UNIVERSITY OF CAPE COAST

THE CONJECTURE OF GROUP STRUCTURE: THE RELATIONSHIP BETWEEN THE ALPHA INVARIANT AND NILPOTENCY IN FINITE GROUPS

BY

BERNARD BONSU-BANDOH JNR

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DECLARATION

Candidate's Declaration

I hereby declare that this thesis is the result of my own original research and that no part of it has been presented for another degree in this university or elsewhere.

Candidate's Signature .... Date ....

Name: Bernard Bonsu-Bandoh Jnr

Supervisors' Declaration

We hereby declare that the preparation and presentation of the thesis were supervised in accordance with the guidelines on supervision of thesis laid down by the University of Cape Coast.

Principal Supervisor's Signature .... Date ....

Name: Prof. Natalia Mensah

Co-Supervisor's Signature .... Date: ....

Name: Dr. Albert Lanor Sackitey

ABSTRACT

In this research work, we acknowledge and explore the relation between the alpha value and non-nilpotent groups, leading to the proof of a conjecture put forward in research by Cayley (2021). We demonstrate that if G is non-nilpotent and $\alpha(G) = \frac{3}{4}$ then $G \cong D_{24} \times C_{2^n}$, with a nontrivial centre, where $n \in \{0, 1\}$. Furthermore, we conclude that the conjecture holds for $G \cong D_{24} \times C_{2^n}$ as well. We again prove, using both computational and theoretical techniques, that a subgroup which is non-trivial in G exists with both normal and characteristic properties. We finally prove a theorem related to the count involving subgroups, cyclic in nature, of finite groups G where $|C(G)| = |G| - 6$. Thus, we demonstrate that if G is one of the groups D_{24}, C_{12}, C_9, C_{10}, D_{18}, or D_{20}, then $|C(G)| = |G| - 6$.

KEY WORDS

Alpha invariant

Cyclic subgroup

Dihedral group

Group theory

Nilpotent group

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DEDICATION

To my family

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LIST OF ABBREVIATIONS

AUT

INN

SYL

GL

Alt

Cl

gcd

Ker

SL

CHAPTER ONE

INTRODUCTION

This chapter covers the study's background, the problem statement, the purpose of the study, its scope, significance, delimitation, limitation, and organisation of the rest of the study.

Background to the Study

Group theory is a field in mathematics that encompasses or deals with the properties and structure of groups. More formally, an algebraic structure is usually defined to be a nonempty set together with a collection of operations, typically binary, such as addition and multiplication, with a finite set of identities called axioms that these operations must satisfy. Algebraic structures include simple structures, group-like structures, ringlike structures (ringoids), lattice structures, modules, vector spaces, inner product spaces and so on. A group can be defined mathematically as a set that possesses a binary operation, which takes any two of its members and combines them to obtain another member belonging to the same set. Thus, a group G consist of a nonempty set and a binary operation $*$ for which closure, associativity, presence of identity, and inverse are held to be true. A collection of groups that share some common property or structure is known as a family of groups. For example, the family of cyclic groups consists of all groups that are isomorphic to the integers modulo n, a positive integer, under the addition operation. Abelian groups, symmetric groups, and dihedral groups are some more families of groups.

Our focus in this research is exclusively on finite groups. A group is said to have an order that is equal to the count of elements if it posses a finite count of members. However, if the members are infinitely countable in a group, then the period claimed to be infinite. An elements period within a group defines the period of the subgroup formed by that element. This period is frequently referred to as an element's period length or just order. For a group with multiplication containing an element a, its order is referred to as the minimum non-negative integer m for which am = e, with e being neutral element. In the event that such m does not exist, then the period of a is considered infinite. The size of the entire group G is marked either $ord(G)$ or $|G|$, whereas an element's order is indicated as $ord(a)$ or $|a|$, rather than $ord(\langle a \rangle)$, where the angle brackets represent the subgroup generated by a. According to Ledermann (1964) the period of a group's element need to meet the following requirements.

- $|e| = 1$. Thus, no other element has order 1 except the identity.
- Within a group, the inverse and its element's order are the same. Stated otherwise, $o(a) = o(a^{-1}) \forall a \in G$.
- Every element in the group has finite period and divides the period. Therefore, in a finite group, no member's period is larger than the group's period.
- If $o(a) = k$ and $a^n = e$, then k divide n.
- Suppose $o(a) = k$. Then $o(a^n) = k$ for every integer n coprime to k.
- For each integer n, $o(a^n)$ is also infinite if $o(a)$ is infinite.

- Both For $a, g \in G$, the orders of a and $g^{-1}ag$ are the same. We refer to these two components as each other's conjugate elements.
- For any $a, b \in G$, $o(ab) = o(ba)$ holds true, meaning that the orders of ab and ba are the same. Since $ab = a(ba)a^{-1}$ is the conjugate of ba and ab.

One of group theory's primary objective is to categorize and understand the properties of different groups. This can be done by studying the group's subgroups, normal subgroups, homomorphisms, automorphisms, and other algebraic properties. Thus, in group theory, there are several ways to classify groups based on different properties and characteristics. Some of the most common classification methods are: Classification by order, classification by structure Group, classification by isomorphism, classification by subgroup, classification by representation and so on that are used in specific contexts and applications.

Theorem 1. *A group G posses just one identity member e* (Fraleigh, 2003).

Theorem 2. *Given a group G. A single member x' exists if, given $x \in G$, $x * x' = x' * x = e$* (Dummit & Foote, 1991).

A subgroup is a fundamental concept that refers to a subset of a given group that itself, with relation to identical binary operation as the original group, generates a group. We can state formally that in $\langle G, * \rangle$, a subgroup of G is seen as a non-empty subset H of G if it also satisfies the group axioms and thus written as $H \leq G$.

Theorem 3. *Consider that $H \leq G$. For any $a \in H$, we will get $a^{-1} \in H$, since H is operation-closed in G, includes e, and has an inverse element* (Baumslag & Chandler, 1968).

For every group G, the trivial subgroups are G and $\{e\}$, while a proper subgroup H exists in G if $H \neq G$. We again note that $\langle Z, +\rangle \leq \langle Q, +\rangle \leq \langle R, +\rangle \leq \langle C, +\rangle$.

Theorem 4. *Consider a group G of finite-period with $P, Q \leq G$. It holds that $P \cap Q \leq G$.* For a proof, see Rotman (2006).

Remark: If $\{P\alpha\}_{\alpha\in I}$ represents subgroups, then likewise $\cap_{\alpha\in I} P_\alpha$.

Theorem 5. *Let H_1 and H_2 be nontrivial groups. Then $H_1 \times H_2$ is cyclic $\Leftrightarrow$ H_1 and H_2 are finite cyclic groups with $gcd(|H_1|, |H_2|) = 1$* (Anderson & Camillo, 2009).

Theorem 6. *Consider G_1 and G_2 as nontrivial groups. Therefore, each subgroup of $G_1 \times G_2$ is a sub-product $\Leftrightarrow$ $g_i \in G_i$, g_i has finite order $o(g_i)$ and $gcd(o(g_1),$ $o(g_2)) = 1$* (Anderson & Camillo, 2009).

We also examine the notion of a group G centre and centralizer. These ideas constitute subgroups of G. In this context, the centralizer of an element $a \in G$ indicates the collection of elements in G commuting with a, whereas any component in G that commutes with any other member in G is said to be the centre of the group G.

Theorem 7. *Let a group G be finite-ordered, $Z(G) \leq G$.* For a proof, see Fraleigh (2003)

Theorem 8. *Let $a \in G$. Then $C(a) \leq G$.* For a proof, see Isaacs (2008).

Theorem 9. $Z(G) = \cap_{a\in G} C(a)$, *$C(a)$ is the centralizer of a in G.* For a proof, see Gallian (2010).

