved.

These are the independent parameters:

1. We obtain one scalar, $k$, for the $7\times7$ singular symmetric matrix; thus, the independent variables $b_{13}$, $b_{14}$, $b_{15}$, $b_{16}$, $b_{17}$, $b_{34}$, $b_{35}$, $b_{36}$, $b_{37}$, $b_{45}$,$b_{46}$, $b_{47}$,$b_{56}$, $b_{57}$.

2. We obtain two scalars, $k_1$ and $k_2$, for the $8 \times 8$ singular symmetric matrix; these are the independent variables. $b_{14}$, $b_{15}$, $b_{16}$, $b_{17}$, $b_{18}$, $b_{34}$, $b_{35}$, $b_{36}$, $b_{37}$, $b_{38}b_{45}$,$b_{46}$ $b_{47}$, $b_{48}$, $b_{56}$, $b_{57}$, $b_{58}$, $b_{67}$, $b_{68}$, and $b_{78}$.

3. The independent variables come from the one scalar, $k$, that we obtain for the $n \times n$ singular symmetric matrix. These are:

$$b_{(n-1)n}, b_{1(n-2)n}, b_{1(n-2)(n-1)}, b_{1n}, b_{1(n-1)} \text{ and } b_{1(n-2)}.$$

**Theorem 4.2.** It is possible to solve the inverse eigenvalue problem for the $n \times n$ singular Hermitian matrix of rank five, given the range of values and the number of variables $k_i, i = 1, 2, 3, \cdots, n-5$.

Proof: Since the rank of $\Lambda_5$ is $\lambda_n = \{\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n\}$, we may consider the range for $i = 6, 7, 8, \cdots, n-1$ $\lambda_i = 0$ ,$\lambda_5 \neq 0$, $\lambda_4 \neq 0$,$\lambda_3 \neq 0$,$\lambda_2 \neq 0$ and $\lambda_1 \neq 0$

However, since $r$ =5, it follows that the number of scalars is $n-5$. Consequently, we swap out the column or row for

$$b_{11} \begin{bmatrix} 1 & k_1 & k_1k_2 & k_1k_2k_3 & \cdots & k_{n-5} \end{bmatrix}$$

then continue the row procedure until all of the variables are utilized to create the following rows. Ultimately, we acquire as the entries are symmetric about the major diagonal.

$B_{(n,5)} =$

$$\begin{bmatrix} b_{11} & k_1^- b_{11} & k_1^- k_2^- b_{11} \cdots & k_1^- k_2^- \cdots & b_{1n}^- \\ k_1 b_{11} & k_1^2 b_{11} & k_1^2 k_2^- b_{11} \cdots & k_1^2 k_2^2 \cdots & k_1 b_{1n}^- \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_1 k_2 \cdots k_{n-5} b_{11} & k_1^2 k_2 \cdots k_{n-5} b_{11}^- & k_1^2 k_2^2 \cdots k_{n-5}^2 b_{11} & k_1 k_2 \cdots & b_{1n}^- \\ b_{1(n-2)} & k_1^- b_{1(n-2)} & k_1 k_2 b_{1(n-2)} & \cdots & b_{1(n-2)}^- \\ b_{1(n-1)} & k_1^- b_{1(n-1)} & k_1 k_2 b_{1(n-1)} & \cdots & b_{1(n-1)}^- \\ b_{1n} & k_1^- b_{1n} & k_1^- k_2^- b_{1n} & k_1^- k_2^- k_{n-5}^- & \cdots b_{nn} \end{bmatrix}$$

that is

$$tr(B_{n,5}) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 =$$
$$b_{11} \begin{bmatrix} 1+ & k_1^2+ & k_1^2 k_2^2 + \cdots + & k_1^2 k^2 \times & \cdots & k_{n-5}^2 \end{bmatrix}$$
$$+ b_{(n-2)(n-2)} + b_{(n-1)(n-1)} + b_{nn}$$

It gives us the formula for quarts.

