UNIVERSITY OF CAPE COAST

GLOBAL STABILITY OF A PREDATOR-PREY FISHERY MODEL WITH NON-SELECTIVE HARVESTING: A STUDY OF LINEAR OPTIMAL CONTROL

BY

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DECLARATION

Candidate's Declaration

I hereby declare that this thesis is the result of my own original research and that no part of it has been presented for another degree in this university or elsewhere.

Candidate's Signature .... Date ....

Name: Cephas Tay-Suka

Supervisor's Declaration

I hereby declare that the preparation and presentation of the thesis were supervised in accordance with the guidelines on supervision of thesis laid down by the University of Cape Coast.

Supervisor's Signature .... Date............................

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ABSTRACT

A proposed two-dimensional modified Lotka-Volterra fishery model in terms of predator-prey aims to explore the effect of non-selective harvesting on the predator and the prey populations. The study delves into various essential aspects of the dynamical system, comprising positivity, uniform boundedness and persistence. Points of equilibrium are identified. The system's local and global stability are thoroughly examined and discussed. Moreover, the research explores the concept of bionomic equilibrium, a scenario where economic rent is entirely dissipated. Extending the bioeconomic model, the study investigates a linear optimal control problem to determine the most effective harvesting strategy. Utilising Pontryagin's maximum principle, the optimal control is characterised. The findings indicate that maximum allowable effort should be exerted whenever the net revenue per unit effort surpasses the total net marginal revenue of predator and prey stocks. Numerical simulations, with data on the marine artisanal fishery in Ghana, are conducted to validate the theoretical discoveries. The outcomes reveal that the fishery can support sustainable harvesting of both predator (tuna) and prey (sardinella) populations, so far as the optimal harvesting effort is set at 100,000 fishing trips annually.

KEY WORDS

Global stability

Marine artisanal fishery

Numerical simulation

Optimal harvesting effort

Pontryagin's maximum principle

Predator-prey model

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DEDICATION

To my family

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MNR Marginal Net Revenue

NRPUE Net Revenue Per Unit Effort

CHAPTER ONE

INTRODUCTION

Fisheries play an essential role in the economy and food security of many countries worldwide. However, overexploitation of fish stocks has become a major concern, leading to the depletion of fish populations and its negative effects on the ecosystem. To address this issue, mathematical models in fishery have been proposed to study predator-prey dynamics, interactions and the impact of harvesting on fish populations. This research work focuses on a predator-prey fishery model with non-selective harvesting and investigates the effectiveness of linear optimal controls in maintaining the fishery system's stability.

Background to the Study

The predator-prey fishery model is mathematically formulated to examine the relationship between two or more species in an ecosystem: the predators and their prey. This model was developed as a way to understand the population dynamics concerning fishing activities. The predator-prey fishery model can be traced back to the mid-20th century when fisheries around the world were being overexploited due to the lack of proper management and regulation. This resulted in a decline in fish populations, which had serious economic, social, and ecological consequences. To address this problem, scientists began developing mathematical models that could help them understand the fish population dynamics and the effect of fishing activities on these populations. One of the most influential models was the fishery model in terms of predator and prey, which was initiated by A. J. Lotka and V. Volterra. The predator-prey fishery model was centred on the equations of Lotka and Volterra, which explains predator-prey interactions and their dynamics in terms of population growth rates. Some scientists have extended these equations to include fishing activities as a factor that affects the swing in the population of both the predators and the prey. The model showed that fishing activities could have a significant impact on the population sizes of both species and that the effort of fishing at an optimal level would border on the biological characteristics of the species and the ecological context. Since its introduction, the predator-prey fishery model has been widely used in fisheries management and conservation. It has helped scientists and policymakers understand fishing activities and their impact on fish populations, has served as a guide to the development of lasting fishing practices that balance economic, social, and ecological objectives. The explosive growth of the human population, combined with rapid technological advancements, has unleashed an insatiable demand for resources, leading to ruthless and excessive exploitation. As highlighted by Clark (2010), the depletion of marine fish stocks has transcended the boundaries of easily accessible near-shore populations and ruthlessly encroached upon remote off-shore regions. Valuable species of significant size have fallen victim to relentless overfishing, a process aptly coined as "fishing down the food web" by Pauly et al. (1998) Alarming statistics by the Food and Agriculture Organization of the United Nations (FAO) paint pictures of grim. Their analysis of assessed fish stocks reveals a distressing downward trend, with the share of stocks maintained at biologically sustainable levels dwindling from 90% in 1974 to a meagre 68.6% by 2003. This leaves a staggering 31.4% of fish stocks teetering on the brink of collapse, victims of overfishing and the biological havoc it wreaks. Shockingly, a mere 10.5% of the assessed fish stocks in 2013 were classified as underfished, while a disheartening 58.1% were mercilessly exploited to their full extent (FAO, 2016). The concerns voiced by Mullon, Freon, and Curry (2005) are undeniably justified as their analysis of the FAO's world fisheries database spanning 80 years reveals a terrifying reality: a staggering 366 fisheries collapsed, accounting for nearly a quarter of all fisheries examined. These dire figures, as corroborated by the Ministry of Fisheries and Agriculture Development (MOFAD, 2016), serve as a clarion call to the impending catastrophe looming over our oceans. The evidence unequivocally asserts that the unsustainable depletion of fish stocks due to unchecked over-exploitation has instilled a mounting sense of fear. The urgency of the situation cannot be understated as we grapple with the remorseless annihilation of vital marine ecosystems, threatening both the delicate balance of nature and the livelihoods of countless communities that depend on healthy fisheries. Global stability of a model with non-selective harvesting of two species refers to the long-term behaviour of the populations of the two species. The model's global stability means that, under certain conditions, the populations of both species will settle into a stable equilibrium point and continue to stay around this point indefinitely. The stability of the steady-state point depends on the model's parameters, such as the predation rate, the rate of growth of the prey, and the rate of harvesting.

Statement of the Problem

This research addresses the need to balance economic and ecological objectives in the management of fisheries. The predator-prey model for fishery is used to study the predator and prey populations dynamics and the impact on fishing activities on these populations. The model also helps identify the optimal fishing effort levels that maximize economic benefits while ensuring the long-term sustainability of the fishery. The research is approached by examining different fishing strategies, such as varying fishing effort levels and their impact on predator and prey populations. The aim is to identify the fishing strategy that maximizes the economic benefits while minimizing the negative impact on the ecosystem. This would require a comprehensive analysis of the biological, ecological, and socio-economic factors that affect the fishery.

Purpose of the study

The ultimate purpose of researching a predator-prey model for fishery with non-selective harvesting is to contribute to the conservation and sustainability of fishery resources. By understanding the impacts of non-selective harvesting and developing effective management strategies, the research aims to promote sustainable fishing practices that maintain healthy fish stocks, preserve biodiversity, and support the long-term livelihoods of fishing communities.

Research objectives

To attain the aim of ensuring the permanence of fishery resources, the research aimed to determine optimal control techniques and procedures for sustainable harvesting strategies. To achieve this aim, the following specific objectives were considered:

- To formulate a mathematical model to optimize the strategies for harvesting fishery resources.
- To ascertain the model's points of equilibrium, local and global stability.
- To develop a linear optimal control problem with non-selective harvesting.
- To analyze and derive conditions necessary for the optimality of the control problem.
- To validate the theoretical findings by performing numerically simulations on data relating to marine artisanal fishery in Ghana.

Research Questions

- What is a predator-prey fishery model with harvesting dynamics?
- What is the optimal harvesting effort using linear control?
- How can simulations help in the application of the model to the marine artisanal fishery in Ghana?

Significance of the Study

Sustainable fisheries management is essential for maintaining ecological balance and ensuring long-term economic viability. Overexploitation and unregulated harvesting often leads to population collapse, disrupting aquatic ecosystems and threatening food security. This study provides a mathematical framework to analyze and control predator-prey dynamics, ensuring sustainable harvesting while maintaining ecological stability. This research makes significant contributions to mathematical ecology, fisheries management, and optimal control theory. By analyzing the global stability of the predator-prey model, the study enhances understanding of species interaction under harvesting pressures.

The incorporation of linear optimal control allows for the development of strategies that maximize yields while preventing species extinction and maintaining ecosystem stability. This findings will serve as a scientific basis for regulating long-term resource sustainability. Policymakers and fisheries managers can use the findings to develop sustainable fishing regulations and quota-based systems that balance the economic interest with conservation efforts. Marine biologist and ecologists will again a quantitative framework to analyze population dynamics and ecosystem-based management strategies.

Economists and fishing industry will benefit from optimized harvesting strategies that maximize long-term fishery profits while preserving marine biodiversity. Additionally, academics in mathematical research will find valuable insights for further exploitation in non-linear dynamics, stability analysis, and modeling. Conservation organizations advocating for sustainable fishing practices will also benefit from the scientific evidence supporting better management of marine resources.

This study contributes to knowledge, policy, and practice by enhancing mathematical understanding of predator-prey interactions, providing scientific guidance for sustainable fisheries management, and offering a practical decision-making tool for fisheries operations. Ultimately, it bridges the gap between theoretical Mathematics and real-world ecological applications, ensuring that fisheries remain productive and ecosystems remain stable for future generations.

Justification of the study

This study intends to equip essential fishing industry stakeholders—such as government entities, fishermen, conservationists and scholars—with a scientific-driven tool to enhance their ability to make well-informed decisions. Unlike conventional approaches, which often rely on discrete-time models, our research embraces the dynamic nature of fisheries by employing a continuous-time framework. By monitoring stocks and related activities at any given moment, our approach unveils new and captivating perspectives that can enhance the understanding of fisheries. This study is poised to inspire a wave of passionate academics and researchers to join forces in combating the alarming decline of fisheries not only along the vibrant coast of Ghana but also worldwide. So, a way can be paved for effective measures that will ensure the sustainable future of fishery resources.

Delimitation

In the realm of the predator-prey fishery relationship, this study involves delineating the crucial elements and variables influencing the dynamics of both predator and prey populations, along with examining how fishing activities exert an impact on these populations. Some possible components and factors would be considered in the scope of the research. The biological characteristics of the predator and prey populations, such as growth rate, mortality rate, and reproductive rate, are important factors that affect their population dynamics. Fishing effort and fishing season duration are important factors that affect the population size of both the prey and predator. The economic benefits and social impacts of fishing activities, such as employment, income, and food security, would be considered.

Limitations

- The predator-prey fishery model faces uncertainty due to the intricate interplay of various factors that make predicting the population dynamics of the predator and prey populations challenging. For example, natural disasters, climate change, and disease outbreaks can all have a significant impact on fish populations.

- The predator-prey fishery model relies on accurate and comprehensive data on the biological, ecological, and socio-economic factors that affect predator and prey populations. However, data on these factors are often limited, notably in developing countries and small-scale fisheries.

- The predator-prey fishery model is grounded on the assumption of a simplified predator-prey relationship, which may not accurately reflect the complexity of real-world ecosystems. In reality, the relationships between predators and prey are often more complex and can involve multiple species and trophic levels.

Definition of Terms

This section defines key terms used in the study to provide clarity on the concepts related to the predator-prey fishery model and optimal control theory.

- Bionomic Equilibrium – A state where the economic rent from harvesting is completely dissipated due to overexploitation.

- Equilibrium Point – A state where the populations of predator and prey remain constant over time.

- Global Stability – A condition where, regardless of initial population sizes, the system stabilizes at an equilibrium.
- Local Stability – A condition where small disturbances in population return to equilibrium over time.
- Lyapunov Function – A mathematical function used to establish the stability of equilibrium points.
- Marginal Net Revenue (MNR) – The additional revenue generated from increasing fishing effort by one unit.
- Net Revenue Per Unit Effort (NRPUE) – A measure of economic return per unit of fishing effort.
- Optimal Harvesting Strategy – A strategy that maximizes long-term economic benefits while sustaining fish populations.
- Pontryagin's Maximum Principle – An optimization technique used to determine the best control strategy for managing fisheries.
- Predator-prey Model – A mathematical model that describes interactions between predator and prey populations in an ecosystem.
- Routh-Hurwitz Criterion – A mathematical method for determining the stability of a system by analyzing characteristic equations.

These definitions provide a foundational understanding of the concepts used in this study and their relevance to the predator-prey fishery model.

Organisation of the Study

This study comprises six captivating chapters that shed light on optimal control theory in the context of a predator-prey fishery model with non-selective harvesting. Chapter One provides an insightful overview of the research background, including the main mathematical tool utilized in our analysis. We also outline the study's objectives, methodology, scope, and limitations, which pave the way for a deeper understanding of the research problem. Chapter Two presents a literature review that delves into the previous work done on optimal control fishery resource management and modelling. The literature review serves as an excellent platform for showcasing how different researchers have approached the challenge of sustainable renewable resources for future generations. Chapter Three explores the mathematical modelling concepts of the study. Chapter Four investigates the model formulation and analysis that underlie our study. In Chapter Five, the optimal control problem is examined and we take a numerical approach to studying the given problem and present our findings and discussions clearly and concisely. This chapter reveals the numerical insights we have gained through our research. Finally, in Chapter Six, we wrap up our research work by presenting an engaging summary of our findings, conclusions, and recommendations. Our recommendations are tailored specifically for the fisheries commission and are aimed at promoting the sustainable management of fishery resources.

CHAPTER TWO

LITERATURE REVIEW

Introduction

In this chapter, an extensive examination of pertinent literature in the field of study will be conducted, with a specific emphasis on optimal control problems within the context of predator-prey mathematical fishery models. This review and discussion aim to provide a thorough understanding of the subject matter.

Optimal Control of a Predator-Prey Fishery Resource Management

(Hairston et al., 1960) findings in the context of predator-prey fishery models highlighted the intricate dynamics between predator and prey populations and the influence of fishing activities on these dynamics. One of their key findings was that selective fishing, particularly targeting larger individuals, can disrupt the age structure and size distribution of fish populations. This alteration in population structure can have cascading effects on the ecological balance within predator-prey systems. Furthermore, they noted that size-selective fishing practices can impact the competitive interactions between predator and prey species, leading to changes in overall population abundance. However, it is important to mention that Hairston, Smith, and Slobodkin's research study focused primarily on simplified models and laboratory experiments, which may not fully capture the complexities of real-world predator-prey interactions in fisheries. Some researchers argued that the findings might not be directly applicable to natural ecosystems due to the inherent complexities and variability present in the field. Schaefer (1954) emphasizes that in the complex biological system of marine fish populations, numerous factors influence their dynamics. However, among these factors, only one—predation by humans—can be significantly controlled or modified through the regulation of fishing activities. Consequently, any meaningful management or control of fisheries must primarily focus on regulating the actions of fishermen. Fishery management is fundamentally driven by the objective of modifying or limiting fisherman activities to achieve desired changes in fish populations and catches, which are deemed more favourable than the outcomes resulting from unregulated fishing operations (Schaefer, 1954). The primary aim of that study was to investigate the economic principles related to the utilization of natural resources within the framework of the fishing industry. The author argued that many of the issues related to conservation, depletion, and overexploitation in fisheries were manifestations of the absence of economic rent in the marine resource. Economic rent refers to the regular income derived from a sustainable resource or the net revenue and profit generated. Fishery resources are commonly treated as common property, which creates various challenges for every collaborator involved in the fisheries industry.