A group that may be produced by exponents of a single element, identified as a generator, is referred to as cyclic. Stated differently, a group is considered cyclic if all of its members may be acquired by performing the group operation repeatedly on a single element, which can be any element in the group. Furthermore, if a cyclic group consist an infinite count of elements, it has infinitely many elements; if it include finite count of members, it has finite elements. According to Shanks (1993), a finite cyclic group C_n, also known as Z_n has period n. Its generator x satisfies $x^n = e$, with the identity element being e. If $a \in S$ and $S \leq G$ then $\langle a \rangle \subseteq S$. Then, we can state that the minimal subgroup within G that holds a is $\langle a \rangle$. Since the group operation is cyclic with generator i, $G = \{1, -1, i, -i\} \subseteq \mathbb{C}^*$ with multiplication. $\langle i \rangle = \{i^0 = 1, i^1 = i, i^2 = -1, i^3 = -i\} = G$, in actuality. Observe that since $\langle -i \rangle = \{(-i)^0 = 1, (-i)^1 = -i, (-i)^2 = -1, (-i)^3 = i\} = G$, $-i$ is likewise a generator for G. As a result, a cyclic group could have several generators. But not every member of G has to be a generator. For instance, -1 is not a generator of G as $\langle -1 \rangle = \{1, -1\} \neq G$. The integers $\mathbb{Z}$, on the other hand, form a cyclic group. $\mathbb{Z}$ does indeed equal $\langle 1 \rangle$ since $k \in \langle 1 \rangle$ and $\langle 1 \rangle = \mathbb{Z}$ since each integer $k = k \cdot 1$ is a multiple of 1. Moreover, $\mathbb{Z} = \langle -1 \rangle$ since for any k in $\mathbb{Z}$, $k = (-k) \cdot (-1)$. $\mathbb{Z}_n$, is, thus, a cyclic group under addition with generator 1. Examining cyclic subgroups configuration and characteristics is a common activity that results when studying finite groups. Each element in a finite group G generates a finite cyclic subgroup, hence one can estimate the number of elemental transformations that are possible within the group by counting the unique cyclic subgroups. Fraleigh (2003), listed some families of finite groups that contains cyclic subgroups:

- Abelian groups: All finite abelian groups have cyclic subgroups as their subgroups.

- Cyclic groups: These are groups with a finite order a, of the form $\langle a \rangle$. These groups contain only cyclic subgroups.
- Dihedral groups: These are groups of regular polygonal symmetries. Each dihedral group posses two cyclic subgroups: one being period 2 and the other of period n, such that the polygon's count of sides is n.
- P-groups: If every element has a period that is some exponent of a given prime integer, p, the group is referred to as a p-group. For each non-trivial p-group, a cyclic subgroup of size p exist.
- Finite fields: There are only cyclic subgroups in finite fields since they are all cyclic groups under addition.
- Symmetric groups: Every finite symmetric group has a cyclic subgroup of period n, with n being the group's element count.
- Alternating groups: All finite alternating groups are composed of order 3 cyclic subgroups.
- Quaternion groups: The quaternion group Q_8 and its higher-order counterparts Q_{16}, Q_{32}, etc., contain cyclic subgroups of orders 2, 4, and 8.
- Generalized quaternion groups: The generalized quaternion groups $Q_{2^n}(n \geq 3)$ contain cyclic subgroups of orders 2, 2^{n-1}, and 2^n.
- Semi-dihedral groups: The semi-dihedral groups $SD_{2^n}(n \geq 3)$ contain cyclic subgroups of orders 2, 2^{n-1}, and 2^n.

We note fundamental properties of cyclic groups that serve as foundational elements throughout our work. As these concepts are already extensively explored, we present their outcomes without providing new proofs. These assertions concerning cyclic groups stem from the authoritative textbook "Abstract Algebra" authored by Dummit and Foote (2004).

Proposition 1. Assume that $G = \langle x \rangle$. $|x^a| = \frac{n}{gcd(n,a)}$ is true if $|x| = n < \infty$ (Dummit & Foote, 2004).

Proposition 2. Assume that $|x| = n < \infty$ and $H = \langle x \rangle$ are cyclic. Then, if and only if $gcd(a, n) = 1$, $H = \langle x^a \rangle$. To be more precise, Euler's phi-function, represented by ϕ (n), has a total of H generators (Dummit & Foote, 2004).

Proposition 3. Let a cyclic group be H. If $|H| = n < \infty$, there exists distinct cyclic subgroup period a in H for each positive integer a dividing n. Furthermore, $\langle x^m \rangle = \langle x^{gcd(n,m)} \rangle$ for each integer m, indicating that the factors of n and the subgroups of H coincide bijectively. Hence, $\tau(n) = |C(H)|$, with $\tau(n)$ representing the divisor count function (Dummit & Foote, 2004)

Theorem 10. *Let the group G be finite. When a prime p yields $G = C_{p^2}$, $|C(G)| = 3$.* For a proof, see Dillstrom (2016).

Theorem 11. *Consider G as finite-ordered. We have $|C(G)| = 3$ if $G = C_{pq}$, $G = C_{p^3}$ or $G = C_2 \times C_2$, are true*. For a proof, see Dillstrom (2016)

Theorem 12. *Consider G as finite-ordered with $|a| = n$. In this case, $\langle a \rangle$ consists of the elements* $\{a^k : 0 \leq k < n\}$ (Dummit & Foote, 2004).

Theorem 13. *A prime order group is cyclic in nature* (Baumslag & Chandler, 1968).

Theorem 14. Let $g \in G$. Then for $\langle g \rangle$:

Possibility 1: Finite cyclic subgroup. In this scenario, $g^n = 1$ occurs for the smallest positive integer n, and hence:

- $n \mid k$ if and only if $g^k = 1$.

- $g^k = g^m \Leftrightarrow k \equiv m \ (mod\ n)$.
- $\langle g \rangle = \{1, g, g^2, \dots, g^{n-1}\}$ and the elements $1, g, g^2, \dots, g^{n-1}$ vary.

Possibility 2: Infinite cyclic subgroup:

- $g^k = 1 \Leftrightarrow k = 0$.
- $g^k = g^m \Leftrightarrow k = m$.
- $\langle g \rangle = \{\dots, g^{-3}, g^{-2}, g^{-1}, 1, g, g^2, g^3, \dots\}$ with varying powers of g (Dummit & Foote, 2004).

Theorem 15. *Suppose a cyclic group $G = \langle g \rangle$ with period n, and* $0 \leq k \leq n - 1$. $o(g^k) = \frac{n}{m}$, *if* $m = gcd(k, n)$ (Dummit & Foote, 2004).

Theorem 16. *Consider a cyclic group G generated by g with order n. The group G is generated by g^k when strictly* $gcd(k, n) = 1$. For a proof, see Fraleigh (2003).

Theorem 17. *All of cyclic group's subgroups are likewise cyclic* (Dummit & Foote, 2004).