$$b_{11}^5 [1 + k_1^2 + k_1^2 k_2^2 + \cdots + k_1^2 k_2^2 \times \cdots \times k_{n-5}^2]^5 - [\textstyle\sum_{i=1}^5 \lambda_i] b_{11}^4 [1 + k_1^2 + k_1^2 k_2^2 +$$
$$\cdots \times k_{n-5}^2]^4 + [\textstyle\sum_{i=1}^4 \lambda_1 \lambda_{i+1} + \sum_{i=1}^3 \lambda_2 \lambda_{i+2} + \sum_{i=1}^2 \lambda_3 \lambda_{i+3}$$
$$+ \textstyle\sum_{i=1}^1 \lambda_4 \lambda i + 4]\, b_{11}^2 [1 + k_1^2 + k_1^2 k_2^2 + \cdots + \cdots k_1^2 k_2^2 \times \cdots \times k_{n-5}^2]^2 - [\sum_{i=1}^3 \lambda_1 \lambda_2 \lambda_{i+2}$$
$$+ \textstyle\sum_{i=1}^2 \lambda_1 \lambda_3 \lambda_{i+3} + \sum_{i=1}^2 \lambda_2 \lambda_4 \lambda_{i+4} + \sum_{i=1}^2 \lambda_2 \lambda_3 \lambda_{i+3} + \sum_{i=1}^1 \lambda_2 \lambda_4 \lambda_{i+4}$$
$$+ \textstyle\sum_{i=1}^2 \lambda_2 \lambda_3 \lambda_{i+3}$$
$$+ \textstyle\sum_{i=1}^1 \lambda_1 \lambda_4 \lambda i + 4]\, b_{11} [1 + k_1^2 + k_1^2 k_2^2 + \cdots + \cdots + k_1^2 k_2^2 \times \cdots \times k_{n-5}^2]$$
$$+ [\textstyle\sum_{i=1}^2 \lambda_1 \lambda_2 \lambda_3 \lambda_{i+3} + \sum_{i=1}^1 \lambda_1 \lambda_2 \lambda_4 \lambda i + 2 +$$
$$\textstyle\sum_{i=1}^1 \lambda_1 \lambda_3 \lambda_4 \lambda_{i+1}] - ] \prod_{i=1}^5 \lambda_i] = 0$$

After resolving the quartic equation above, we get

$$b_{11} = \frac{\lambda_1}{1 + k_1^2 + k_1^2 k_2^2 + \cdots + k_1^2 k_2^2 \times \cdots \times k_{n-5}^2},$$
$$\lambda_2 = b_{(n-3)(n-3)}$$
$$\lambda_3 = b_{(n-2)(n-2)}$$
$$\lambda_4 = b_{(n-1)(n-1)}$$
$$\lambda_5 = b_{nn}$$

It is solved.

For $7 \times 7$ singular Hermitian matrix, we obtain a single scalar, $k$, which accounts for the independent variables,

$$b_{13},b_{14},b_{15},b_{16},b_{17},b_{34},b_{35}\ b_{36},b_{37}\ b_{45}\ b_{46}\ ,b_{47},b_{56},b_{57} \text{ and } b_{67}.$$

Generally, we obtain $n-5$ scalars for a $n \times n$ singular Hermitian matrix:

$$b_{(n-1)n},b_{1(n-2)n},b_{1(n-2)(n-1)},b_{1n},b_{1(n-1)} \text{ and } b_{1(n-2)}.$$

For a $n \times n$ singular Hermitian matrix of rank six, the inverse eigenvalue problem can be solved given the range and the number of scalars

$$k_i, i = 1, 2, 3, \cdots, n-6.$$

Proof: Considering the range, since $\Lambda_6$ has a rank of $\lambda_n = \{\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n\}$, it follows that

$\lambda_1 \neq 0\ \lambda_2 \neq 0\ \lambda_3 \neq 0\ \lambda_4 \neq 0\ \lambda_5 \neq 0\ \lambda_6 \neq 0\ \lambda_i = 0$ For $i = 7, 8, 9, \cdots, n$

However, since $r = 6$, it follows that the number of scalars is equal to $n$–6. Consequently, we swap out the column or row for