Predator-Prey Mathematical Modelling of Fishery Resource Management

Clark and Munro (1975) illustrated that fisheries economics can be effectively analyzed using the theory of optimal control within the framework of capital theory, providing general and comprehensible results. The authors detected shortcomings in the Gordon-Schaefer model which is static and devised a dynamic autonomous linear model. They demonstrated that the static interpretation of the fisheries model in economics constitutes a specific instance within the dynamic autonomous model. To broaden its applicability, they expanded the model by introducing non-autonomous elements concerning price and cost parameters and incorporating non-linearity through a non-linear objective function based on the rate of harvest as the control variable. According to Clark and Munro (1975), an argument was made for analyzing fisheries economics, akin to other sectors of resource economics, through the lens of capital theory. They drew parallels between fish population or biomass and capital stock, emphasizing their potential to sustain consumption over time. Much like traditional capital, decisions related to present consumption impact stock levels and consequently influence future consumption possibilities. Therefore, effective resource management entails selecting an optimal consumption pattern over time, involving the determination of a stock level which is optimal as a function of time. In their examination of the models which are linearly autonomous, the authors pinpointed an optimal stationary steady-state guided by a generalized modified important rule. This principle suggests that the capital stock, encompassing fish populations, should be adjusted—either increased or declined—until the marginal utility aligns with the discount rate or interest. In Dubey et al. (2013), a resource model which is two-dimensional was proposed to investigate the crowding outcomes of a renewable resource and the human population utilizing it. The model aimed to analyze the consequences of crowding and the interplay between resource and population dynamics. The researchers employed the logistic equation to represent the growth of the population. The modelling involved representing the intrinsic growth rate and the carrying capacity of the population as functions that rise in tandem with the resource stock. This implies that the population's carrying capacity and growth rate are influenced by resource availability. To accommodate resource exploitation, the assumption was made that the resource stock undergoes harvesting at a rate proportionate to its size and the effort invested in the harvesting process. This reflected the idea that resource utilization depended on both the stock's abundance and the human effort devoted to harvesting it. The study analyzed and discussed the biological and bioeconomic equilibria of the system. These equilibria represented stable states where the resource and population dynamics reached a balance. The researchers also investigated the global stability characteristics of the non-negative equilibrium point, which provided in-depth knowledge of the behaviour of the system in the long run. Output feedback control techniques were utilized to evaluate the stability of this equilibrium.

The objective function employed in the study sought to maximize the discounted present value of future net revenues. This objective reflected the desire to optimize the utilization of the resource while considering the economic benefits over time. The researchers utilized Pontryagin's maximum principle, a mathematical optimization approach, to analyze the system dynamics and derive optimal control strategies. To validate their theoretical findings, numerical simulations were conducted. These simulations involved running the model under various scenarios and parameter settings to observe how the system responded, and to assess the implications of different control strategies. In a groundbreaking study by Hanson and Ryan (1998), an optimal harvesting model was investigated, accounting for both price and population dynamics . The researchers looked into how large price dynamics in a randomised Schaefer model affected the harvest method. They included background (Wiener) and leap (Poisson) components in prices and population size to add randomization. Interestingly, density-independent population fluctuations were assumed, meaning that relative changes were independent of population size. The model also took the inflationary implications of quadratic costs into account.

The optimal harvesting and discounted net revenue were found using stochastic dynamic programming with realistic bioeconomic data from the Pacific halibut fishery. Even in risky or catastrophic situations, the results showed that inflationary effects had a significant influence on the optimal net revenue. It was discovered that the optimal amounts of harvesting effort were less susceptible to the impacts of inflation.

Five common fishery harvesting strategies were introduced by Idels and Wang (2008) in a related study on fishery management strategies: rotational harvesting, constant harvesting, proportional harvesting, seasonal harvesting, and proportional threshold harvesting. Idel and Wang overcame restrictions by creating a new fishing effort technique based on the density of fish population dynamics in the fishery, in contrast to classic fisheries models that describe fishing effort only as a function of time. Using a canonical differential equation model (Schaefer model), they showed that the control parameter, which measures how much the size of the fish population affects fishing effort, affects both the equilibrium values and the rate at which the population approaches equilibrium. They investigated the effects of several harvesting techniques using both numerical simulations and qualitative studies.

Using a bioeconomic model of a single-species fishery, another study by Kar and Matsuda (2008) examined the effects of marine protected areas from both biological and economic standpoints. Using a two-dimensional model, the study examined the effects of harvesting and protected areas on resource populations. Using a logistic growth model with a carrying capacity proportional to the distribution area, the model took into account variables indicating the nature reserve and harvesting reserve sub-areas of the population's habitat region. The goal of the study was to optimize discounted net revenue from fish harvesting in the sub-area of the harvesting reserve while taking fishing effort limits and population dynamics into account. The researchers obtained and analyzed steady states, and local and global equilibrium, concluding that protected patches were effective in conserving resource populations, although complete extinction could not be ruled out in all cases. The study also discussed the economic and biological interpretations of the optimal equilibrium harvest policy.

The model employed in this current study incorporated linear controls to assess their optimality. Through simulations conducted on the developed model, intriguing and insightful results were obtained. Nevertheless, it is important to highlight that there is a lack of predator-prey fishery models explicitly designed to tackle non-selective harvesting practices in the context of the marine artisanal fishery in Ghana. The exploration of such models is limited, highlighting the need for more research in this area. Furthermore, the study aimed to contribute by introducing original ideas that provide a fresh perspective on understanding and analyzing fishery dynamics. By utilizing a linear optimal control, this research offers a fairly novel approach to investigating the dynamics of predator-prey fishery models. The simulations were conducted to shed light on application aspects of these systems, prompting further exploration and study in this vital field of research.

Chapter Summary

This chapter provides an extensive examination of previous research on predator-prey fishery models, focusing on optimal control strategies. It explores key studies, such as those by Hairston et al. (1960), Schaefer (1954), and Clark & Munro (1975), which analyze predator-prey interactions and the role of fishing activities in resource depletion. The chapter highlights the impact of selective and non-selective harvesting on population dynamics, emphasizing how overfishing can disrupt ecological balance. Additionally, the review identifies gaps in existing literature, particularly the need for models that explicitly address non-selective harvesting in marine artisanal fisheries, with a focus on Ghana.

CHAPTER THREE

RESEARCH METHODS

Introduction

In this chapter, we delve into the fundamental theory and mathematical principles that underpin the optimal control of fishery resources. We start with a comprehensive review of pertinent definitions, ensuring a solid understanding of the key terms and concepts involved. Building on this, we then proceed to formulate an optimal control model tailored specifically for the management of fishery resources. The formulation of this model is grounded in the traditional method of solving problems on optimization dynamics within the context calculus of variations. Additionally, we integrate the principles of dynamic optimization theory, which play a crucial role in formulating and solving dynamic control problems.

Definition and Concepts

This section provides an in-depth exploration of essential definitions and concepts relevant to our research. Consider a set of differential equations describing an autonomous system

$$\frac{dX}{dt} = F(X), \tag{3.1}$$

$X = (x_1, x_2, \ldots, x_n)^T$ and

$F(X) = \big(f_1(x_1, x_2, \ldots, x_n), \ldots, f_n(x_1, x_2, \ldots, x_n)^T\big)$and F does not explicitly depend on t. In our study, we consider problems governed by System (3.1) and subject to a specific initial condition: $X(t_0) = X_0$. In the course of our analysis, we presume the presence of a singular solution to these initial value problems, unless expressly specified otherwise. Furthermore, we work within the interval of $[t_0, \infty)$ for the domain of existence (Nohel and Pego, 1993).

Theorem 3.1. *(Allen, 2007) If* F *and* $\frac{\partial F}{\partial x_i}$ *for* $i = 1, 2, \ldots, n$ *are assumed to be continuous functions of* $x_1, x_2, \ldots, x_n$ *on* R^n*, then the initial value problem has a unique solution.* $X(t_0) = X_0$*, for any initial value* $X_0 \in R^n$*, where* $\frac{dX}{dt} = F(X)$*. The maximal interval of existence,* $[t_0, T)$*, may be finite,* $T < \infty$ *unless the solution is bounded, although a unique solution exists.*

corollary 3.2. *(Allen, 2007) Assume that for each* $i = 1, 2, ...n$*,* F *and* $\frac{\partial F}{\partial x_i}$ *are continuous functions of* $x_1, x_2, ..., x_n$ *on* R^n*. Additionally, if* $X_1(t)$ *and* $X_2(t)$ *are two solutions that satisfy the differential system* $\frac{dX}{dt} = F(X)$*, then* $X_2(t) = X_1(t - t_0)$*, given the initial conditions* $X_1(t_0) = X_0$ *and* $X_2(t_0) = X_0$*.*

Definition 3.1. *(Allen, 2007)* X^* *represents a fixed solution, a steady-state solution, a fixed point, or a critical point of the differential System (3.1), satisfying the condition* $F(X^*) = 0$*.*

Definition 3.2. *(Allen, 2007) When a steady-state solution* X^* *of System (3.1) satisfies the condition that* $\|X(t) - X^*\| < \epsilon$*, for all* $t \geq t_0$*, it is said to be locally stable if, for each* $\epsilon > 0$*, there exists* $\delta > 0$*. This characteristic applies to every solution* $X(t_0)$ *of System (3.1) with initial condition* $X(t_0) = X_0$*,* $\|X_0 - X^*\| < \delta$*. The steady-state solution is deemed unstable if it is not locally stable.*

Definition 3.3. *(Allen, 2007) If a steady-state solution* X^* *is locally stable and there exists a* $\gamma > 0$ *such that* $\|X_0 - X^*\| < \gamma$*, then it is considered locally asymptotically stable. This means that* $lim_{t \to \infty} \|(X(t) - X^*)\| = 0$*.*

Lemma 3.3 (Gronwall's differential form)**.** *(Rudin, 1953) Let* $u(t)$ *be non-negative, differentiable function satisfying the inequality:* $u'(t) \leq \alpha(t)u(t)$ *for some continuous function* $\alpha(t)$ *on an interval* $[t_0, T]$*.Then,* $u(t) \leq u(t_0)e^{\int_{t_0}^{t} \alpha(s)ds}$*, for* $t \in [t_0, T]$*.*

Lemma 3.4 (Gronwall's integral form)**.** *(Rudin, 1953) Let u(t) be a continuous,non-negative function satisfying* $u(t) \leq a + \int_{t_0}^{t} \alpha(s)u(s)ds$*,* *for* $t \geqslant t_0$*, Where*

$a \geqslant 0$ *and* $alpha(t)$ *is a continuous function.Then,* $u(t) \leq ae^{int_{t_0}^{t}\alpha(s)ds}$ *, for* $t \geqslant t_0$.

Logistic Growth Model

The model assumes that the per capita growth rate $b(x)$ will decrease as the population size increases:

$$b(x) = r\left(1 - \frac{x}{K}\right). \tag{3.2}$$

The logistic growth differential equation assumes that for population sizes greater than a certain carrying capacity K, the rate of growth is negative, while for population sizes smaller than K, the rate of growth is positive. The logistic growth differential equation is represented as

$$\frac{dy}{dx} = rx\left(1 - \frac{x}{K}\right), \tag{3.3}$$

where the carrying capacity is denoted by $K > 0$ and the intrinsic growth rate is denoted by $r > 0$. There exist two equilibrium values, $x = 0$ and $x = K$, derived from the equation. The differentiation equation that is solved by separating the variables is as follows:

$$x(t) = \frac{x_0 K}{x_0 + (K - x_0)e^{-rt}}. \tag{3.4}$$

The population size converges to the carrying capacity if $x(0) > 0$; hence, for non-negative beginning conditions, the steady-state $x = K$ is globally asymptotically stable. This is expressed as $lim_{t\to\infty}x(t) = K$. As a result of $f'(0) = r > 0$ and $f'(K) = -r$,

$$f'(x) = r\left(1 - \frac{2x}{K}\right). \tag{3.5}$$

Local Stability in First-order Systems in Two Variables

Consider the following two-dimensional system:

$$\left.\begin{aligned}\frac{dx}{dt} &= f(x,y) \\ \frac{dy}{dt} &= g(x,y)\end{aligned}\right\}. \tag{3.6}$$

The equilibrium solution of the system, (x^*, y^*) satisfies $f(x^*, y^*) = 0$ and $g(x^*, y^*) = 0$. The local stability analysis of a steady-state in a dynamic system is found by looking at the eigenvalues of the Jacobian matrix. To carry out this analysis, the functions of the system are denoted by f and g, which are extended using Taylor's formula centring on the steady-state (x^*, y^*), where $\mu = x - x^*$ and $v = y - y^*$.If f and g have continuous partial second-order in an open set containing (x^*, y^*),

$$\left.\begin{aligned}\frac{d\mu}{dt} &= f(x^*,y^*) + f_x(x^*,y^*)\mu + f_y(x^*,y^*)\nu + \tfrac{1}{2}f_{xx}(x^*,y^*)\mu^2 \\ &\quad + f_{xy}(x^*,y^*)\mu\nu + \tfrac{1}{2}f_{yy}(x^*,y^*)\nu^2 + \dots \\ \frac{d\nu}{dt} &= g(x^*,y^*) + g_x(x^*,y^*)\mu + g_y(x^*,y^*)\nu + \tfrac{1}{2}g_{xx}(x^*,y^*)\mu^2 \\ &\quad + g_{\nu y}(x^*,y^*)\mu\nu + \tfrac{1}{2}g_{yy}(x^*,y^*)\nu^2 + \dots \\ .\end{aligned}\right\}. \tag{3.7}$$

where $f(x^*, y^*) = 0$ and $g(x^*, y^*) = 0$. The linearized system concerning the steady-state (x^*, y^*) is then $\frac{dZ}{dt} = JZ$, where the Jacobian matrix J is evaluated at the steady-state and $Z = (\mu, \nu)^T$. Therefore,

$$J = \begin{bmatrix} f_x(x,y) & f_y(x,y) \\ g_x(x,y) & g_y(x,y) \end{bmatrix}.$$

If the eigenvalues have a negative real portion, the linearized system $\frac{dZ}{dt} = JZ$ converges to zero. Given that $\lambda^2 - Tr(J)\lambda + \det(J)$ is the characteristic polynomial of matrix J, the eigenvalues have a negative real portion if $Tr(J) < 0$ and $det(J) > 0$.Where $Tr(J)$ and $det(J)$ represents the trace and the determinant of the Jacobian matrix (J) respectively.