Theorem 18. *Fundamental Theorem of Finite Cyclic Groups. Take $G = \langle g \rangle$ as a cyclic group of period n.*

- *Assume $H \leq G$, then $H = \langle g^d \rangle$ for some $d|n$.*
- *Assume $H \leq G$ and $|H| = k$, then $k|n$.*
- *The unique subgroup of G of period k is $\langle g^{n/k} \rangle$ if $k|n$.* For a proof, see Gallian (2010)

An alpha invariant as captured in this work is a measure of how 'cyclic' a group is, in the sense that it quantifies the proportion of cyclic subgroups in the group. The alpha invariant is a non-negative real number, more specifically, the alpha invariant $\alpha(G)$ indicates ratio of $G's$ cyclic subgroup count to its period:

$$\alpha(G) = \frac{C(G)}{|G|} \quad (1)$$

with $C(G)$ as the cyclic subgroups count of G up to isomorphism. We note that the alpha invariant is always a non-negative rational number such that $0 < \alpha(G) \leq 1$. Take $\alpha(G)$ close to or approaching 1, the more cyclic the group is. Notably, a group is cyclic if and only if its alpha invariant is 1. A group with a high alpha invariant has a relatively large number of cyclic subgroups, which means that it is more 'cyclic' in structure. Conversely, a group with a low alpha invariant has relatively fewer cyclic subgroups, which means that it is less 'cyclic' in structure. The alpha invariant can also be used to study properties of groups, such as their order, number of generators, and conjugacy classes.

Definition: $\varphi(|x|)$ generators exist for any cyclic subgroup $\langle x \rangle$ of G, where φ is the totient function of Euler. Hence,

$$C(G) = \sum_{x \in G} \frac{1}{\varphi(|x|)}$$

The dihedral group comprises the symmetries associated with a regular polygon, incorporating rotations and reflections. Dihedral groups D_n with $n \geq 3$ are finite groups and whether n is even or odd determines their properties. If n is odd in this scenario, the $Z(D_n)$ is just the identity, but for even n, it comprises two elements: the identity and $r^{\frac{n}{2}}$. In this research we will denote the group order of D_{2n} as $2n$. There are $2n$ symmetries in a regular polygon of side n, which include n rotations and n reflections. The size of the n rotation of a regular polygon is $\theta = \frac{360^0}{n}$ or $\theta = \frac{2\pi}{n}$ radians. Since there are two counts of items in period 2, $(n + 1)$ for n even values and n for odd n values, the identity element in D_{2n} has period 1.

Definition: Given that D_{2n} denotes period $2n$ dihedral group and the vertices in clockwise direction of D_n are $v_1, v_2, v_3, \ldots, v_n$. If r is the rotation of the n-gon by $\frac{2\pi}{n}$ radians and s is the reflection across the line connecting v_1 to the centre of the polygon, then:

- $e, r, r^2, \ldots, r^{n-1}$ are all distinct and $r^n = e$, so $o(r) = n$.
- $o(s) = 2$.
- $s \neq r^i$ for any i.
- $r^i s \neq r^j s$ for all $0 \leq i, j \leq n-1$ with $i \neq j$.

Hence, $D_{2n} = \{e, r, r^2, \ldots, r^{n-1}, s, rs, r^2 s, \ldots, r^{n-1} s\}$.

Theorem 19. *Dihedral groups D_n are non-abelian for integers $n \geq 3$.* For a proof, see Thangarajah (n.d.)

Theorem 20. *Take $n \in N$ with $n \geq 3$. D_n, dihedral group, has a period $2n$, and with presentation: $D_n = \langle a, b \mid a^n = b^2 = e, ab = ba^{-1} \rangle$ We use $Z(D_n)$ to refer to the centre of D_n. Following this:*

$$Z(D_n) = \begin{cases} \{e\} & if\ n\ s\ odd \\ \left\{e, a^{\frac{n}{2}}\right\} & if\ n\ is\ even \end{cases}$$

For a proof, see Clark (1971).

Theorem 21. *The count of all n divisors, including the endpoints 1 and n, is denoted $\tau(n)$, where $\alpha(Cn) = \frac{\tau(n)}{n}$* (Cayley, 2021).

In D_{2n}, the identity element e and each rotation r^k where k is a divisor of n, form a cyclic subgroup $\langle r^{n/d} \rangle$. In addition, each reflection s and combinations $r^k s$ (where k ranges from 0 to $n-1$) form a period 2 cyclic subgroup.

Theorem 22. $|C(D_{2n})| = n + \tau(n)$, *for every* $n \in N$. For a proof, see Dillstrom (2016)

Theorem 23. $\alpha(D_{2n}) = \frac{\text{n+ } \tau \text{ (n)}}{2n}$

Theorem 24. *The cardinality of subgroups in a dihedral group* $|D_n| = n$ *is* $\tau\ (n) + \sigma(n)$, (Bardhan et al., 2023).

The Klein four-group is the smallest non-cyclic group and it has 4 elements, which are all self-inverse, and any two of the three non-identity elements multiply to the third one. The elements of K_4 are often written as e, a, b, c, where e is the identity element and has period length 1 but the other three non-identity elements having a higher period length. The Klein group has order 2 for all other instances of e. The Klein four-group is the smallest non-cyclic group. However, it is an abelian group, and structurally equivalent to the dihedral group of period 4 that is $D_{2(2)}$. The Klein four-group has the description: $K_4 = \langle a, b | a^2 = b^2 = (ab)^2 = e \rangle$

An element's conjugacy class means collection of all members that are conjugate to one another. Let G be a group. When something like $gag^{-1} = b$ holds true, then the element $a, b \in G$ are conjugate with each other such that b is known as a conjugate of a and in same manner a is called as a conjugate to b. The conjugacy class is a quotient of which (G) which also divided into several classes of equivalence the component group associate to each exactly one conjudate class when its equivalent classes $Cl_G(a)$ and $Cl_G(b)$ are same exactly when a and b are conjuagted or else different. The conjugacy class $a \in G$ is denoted as $Cl_G(a) = \{gag^{-1} : g \in G\}$. If G is abelian, consequently, for every $a, g \in G$, $Cl_G(a) = \{a\}$. If all conjugacy classes are singletons, then G is abelian. An element $a \in G$ is in

$Z(G)$ if and only if its conjugacy class consists solely of a. The identity element is the only member of its conjugacy class, that is, $Cl_G(e) = \{e\}$.

Proposition 4. $Cl_G(x) = \{x\}$ if and only if $x \in Z(G)$. For a proof, see Macauley, (2014).

Proposition 5. Conjugacy class is an equivalence relation. For a proof, see Macauley (2014).

Proposition 6. In a group, for all positive integers n, $(xgx^{-1})^n = xg^n x^{-1}$. For a proof, see Miller (1911).

Theorem 25. *Every element within a conjugacy class shares the identical period*. For a proof, see Miller (1911).

Proposition 7. Every normal subgroup is the union of conjugacy classes. For a proof, see Macauley (2014).

A subset of a group called a coset is produced by multiplying each element of a subgroup by a fixed group's element. The collection of elements gH, where $gH = \{gh|h \in H\}$, is defined as the left coset of an element $g \in G, H$, represented by gH, given a group G and a subgroup H of G. This is done by multiplying g with each element of H. Similarly, all items that belong to the set produced by multiplying g by each element in H are designated as the right coset of $g \in G$ regarding H, which is represented as Hg: $Hg = \{hg|h \in H\}$. Cosets are a naturally occurring class of subsets of a group. Take an abelian group G of integers, whose subgroup H is composed of even integers, and whose operation is specified by the standard addition. Then, precisely two cosets exist: odd integers, $1 + H$, and even integers, $0 + H$. It should be noted that the cosets are subsets of the group G, not necessarily subgroups in and of themselves. Still, they have certain characteristics in common with subgroups, (Rotman, 2006).

- Coset Closure: Given that $H \leq G$, the coset gH for any $g \in G$ is closed under group operation. In other words, if h_1 and h_2 belong to H, then $h_1 * h_2$ will lie in gH.
- Coset Equality: There exist two cosets g_1H and g_2H of the same subgroup H, which are either identical or disjoint.
- Coset Partition: The set G is expressed in the form of distinct union of cosets of H. That is, each element in G belongs to distinct coset of H.
- Coset Cardinality: The size of every cosets of H within G is identical to $H's$ size.