$B_{(n,6)} =$

$$\begin{bmatrix} b_{11} & k_1^- b_{11} & k_1^- k_2^- b_{11} \cdots & k_1^- k_2^- \cdots & b_{1n}^- \\ k_1 b_{11} & k_1^2 b_{11} & k_1^2 k_2^- b_{11} \cdots & k_1^2 k_2^2 \cdots & k_1 b_{1n}^- \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_1 k_2 \cdots k_{n-6} b_{11} & k_1^2 k_2 \cdots k_{n-6} b_{11} & k_1^2 k_2^2 \cdots k_{n-6}^2 b_{11} & k_1 k_2 \cdots & b_{1n}^- \\ b_{11(n-2)} & k_1^- b_{11(n-2)} & k_1^- k_2^- b_{(n-2)n} & \cdots & b_{(n-2)n}^- \\ b_{1(n-1)} & k_1^- b_{11(n-1)} & k_1^- k_2^- b_{11(n-1)} & \cdots & b_{(n-1)n}^- \\ b_{1n} & k_1^- b_{1n} & k_1^- k_2^- b_{1n} & k_1^- k_2^- k_{n-6}^- & \cdots b_{nn} \end{bmatrix}$$

since $tr(B_{(n,6)}) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 =$

$$b_{11}\left[1+\quad k_1^2+\quad k_1^2k_2^2+\cdots+\quad k_1^2k^2\times\quad\cdots\quad k_{n-6}^2\right]$$

$$+b_{(n-2)(n-2)}+b_{(n-1)(n-1)}+b_{nn}$$

leading to quartic equation.

$b_{11}^6 1+k_1^2+k_1^2k_2^2+\cdots+k_1^2k_2^2\times\cdots\times k_{n-6}^2]^6-[\sum_{i=1}^6\lambda_i]b_{11}^5[1+k_1^2+k_1^2k_2^2+\cdots\times k_{n-6}^2]^5[\sum_{i=1}^4\lambda_1\lambda_{i+1}+\sum_{i=1}^3\lambda_2\lambda_{i+2}+\sum_{i=1}^2\lambda_3\lambda_{i+3}+\sum_{i=1}^1\lambda_4\lambda_{i+4}]b_{11}^2[1+k_1^2+k_1^2k_2^2+\cdots+\cdots k_1^2k_2^2\times\cdots\times k_{n-6}^2]^2-[\sum_{i=1}^3\lambda_1\lambda_2\lambda_{i+2}+\sum_{i=1}^2\lambda_1\lambda_3\lambda_{i+3}+\sum_{i=1}^2\lambda_2\lambda_4\lambda_{i+4}+\sum_{i=1}^2\lambda_2\lambda_3\lambda_{i+3}+\sum_{i=1}^1\lambda_2\lambda_4\lambda_{i+4}+\sum_{i=1}^2\lambda_2\lambda_3\lambda_{i+3}$
$+\sum_{i=1}^1\lambda_1\lambda_4\lambda i+4]b_{11}[1+k_1^2+k_1^2k_2^2+\cdots+\cdots+k_1^2k_2^2\times\cdots\times k_{n-6}^2]+[\sum_{i=1}^5\lambda_1\lambda_2\lambda_3\lambda_{i+3}+\sum_{i=1}^4\lambda_1\lambda_2\lambda_4\lambda_{i+2}+\sum_{i=1}^3\lambda_1\lambda_2\lambda_5\lambda_{i+3}+\sum^2\lambda_2\lambda_6\lambda_{i+4}+\sum_{i=1}^1\lambda_1\lambda_3$

$$\lambda_4\lambda_{i+1}]b_{11}^2[1+k_1^2+k_1^2k_2^2+\cdots\times k_{n-6}^2]^2]\prod_{i=1}^6\lambda_i]=0$$

After resolving the quartic equation above, we arrived at

$$b_{11}=\frac{\lambda_1}{1+k_1^2+k_1^2k_2^2+\cdots+k_1^2k_2^2\times\cdots\times k_{n-5}^2},$$
$$\lambda_2=b_{(n-4)(n-4)}$$
$$\lambda_3=b_{(n-3)(n-3)}$$
$$\lambda_4=b_{(n-2)(n-2)}$$
$$\lambda_5=b_{(n-1)(n-1)}$$
$$\lambda_6=b_{nn}$$

It is solved.

The independent variables are;

1. We obtain one scalar, $k$, for the $7\times 7$ singular Hermitian matrix; thus, the independent variables $b_{13}$, $b_{14}$, $b_{15}$, $b_{16}$, $b_{17}$, $b_{34}$, $b_{35}$, $b_{36}$, $b_{37}$, $b_{45}$,$b_{46}$,$b_{47}$ $b_{56}$, $b_{57}$.