Theorem 3.5. *Assume that the first-order partial derivatives of f and g are continuous in an open set that contains the steady-state $(x*, y*)$ of the system (3.8). If $Tr(J) < 0$ and $det(J) > 0$, where the Jacobian matrix J is evaluated at steady-state, then the steady-state is locally asymptotically stable. If $Tr(J) > 0$ or $det(J) < 0$, the steady-state is unstable. There are three situations when the nonlinear system may behave differently from the linear system.*

- When $Det(J) = 0$, at least one zero eigenvalue is present. There is no isolation of stable states in the linearized system. If there is an isolated steady-state in the nonlinear system—which could be a node, spiral, or saddle—then this could also be the case.
- If $Tr(J) = 0$ and $det(J) > 0$ the eigenvalues are entirely imaginary. In a linear system, the steady state is a centre; in a nonlinear system, it can be a spiral or a centre.
- $Tr(J)^2 = 4\det(J)$. Consequently, in the nonlinear system, the steady state might be either a node or a spiral.

Definition 3.4. *(Allen, 2007)*

Let J represent the Jacobian matrix of $F(X)$ evaluated at X^*, $\frac{dX}{dt} = F(X)$, and X^* represent the system's steady-state. If the eigenvalues of the Jacobian matrix J have a nonzero real portion, the steady-state X^* is considered hyperbolic; otherwise, it is considered nonhyperbolic.

When $Tr(J) = 0$ and $det(J) > 0$, or when $det(J) = 0$, the local stability conditions are uncertain in the case of a nonhyperbolic steady-state. Matrix J either contains zero eigenvalues or a complex conjugate eigenvalues with zero real parts. The steady state is not hyperbolic in any scenario.

Phase Plane Analysis of the Autonomous Systems

Let us examine the two-dimensional autonomous systems represented by System (3.6). We presume that the functions f and g have continuous first-order partial derivatives, which guarantees the uniqueness and existence of solutions to initial value problems with the given scenario. In the $x - y$ plane, each point (x_0, y_0) corresponds to a single solution curve or trajectory. Since the direction of the flow in three dimensions $(t, x(t), y(t))$ is independent of time t, we may concentrate only on the solution curve $(x(t), y(t))$, on the $x - y$ phase plane.

An autonomous differential equation

$$\frac{dx}{dt} = f(x), \tag{3.8}$$

can be studied by phase line diagramming, which is derived from simplifying the direction fields in the $t - x$ phase plane. Similar to this, we can further extend the dynamics to a phase plane in the situation of two autonomous differential equations, where the direction field is represented in the $x - y$ plane. With t serving as the parameter, the solution trajectories $(x(t), y(t))$, which depict the evolution of the system's state variables x and y over time, can be written as parametric equations. For every point (x, y),

$$\frac{dy}{dx} = \frac{g(x, y)}{f(x, y)} \tag{3.9}$$

provides the gradient of the trajectory in the $x - y$ plane, and the tangent vector $(f(x, y), g(x, y))^T$ provides the path of the solution curve, except at the steady-state (x^*, y^*), where $f(x^*, y^*) = 0 = g(x^*, y^*)$. Each point (x, y) has a unique direction, indicated by the vector $(f(x, y), g(x, y))^T$. At singular points, the flow in the system comes to a standstill as these points represent fixed points where the derivatives become zero.

The arrangement of vectors showing the path of the system's evolution across the entire phase plane is referred to as the direction field. This direction field is a crucial graphical depiction in sketching a family of solution trajectories, commonly known as a phase plane image. Constructing the field of direction for two autonomous equations can be a laborious task unless it is generated with the aid of computers. An analysis of the flow path along the x and y zero isoclines or the nullclines yields a more effective way to determine the flow path in the system.

The zero x and y isoclines represent sets of points where either derivative becomes zero respectively concerning x or y. Along these nullclines, the direction of flow changes, providing critical information about the system's behaviour. Without generating the whole direction field, we can infer the nature of trajectories in the phase plane by looking at the route of vectors in the path of the x and y zero isoclines.

Definition 3.5. *(Allen, 2007) The set of all points in the $x - y$ plane satisfying $f(x, y) = 0$ is the x-zero isocline or nullcline for System (3.6). Similarly, the collection of all points satisfying $g(x, y) = 0$ is known as the y-zero isocline or nullcline.*

In general, a y isocline is a curve satisfying $g(x, y) = c_2 = constant$, while a x isocline is a curve on the $x - y$ phase plane satisfying $f(x, y) = c_1 = constant$. The unique curves where the constants c_1 and c_2 are zero are known as null isoclines. The tangent vector $(0, g)$ is parallel to the y-axis on a x-nullcline. The tangent vector $(f, 0)$ is parallel to the x-axis on the y-nullcline.We must confirm the flow path on the nullclines to choose the best control for the given situation. An equilibrium exists when the x and y nullclines overlap, indicating that the tangent vector's course remains unchanged.

Benedixson and Dulac Criteria

These essential two mathematical outcomes provide adequate conditions that rule out the feasibility of periodical solutions.

Definition 3.6. *(Allen, 2007) An example of a simply connected set without any holes is the entire phase plane R^2. A connected simple set $D \subset R^2$ is a set connected with the property that all simple closed curves in D may be continuously shrunk (with D) to a point.Geometrically, a set which is simply connected, is one without any holes.*

Theorem 3.6 (Bendixson criterion)**.** *Assume D is a simply connected open subset of R^2. If the expression $div(f, g) \equiv \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$ is not identically zero and does not change sign in D, then there are no periodic orbits of the autonomous system (3.8) in D.*

Theorem 3.7 (Dulac criterion)**.** *Suppose D is a simply connected open subset of R^2 and $B(x, y)$ is a real-valued function C' in D. If the interpretation, $div(Bf, Bg) = \frac{\partial(Bf)}{\partial x} + \frac{\partial(Bg)}{\partial y}$, is not identically zero and never changes sign D, then there are no periodical solutions of the autonomous system (3.6) in D.*

Dulac's criterion serves as a simplification of Bendixson's criterion and involves the use of a function B, known as the Dulac function. In the unique scenario in which $B(x, y) \equiv 1$. The criterion is further simplified. However, there exists no universal approach for finding a suitable Dulac function for a defined system, which poses a challenge in practical applications. When attempting to solve differential equations, finding an appropriate ”integrating factor,” represented by the Dulac function B, can be difficult and non-trivial (Koçak and Hale, 1991). Both Bendixson and Dulac criteria offer sufficient but not necessary conditions for the absence of periodical solutions in a dynamical system. This implies that if either of these criteria is satisfied, then periodic solutions do not exist. However, the non-satisfaction of these criteria does not definitively determine the presence or absence of periodic solutions.

Routh-Hurwitz Criteria

In assessing the local asymptotic stability of steady-state points in non-linear differential equation systems, the Routh-Hurwitz criteria are crucial tests that provide both necessary and sufficient conditions for guaranteeing all roots of the characteristic polynomials with real coefficients remain in the half left of the complex phase plane.

Theorem 3.8 (Routh-Hurwitz criteria)**.** *(Allen, 2007) Let $P(\lambda) = \lambda^n + a_1\lambda^{n-1} +, \ldots a_{(n-1)}\lambda + a_n$ be the polynomial function. The coefficients, a_i, $i = 1, 2, \ldots n$, are real constants formed with n Hurwitz matrices using the coefficients, a_i, of the characteristic polynomial function,*

$$H_1 = (a_i),$$
$$H_2 = \begin{bmatrix} a_1 & 1 \\ a_3 & a_2 \end{bmatrix},$$
$$H_3 = \begin{bmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{bmatrix},$$
$$H_n = \begin{bmatrix} a_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_n \end{bmatrix}.$$

$detH_j > 0$, $j = 1, 2, \ldots n$. Where $a_j = 0$, if $j > n$, then all of the roots of the polynomial $P(\lambda)$ are negative or have a negative real part if and only if the determinants of all the Routh-Hurwitz matrices are positive.

corollary 3.9. *Assume the coefficients of the characteristic polynomial are real.*

If all the roots for the characteristic polynomial $P(\lambda) = \lambda^n + a_1\lambda^{n-1} +, \ldots a_{(n-1)}\lambda + a_n$*, are negative or have a negative real part ,then the coefficients,* $a_i > 0$ *for* $i = 1, 2, \ldots n$.

Global Stability and Lyapunov Functions

The "direct method of Lyapunov" is a crucial technique in stability theory for differential equations. It involves constructing a Lyapunov function with specific properties to establish the stability or asymptotic stability of an equilibrium point within a defined region. This method is practically significant as it allows us to obtain estimates for the basin of attraction of the steady-states. The basin of attraction refers to a subset U in R^n that contains the equilibrium point and possesses the property that solutions starting within U approach the equilibrium point. By employing Lyapunov functions and the direct method, we can rigorously analyze the stability properties of dynamic systems and gain valuable insights into the characteristics of solutions closer to the steady states.

Definition 3.7. *(Allen, 2007) Let* U *be an open subset of* R^2 *containing the origin. A real-valued continuously function* V*,* V*:* $U \to R$*,* $[(x, y) \in U, V(x, y) \in R)]$ *is said to be positive definite on the set* U *if the following two conditions hold.* $V(0, 0) = 0$*,* $V(x, y) > 0$ *for all* $(x, y) \in \mu$ *with* $(x, y) \neq (0, 0)$*. The function is said to be negative definite if* V *is positive definite.*

Definition 3.8. *(Allen, 2007) A positive definite function* V *in an open neighbourhood of the origin is said to be a Lyapunov function for the autonomous differential System (3.6), if* $\frac{dV(x,y)}{dt} \leqslant 0$ *for all* $(x, y) \in U - (0, 0)$*. If* $\frac{dV(x,y)}{dt} < 0$ *for all* $(x, y) \in U - (0, 0)$*, the function* V *is called a strict Lyapunov function.*

Theorem 3.10 (Lyapunov's stability theorem)**.** *Let (0,0) be an equilibrium of the autonomous system (3.6) and let* V *be a positive definite, function in the neighbourhood* U *of origin.*

- *If* $\frac{dV(x,y)}{dt} \leqslant 0$, *for* $(x, y) \in U - (0, 0)$,*then* $(0, 0)$ *is stable.*
- *If* $\frac{dV(x,y)}{dt} < 0$, *for* $(x, y) \in U - (0, 0)$,*then* $(0, 0)$ *is asymptotically stable.*
- *If* $\frac{dV(x,y)}{dt} > 0$, *for* $(x, y) \in U - (0, 0)$,*then* $(0, 0)$ *is unstable.*

Optimal Control Theory

According to Chiang (1992), the classical variational calculus, a conventional dynamic method for optimization, relies on the differentiability of functions involved, limiting its applicability to interior solutions only. In contrast, a more modern approach, found in optimal control theory, addresses non-classical features like corner or boundary solutions. Optimal control theory focuses on one or more control variables as the key instruments for optimisation. The main goal of the theory of optimal control is to identify the ideal path for the control variable E, which leads to the associated optimal control path $x_\delta(t)$. This is in contrast to the calculus of variations, which seeks to evaluate the optimal time direction for a stable x variable. The optimal paths for $x_\delta(t)$ and $E_\delta(t)$ are typically obtained together in a single procedure. Control variables' central focus in the theory of optimal control significantly alters the orientation of the optimisation problem dynamics.

The problem of optimal control is considered with n state variables and m control variables with a payoff term ϕ. This formulation allows for a more versatile and comprehensive approach to handling many dynamic optimization scenarios.

$$\begin{aligned}
&\underset{E_1,\dots E_m}{\text{Maximize}} \quad \int_0^T f(t, x_1(t), \dots, x_n(t), E_1(t), \dots, E_m(t))\, dt + \phi(x_1(T), \dots x_n(T)),\\
&\text{Subject to} \quad x'_1(t) = g_1(t, x_1(t), \dots, x_n(t), E_1(t), \dots, E_n(t)),\\
&\qquad \vdots\\
&\qquad x'_n(t) = g_n(t, x_1(t), \dots, x_n(t), E_1(t), \dots, E_m(t)),\\
&\qquad x_1(0) = x_{1_0}, \dots, x_n(0) = x_{n_0},
\end{aligned} \tag{3.10}$$

Consequently, in every variable, the functions f and g are continuously differentiable. The relative magnitude of m and n is unrestricted. It is all suitable for $m < n$, $m = n$, and $m > n$.To have a more compact statement of the posed optimal problem, the following is defined using a vector notation:

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, E = \begin{bmatrix} E_1 \\ \vdots \\ E_m \end{bmatrix}, g = \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}.$$

Subsequently, the optimal control issue can be stated as follows:

$$\begin{aligned}
&\text{Maximize} \quad Z(E) = \int_0^T f(t, x(t), E(t))\, dt + \phi(x(T)),\\
&\text{Subject to} \quad x'(t) = g(t, x(t), E(t)),\\
&\qquad x(0) = x_0.
\end{aligned} \tag{3.11}$$

Lebesgue integration and continuous piecewise of the control variable $E \in R^m$ is assumed, as is the state variable $x \in R^n$, where 0 denotes the starting time and T denotes the terminal time. There are two possible time frames: finite and infinite $(T \to \infty)$. Additionally, it is assumed that, for $0 \leqslant t \leqslant T$, $E \in U_t$. Acceptable controls are those in which U_t, the control set, is stated and may vary over time t. Finding the optimal control, $E_\delta(t)$, and the corresponding optimal direction, $x_\delta(t)$, that maximizes the objective functional while abiding by all constraints, including the state system x' with its initial condition x_0, is the main goal of the control problem.

To address this optimization problem, a basic method is to find solutions to a set of necessary conditions that both the interrelating state and the optimal control need to meet. Distinguishing between necessary and sufficient conditions of solution sets is vital.