For each member $g \in G$ and each subgroup H of an abelian group G, the equation $g + H = H + g$ holds. In generic groups, $Hg = g(g^{-1}Hg)$ is the right coset of H with regard to g and the left coset of the conjugate subgroup $g^{-1}Hg$ with respect to g. A subgroup H and an element g of a group G are supplied to this. Cosets are essential group theory studies because they let us examine a group's coset decomposition and the connections between cosets and subgroups to examine a group's structure and characteristics.

A subset of a group that remains unchanged when conjugated by components of the larger group is known as a normal subgroup. Put otherwise, this group doesn't change when any element from the bigger group conjugates with it. To be more precise, if $N \leq G/N$ is a subgroup of a group G, then N is normal in G if the conjugate of N by g lies within N. This is because element g belongs to G in every instance. Therefore, if $gNg^{-1} = N$ is true for any g in G, then a subgroup N of a group G is normal. Accordingly, it is denoted as $N \trianglelefteq G$. There are several comparable standards that can be used to assess if a subgroup is normal:

- Right and left cosets equality: In each and every $g \in G$, Ng is the same as gN (Hungerford, 2003).
- Quotient group: G/N is the representation of the resultant group obtained by applying coset multiplication to the set of left cosets of N in G. N is considered normal in this case if and only if G/N is a group.
- Kernel of a homomorphism: If N is the kernel of a group homomorphism from G to a different group, then it is normal.
- A union of $G's$ conjugacy classes is N (Cantrell, 2000).
- There is a group homomorphism $G \to H$ with kernel N (Cantrell, 2000).
- $G's$ inner automorphism preserves N (Fraleigh, 2003).

The trivial subgroup $\{e\}$ of G is self-normal in G for every group G. Similarly, G is normal within its own structure. According to Robinson (1996), the group G is defined as simple if its normal subgroups consist solely of $\{e\}$ and G. Furthermore, it was shown by Hungerford (2003) and Hall (1999) that the centre of an arbitrary group and the commutator subgroup $[G, G]$ are included in the normal subgroups of that group. More broadly, any characteristic subgroup is a normal subgroup since conjugation is an isomorphism (Hall, 1999). In group theory, normal subgroups are significant. They allow for the definition of quotient groups, which capture the algebraic structure of a group modulo a normal subgroup.

Additionally, they can be found in a number of theorems and notions, including the notion of simple groups, the normalizer of a subgroup, and isomorphism theorems.

Proposition 8. If the index of H in G is 2, then the subgroup H of G is normal. For a proof, see Humphreys (1996).

Theorem 26. *Given that* $G = M + N$ *and* $M \cap N = \{0\}$ *are additive groups with* M *and* N *as normal subgroups, then* $G = M \oplus N$. For a proof, see Hungerford (2003).

An essential idea in group theory and abstract algebra is a quotient group, sometimes referred to as a factor group. It is formed by considering a new group whose components constitute the cosets of the provided Group's normal subgroup after determining the group's normal subgroup. The quotient group G/N, read as 'G modulo N' or 'G quotient N' emphasizes the idea that G/N is obtained via 'modding out' or 'dividing' the group G by the normal subgroup N. It represents the collection of distinct cosets formed by partitioning G based on the equivalence relation defined by N. The quotient group G/N is structurally equivalent to the trivial group, and $G/\{e\}$ is isomorphic to G. $|G: N|$, or the index of N in G, is the size of G/N. The index is likewise equivalent to the period of G divided by the period of N if G is finite. If G and N are not finite, the set G/N may be, for example, $\mathbb{Z}/2\mathbb{Z}$.

An isomorphism is a type of mapping between two mathematical structures that preserves their structure. In other words, if two structures are isomorphic, they are essentially the same structure, but may differ in their notation or presentation. If there is a bijective homomorphism (a function that maintains the group structure) between two groups, say G and H, then the two groups are deemed to be structurally equivalent or isomorphic. The implication is that the elements of G and H behave same way under the group operations, and the only difference is the way they are labeled or presented. Isomorphisms are important in mathematics because they allow us to study and understand complex structures by comparing them to simpler, more familiar structures. For instance, if we can show that two groups are isomorphic, we can use our knowledge of one group to understand the other group. Moreover, isomorphisms are useful for constructing new examples of mathematical structures by finding isomorphic copies of known structures. For instance, if we know that two groups are structurally equivalent, we can create new groups that share the same structure. We also have the isomorphism theorems which are a set of three important theorems in group theory. They relate the concepts of homomorphisms and quotients of groups, and they aid in the comprehension of structure of groups. The first isomorphism theorem states that if we have a homomorphism between two groups, we can find a natural isomorphism between the image of homomorphism and the quotient group. The function $\phi_a: G \to G$ is defined by $\phi_a(g) = aga^{-1}$ for all $g \in G$ when we examine the inner automorphism theorem. Put differently, the map ϕ_a represents an isomorphism between G and itself. This theorem states that for every a, the inner automorphism ϕ_a preserves the group structure in a group G, where $\phi_a(g) = aga^{-1}$. Since it is an isomorphism from G to itself, the map is bijective and preserves the group's functionality. It is represented by $Inn(G)$.

Theorem 27. *Let G be a group, $x \in G$, and φ denote the inner automorphism induced by x in G. Hence, an automorphism of G is φ* (Schupp, 1987).

Theorem 28. *Let* $f: G \to \tilde{G}$ *represent a homomorphism of groups. Let* $K = ker(f)$ *and* $\widetilde{H} = im(f)$ *be the kernel and image of f, respectively, where K is a normal subgroup in G and* $\widetilde{H}$ *is a subgroup of* $\tilde{G}$*. Therefore we have a natural isomorphism.* $\tilde{f}: G/K \to \widetilde{H},\ gK \longmapsto f(g)$. For a proof, see Schupp (1987).

Theorem 29. *Consider G as a group, where* $H \leq G$ *and* $K \lhd G$*. Then there is a natural isomorphism* $HK/K \to \widetilde{H}/(H \cap K),\ hK \longmapsto h(H \cap K)$. For a proof, see Schupp (1987).

Theorem 30. *Consider the group* G*, where* $K \lhd G$ *and* $N \lhd K$ *and* $N \lhd G$ *are present. Then* $K/N \lhd G/N$ *and* $(G/N)/(K/N) \cong G/K$*, with the isomorphism* $(gN) \cdot (K/N) \longmapsto gK$. For a proof, see Schupp (1987).

In the concept of isomorphism and count of cyclic subgroups, Dillstrom (2016) presented two tables. We first analyze the count of cyclic subgroups in small groups (Table 1) before listing their isomorphism types (Table 2).

Table 1: Count of Cyclic Subgroups

$\boldsymbol{G}$	$\lvert\boldsymbol{C(G)}\rvert$	Note	$\boldsymbol{G}$	$\lvert\boldsymbol{C(G)}\rvert$	Note
$\langle \boldsymbol{e} \rangle$	1	$\lvert C(G)\rvert = 1$	C_8	4	$\lvert C(G)\rvert = \lvert G\rvert/2$
$\boldsymbol{C_2}$	2	$\lvert C(G)\rvert = \lvert G\rvert$	$C_4 \times C_2$	6	$\lvert C(G)\rvert = \lvert G\rvert - 2$
$\boldsymbol{C_3}$	2	$\lvert C(G)\rvert = \lvert G\rvert - 1$	C_{2^3}	8	$\lvert C(G)\rvert = \lvert G\rvert$
$\boldsymbol{C_4}$	3	$\lvert C(G)\rvert = \lvert G\rvert - 1$	D_8	7	$\lvert C(G)\rvert = \lvert G\rvert - 1$
$\boldsymbol{C_2 \times C_2}$	4	$\lvert C(G)\rvert = \lvert G\rvert$	Q_8	5	$\lvert C(G)\rvert = \lvert G\rvert - 3$
$\boldsymbol{C_5}$	2	$\lvert C(G)\rvert = 2$	C_9	3	$\lvert C(G)\rvert = \lvert G\rvert/3$
$\boldsymbol{C_6}$	4	$\lvert C(G)\rvert = \lvert G\rvert - 2$	$C_3 \times C_3$	5	$\lvert C(G)\rvert = \lvert G\rvert - 4$
$\boldsymbol{D_6 \cong S_3}$	5	$\lvert C(G)\rvert = \lvert G\rvert - 1$	C_{10}	4	$\lvert C(G)\rvert = \lvert G\rvert - 6$
$\boldsymbol{C_7}$	2	$\lvert C(G)\rvert = 2$	D_{10}	7	$\lvert C(G)\rvert = \lvert G\rvert - 3$