2. For $8 \times 8$ singular Hermitian matrix we get two scalars that is $k_1$ and $k_2$, hence the independent variables $b_{14}$,$b_{15}$, $b_{16}$, $b_{17}$,$b_{18}$,$b_{34}$,$b_{35}$,$b_{36}$,$b_{37}$,$b_{38}$$b_{45}$,$b_{46}$ $b_{47}$,$b_{48}$,$b_{56}$,$b_{57}$,$b_{58}$,$b_{67}$,$b_{68}$, and $b_{78}$ .

3. The independent variables come from the one scalar, $k$, that we obtain for the $n \times n$ singular Hermitian matrix $b_{(n-1)n}$, $b_{1(n-2)n}$, $b_{1(n-2)(n-1)}$, $b_{1n}$, $b_{1(n-1)}$ and $b_{1(n-2)}$.

**Theorem 4.3.** If $n - r$ variable parameters are given, the inverse eigenvalue problem for a $n \times n$ singular symmetric and Hermitian matrix of ranks five or six may be solved.

Proof: $B_{(n,r)} = b_{ij}$, where $n \geq 2$, is the case. It is evident that,

$$b_{i,j} = b_{11} \begin{Bmatrix} \prod_{s=0}^{j=1} k_s, & i < j \\ (\prod_{s=0}^{i-1} k_s)^2, & i = j \\ \prod_{s=0}^{i-1} K_s i > j & \end{Bmatrix}$$

moreover

$$b_{11} = \frac{\lambda_1}{1 + k_1^2 + k_1^2 k_2^2 + \cdots + k_1^2 k_2^2 \cdots k_{n-r}^2}.$$

and

$$\begin{aligned} b_{i,j} &= \lambda_2, \quad where \quad i = j = n - r + 2 \\ \vdots &= \vdots \\ b_{n,n} &= \lambda_r \end{aligned}$$

$\lambda_1, \lambda_2, \cdots, \lambda_r \in R$ such that $R_i = k_{i-1} R_{i-1}$ and $R_i$ is the $ith$ row of $B$.

$$Tr(B) = 1 + \sum_{s=1}^{n} (\prod_{s=1}^{i-1} k_s)^2$$

We see that

$$r = 5:$$
$$b_{11} = \frac{\lambda_1}{1 + k_1^2 + k_1^2 k_2^2 + \cdots + k_1^2 k_2^2 \cdots k_{n-5}^2}.$$
$$r = 6:$$
$$b_{11} = \frac{\lambda_1}{1 + k_1^2 + k_1^2 k_2^2 + \cdots + k_1^2 k_2^2 \cdots k_{n-6}^2}.$$

**Chapter Summary**

The inverse eigenvalue problem relating to ranks five and six singular symmetric and Hermitian matrices are solved in this chapter. Building Hermitian and singular symmetric matrices using a modified procedure to come out with the results.

# CHAPTER FIVE

# SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

## Overview

The summary, conclusions and recommendations are looked at in this chapter.

## Summary

In this study, ranks five and six singular symmetric Hermitian matrices with an inverse eigenvalue problem are to be solved. Upon knowledge of the matrices' eigenvalues, the primary objective is to locate the original matrices. Matrix features exhibiting singularity or Hermitian characteristics receive special attention (Baah, 2012). The research extended a method for the inverse eigenvalue problem for singular symmetric matrices to include rank four and provided multiple examples (Aidoo et al., 2013). The lone Hermitian matrix is another thing the researcher looks at.

## Conclusions

Complex mathematical methods, which may involve advanced numerical algorithms, are needed to solve the inverse eigenvalue problem for singular symmetric and Hermitian matrices of ranks five and six. The work provides comprehensive knowledge about these matrices and their manipulation, which opens up new possibilities for applications in various disciplines, including quantum physics and signal processing. More research and development of these solutions may increase our ability to address complex problems in a variety of scientific and engineering domains.

## Recommendations

Regarding the study results, it is advised that the following be given careful consideration.

1. Singular symmetric matrices of ranks five and six should have their inverse eigenvalue problem handled with mathematical technology and software.

2. Analyze the applicability of the solution to the inverse eigenvalue problem for singular Hermitian matrix of ranks five and six in the field of physics, especially quantum mechanics.