Necassary Conditions: If $E_\delta(t)$, $x_\delta(x)$ are optimal, then the following conditions hold

Sufficient Condition: If $E_\delta(t)$, $x_\delta(t)$ satisfy the following conditions, then $E_\delta(t)$,$x_\delta(t)$ are optimal

Usually, the necessary conditions are met by using an optimization technique known as Pontryagin's maximum principle, which was put forth by Pontryagin et al. (1962).

Theorem 3.11 (Pontryagin's Maximum Principle)**.** *(Allen, 2007) If $E_\delta(t)$ and $x_\delta(t)$ are optimal for problem (3.12),then there exists a piecewise differential adjoint function $\lambda \in \mathbb{R}^n$ such that*

$H\left(t, x_\delta(t), E\delta(t), \lambda(t)\right) \geq H\left(t, x_\delta(t), E(t), \lambda(t)\right)$*,*

for all controls E at each time t,where the Hamiltonian

$H = f(t, x(t), E(t)) + \lambda^T(t) g(t, x(t)E(t))$*,*

$$\textit{and} \quad \lambda'(t) = -\left[\frac{\partial H(t, x_\delta(t), E_\delta(t), \lambda(t))}{\partial x(t)}\right]^T,$$
$$\lambda(T) = \left[\frac{\partial \phi(x_\delta(T))}{\partial x(T)}\right]^T.$$

The optimal state system dynamics is given as

$$x'(t) = \left[\frac{\partial H(t, x_\delta(t), E_\delta(t), \lambda(t))}{\partial \lambda(t)}\right]^T. \tag{3.12}$$

where $x(0) = x_0$. Furthermore, the optimality condition becomes, if the Hamiltonian is differentiable in $E(t)$ and $E_\delta(t)$ is in the interior.

$$\left[\frac{\partial H(t, x_\delta(t), E_\delta(t), \lambda(t))}{\partial E(t)}\right]^T = 0 \quad (\text{zero row vector}, 0 \in R^n).$$

In optimal control theory, the transversality condition is an essential component. With the given problem, the transversality condition reduces to $\lambda(T) = 0$ when the payoff term $\phi(x(T)) = 0$. (Chiang, 1992).

Bounded controls of the system

The development of an alternate condition is required to handle optimal control problems with control bounds.

$$\begin{aligned} \max \quad & \int_0^T f(t, x(t), E(t))\, dt, \\ \text{Subject to:} \quad & x'(t) = g(t, x(t), E(t)), \\ & x(0) = x_0, \quad a \leq E \leq b. \end{aligned} \tag{3.13}$$

In this case, $b > a$ with fixed, real constants a and b. These conditions are necessary to deal with this particular situation. Building with the Hamiltonian

$$H(t, x, E, \lambda) = f(t, x, E) + \lambda(t) g(t, x, E). \tag{3.14}$$

The necessary condition for both λ and x_δ remain unchanged, specifically:

$$\begin{aligned} x'(t) &= \frac{\partial H}{\partial \lambda}, \quad x(0) = x_0, \\ \lambda'(t) &= -\frac{\partial H}{\partial x}, \quad \lambda(T) = 0. \end{aligned} \tag{3.15}$$

Nevertheless, if a non-linear control is used, the optimality condition

$$\frac{\partial H}{\partial E} = f_E(t, x, E) + \lambda(t) g_E(t, x, E). \tag{3.16}$$

gives to the characterization of the optimal control as

$$E_\delta = \begin{cases} a, & \text{if } \frac{\partial H}{\partial E} < 0, \\ [a,b], & \text{if } \frac{\partial H}{\partial E} = 0, \\ b, & \text{if } \frac{\partial H}{\partial E} > 0. \end{cases} \tag{3.17}$$

Another characterization of the optimal control is

$$E_\delta = \begin{cases} a, & \text{if } \frac{\partial H}{\partial E} \leq 0, \\ (a,b), & \text{if } \frac{\partial H}{\partial E} = 0, \\ b, & \text{if } \frac{\partial H}{\partial E} \geq 0. \end{cases} \tag{3.18}$$

For a more comprehensive understanding, refer to the work of Kamien and Schwartz (1991).

Linear controls

Optimal control problems exhibiting linear dependence on the control variable commonly manifest in two distinct forms: bang-bang controls and singular controls. Now, let's delve into the specifics of the optimal control problem:

$$\begin{aligned} \underset{E}{\text{Maximize}} \quad & \int_0^T f_1(t,x) + E(t) f_2(t,x) dt, \\ \text{Subject to} \quad & x'(t) = g_1(t,x) + E(t) g_2(t,x), \\ & x(0) = x_0, \\ & a \leqslant E \leqslant b. \end{aligned} \tag{3.19}$$

The integrand function f and the right-hand side of the state system in problem (3.20) are both linear functions of the control variable E. Consequently, the Hamiltonian is also linear in E and can be expressed as

$$H = [f_1(t,x) + \lambda(t) g_1(t,x)] + E(t)[f_2(t,x) + \lambda(t) g_2(t,x)].$$

As expected, the necessary condition $\lambda'(t) = -\frac{\partial H}{\partial x}$ is applicable. However,the optimality condition

$\frac{\partial H}{\partial E} = f_2(t, x) + \lambda(t)g_2(t, x),$

gives no information on the control.To find the characterization for the optimal control E,a switching function is defined

$\psi(t) = f_2(t, x) + \lambda(t)g_2(t, x).$

$$\begin{cases} E = a & \text{if } \psi(t) < 0, \\ a < E < b & \text{if } \psi(t) = 0, \\ E = b & \text{if } \psi(t) > 0. \end{cases} \tag{3.20}$$

Bang-bang controls

If the condition $\psi(t) = 0$ cannot be sustained over a time interval but instead occurs only at finitely many points within the given time interval, then the control is termed bang-bang. In such instances, the control is a piecewise constant function, switching between upper and lower bounds. These switches coincide with the places where ψ changes signs (so that $\psi = 0$), hence earning the name switching function. The specific points where this occurs are known as switching times. Consequently, the expressions in condition (3.20) simplifies to

$$E_\delta(t) = \begin{cases} a & \text{if } \psi(t) < 0, \\ b & \text{if } \psi(t) > 0. \end{cases} \tag{3.21}$$

Numerically solving a bang-bang problem often involves employing the forward-backward sweep method. However, prior to employing this method, it is imperative to establish analytically the bang-bang problem, verifying that $\psi = 0$ over an interval is unattainable.

Singular controls

If $\psi(t)$ is identically zero in some interval of time, $I \subseteq [0, T]$ ($\psi(t) = 0$ for all $t \in I$), we say E_δ is singular on that interval. A characterization of E_δ in this interval must be attained using other information. The endpoints of this interval are also referred to as switching times. It is important to emphasize that if the problem involves singular control or a blend of singular and bang-bang controls and the interior solution, the second line of expressions in condition (3.21) cannot be dismissed.

Methods

In this study, the functions of growth employed considering the state dynamics of the resources were the logistic and exponential functions. The objective function aims to maximize the discounted future value of net revenues or profits. The analysis involved both direct computation and the utilization of computer resources. Several software tools were utilized in the analysis, including MAPLE and MATLAB. The phase portraits were addressed using the pplane8 interface software developed by John Polking. Simulations were conducted employing the forward-backwards sweep method in MATLAB to solve forward in time the state equations, and the adjoint equations backwards in time (Lenhart and Workman, 2007). The updated control at each iteration employs the formula derived for the optimality of control, and the methodology could be extended to cover various scenarios. The codes used in the simulations were initially developed by Lenhart and Workman (2007), which were adapted to fit the specific model examined in this research. The TRAPZ function in MATLAB was employed to compute the performance criterion, which was the value of the objective function.

Chapter Summary

This chapter details the mathematical and computational techniques used to analyze optimal control in fisheries management. It introduces fundamental definitions, including differential equations governing population dynamics and stability conditions. The research employs logistic and exponential growth models to describe fish population changes and applies dynamic optimization theory to derive optimal harvesting strategies. The chapter also outlines numerical methods, including the forward-backward sweep method in MATLAB, which was used to simulate population trends and economic outcomes. Key mathematical tools such as Lyapunov functions, the Routh-Hurwitz criterion, and Pontryagin's Maximum Principle are employed to ensure theoretical rigor in model validation.

CHAPTER FOUR

RESULTS AND DISCUSSION

Introduction

This chapter presents the findings from the mathematical analysis and numerical simulations of the predator-prey fishery model with non-selective harvesting. It examines equilibrium points, stability conditions, and optimal harvesting strategies to assess their impact on predator and prey populations. Numerical simulations based on Ghanaian marine artisanal fishery data validate the theoretical results. The discussion highlights the ecological and economic implications of different harvesting efforts, emphasizing strategies for sustainable fishery management.

Model Formulation and Analysis

We give the model's definition, formulation, and analysis in this section. Imagine a predator-prey model where $x(t)$ and $y(t)$ represent the prey and predator population densities at any given time t.The model has two autonomous ordinary differential equations describing how the population densities of the two species would vary with time.

Assumptions of the model

- The only populations in the ecosystem that are thought to have an impact on the system are the populations of predators and prey.
- The prey population experiences logistic growth, implying that its rate of growth is proportionate to the population size and constrained by the carrying capacity of the ecosystem.
- It is hypothesised that the prey population is the only source of food for the predator population, and that the rate of growth of the predator popu-

lation is proportionate to the combined densities of the prey and predator populations.

- It is believed that the rate of predation is proportionate to the numbers of prey and predators.
- It is assumed that the mortality rate of the predator population is proportional to the size of the predator population.
- The model assumes that the total population sizes in the specified ecosystem stay constant and that there is no migration, emigration, or immigration of the predator or prey populations.
- Positive initial populations are assumed for both the predator and prey.

Model formulation

The following is a modified Lotka-Volterra model:

$$\begin{aligned} \frac{dx}{dt} &= rx\left(1 - \frac{x}{K}\right) - mxy - q_1Ex, \\ \frac{dy}{dt} &= nxy - sy - q_2Ey, \end{aligned} \tag{4.1}$$

under the initial assumptions that $y(0) = y_0 > 0$ and $x(0) = x_0 > 0$. Positive constants make up the parameters r, K, m, n, s and the combined harvesting effort is given by E.

Table 1 provides a thorough explanation of the model's parameters as well as that of the ensuing optimal control problem.

Model Dynamics

The dynamics of the system which consist of positivity, uniform boundedness, equilibrium points along with their local and global stability, and uniform persistence are examined.

Table 1: Description of Parameters

Parameter	Description
r	Rate of intrinsic growth of the population of prey
m	Rate at which the predator predates the prey
K	The prey population's carrying capacity
n	Rate of converting prey biomass into predator births
s	Natural rate of mortality of the predator
δ	Discount rate
q_1, q_2	Coefficients of catchability of prey and predator populations respectively

Positivity and boundedness of solutions

We shall establish a compact, positively invariant region where the model is well-posed ecologically and mathematically. The theorem below ensures that every solution of the system admits non-negative values given positive initial conditions.

Theorem 4.1. *Starting from the interior of the quadrant, the system's solutions $x(t)$ and $y(t)$ will stay in the first quadrant of the xy plane.*

proof 4.1.1. *Given $x(0) = x_0 > 0$ and $y(0) = y_0 > 0$, the system's first equation becomes*

$$\frac{dx}{x} = \left[r\left(1 - \frac{x}{K}\right) - my - q_1E\right] dt,$$

which when integrated yields the following:

$$x(t) = x_0 \exp\left\{\int_0^t \left[r\left(1 - \frac{x(u)}{K}\right) - my(u) - q_1E\right] du\right\}.$$

Similarly, the system's second equation provides

$$\frac{dy}{y} = [nx - s - q_2E]\, dt,$$

which produces the following upon integration:

$$y(t) = y_0 \exp\left\{\int_0^t [nx(u) - s - q_2E]\, du\right\}.$$

Therefore, all the system's solutions are limited to the first quadrant and hence non-negative.

The subsequent theorem demonstrates the uniform boundedness of the solutions to the system:

Theorem 4.2. *For* $0 < \eta \leq s + q_2E$ *and* $\mu = \frac{K}{4r}(r + \eta - q_1E)^2$*, the set,*

$\Omega = \left\{(x, y) \in \mathbb{R}^2_+ : x \leq K, x \leq \frac{\mu}{\eta}\right\}$

is positively invariant for every solution initiated in the first quadrant's interior.

proof 4.2.1. *The differential equations comparison theorem is applied to the first equation of the system,*

$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) - mxy - q_1Ex \leq rx\left(1 - \frac{x}{K}\right)$. *Thus,*

$\lim_{t\to\infty} \sup x(t) \leq K$.

Now we define $N(t) = x(t) + \beta y(t)$*, where* $\beta = \frac{m}{n}$*. Then*

$$\begin{aligned}
\frac{dN}{dt} + \eta N &= \frac{dx}{dt} + \beta\frac{dy}{dt} + \eta x + \beta\eta y, \\
&= rx\left(1 - \frac{x}{K}\right) - q_1Ex + \eta x - \beta sy - \beta q_2Ey + \beta\eta y, \\
&= (r + \eta - q_1E)x - \frac{rx^2}{K} - \beta(s + q_2E - \eta)y, \\
&\leq \frac{K}{4r}(r + \eta - q_1E)^2 - \frac{r}{K}\left[x - \frac{K}{2r}(r + \eta - q_1E)\right]^2 \\
&\leq \frac{K}{r}(r + \eta - q_1E)^2 = \mu.
\end{aligned}$$

By Gronwall's lemma, then

$$0 \leq N(t) \leq N(0)e^{-\eta t}$$

Therefore,

$$\lim_{t\to\infty} \sup N(t) \leq \frac{\mu}{\eta}.$$

Equilibrium points and stability analysis

The system's equilibrium points are examined, and an analysis is conducted to determine the stability, whether it is local or global, around the points of equilibrium.

Finding the system's three equilibrium positions requires solving $\frac{dx}{dt} = \frac{dy}{dt} = 0$, and are as follows:

(i) The trivial equilibrium point, denoting the mutual extinction of species, is expressed as $P_0 = (0, 0)$.

(ii) The axial equilibrium point, representing a predator-free scenario, is indicated as $P_1 = (\hat{x}, 0)$, where $\hat{x} = \frac{K}{r}(r - q_1 E)$ exists if $E < \frac{r}{q_1}$.