Source: Dillstrom (2016)

Table 2: Small Order Groups

$\|G\|$	Number of Groups	Isomorphism Tyes
1	1	$\langle e \rangle$
2	1	C_2
3	1	C_3
4	2	$C_4, C_2 \times C_2 \cong V_4$
5	1	C_5
6	2	$C_6, D_6 \cong S_3$
7	1	C_7
8	5	$C_8, C_4 \times C_2, C_{2^3}, D_8, Q_8$
9	2	$C_9, C_3 \times C_3$
10	2	C_{10}, D_{10}

Source: Dillstrom, (2016)

From Table 1, Dillstrom (2016) proposed and established the following propositions and theorems:

Proposition 9. Consider G as finite-ordered. $|C(G)| = 1$ if $G = \langle e \rangle$ is true. For a proof, see Dillstrom (2016).

Proposition 10. Consider G as finite-ordered. If, a certain prime p, $G = C_p$, then $|C(G)| = 2$. For a proof, see Dillstrom (2016).

The set of all elements of G that are mapped to the identity element of H is known as the kernel of a group homomorphism $f: G \to H$. The identity element of G is always present in the kernel, which is a subgroup normal in G. Consequently, the preimage of the singleton set $\{e_H\}$, or the subset of G consisting of all the elements of G that are mapped by f to the element e_H, is the kernel of f. This is the case if e_H is the identity element of H. Typically represented by $ker\ f$:

$$ker\ f = \{g \in G \mid f(g) = e_H\}$$

Identity elements are preserved by group homomorphisms, hence the identity element e_G of G must be a member of the kernel. If and only if the homomorphism f has the singleton set $\{e_G\}$ as its kernel, then it is injective.

Proposition 11. If there exists a homomorphism $\varphi: G \to H$. Then φ is injective whenever $ker\ \varphi = \{e\}$ (Dummit & Foote, 2004).

The direct product is defined as the natural component-wise operation on the Cartesian product

$$\prod_{k=1}^{n} G_k = G_k \times \cdots \times G_n, (x_1, \cdots, x_n) \cdot (y_1, \cdots, y_n) \coloneqq (x_1 y_1, \cdots, x_n y_n)$$

which defines a group structure. The order of a direct product, being a Cartesian product, is exactly the product formed by the orders of its components. The algebraic object that is produced satisfies a group's axioms. If a group P satisfies the following three conditions, it is structural similarity to the direct product of G and H.

- $G \cap H$ is trivial.
- Each member of P is uniquely expressed as product of members of G and H.
- Both H and G are normal in P.

One generalization of a direct product of groups is the semi-direct product. Denoted as $P = G \rtimes H$ or $P = H \ltimes G$, we say P is the semi-direct product of G and H. As a result, when H acts on G, P is a semi-direct product. For clarity, it is recommended to indicate the normal subgroup. The two concepts associated with the semi-direct product are the inner semi-direct product and outer semi-direct product. In order to achieve the inner semidirect product of G and H, the third requirement can be relaxed such that out of the two subgroups G and H, only one needs to be normal. Suppose that G is a normal subgroup and as a group under composition, let $Aut(G)$ represent the group of all automorphisms of G. We construct a group homomorphism $\phi: H \to Aut(G)$ specified by conjugation, $\phi h(g) = hgh^{-1}$, for every $h \in H$ and $g \in G$. This allows us to form a group $P' = (G, H)$ having as its definition the group operation $(g_1, h_1) \cdot (g_2, h_2) = (g_1 \phi h_1(g_2), h_1 h_2)$ for $g_1, g_2 \in G$ and $h_1, h_2 \in H$. The subgroups G and H determine P up to isomorphism. According to Dummit and Foote (1991), the construction of the group P from its subgroups is known as the internal semi-direct product or inner semi-direct product. The outer semi-direct product considers two groups N and H which are combined in a specific way to form a new group. The construction of a new group $N \rtimes_\phi H$, also known as the outer semi-direct product of N and H with regard to ϕ, is possible given any two groups N and H and a group homomorphism $\phi: H \to Aut(N)$ (Robinson, 2003):

- The set underlying the structure is $N \times H$, the Cartesian product of N and H.
- The homomorphism ϕ determines the group operation:

  $(N \rtimes_\phi H) \times (N \rtimes_\phi H) \to N \rtimes_\phi H$

  $(n_1, h_1) \cdot (n_2, h_2) = (n_1 \phi h_1(n_2), h_1 h_2)$

  for $n_1, n_2 \in N$ and $h_1, h_2 \in H$.

The direct product of distinct normal subgroups respectively from distinct groups is also a normal subgroup of the new group formed by the direct product of the distinct groups.

Theorem 31. *Given the groups G and G'. Let $H \lhd G$ and $H' \lhd G'$. This yields* $(H \times H) \lhd (G \times G')$. For a proof, see Robinson (2003).

Theorem 32. *For a finite group G, if $H \leq G$ and $K \leq G$, and both H and K are normal subgroups, and $H \cap K = \{e\}$, then $|G| = |HK| = |H| \times |K|$.* For a proof, see Robinson (2003).

By using Lagrange's theorem, $|H|$ divides $|G|$ for a finite group G if H is a subgroup of G.

Theorem 33. $|G| = [G: H] \cdot |H|$ *when* $H \leq G$ (Dummit & Foote, 2004).

Remarks: If G is infinite, then Theorem 33 is still valid as long as $|G|$, $|H|$, and $[G: H]$ are taken to be cardinal integers.

Theorem 34. $[G: K] = [G: H][H: K]$ *if H is a subgroup of G and K is a subgroup of H*. For a proof, see Robinson (2003).

Remarks: If $K = \{e\}$ where $e \in G$, then $[G: \{e\}] = |G|$ and $[H: \{e\}] = |H|$. Hence $|G| = [G: H] \cdot |H|$.

Theorem 35. *Given a prime number p and a finite group G, if p divides the period of the group, then G has to contain an element with period p (Groups abelian to Cauchy's theorem)*. For a proof, see Robinson (2003)

The Sylow theorems constitute a series of important theorems in group theory that provide information about finite group structure. They provide insight about the existence and properties of certain subgroups, known as Sylow subgroups, within a finite group. The theorem as it is explained, if G is a finite group with prime number p, then G contains subgroups of size p^k for every positive integer k for which p^k is a divisor of $|G|$.

Definition: Let G be a finite group with $|G| = p^k m$, where p is a prime dividing $|G|$, $k \geq 1$, and $p \nmid m$.

Theorem 36. *If $H \leq G$ of size p^k. Then we have Sylow p-subgroup being H* (Kämmerer & Paulson, 1999).

Definition: A Sylow p-subgroup of G is a subgroup $H < G$ with order p^n, where $|G| = p^k \cdot m$ and p does not divide m. The collection of such subgroups is denoted by $Sylp(G)$.