# REFERENCES

Aidoo, A. Y., Gyamfi, K. B., Ackora-Prah, J., & Oduro, F. T. (2013). Solvability of Inverse Eigenvalue Problem for Dense Singular Symmetric Matrices. *Advances in Pure Mathematics*, *03*(01), 14–19. doi: 10.4236/apm.2013.31003

Andrea, S. A., & Berry, T. (1992). Continued fractions and periodic jacobi matrices. *Linear algebra and its applications*, *161*, 117–134.

Baah, G. K. (2012). Solution of inverse eigenvalue problem of certain singular hermitian matrices.. Retrieved from `https://api.semanticscholar.org/CorpusID:118995509`

Boley, D., & Golub, G. H. (1987). A survey of matrix inverse eigenvalue problems. *Inverse Problems*, *3*(4), 595–622. doi: 10.1088/0266-5611/3/4/010

Bronshtein, I. N., & Semendyayev, K. A. (2013). *Handbook of mathematics*. Springer Science & Business Media.

Chu, M. T. (1998). Inverse eigenvalue problems. *SIAM review*, *40*(1), 1–39.

Chu, M. T., Diele, F., & Ragni, S. (2005). On the inverse problem of constructing symmetric pentadiagonal toeplitz matrices from their three largest eigenvalues. *Inverse problems*, *21*(6), 1879-1883.

Chu, M. T., & Golub, G. H. (2002). Structured inverse eigenvalue problems. *Acta Numerica*, *11*, 1–71.

Connell, E. H. (2002). *Elements of abstract and linear algebra*. Gainesville, Florida, USA:University Press of Florida.

Deakin, A., & Luke, T. (1992). On the inverse eigenvalue problem for matrices (atomic corrections). *Journal of Physics A: Mathematical and General*, *25*(3), 635-640.

Dehghan Niri, T., Shahzadeh Fazeli, S., & Heydari, M. (2020). A two-step improved newton method to solve convex unconstrained optimization problems. *Journal of Applied Mathematics and Computing*, *62*, 37–53.

Downing Jr, A., & Householder, A. S. (1956). Some inverse characteristic value problems. *Journal of the ACM (JACM)*, *3*(3), 203–207.

Draper, N. R., & Smith, H. (1998). *Applied regression analysis* (Vol. 326). John Wiley & Sons.

Dumond, P., & Baddour, N. (2016). Structured approaches to general inverse eigenvalue problems. In *Applied linear algebra in action.* IntechOpen.

Efron, B., & Morris, C. (1976). Multivariate empirical bayes and estimation of covariance matrices. *The Annals of Statistics*, 22–32.

Egleston, P. D., Lenker, T. D., & Narayan, S. K. (2004). The nonnegative inverse eigenvalue problem. *Linear Algebra and Its Applications*, *379*(1-3 SPEC. ISS), 475–490. doi: 10.1016/j.laa.2003.10.019

Fallat, S. M., & Hogben, L. (2007). The minimum rank of symmetric matrices described by a graph: A survey. *Linear Algebra and its Applications*, *426*(2-3), 558–582.

Fiedler, M. (1974). Eigenvalues of nonnegative symmetric matrices. *Linear Algebra and its Applications*, *9*, 119–142.

Friedland, S. (1977). Inverse eigenvalue problems. *Linear Algebra and Its Applications*, *17*(1), 15–51. doi: 10.1016/0024-3795(77)90039-8

Friedland, S., Nocedal, J., & Overton, M. L. (1987). The formulation and analysis of numerical methods for inverse eigenvalue problems. *SIAM Journal on Numerical Analysis*, *24*(3), 634–667.

Hadeler, K. (1969). Multiplikative inverse eigenwertprobleme. *Linear Algebra and Its Applications*, *2*(1), 65–86.

Hald, O. H. (1976). Inverse eigenvalue problems for jacobi matrices. *Linear Algebra and Its Applications*, *14*(1), 63–85.

Hald, O. H. (1983). Inverse eigenvalue problems for the mantle—ii. *Geophysi-*

*cal Journal International*, *72*(1), 139–164.

Herstein, I. N., & Winter, D. J. (1988). *Matrix theory and linear algebra*. Macmillan Publishing Co., Inc.

Johnston, N. (2021). *Introduction to linear and matrix algebra*. Springer Nature.