(iii) The unique interior equilibrium point depicting the coexistence of species is denoted by $P_2 = (x^*, y^*)$,

where

$$r\left(1 - \frac{x^*}{K}\right) - my^* - q_1 E = 0,$$

$$nx^* - s - q_2 E = 0.$$

Therefore, solving simultaneously, the unique positive solution is

$$x^* = \frac{s + q_2 E}{n} \quad \text{and} \quad y^* = \frac{Kn(r - q_1 E) - r(s + q_2 E)}{Kmn},$$

exists if $E < \frac{r(Kn-s)}{q_1 Kn + q_2 r}$, with $s < Kn$.

Local stability of the system

Eigenvalue analysis in the phase plane is used to determine the system's local stability at its equilibrium points. The definition of the Jacobian matrix of the system surrounding any given point $P = (x, y)$ is as follows:

$$J = \begin{bmatrix} r - \frac{2rx}{K} - my - q_1E & -mx \\ ny & nx - s - q_2E \end{bmatrix}.$$

Theorem 4.3. *The mutual extinction equilibrium point $P_0 = (0, 0)$ always exists and can be classified as:*

(i) Locally asymptotically stable node when $E > \frac{r}{q_1}$.

(ii) Unstable saddle when $E < \frac{r}{q_1}$.

proof 4.3.1. *The result of finding the matrix J at $P_0 = (0, 0)$ is*

$$J_0 = \begin{bmatrix} r - q_1E & 0 \\ 0 & -s - q_2E \end{bmatrix}.$$

As the matrix is diagonal, The elements along the major diagonal are the corresponding eigenvalues: $\lambda_1 = r - q_1E$ and $\lambda_2 = -s - q_2E$. Therefore, Condition (i) ensures that λ_1 and λ_2 are both negative, thereby making P_0 locally asymptotically stable. On the other hand, if Condition (ii) holds, the eigenvalues possess opposite signs, ensuring that the trivial equilibrium point functions as a saddle.

Theorem 4.4. *The predator-free equilibrium point $P_1 = (\hat{x}, 0)$ can be categorized as:*

(i) Locally asymptotically stable node when $\frac{r(Kn-s)}{q_1Kn+q_2r} < E < \frac{r}{q_1}$.

(ii) Unstable saddle when $E < \min\{\frac{r(Kn-s)}{q_1Kn+q_2r}, \frac{r}{q_1}\}$.

proof 4.4.1. *At the point P_1, the Jacobian becomes*

$$J_1 = \begin{bmatrix} q_1E - r & \frac{Km}{r}(q_1E - r) \\ 0 & -\frac{Kn(r-q_1E)-r(s+q_2E)}{r}. \end{bmatrix}.$$

The matrix is upper triangular, thus: $\lambda_1 = q_1 E - r$ *and* $\lambda_2 = -\frac{Kn(r-q_1E)-r(s+q_2E)}{r}$. *Therefore, the first condition makes both eigenvalues negative, which translates to local asymptotic stability. Concerning the second condition, since the eigenvalues have opposite signs,* P_1 *is a saddle.*

Theorem 4.5. *The coexistence equilibrium point, if it exists, is locally asymptotically stable.*

proof 4.5.1. *By virtue of the fact that the point* $P_2 = (x^*, y^*)$ *is the coexistence equilibrium point, the Jacobian simplifies to*

$$J_2 = \begin{bmatrix} -\frac{r(q_2E+s)}{Kn} & -\frac{m(q_2E+s)}{n} \\ \frac{Kn(r-q_1E)-r(s+q_2E)}{Km} & 0 \end{bmatrix}.$$

Then the trace tr(J_2) = -$\frac{r(q_2E+s)}{Kn} < 0$ *and the determinant det*(J_2) = $\frac{(q2E+s)(Kn(r-q_1E)-r(s+q_2E))}{nK}$ 0 *if* P_2 *exists. Since* J_2 *has its trace to be negative and the determinant to be positive,* P_2 *locally asymptotically stable.*

Global stability

Through the utilization of the Bendixson-Dulac criterion and an appropriate Lyapunov function, the system's equilibrium points are proven to be asymptotically globally stable. For any equilibrium point $P(x_1, y_1)$, introduce a positive definite function $V(x, y)$ such that $V(x, y) > 0$ for every $(x, y) \neq (x_1, y_1)$ and $V(x, y) = 0$ if and only if $(x, y) = (x_1, y_1)$.

Theorem 4.6. *When it is locally stable, the mutual extinction equilibrium point* $P_0 = (0, 0)$ *is asymptotically globally stable.*

proof 4.6.1. *Examine a positively definite function regarding* $P_0 = (0, 0)$*:*

$$V_0(x, y) = x + \beta_0 y, \quad \beta_0 = \frac{m}{n}.$$

$$\frac{dV_0}{dt} = \frac{dx}{dt} + \beta_0 \frac{dy}{dt},$$

$$= rx\left(1 - \frac{x}{K}\right) - mxy - q_1 Ex + mxy - \beta_0 sy - \beta_0 q_2 Ey,$$

$$= rx\left(1 - \frac{x}{K}\right) - q_1 Ex - \beta_0 sy - \beta_0 q_2 Ey,$$

$$= (r - q_1 E)\, x - \frac{rx^2}{K} - \beta_0 sy - \beta_0 q_1 Ey.$$

.

The derivative is negative definite under the conditions of local stability, $E > \frac{r}{q_1}$. Therefore, by Lyapunov's theorem on stability, P_0 is asymptotically globally stable.

Theorem 4.7. *Whenever $\hat{x} < \frac{s+q_2E}{n}$, the predator-free equilibrium point $P_1 = (\hat{x}, 0)$ is globally asymptotically stable.*

proof 4.7.1. *Let a positive definite function about $P_1 = (\hat{x}, 0)$:*

$$V_1(x, y) = \left[x - \hat{x} - \hat{x}\ln\frac{x}{\hat{x}}\right] + \beta_2 y,$$

where $\beta_2 = \frac{m}{n}$.

$$\frac{dV_1}{dt} = \left[1 - \frac{\hat{x}}{x}\right]\frac{dx}{dt} + \beta_2 \frac{dy}{dt},$$

$$= (x - \hat{x})\left[r\left(1 - \frac{x}{K}\right) - my - q_1 E\right] + \beta_2 nxy - \beta_2 sy - \beta_2 q_2 Ey,$$

$$= (x - \hat{x})\left[r\left(1 - \frac{x}{K}\right) - my - r\left(1 - \frac{\hat{x}}{K}\right)\right] + \beta_2 nxy - \beta_2 sy - \beta_2 q_2 Ey,$$

$$= -\frac{r}{K}(x - \hat{x})^2 - mxy + m\hat{x}y + \beta_2 nxy - \beta_2 sy - \beta_2 q_2 Ey,$$

$$= -\frac{r}{K}(x - \hat{x})^2 + (m\hat{x} - \beta_2 s - \beta_2 q_2 E)y,$$

$$\leq -\frac{r}{K}(x - \hat{x})^2.$$

The derivative is negative definite. Therefore, according to Lyapunov's stability theorem, P_1 is globally asymptotically stable.

Theorem 4.8. *The coexistence equilibrium point $P_2 = (x^*, y^*)$ is globally asymptotically stable whenever it exists.*

proof 4.8.1. *Let*

$$f(x, y) = rx\left(1 - \frac{x}{K}\right) - mxy - q_1Ex,$$

$$g(x, y) = nxy - sy - q_2Ey,$$

$$\phi(x, y) = \frac{1}{xy} > 0,$$

$$\Delta(x, y) = \frac{\partial}{\partial x}(\phi f) + \frac{\partial}{\partial y}(\phi g),$$

$$= \frac{\partial}{\partial x}\left[\frac{r}{y}\left(1 - \frac{x}{K}\right) - m - \frac{q_1E}{y}\right] + \frac{\partial}{\partial y}\left[n - \frac{s}{x} - \frac{q_2E}{x}\right],$$

$$= -\frac{r}{yK}.$$

According to the Bendixson-Dulac criterion, the system lacks limit cycles since $\Delta(x, y)$ does not change sign within the first quadrant. Hence P_2 is globally asymptotically stable.

Numerical Results

This section will present the numerical results obtained from our mathematical model for fishery management.The model is designed to analyze the dynamics of fish population growth,harvesting strategies, and economic sustainability. Numerical simulations will be conducted to explore:

(i) Popuation dynamics – Evaluating how different harvesting rates affect fish population sustainability over time.

(ii) Optimal harvesting strategy – Identifying the best fishing effort that maximizes long-term yield while preventing over-exploitation.

(iii) Economic optimization – Examining the impact of discounting future profits on harvesting decision.

Parameter values for the model

The biological and economic parameter values shown in Table 2 are used in the study.

Table 2: Biological and Economic Parameter Values

Parameter	Value	Units	Source
r	1.42	year^{-1}	(Ibrahim, 2021)
m	3.8×10^{-5}	(days $\times$ Tonnes)$^{-1}$	(Demir, 2023)
K	$1,000,000$	Tonnes	(Ibrahim, 2021)
n	3.4×10^{-7}	(days $\times$ Tonnes)$^{-1}$	(Demir, 2023)
s	0.001	year^{-1}	Assumed
δ	0.10	year^{-1}	Assumed
q_1	1.8×10^{-6}	Trip^{-1}year^{-1}	(Ibrahim, 2021)
q_2	1.5×10^{-6}	Trip^{-1}year^{-1}	Assumed
p_1	1000	\$Tonnes^{-1}	Assumed
p_2	2000	\$Tonnes^{-1}	Assumed
c	400	\$Trip^{-1}year^{-1}	Assumed

The state equations slope fields and solution curves are graphed to emphasize the equilibrium and stability characteristics of the model. To facilitate analysis, the parameter values are scaled to a thousand (or thousandths) units. Since the data relate to the marine artisanal fishery in Ghana, the prey can be considered as the round sardinella and the predator as the tuna (comprising the bigeye, yellowfin and skipjack).

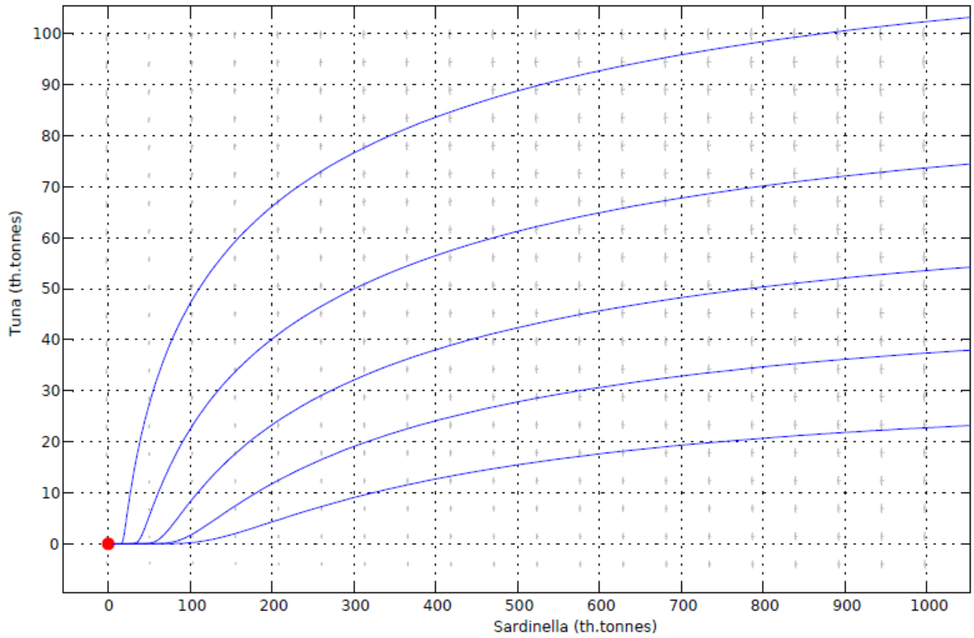

Figure 1: Phase-plane Portrait for the Mutual Extinction Equilibrium Point $P_0 = (0, 0)$ with $E = 800,000$

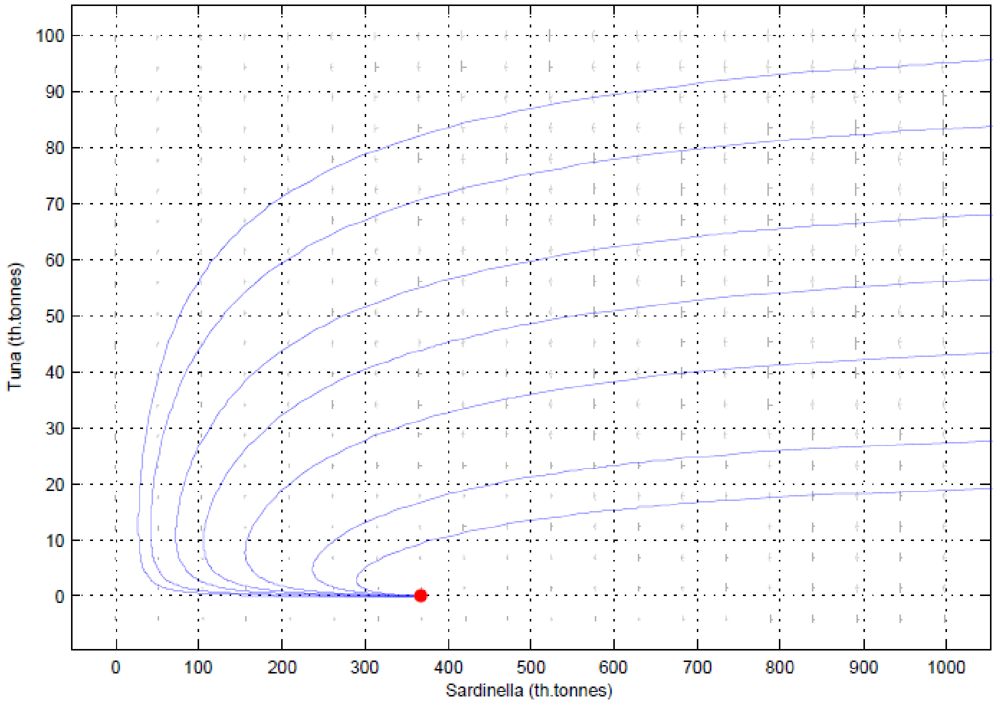

Figure 2: Phase-plane Portrait for the Predator-free Equilibrium Point $P_1 = (366200, 0)$ with $E = 500,000$

The equilibrium point P_0 is globally asymptotically stable when the harvesting effort exceeds the biotechnical productivity of only sardinella species (refer to Figure 1).The equilibrium point p_1 is globally asymptotically stable when the harvesting effort is greater than the biotechnical productivity of the sardinella (refer to Figure 2). Additionally, the coexistence equilibrium point P_2 is globally asymptotically stable whenever it exists (refer to Figure 3).