Theorem 37. (Second Sylow theorem). *Every pair of Sylow p-subgroups is conjugate, implying that they are structurally equivalent: if H and K are Sylow p-subgroups, there is an element $g \in G$ such that $g^{-1}Hg = K$* (Kämmerer & Paulson, 1999).

Claim that G acts on a set X. The orbit of x for each $x \in X$ is defined as $Orbit(x) = \{g \cdot x | g \in G\}$, denoting the subset of X that may be reached from x by means of the action of elements in G. Furthermore, all components of G that leave x unchanged under the group action constitute the subgroup $Stab(x) = \{g \in G | g \cdot x = x\}$, which stabilizes x. These ideas are essential to comprehending group dynamics.

Proposition 12. (Orbit-Stabilizer Theorem). If G is a group acting on a finite set X, and $x \in X$, then $|Orbit(x)| \cdot |Stab(x)| = |G|$. For a proof, see Macauley (2014).

Proposition 13. If G, a p-group, acts on a set S via $\varphi: G \rightarrow Perm(S)$, then the condition $|Fix(\varphi)| \equiv_p |S|$ is satisfied. For a proof, see Macauley (2014).

Proposition 14. For a p-subgroup H of G, it holds that $[N_G(H): H] \equiv_p [G: H]$. For a proof, see Macauley (2014).

Theorem 38. *Suppose* $|G| = p^n m$. *Let* $H \leq G$ *where* $|H| = p^i < p^n$, *we have* $H \ntrianglelefteq N_G(H)$, *where* p *divides the quotient* $[N_G(H): H]$. For a proof, see Macauley (2014)

Theorem 39. *(Third Sylow Theorem). Define* n_p *as the count of Sylow* p*-subgroups where,*

- $n_p | m$
- $n_p \equiv 1 \ mod \ p$
- $n_p = \frac{|G|}{|N_G(H)|}$ *where* H *is a Sylow* p*-subgroup and* $N_G(H)$ *denotes the normalizer of* H*, the largest subgroup of* G *in which* H *is normal* (Kämmerer & Paulson, 1999).

Remark: This theorem implies that $n_p | m$, where $|G| = p^n \cdot m$. For a proof, see Macauley (2014).

Theorem 36 asserts that there is at least one subgroup of G, known as a Sylow p-subgroup of period p^k, for each prime p dividing the period of G. Put differently, there is a subgroup H satisfying $|H| = p^k$ withing G, where p does not divide $|G/H|$ (the index of H in G). This theorem guarantees the existence of subgroups of certain orders within a finite group. The second theorem establishes the conjugacy relationship between Sylow p-subgroups. It states that for a finite group G, any pair of Sylow p-subgroups is conjugate. In other words, an element g exists in G such that conjugating H by g yields K. This result provides information about the structure and interrelations between Sylow p-subgroups within a group.

The third theorem assert, Sylow p-subgroups count in G is 1 $mod\ p$, which means if it is divided by p, a residue of 1 remains. Furthermore, this number divides the index m, which is the factor of $|G|$ not divisible by p. It provides information about the counting and divisibility properties of Sylow p-subgroups.

The Burnside theorem is an important theorem for comprehending finite group structure. Thus, it provides insight into the simplicity of groups whose orders follow a specific factorization pattern involving two distinct primes.

Theorem 40. (Burnside). *Let a group G posses period $p^a q^b$ such that p, q primes. Hence, without being a prime period cyclic, G is not simple*. For a proof, see Bender (1972).

Nilpotency is a concept that describes how 'nilpotent' a group is. Intuitively, a nilpotent group is one that is 'close' to being abelian, meaning that its commutator subgroup (subgroup formed by all commutators of group elements) is 'small'. Specifically, if G is finite-ordered and a sequence of subgroups exists, then the group is nilpotent

$$\{1\} = G_0 \lhd G_1 \lhd G_2 \lhd \cdots \lhd G_n = G$$

so that every factor group G_i/G_{i-1} is a normal subgroup of G/G_{i-1} and is also an abelian group. We also look at the nilpotency class of a group which measures how many steps are required in the nilpotent sequence to reach the trivial subgroup. It quantifies the extent to which the group deviates from being abelian. Formally, in a group G the nilpotency class is referred to as the length of the shortest possible nilpotent sequence for G. Some important examples of nilpotent groups include abelian groups (which have nilpotency class 1), p-groups (which have nilpotency class at most $|G|$), and the Heisenberg group (which has nilpotency class 2). Nilpotent groups have many special properties and are of interest in numerous mathematical disciplines, including as number theory, geometry, and physics. For example, finite p-groups are nilpotent and are crucial to algebraic number theory and Galois theory. Moreover, nilpotent Lie groups (consist of continuous groups that are 'almost' abelian) are fundamental objects in differential geometry and mathematical physics.

Theorem 41. *Consider a group G with $N \lhd G$. Then G is solvable if and only if both N and G/N are solvable*. For a proof, see Rotman (2006)

Proposition 15. If A and B have coprime periods, then $c(A \times B) = c(A)c(B)$, and subsequently $\alpha(A \times B) = \alpha(A)\alpha(B)$ (Dummit & Foote, 2004).

Theorem 42. *For any $n \geq 0$, $\alpha(G) = \alpha(G \times C_{2^n})$ is true if G represent any finite group*. For a proof, see Dillstrom (2016).

Remark: If $G \cong C_{2^n}$, consequently $\alpha(G) = 1$.

The relationship between the alpha invariant and nilpotency of finite groups is that the alpha invariant can be used to provide a standard for a finite group to be nilpotent. In other words, If and only if the ratio of $G's$ cyclic subgroup count to its order is smaller than or equal to $\frac{1}{2}$, then a finite group G is nilpotent.

Theorem 43. *Consider a finite group G. G is nilpotent if and only if $\alpha(G) \leq \frac{1}{2}$* (Garonzi & Lima, 2018).

Similarly, the alpha invariant provides a useful criterion to ascertain if a finite group is solvable, and has important impact on the structure and properties of finite groups. Moreover, the Feit-Thompson Theorem provides a deep connection between the alpha invariant and the solvability of finite groups, and is a crucial result in the field of finite groups.

Theorem 44. *Consider a finite group G. If $\alpha(G) \leq \frac{3}{4}$, then G is solvable* (Garonzi & Lima, 2018).

Statement of the Problem

The research seeks to contribute to the field of group theory by considering a conjecture and its possible extension on the relationship between finite groups, nilpotency, the alpha invariant, and the structure of the group.

Research Objectives

The objectives of this thesis includes:

- To prove the conjecture on the condition that $\alpha(G) \leq \frac{3}{4}$ and G is not nilpotent, will yield $G \cong D_{24} \times C_{2^n}$, and specify the values of n for which the conjecture holds.
- To examine the group $G \cong D_{24} \times C_{2^n}$ when $\alpha(G) = \frac{3}{4}$ and G is not nilpotent, with a focus on identifying if it possesses a non-trivial centre.
- To demonstrate that under these conditions of $G \cong D_{24} \times C_{2^n}$, with the alpha invariant being $\frac{3}{4}$ and G not being nilpotent, then G necessarily contains subgroup, non-trivial in nature, that is both normal and characteristic.
- To investigate normal subgroups within the group $D_{24} \times C_{2^n}$ and analyze their structural properties up to isomorphism.

- To characterize finite groups G for which the count of cyclic subgroups $|C(G)| = |G| - 6$ is achieved, exploring the structural properties and characteristics that lead to this cardinality.