Kandić, D., Reljin, B., et al. (2008). Explicit solution of the inverse eigenvalue problem of real symmetric matrices and its application to electrical network synthesis. *Mathematical Problems in Engineering*, *2008*.

Lipovetsky, S. (2013). Priorities in thurstone scaling and steady-state probabilities in markov stochastic modeling. *Journal of Modern Applied Statistical Methods*, *12*(1), 12.

Marcus, M., & Minc, H. (1988). *Introduction to linear algebra, 182*. Dover Publications Inc., Mineola, New York.

Marijuán, C., Pisonero, M., & Soto, R. L. (2007). A map of sufficient conditions for the real nonnegative inverse eigenvalue problem. *Linear Algebra and Its Applications*, *426*(2-3), 690–705. doi: 10.1016/j.laa.2007.05.046

Mirsky, L. (1964). Inequalities and existence theorems in the theory of matrices. *Journal of Mathematical Analysis and Applications*, *9*(1), 99–118.

Oduro, F. T. (2014). Solution of the inverse eigenvalue problem for certain (anti-) hermitian matrices using newton's method. *Journal of Mathematics Research*, *6*(2), 64-70.

Oliveira, G. (1972). Matrix inequalities and the additive inverse eigenvalue problem. *Computing*, *9*(2), 95–100.

Radwan, N. (1996). An inverse eigenvalue problem for symmetric and normal matrices. *Linear Algebra and Its Applications*, *248*, 101–109. doi: 10.1016/0024-3795(95)00162-X

Riondato, M., García-Soriano, D., & Bonchi, F. (2017). Graph summarization with quality guarantees. *Data mining and knowledge discovery*, *31*, 314–349.

Rothblum, U. G. (2006). Nonnegative matrices and stochastic matrices. In

*Handbook of linear algebra* (pp. 9–1). Chapman and Hall/CRC.

Sasikumar, S., Karthikeyan, S., Suganthi, M., & Madheswaran, M. (2009). Efficient estimation algorithm for arma, exponential and other trigonometric model with quantum constraints. *IET Signal Processing*, *3*(1), 64–73.

Shapiro, A. (1983). Asymptotic distribution theory in the analysis of covariance structures. *South African Statistical Journal*, *17*(1), 33–81.

Sharma, D., & Sen, M. (2018). Inverse eigenvalue problems for acyclic matrices whose graph is a dense centipede. *Special Matrices*, *6*(1), 77–92.

Simovici, D. A. (2012). *Linear algebra tools for data mining*. World Scientific.

Sing, F.-Y. (1976). Some results on matrices with prescribed diagonal elements and singular values. *Canadian Mathematical Bulletin*, *19*(1), 89–92.

Soto, R. L. (2016). Nonnegative inverse elementary divisors problem. In *Applied linear algebra in action.* London, United Kingdom: IntechOpen .

Soto, R. L., & Rojo, O. (2006). Applications of a Brauer theorem in the nonnegative inverse eigenvalue problem. *Linear Algebra and Its Applications*, *416*(2-3), 844–856. doi: 10.1016/j.laa.2005.12.026

Strang, G. (2012). *Linear algebra and its applications*.

Thompson, R. C. (1977). Singular values, diagonal elements, and convexity. *SIAM Journal on Applied Mathematics*, *32*(1), 39–63.

Voyevodin, V. V. (1983). *Linear algebra*. Mir Publishers.

Weiss, J. (2019). A tutorial on the proper orthogonal decomposition. In *Aiaa aviation 2019 forum* (p. 3333).

Wu, J.-L. (2011). On Open Problems of Nonnegative Inverse Eigenvalues Problem. *Advances in Pure Mathematics*, *01*(04), 128–132. doi: 10.4236/apm.2011.14025

Xu, Y.-H., & Jiang, E.-X. (2006). An inverse eigenvalue problem for periodic jacobi matrices. *Inverse problems*, *23*(1), 165.

Zhang, T., & Golub, G. H. (2001). Rank-one approximation to high order tensors. *SIAM Journal on Matrix Analysis and Applications*, *23*(2), 534–

550.

Zhao, L., Hu, X., & Zhang, L. (2011). Inverse eigenvalue problems for bisymmetric matrices under a central principal submatrix constraint. *Linear and Multilinear Algebra*, *59*(2), 117–128.