Persistence of the system

In ecology, persistence is the long term survival of a species given any initial population density.

Theorem 4.9. *The system is uniformly persistent provided the following sufficient condition holds:* $E < \min\{\frac{r}{q_1}, \frac{r(Kn-s)}{q_1Kn+q_2r}\}$.

proof 4.9.1. *Consider the average Lyapunov function* $\psi(x, y) = x^a y^b$, *here* a

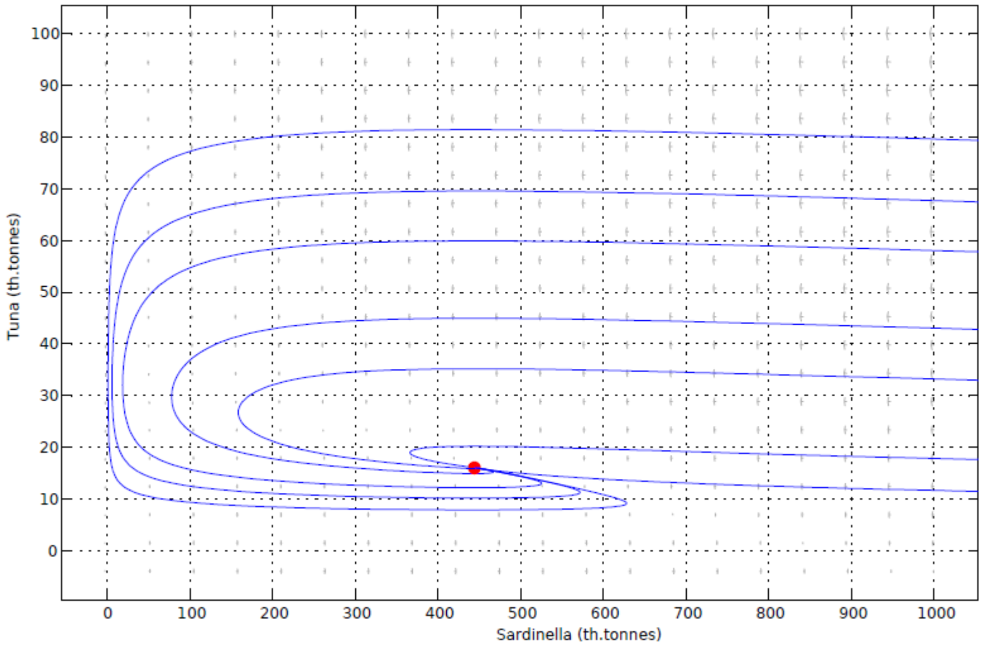

Figure 3: Phase-plane Portrait for the Coexistence Equilibrium Point $P_2 = (444120, 16040)$ with $E = 100,000$

and b *are arbitrary constants which are positive. It is clear that* $\psi(x, y) > 0$ *for every* (x, y) *in the interior of the first quadrant and* $\psi(x, y) = 0$ *on the boundary of the quadrant. Now, let* $\psi'(x, y) = \frac{d}{dt}\psi(x, y)$*. Then,*

$$\Lambda(x, y) = \frac{\psi'(x, y)}{\psi(x, y)} = \frac{a}{x}\frac{dx}{dt} + \frac{b}{y}\frac{dy}{dt},$$

$$= a\left[r\left(1 - \frac{x}{K}\right) - my - q_1E\right] + b\left[nx - s - q_2E\right],$$

$$\Lambda(E_0) = a\left[r - q_1E\right] + b\left[-s - q_2E\right],$$

$$\Lambda(E_1) = \frac{b}{r}\left[nrK - rs - q_1nEK - q_2rE\right].$$

Clearly, $\Lambda(E_0) > 0$ *whenever* $E < \frac{r}{q_1}$ *with a large value of the constant* a*. Similarly* $E < \frac{r(Kn-s)}{q_1Kn+q_2r}$ *ensures that* $\Lambda(E_1) > 0$.

Bionomic Equilibrium

The biological model's integration of economic parameters, as indicated in the system, results in the development of the bioeconomic model. Bioeconomic equilibrium is reached when the total sustainable revenue from harvests is equal to the total effort cost incurred during the harvesting process. In other words, this indicates an instance in which the total economic rent from the harvesting activity is completely dissipated. The sustainable net revenue can be written as follows, assuming that p_1 represents the price per unit harvest of the sardinella species, p_2 represents the price per unit harvest of the tuna species, and c represents the harvesting cost per unit effort:

$\pi(x, y, E) = (p_1 q_1 x + p_2 q_2 y - c)E,$

provided that $c < p_1 q_1 x + p_2 q_2 y$.

The bionomic equilibrium can be determined by finding a solution to

$\frac{dx}{dt} = \frac{dy}{dt} = \pi = 0.$

This implies solving the homogeneous system of equations:

$$r\left(1 - \frac{x}{K}\right) - my - q_1 E = 0. \tag{4.2}$$

$$nx - s - q_2 E = 0. \tag{4.3}$$

$$p_1 q_1 x + p_2 q_2 y - c = 0. \tag{4.4}$$

From Equation (4.2)

$$E = \frac{1}{q_1}\left(r\left(1 - \frac{x}{K}\right) - my\right). \tag{4.5}$$

And from Equation (4.3)

$$E = \frac{1}{q_2}(nx - s), \tag{4.6}$$

Equating (4.5) to (4.6) and simplifying,

$$\frac{1}{q_1}\left(r\left(1 - \frac{x}{K}\right) - my\right) - \frac{1}{q_2}(nx - s) = 0. \tag{4.7}$$

From Equation (4.4)

$$y = \frac{c - p_1 q_1 x}{p_2 q_2}. \tag{4.8}$$

Substituting (4.8) into (4.7) gives

$$Ax^2 + Bx + C = 0, \tag{4.9}$$

where,

$A = q_1 p_1 p_2 (q_2 r + q_1 K n)$,

$B = q_1 p_2 (q_1 p_1 K s - q_2 c r - K n c - q_2 p_1 K r)$, and

$C = q_1 q_2 p_2 K c r - q_1 p_2 K c s - K n c.$

Equation (4.9) offers a positive unique solution x_∞ provided that $C < 0$. Thus, the unique bionomic equilibrium point is given by

$$p_\infty = (x_\infty, y_\infty, E_\infty), \tag{4.10}$$

where,

$$x_\infty = \frac{-B + \sqrt{B^2 - 4AC}}{2A}. \tag{4.11}$$

$$y_\infty = \frac{c - p_1 q_1 x_\infty}{p_2 q_2}. \tag{4.12}$$

$$E_\infty = \frac{1}{q_1}\left[r\left(1 - \frac{x_\infty}{K}\right) - m y_\infty\right], \tag{4.13}$$

or

$$E_\infty = \frac{1}{q_2}[nx - s]. \tag{4.14}$$

It is crucial to observe that the bioeconomic harvesting effort E_∞ declines as the rate of predation m increase, since $\frac{\partial E_\infty}{\partial m} = -\frac{y_\infty}{q_1} < 0.$

Linear Optimal Control

In this section, we examine the best control strategy for the system, subject to certain constraints. The objective is to maximize a profit function that depends on the system's control and state. Since both species are prone to harvesting, the issue of the optimal control problem is to maximize the current value of the net revenue or profit.

Linear optimal control problem

The goal is to maximize the economic benefits of harvesting while ensuring the ecological sustainability of both species. Therefore, the analysis is centred on the coexistence equilibrium point where both species persist and are permanent. Taking the harvesting effort as a time-dependent control variable $E = E(t)$, the optimal control problem is written to maximize the current value of the net profits from the harvests:

$$\begin{aligned} \max \mathbb{J}(E) &= \int_0^T e^{-\delta t}(p_1 q_1 E x + p_2 q_2 E y - cE)dt, \\ \text{subject to} \quad \frac{dx}{dt} &= rx\left(1 - \frac{x}{K}\right) - mxy - q_1 E x, \\ \frac{dy}{dt} &= nxy - sy - q_2 E y, \end{aligned} \tag{4.15}$$

considering initial conditions $x(0) = x_0$, $y(0) = y_0$ and the harvesting effort constraint, $0 \leq E \leq E_{\max}$.

The discount factor $e^{-\delta t}$ ensure that the integral converges properly when optimizing over an infinite or finite time horizon.The discount factor serves to appropriately discount future payoffs,making the optimization problem realistic and aligned with economic and ecological principles.The exponent in the discount factor is negative because it represent a decline in the present value of future payoffs over time.

The discount rate is δ, the terminal time is T, and the highest permissible harvesting effort is E_{max} in the optimal control problem (5.1). The control E linearly affects the state system and objective functional. Additionally, the unboundedness among the state system solutions has been proven (refer to Theorem 4.2).

Existence of optimal controls

The goal is to maximize future net revenue discounted present value, hence the optimal control E_δ is sought after in such a way that;

$\mathbb{J}(E_\delta) = \sup\{\mathbb{J}(E) : E \in \mathbb{U}\}$,

where the control set is Lebesgue measurable and is defined by

$\mathbb{U} = \{E(t) : 0 \leq E(t) \leq E_{\text{max}}, t \in [0, T]\}$ As previously mentioned, investigating and confirming the problem's necessary and sufficient conditions is a prerequisite for solving an optimal control problem. Consequently,conditions sufficient for the existence of an optimal control are satisfied since the control is linear in the objective functional and state system.

Pontryagin's maximum principle is used to determine the necessary conditions for the optimal control and the related states. For Problem (5.1), the current value Hamiltonian is given by

$$H = p_1q_1Ex + p_2q_2Ey - cE + \lambda_1\left[rx\left(1 - \frac{x}{k}\right) - mxy - q_1Ex\right] + \lambda_2\left[nxy - sy - q_2Ey\right]. \tag{4.16}$$

Having established the existence of an optimal control $E_\delta(t)$, corresponding to the state variables $x(t)$ and $y(t)$ exist adjoint variables $\lambda_1(t)$ and $\lambda_2(t)$

satisfying the following system:

$$\begin{aligned}\lambda_1' &= \delta\lambda_1 - \frac{\partial H}{\partial x}\\ &= (\delta - r + \frac{2rx}{K} + my + q_1 E)\lambda_1 - ny\lambda_2 - p_1 q_1 E,\\ \lambda_2' &= \delta\lambda_2 - \frac{\partial H}{\partial y}\\ &= (\delta - nx + s + q_2 E)\lambda_2 + mx\lambda_1 - p_2 q_2 E,\end{aligned} \tag{4.17}$$

with transversality conditions $\lambda_1(T) = 0$ and $\lambda_2(T) = 0$. The Hamiltonian maximizes the optimal control E_δ and since the control is linear in the Hamiltonian, the possibility of singulars is unable to be ruled out. The condition of optimality is defined by

$$\frac{\partial H}{\partial E} = p_1 q_1 x + p_2 q_2 y - c - q_1 x\lambda_1 - q_2 y\lambda_2. \tag{4.18}$$

and the characterisation of optimal control is

$$\begin{cases} E_\delta = 0 & \text{if } \frac{\partial H}{\partial E} < 0,\\ E_\delta = E_{singular}(t) & \text{if } \frac{\partial H}{\partial E} = 0,\\ E_\delta = E_{\text{max}} & \text{if } \frac{\partial H}{\partial E} > 0.\end{cases} \tag{4.19}$$

Thus, the optimal control can assume the lower and upper bounds of the control according as the optimality condition is negative or positive, giving rise to the bang-bang controls. On the other hand, the singular optimal control occurs when the optimality condition is identically zero, which implies

$0 < E_{\text{singular}}(t) < E_{\text{max}}$.

To investigate the singular control, we presume that there is a time interval such that $\frac{\partial H}{\partial E} = 0$.

Further, employing Systems (4.1) and (5.3) we compute,

$\frac{d}{dt}\left(\frac{\partial H}{\partial E}\right)$ and set it to zero, which simplifies to

$$\frac{d}{dt}\left(\frac{\partial H}{\partial E}\right) =\frac{1}{K}(-2rq_1x(\lambda_1+p_1)-(q_2\lambda_1 m-q_2np_2-nq_1\lambda_2+mp_1q_1)y$$
$$+Krxq_1(r\lambda_1-rp_1)(1-\frac{x}{K})-mxy-q_1Ex)$$
$$-(q_2\lambda_1 m-q_2np_2-nq_1\lambda_2+mp_1q_1)x+Kyq_2(\delta\lambda_2+p_2s)$$
$$(-Eq_2+nx-s)-rq_1x-Kx\delta\lambda_1q_1E(q_2ym-\delta q_1)$$
$$-\lambda_1(r(1-\frac{x}{K})-\frac{rx}{K}-my-q_1E-\lambda_2my)-Ky(\delta\lambda_2-p_2q_2E$$
$$-p_2q_2+\lambda_1mx)-\lambda_2(-Eq_2+nx+s)+(xnq_1-q_2\delta)=0.$$

It is obvious that explicitly the control does not occur in the preceding equation, hence the second derivative with respect to time of the function of optimality is computed and set to zero. This simplifies to,

$$\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial E}\right)=\alpha(t)E(t)+\beta(t)=0. \tag{4.20}$$

Therefore, the expressions for $\alpha(t)$ and $\beta(t)$ are given as

$$\alpha(t)=\frac{1}{k^2}((\lambda_1-p_1)nmyq_1+q_2x^2(-\lambda_1m+np_2)-q_1(\lambda_2n-mp_1)$$
$$+q_2n(p_2-\lambda_2my^2)-q_1^2(\lambda_2n-mp_1)-2Eq_2(-mp_1+np_2)$$
$$-2p_1m(r-\frac{1}{2}s)+2n\lambda_2q_1(\delta+\frac{1}{2}r)-q_2^2E(-\lambda_1m+np_2)+2\lambda_1m(\delta$$
$$-\frac{1}{2}s)-p_2n(r-2s)y+q_1x(Ep_1q_1(\delta-r)-\delta^2\lambda_1+r^2p_1)$$
$$+K^2yq_2(Ep_2q_2(\delta+s)+p_2s^2-\delta^2\lambda_2)+rx^2mq_1(\lambda_1+3p_1)$$
$$-q_2y(\lambda_1m+np_2)+q_1^2E(\lambda_1+3p_1)-(\lambda_1+3p_1)r$$
$$-2K\delta\lambda_1+2q_1r^2x^3p_1),$$

$$
\begin{aligned}
\beta(t) =& \frac{1}{K^2}(-\lambda_2 + mp_1)q_1^2 - (mP_1 + nP_2)2q_2q_1 - q_2^2y(-\lambda_1 m + np_2) \\
&+ (\delta - r)p_1q_1^2x + (\delta + s)K^2yq_2^2p_2 + (\lambda_1 + 3p_1)q_1^2Krx^2E \\
&+ \frac{1}{k^2}(\lambda_1 - p_1)q_1nym + q_2x^2(-\lambda_1 m + np_2) - (\lambda_2 n + mp_1)q_1 \\
&+ (p_2 - \lambda_2)q_2nmy^2 - (r + \frac{1}{2})2p_1m + 2n\lambda_2q_1(\delta + \frac{1}{2}r) \\
&- q_22\lambda_1 m(\delta - \frac{1}{2}s) - p_2ny(r - 2s) + q_1x(-\delta^2\lambda_1 + p_1r^2) \\
&+ yq_2^2k^2(-\delta^2\lambda_2 + p_2s^2) + rx^2q_1m(\lambda_1 + 3p_1) \\
&- q_2y(\lambda_1 m + np_2) + Kq_1r(-\lambda_1 - 3p_1 - 2\delta\lambda_1) + 2q_1r^2x^3p_1).
\end{aligned}
$$

Equation (5.6) is linear in the control variable $E(t)$ and so the singular control solution is

$E_{\text{singular}}(t) = -\frac{\beta(t)}{\alpha(t)}$. provided that, $\alpha(t) \neq 0$ and $0 < -\frac{\beta(t)}{\alpha(t)} < E_{max}$.