Significance of the Study

The research will provide existence and uniqueness of non-nilpotent group with a high proportion of cyclic subgroups, and highlights the importance of understanding the relationship between the group structure and characteristics of its subgroups as it relates to the conjecture. The significance of examining cyclic subgroups and its count in finite group lies in its implications for understanding the properties and structure of finite groups. This area of research holds fundamental position in group theory and finds practical applications in various fields. By investigating the distinct cyclic subgroups count in a finite group, we can gain insights into the group's structure and its relationship to other groups. This knowledge is essential for various applications, ranging from cryptography in computer science to representation theory in chemistry, and even in advanced theories and implementations in computer science (Wen, 2022). Secondly, the significance of this research on the value $\frac{3}{4}$ in the classification of non-nilpotent groups is that it has been shown that if a group has $\alpha(G) = \frac{3}{4}$ and is nilpotent, so G is a 2-group and every group that demonstrates $\alpha(G) > \frac{3}{4}$ has been categorized (Garonzi & Lima, 2018) as cited by Cayley (2021). However, for non-nilpotent groups, the significance of $\alpha(G) = \frac{3}{4}$ is not well-documented according to the researcher's perspective. Therefore, the study will lead to the full appreciation of the conjecture and its possible extensions into gray areas that will contribute to knowledge in the academic field.

Delimitation

To address the research problems, the study acknowledges that examining infinite groups could be relevant. However, the focus is deliberately restricted to finite groups that meet the criteria of the research problem and its potential extensions. This delimitation ensures a more manageable scope and permits a more thorough investigation of finite groups within the context of the conjecture.

Limitation

A potential restriction of this research is the availability or accessibility of certain groups that may meet the specified criteria for testing the conjecture, as well as our scope in terms of the order of groups which will be finite.

Definitions of Terms

We present key ideas and concepts that will be employed throughout this thesis. For additional information, we recommend referring to the works by Dummit and Foote (1991; 2004), Fraleigh (2003), Hall (1999), and Isaacs (2008).

Definition 1 (A Group)

A group G is endowed with an operation (denoted as $*$) and a nonempty set that meet the subsequent assumptions

- Closure: If a, b is any element in G, then the operation $a * b$ is also in G.
- Associativity: For any elements a, b, c in G, $(a * b) * c = a * (b * c)$.
- Identity element: There is an element e in G where $a * e = e * a = a$ exists for every element a in G.

- Inverse element: For every element a in G, $a * b = b * a = e$ can be expressed for some element b in G.

Definition 2 (Period of element)

Considering the group G and $a \in G$, a has finite period if n and a positive integer with $a^n = e$, where $n > 0$. The order or period of a is expressed as $o(a)$ or $|a|$. It is claimed that a has infinite period if $\forall n \in \mathbb{N}^+$, $a^n \neq e$. In this case, we write $o(a) = \infty$.

Definition 3 (Subgroup of a Group)

Considering the group G. If H forms a group under the operation induced by G, then a subset H of G is called a subgroup of G. We designate $H \leq G$ if H is a subgroup of G.

Definition 4 (Centre and centralizer)

Let G be finite-ordered:

- $Z_G = Z(G) = \{z \in G |\ za = az, \forall a \in G\}$.
- If $a \in G$, the $C_G(a) = C(a) = \{z \in G | za = az\}$ is called the centralizer of a.

Definition 5 (cyclic subgroup)

The subgroup $\langle g \rangle = \{g^k : k \in \mathbb{Z}\}$ is a cyclic subgroup of G formed by g; if $G = \langle g \rangle$, consequently we claim that G is a cyclic group in which g is a generator of G. Thus, suppose G be a group with multiplication as the group operator. We state that G is a cyclic group only if $\exists a \in G$ and $G = \langle a \rangle$, and $\langle a \rangle = \{a^k : k \in \mathbb{Z}\}$. Also, if G be a group with addition as the group operator. Then, G is considered cyclic only when $\exists a \in G$ resulting in $G = \langle a \rangle$, and $\langle a \rangle = \{ka : k \in \mathbb{Z}\}$.

Definition 6 (Euler's phi function)

The function known as Euler's totient determines the relative primeness (represented as p) of a set of positive integers less than or equal to n. Thus,

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

Definition 7 (Class equation)

Suppose, in finite group G, the class equation takes the form

$$|G| = |Z(G)| + \sum |Cl_G(x_i)|,$$

with the sum ranging over distinct conjugacy classes of size greater than one.

Definition 8 (Normal subgroup)

When a subgroup N of G remains intact when conjugated, meaning that any element in N that has a conjugate in G also belongs to N. This is known as normality.

Definition 9 (Quotient group)

Let $N \lhd G$. The collection of all left cosets of N in G such that $G/N = \{aN: a \in G\}$ is the quotient group, which we refer to as G/N.

Definition 10 (Isomorphism)

An isomorphism is defined as a map $\phi: G \to H$, and G and H are said to be isomorphic or to be of the same isomorphism type $G \cong H$, if

- ϕ is a homomorphism $\big(\phi(xy) = \phi(x)\phi(y)\big)$, for all $x, y \in G$
- ϕ is a bijection

where in G, the product xy is calculated on the left, and in H is the product $\phi(x)\phi(y)$ on the right. It makes sense that if a map ϕ preserves the group structures of both its domain and codomain, it is a homomorphism.

Definition 11 (Automorphism)

An automorphism $\phi: G \to G$ denote bijective map, in which for any $x, y \in G$:

- ϕ is a homomorphism $\left(\phi(xy) = \phi(x)\phi(y)\right)$
- $\phi(e) = e$, where $e \in G$
- $\phi(x - 1) = (\phi(x))^{-1}$

Definition 12 (Characteristic subgroup)

If H satisfies $\phi(H) \subseteq H$ for any automorphism ϕ of G, it is a characteristic subgroup and a subgroup of G. Since $\phi^{-1}(H) \subseteq H$ implies an inverse inclusion $H \subseteq \phi(H)$, $\phi(H) = H$ holds true for every automorphism ϕ of G.

Definition 13 (Inner Automorphism theorem)

Suppose G represent the group where $a \in G$. If $\phi_a: G \to G$ by $\phi_a(g) = aga^{-1}$ for all $g \in G$. Then, ϕ_a is an automorphism of G.

Definition 14 (Direct product)

Given the groups (H, Δ) and $(G,*)$, $G \times H$ has the following definition:

- The Cartesian product, $H \times G$, is the underlying set. In other words, in the ordered pairs (h, g), $g \in G$ and $h \in H$.
- Component-wise, the binary operation on $H \times G$ is defined as follows: $(h_1, g_1) \cdot (h_2, g_2) = (h_1 \Delta h_2, g_1 * g_2)$.

Definition 15 (Nilpotent class)

A group G is called nilpotent of class k if the kth term of its lower central series is the trivial subgroup $\{1\}$, where the kth term is recursively defined in this way: $G_1 = G$ and $G_{k+1} = [G_k, G]$ where $[G_k, G]$ denotes the subgroup formed by the commutators $[g, h]$ with $g \in G_k$ and $h \in G$.

Organisation of the Study

The research consists of five chapters, with Chapter One already discussed. The succeeding Chapters are organized as follows: In the second chapter, relevant group structure literature is reviewed and also the relationship between alpha invariant and nilpotency in finite groups. Description of the research methodology and approach is the focus of Chapter Three. The key findings of the study are captured in Chapter Four, while Chapter Five summarizes, concludes, and makes recommendations.

CHAPTER TWO

LITERATURE REVIEW

Introduction

In this chapter, there is a review of the work of several authors whose research in the field of group theory have laid the foundation for further studies in this area. First, there is a notional look at the seminal works of foundational authors in the field group theory and then a careful review of research works in terms of contributions and theories that are directly related to the achievement of the objectives of the research work.