For the control to be singular and optimal, the generalised Legendre-Clebsch condition needs to be met. Thus, $\alpha(t) = \frac{\partial}{\partial E}\left[\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial E}\right)\right]$

must be positive, since the order of singularity is one. Hence, the optimal harvesting effort is given by

$$
E\delta = \begin{cases} 0 & \text{if } \lambda_1q_1x + \lambda_2q_2y > p_1q_1x + p_2q_2y - c, \\ -\frac{\beta(t)}{\alpha(t)} & \text{if } \lambda_1q_1x + \lambda_2q_2y = p_1q_1x + p_2q_2y - c, \\ E_{\text{max}} & \text{if } \lambda_1q_1x + \lambda_2q_2y < p_1q_1x + p_2q_2y - c. \end{cases} \tag{4.21}
$$

Therefore, the bang-bang controls indicate that harvesting at the maximum allowable effort should only be initiated whenever the net revenue per unit effort (NRPUE) exceeds the total marginal net revenue of the sardinella and tuna stocks (MNR). Otherwise, it is optimal to exert zero effort, which translates to no harvesting of the stocks. The singular control is only applicable when the net revenue per unit effort exactly equals the total net marginal revenue, as long as the generalised Legendre-Clebsch condition is met.

Numerical Simulations

The optimality system is numerically solved using the Forward-Backward Sweep (Runge-Kutta) method. Singular controls are permissible only when the optimal control function is identically zero on the interval, with $\alpha(t) > 0$. The discount rate $\delta = 0.10$ per year, initial sardinella population $x_0 = 150,000$ tonnes, initial tuna population $y_0 = 5,000$ tonnes and terminal time $T = 100$ years with previously defined biological and economic parameter values.

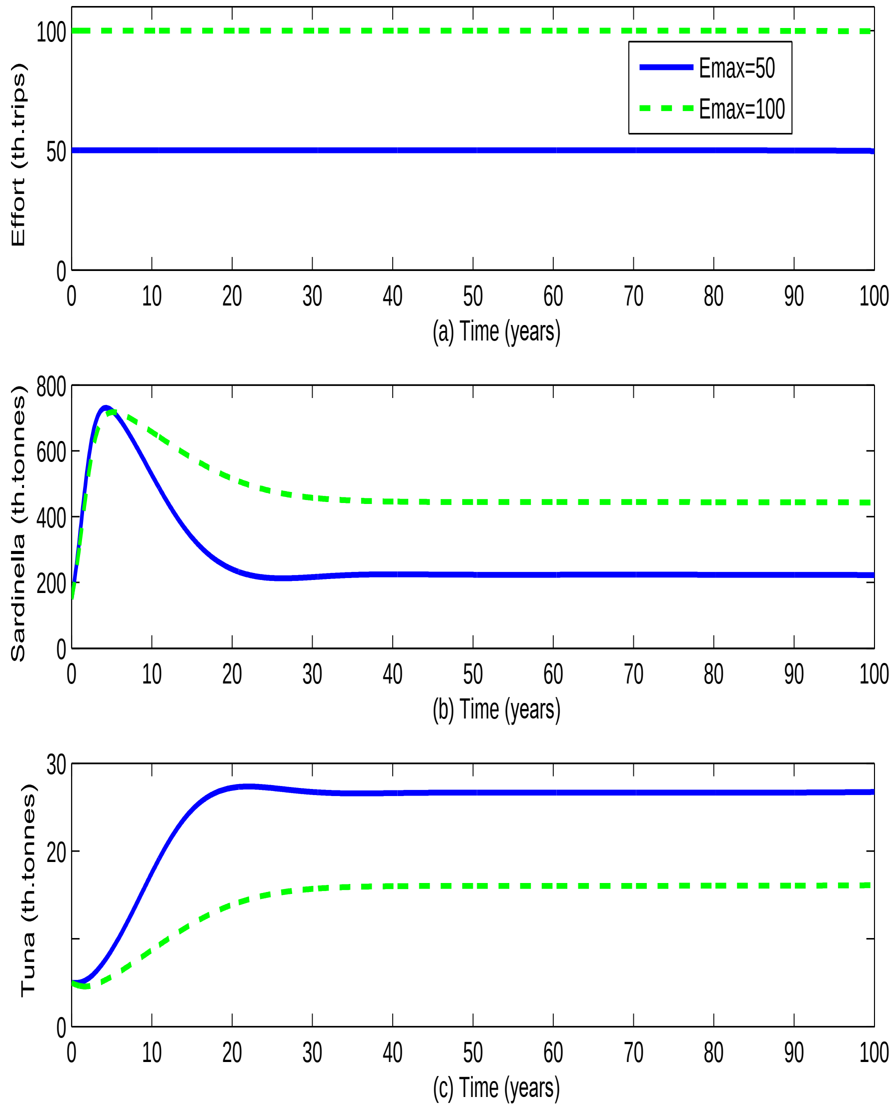

Figure 4: (a) Effort level, (b) Sardinella population and (c) Tuna population for $E_{\text{max}} = 50,000$ versus $E_{max} = 100,000$

In Figure 4, it is observed that when the system is subjected to two different effort rates with a fixed predation rate, the higher effort rate corresponding to 100,000 fishing trips results in the sardinella population settling at a higher steady state of around 450,000 tonnes whereas the tuna population attains a lower steady state of around 16,000 tonnes. It is interesting to note that this harvesting regime produces an annual catch (or sustainable yield) of 80,000 tonnes of sardinella and 24,000 tonnes of tuna. Conversely, with a lower effort rate corresponding to 50,000 fishing trips, the sardinella population settles at a lower steady state of around 210,000 tonnes, while the tuna population settle at a higher steady state of around 28,000 tonnes. The net revenue corresponding to the higher effort rate is $1,638,500,000, whereas the lower effort rate yields a net revenue of $669,600,000. This indicates that the higher the effort rate the greater the net revenue accrued.

In Figure 5, the examination focuses on an effort rate set at 100,000 fishing trips, while systematically varying the growth rate of the sardinella. A higher growth rate $(r = 2.84)$ prompts the sardinella population to stabilize at the steady state of approximately 450,000 tonnes. Concurrently, the tuna population experiences a surge, reaching a higher steady state.

Interestingly, a lower growth rate $(r = 1.42)$ results in the sardinella population settling at the same steady state of around 450,000 tonnes, while the tuna population exhibits restrained growth and settling at a lower steady state of approximately 16,000 tonnes. The net revenue corresponding to the higher growth rate totals $2,106,000,000, while that associated with the lower growth rate is $1,638,500,000.

This analysis underscores a crucial relationship: increasing the growth rate of the sardinella leads to an augmented availability of tuna resources for harvest, consequently contributing to an increase in net revenue.

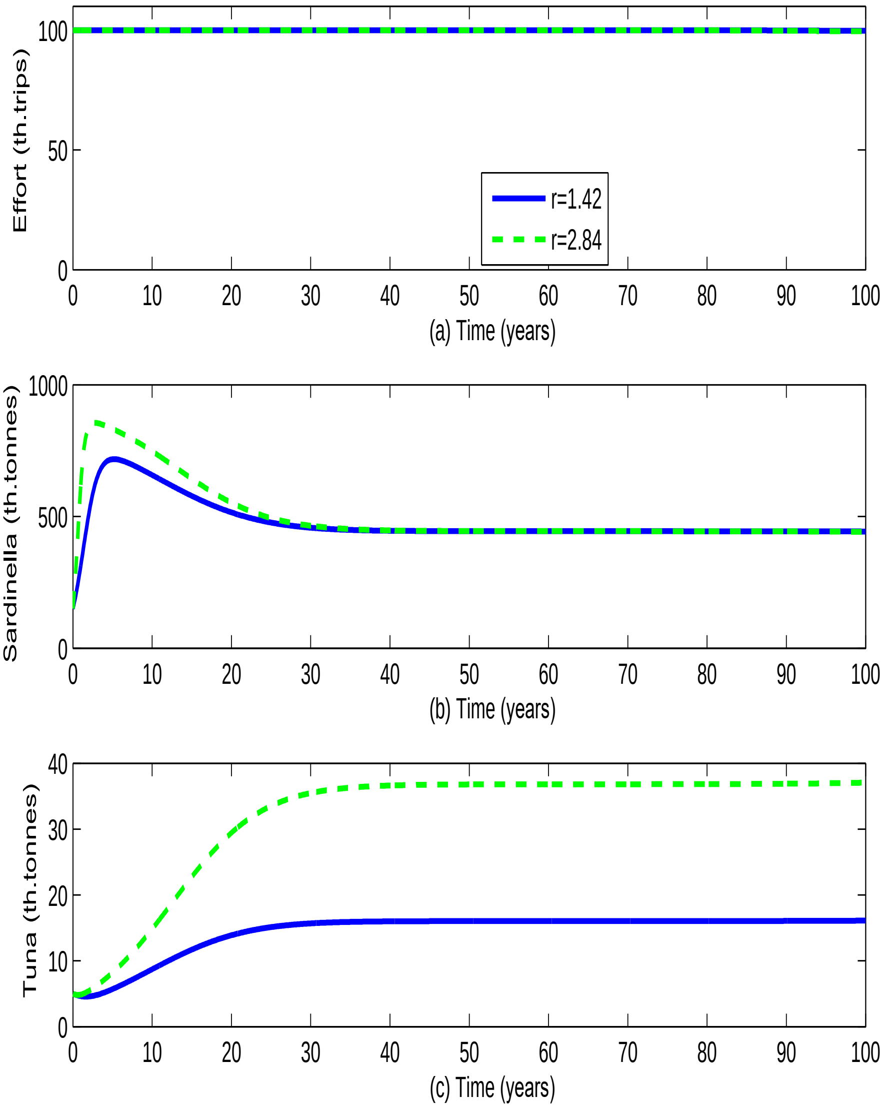

Figure 5: (a) Effort level, (b) Sardinella population and (c) Tuna population for $E_{\text{max}} = 100,000$ and $r = 1.42$ versus $r = 2.84$

In Figure 6, the investigation focuses on a fixed effort rate of 100,000 fishing trips while systematically altering the predation rate. Under a higher predation rate $(m = 3.8 \times 10^{-5})$, the sardinella population initially experiences an increase, settling at an elevated steady state of approximately 450,000 tonnes. Concurrently, the tuna population grows but reaches a lower steady state, stabilizing around 16,000 tonnes.

Conversely, when the system encounters a lower predation rate $(m = 1.9 \times 10^{-5})$, the sardinella population initially rises to 800,000 tonnes but set-

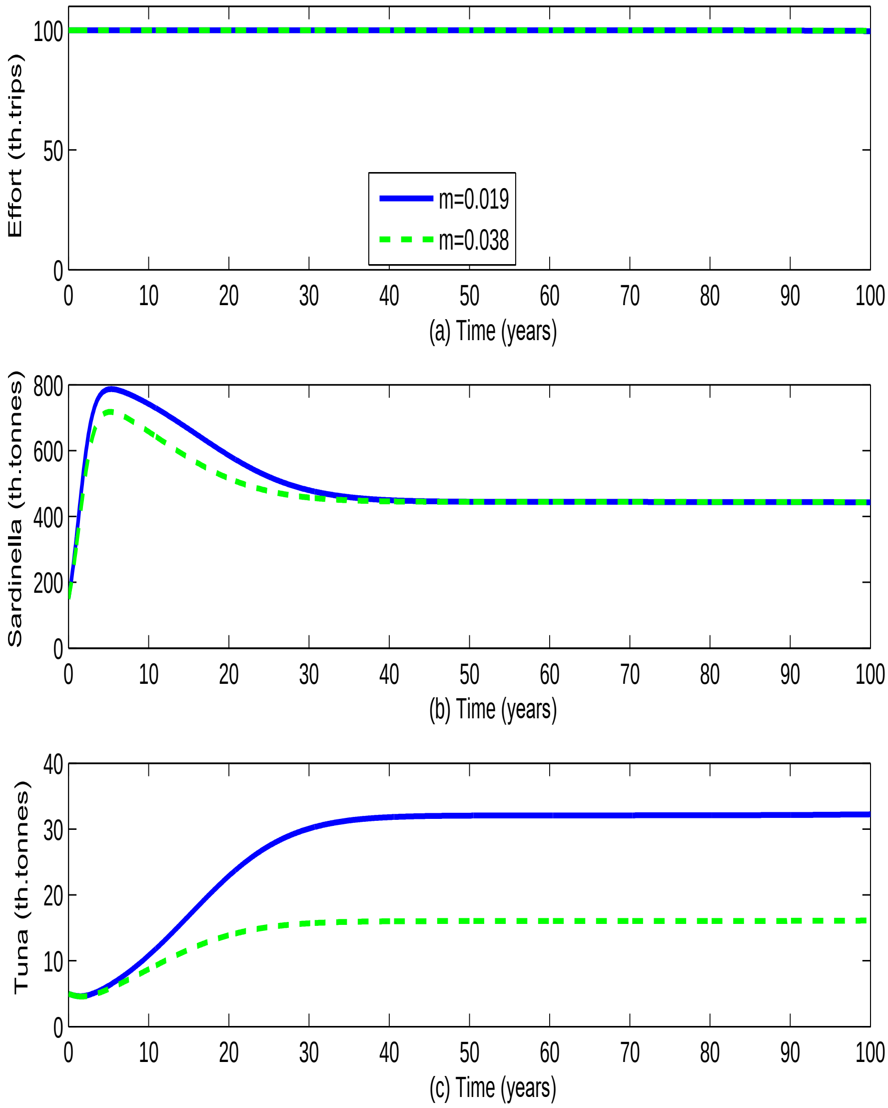

Figure 6: (a) Effort level, (b) Sardinella population and (c) Tuna population for $E_{\text{max}} = 100,000$ and $m = 1.9 \times 10^{-5}$ versus $m = 3.8 \times 10^{-5}$

tles at a lower steady state of around 450,000 tonnes. Notably, the tuna population exhibits accelerated growth, attaining a higher steady state of approximately 32,000 tonnes. The net revenue associated with the higher predation rate amounts to $1,638,500,000, while the lower predation rate yields a net revenue of $1,862,5000,000.

This analysis underscores a noteworthy trend: a lower predation rate is correlated with an increase in net revenue.

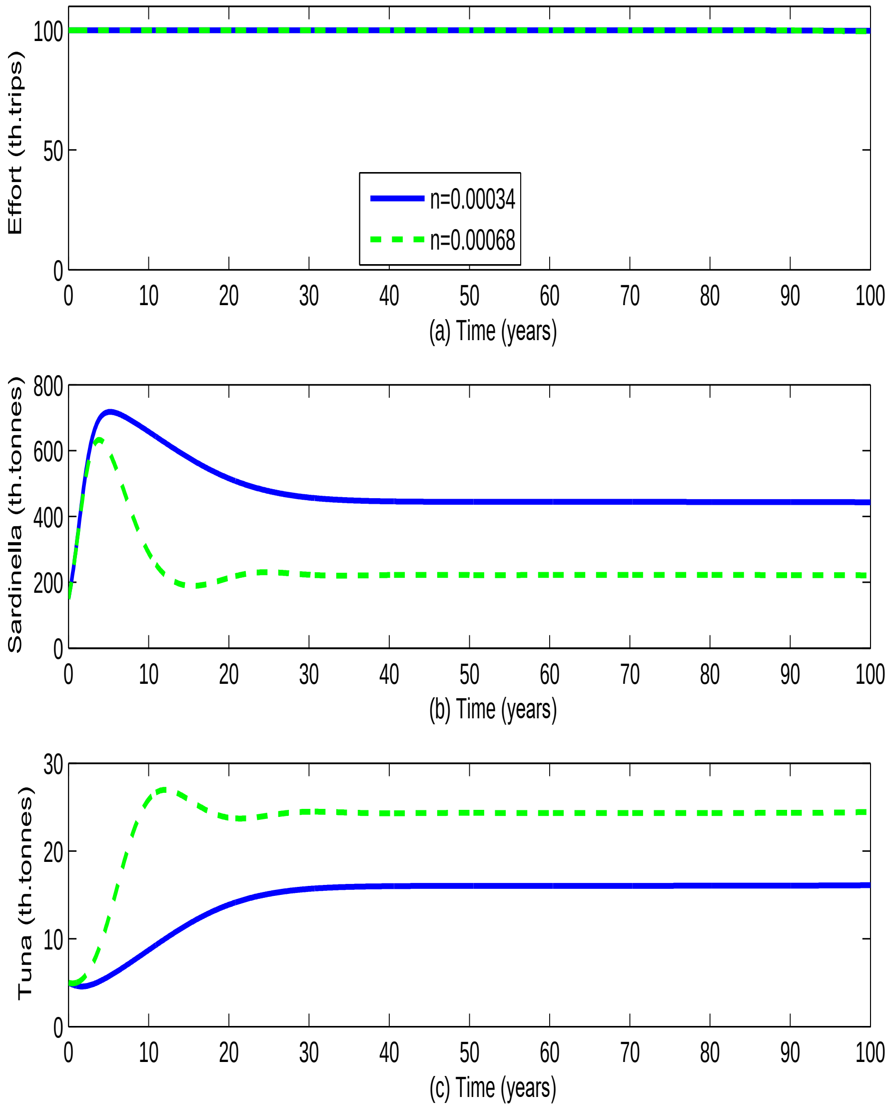

Figure 7: (a) Effort level, (b) Sardinella population and (c) Tuna population for $E_{\max} = 100,000$ and $n = 3.4 \times 10^{-7}$ versus $n = 6.8 \times 10^{-7}$

In Figure 7, the exploration centers on a consistent effort rate of 100,000 fishing trips, with a variation in the rate of converting sardinella biomass into tuna births. When employing a higher rate $(n = 6.8 \times 10^{-7})$, the sardinella population stabilizes at a diminished steady state of approximately 210,000 tonnes. Simultaneously, the tuna population experiences accelerated growth, settling at a higher steady state of around 25,000 tonnes.

Conversely, a lower rate $(n = 3.4 \times 10^{-7})$ results in the sardinella population settling at an elevated steady state of around 450,000 tonnes, while the tuna population grows steadily, settling at a lower steady state of approximately 16,000 tonnes. Remarkably, the net revenue associated with $n = 6.8 \times 10^{-7}$ and $n = 3.4 \times 10^{-7}$ amounts to $963,660,000 and $1,638,500,000, respectively. This intriguing observation suggests that a lower rate of converting sardinella biomass into tuna births is associated with an increase in net revenue.

It is evident that in the simulation process, variations in the parameters significantly influence the growth rates and steady states of both sardinella and tuna populations within the ecosystem, consequently impacting the net revenues accrued.

The numerical simulations encapsulate a holistic approach to the ecological system, integrating stability, optimization, and control measures within the constraints imposed by maximum harvesting efforts. This framework seeks to balance ecological health and economic productivity while accounting for the dynamic nature of sardinella-tuna coexistence in the long term.

Chapter Summary

This chapter formulates a predator-prey fishery model based on modified Lotka-Volterra equations, incorporating variables for prey (sardinella) and predator (tuna) populations. The model assumes logistic growth for prey, proportional predation rates, and mortality rates for predators. Fishing efforts are represented as control variables impacting both species. The analysis includes equilibrium points, stability conditions, and the derivation of optimal control strategies. Numerical simulations validate the theoretical findings, illustrating how different harvesting efforts affect fish populations and economic returns. The results indicate that sustainable management requires setting an optimal fishing effort threshold (100,000 trips annually), ensuring ecological stability while maximizing revenue.

CHAPTER FIVE

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

Overview

This research presented a comprehensive synopsis and significant conclusions were drawn, based on investigations into applications of optimal control strategies to determine the most effective efforts of harvesting for the management of fishery resources. The findings offer valuable insights into the sustainable utilization of these resources. As the study is concluded, recommendations are extended to stakeholders involved in fisheries management. These recommendations are designed to inform and guide their decision-making processes, aiding in fishery resource conservation and responsible utilization. Furthermore, suggestions for prospective research areas that require additional exploration and investigation are identified by outlining research paths. These suggestions aim to contribute to the ongoing advancement of knowledge and practices in fisheries management and optimal resource utilization.

Summary

In this study, the primary focus was on applying linear optimal control techniques to discern sustainable harvesting strategies in fisheries management. A thorough review of various standard fishery models, alongside the introduction of a model tailored to the study's objectives, formed a crucial part of the investigation. The model underwent detailed analysis, identifying static equilibrium points and delving into dynamic equilibrium reference points. Emphasizing the significance of effective renewable resource utilization, an innovative two-dimensional model was proposed. The study aimed to address the current low yields of fishery resources and the role played by fishermen in the process.

Considering catchability as a pivotal parameter in the standard model, the study acknowledged its constant assumption and established the distinctiveness and persistence of the optimal control. The state dynamics of the model, portraying a relationship between the tuna and the sardinella interaction, centered around the objectives of total net revenue extrapolated from the harvested ratio of both tuna and sardinella biomass.In addition, both local and global stability were examined and discussed.

To ascertain optimality, the study employed the condition of the Generalized Legendre-Clebsch to analyze the existence of a singular route. Numerical simulations were conducted to provide practical insights and validate theoretical findings, bringing an application-oriented dimension to the study's outcomes.

Therefore, the main findings of the research are as follows:

1. In the dynamic model, the optimal fishing effort rate is determined to be $E_{\text{max}} = 100,000$ trips annually, considering a 10% discount rate. This rate aims to sustain the resource, requiring an initial biomass of at least 150,000 tonnes for sardinella and 5,000 tonnes for tuna.

2. As the fishing effort rate rises, more resources are allocated to fishing activities, leading to increased catch or yield. Consequently, with higher catch volumes, the net revenue generated from fishing operations also increases.

3. Augmenting the growth rate of sardinella enhances the pool of available resources for harvest, thereby fostering a surge in net revenue. This occurs because a higher growth rate translates to a larger population of sardinella, consequently bolstering the potential catch and revenue from fishing activities.

4. Correlated with a decrease in predation rate is an upsurge in net revenue. This relationship arises because a lower predation rate means fewer fish are consumed by predators, allowing for a larger population of fish to be available for harvest. Consequently, this leads to higher yields and increased revenue for fishing operations.

5. Decreasing the rate of converting sardinella biomass into tuna births is linked with a rise in net revenue. This connection stems from the fact that

a lower conversion rate means more sardinella biomass remains available for harvest rather than being utilized for tuna reproduction. Consequently, this results in higher sardinella yields and, ultimately, increased revenue for fishing operations.

Conclusions

This research delved into the effective and sustainable harvesting strategies within a predator-prey fishery model incorporating optimal control. The study encompasses a comprehensive qualitative analysis, scrutinizing the positivity and uniform boundedness of the model's solutions. The investigation extends to understanding the local and global behaviour of the system around equilibrium points. The coexistence equilibrium is found to be locally and globally asymptotically stable under specific conditions, with a recognition of the potential for mutual extinction. The phase-portrait analysis of the model across varying harvesting rates revealed the presence of three equilibrium points. At a harvesting rate of 800,000 trips, both sardinella and tuna faced extinction, indicating an unsustainable and environmentally unfriendly fishing rate. Similarly, a harvesting rate of 500,000 trips led to the extinction of tuna, undermining the sustainability of this vital resource and jeopardizing the coexistence of sardinella and tuna, both crucial for the ecosystem.

Contrastingly, the phase-portrait analysis of a fishing effort of 100,000 trips demonstrated the coexistence of sardinella and tuna, suggesting that at this rate, the ecosystem is more sustained, preserving the resources in the long term and fostering enhanced economic benefits.

The simulation results underscore the effectiveness of a two-species fishery model in fostering the sustainability of fishery resources and preserving ecosystem integrity. Conversely, reliance on a one-species dynamic fishery model tends to overestimate biomass of the single species, inadvertently exacerbating the risk of overfishing rather than mitigating it. Relying solely on single-species models in fishery management poses significant risks, as it overlooks crucial predator-prey dynamics within ecosystems.

Persistency and permanence of the system were established through the average Lyapunov approach. The bionomic equilibrium, integrating biological and economic parameters, was presented and thoroughly discussed. Optimal control characterization revealed the existence of both bang-bang and singular controls, with the generalized Legendre-Clebsch condition offering insights into the optimality of singular control.

Numerical simulations focused on the coexistence equilibrium, confirming theoretical findings. Varied harvesting efforts demonstrate a significant impact on sardinella and tuna populations, with higher efforts resulting in lower population levels. The introduction of a predation rate amplifies the effect, showcasing a greater loss in population density with higher harvesting efforts. The net revenue aligns with expectations, reflecting higher returns for increased harvesting efforts.

Simulations involving varying harvesting efforts, varying predation rates and growth rates of both the sardinella and the tuna revealed the destabilizing effect of higher predation rates on sardinella populations. Interestingly, the tuna population remains unaffected at a higher predation rate. Moreover, net revenue is inversely related to predation rates.

In simulations employing bang-bang controls, the upper bound is consistently optimal during steady states, emphasizing the non-optimality of zero effort.

Recommendations

In pursuit of the study's objectives, a two-species model with finite time horizons, incorporating bounded controls, was utilized. Qualitative and quantitative methods, along with numerical simulations, were employed for analysis. Pontryagin's maximum principle established necessary conditions, supplemented by sufficiency conditions. Various harvesting strategies were applied to identify optimal control approaches. Given the current state of fishery resources, decisive measures are imperative for sustainability.

1. To enhance the reliability of fishery management policies concerning targeted fish populations, the adoption of two-species models is recommended. These models offer a more comprehensive understanding of the ecosystem dynamics, allowing for more accurate and effective strategies in fisheries management.

2. To foster long-term sustainability and effective management of sardinella and tuna populations, the introduction of a licensing system is essential. This system would grant select fishers exclusive rights to target these species, facilitating the implementation of tailored fishing efforts. By enabling more precise monitoring and control, this approach enhances the overall management of these resources, ensuring their continued viability.

3. Regulation of Fishing Efforts:

• Acknowledge the inherent risk of overcapacity due to open access.

• Impose a cap on fishing efforts within the fishery sector to ensure resource sustainability.

Future research

While the study extensively explored optimal control harvesting strategies for fishery resources, certain aspects crucial for the sustainable harvesting of renewable resources remain uncharted. Future research endeavours could focus on the following areas of interest:

1. Incorporation of Delay Equations and Allee Effect: Explore models that incorporate delay equations, shedding light on temporal dynamics in har-

vesting strategies. Investigate the impact of the Allee effect on optimal controls, addressing its implications for sustainable resource management.

2. Stochastic and Gompertz Models: Delve into stochastic models to capture the inherent uncertainty in fishery dynamics. Consider Gompertz models to enhance understanding of non-linear growth patterns and their influence on optimal harvesting strategies.

3. Three-dimensional Predator-Prey Model with age-structure for predator: Extend research to encompass a three-dimensional predator-prey model. Incorporate the age-structure to examine intricate interactions and their implications for optimal control strategies in fisheries.

By exploring these unexplored facets, future research can contribute to a more nuanced understanding of optimal control dynamics in the sustainable harvesting of renewable resources.

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