The Evolution and Advancement of Group Theory

The evolution of group theory can be traced back to four main sources: Poincaré and Klein (1876), cited by Kleiner (1986); the classical algebra Lagrange (1770), number theory Gauss (1801), geometry Klein (1874), and analysis Lie (1874). In the paper (Lagrange, 1770), contributions to the study of algebraic equations and permutations, introducing fundamental concepts that later influenced the formal development of group theory were made. The development of group theory was greatly aided in 1801, by Carl Friedrich Gauss's investigation of modular arithmetic and additive and multiplicative groups in number theory. In his Erlangen Programme (Klein, 1872), Felix Klein proposed a framework for understanding various geometries through transformation groups, strengthening the connection between geometry and group theory. In 1874, Sophus Lie introduced the concept of continuous transformation groups, later known as Lie groups, establishing a fundamental link between differential equations and group theory

(Lie, 1874). Additionally, Henri Poincar´e and Felix Klein made significant contributions to the application of Lie groups in analysis, further advancing group theory (Poincaré & Klein, 1890). Group theory, a subfield of abstract algebra, explores the properties and structures of mathematical groups, thus sets equipped with an operation that satisfies specific axioms. As we delve into the literature, we encounter the profound insights and groundbreaking work of several influential authors who have significantly advanced the understanding of group theory.

One of the foundational figures in the development of group theory is Évariste Galois, a French mathematician from the 19th century cited by Kleiner (1986). The paper made remarkable contributions to the study of group theory, particularly through exploration of solvability by radicals and the concept of Galois groups. The work also laid the groundwork for understanding the relationships between symmetries, permutations, and equations, forming the basis for modern group theory. Building on Galois's foundation, Fraleigh (2003) noted that Augustin-Louis Cauchy made significant strides in the early 19th century by formalizing a systematic approach to the study of groups. His work led to Cauchy's theorem, which states that every finite group with prime order is cyclic. This result marked a crucial milestone in the development of group theory and spurred further investigations into the properties of finite groups.

In the later nineteenth and early twentieth centuries, the German mathematician Georg Cantor and British mathematician Arthur Cayley made substantial contributions to group theory, (Robinson, 1996). Cantor's work on infinite sets and cardinality influenced the study of infinite groups, while Cayley formalized the concept of abstract groups and introduced the notion of group presentations. Their insights paved the way for the generalization of group theory beyond finite structures. In the mid-20th century, Hungarian mathematician Paul Erdós and Paul Turán made significant contributions to the study of permutation groups and their applications (Miller, 1922). Their work, along with that of Hall (1999) focused on the representation theory associated with finite groups, providing essential tools for understanding the structure of groups. Moving into more recent times, in the area of classification theorems, contributions of contemporary mathematicians who have expanded the boundaries of group theory such as Aschbacher et al. (2011), through their research developed the Classification of Finite Simple Groups. The monumental achievement helped to classify all finite simple groups and provided a comprehensive framework for understanding their structure. However, a note must be taken that two volumes by Gorenstein (1982, 1983) cover the low rank and odd characteristic parts of the proof for the classification theorem.

The research study by Cayley (2021) delves into the properties of finite groups concerning the ratio of cyclic subgroups to the group's order. The study presents a comprehensive analysis of various aspects related to this ratio, with a focus on the parameter $\alpha(G)$. The fundamental objectives of the research, which include an investigation into the basic properties ofα, computation of $\alpha(G)$ for different groups, and classification of Dihedral groups exhibiting $\alpha(G) = \frac{3}{4}$. Additionally, the study explores the minimum number of involutions required in groups with $\alpha(G) = \frac{3}{4}$, thus groups satisfying $\alpha(G) = \frac{3}{4}$ must have a minimum number of involutions, specifically $\frac{|G|}{2} - 1$. The research emphasizes the significance of understanding groups where the ratio of cyclic subgroups to group order is $\frac{3}{4}$. It discusses previous works by Garonzi and Lima (2018) and the classification efforts by Wall (1970) and Miller (1920), providing a context for the current study. Cayley (2021) research highlights a partial nilpotent group classification with $\alpha = \frac{3}{4}$, conducted by Tărnăuceanu (2015). It introduces the concept of nilpotent groups and presents a useful characterization theorem along with a proposition stating that if a nilpotent group has $\alpha = \frac{3}{4}$, then it must be a 2-group. The proof of this proposition is provided, showing that for a nilpotent group, the Sylow p-subgroups must have certain properties regarding their cyclic subgroups, leading to the conclusion that the group must be a 2-group. The research discusses the classification of groups based on the number of cyclic subgroups. It begins by noting that all groups with alpha greater than $\frac{3}{4}$ have been classified, referring to the works of Garonzi and Lima (2018) and Wall (1970) suggesting that it is natural to investigate groups where alpha equals $\frac{3}{4}$ (Cayley, 2021). A computational analysis is also undertaken to determine a comprehensive group classification with $\alpha = \frac{3}{4}$, (Group, 2021). The investigation into the properties of finite groups, particularly in relation to their cyclic subgroups, simplicity, involutions, and commuting probabilities, has been a subject of interest among mathematicians seeking to understand fundamental characteristics of group structures. Cayley (2021) work introduces a series of propositions and definitions that shed light on various aspects of finite groups in his preliminary findings.

Proposition 5.4 (Cayley, 2021), serves as a foundational insight, establishing that among dihedral groups, only D_{16} and D_{24} possess α values equal to $\frac{3}{4}$. This proposition lays the groundwork for subsequent inquiries into groups exhibiting specific values of α, highlighting the significance of these groups within the broader context of finite group theory. It is noted that the count of group G elements satisfying $g^2 = 1$ and represented by $I(G)$, is equal to or greater than twice the ratio of cyclic subgroups to the order of the group ($\alpha(G)$), minus one. Additionally, $\alpha(G)$ equals one, holds exactly when G is an elementary abelian 2-group. This provides valuable understanding into the relationship between the count of involutions in a group and its α value. By establishing a lower bound on the count of involutions based on alpha value, thereby offering a method for identifying elementary abelian 2-groups and enriching our understanding of group classification.

Moving forward, the findings introduce the concept of commuting probability, a measure of the likelihood of elements commuting within a group. This concept sets the stage for analysing the dynamics of group elements and their interactions, offering valuable insights into group behavior. On the other hand, if the ratio of cyclic subgroups to the period of the group ($\alpha(G)$) is greater than or equal to $\frac{1}{2}$, then the commuting probability ($cp(G)$) is greater than or equal to the square of the ratio of elements in the group that satisfy $g^2 = 1$ to the period of the group ($I(G)/|G|$), squared. This establishes a lower bound on the commuting probability of a group, further elucidating the relationship between α and the commuting behavior of group elements. By connecting α values to commuting probabilities and the number of involutions, it contributes to a deeper understanding of group structures and dynamics.

The research then moves into a different gear particularly focusing on finite groups and their representations. The discussion begins with the introduction of the general linear group (GL) of a vector space V over a field F. By defining $GL(V)$ as the group of automorphisms of V and $GL(n, F)$ as the group of invertible $n \times n$ matrices over F, this definition establishes a fundamental connection between linear transformations and group theory. The subsequent note emphasizes the structural equivalence of $GL(V)$ and $GL(n, F)$ under certain conditions, highlighting the versatility of GL in mathematical contexts involving linear algebra.

The research extends the discussion to representations of groups on vector spaces, introducing group homomorphisms from G to $GL(V)$. This definition establishes the foundation for comprehending the manner in which groups operate on vector spaces via linear transformations, preparing the groundwork for delving deeper towards the investigation of group representations. There is also the introduction of faithful representations, characterizing representations where the group homomorphism from G to $GL(V)$ is injective. Faithful representations are pivotal in understanding group actions on vector spaces, providing insights into the structure and behavior of groups. Proposition 5.12 Shifts the focus to non-abelian simple groups, establishing an upper bound on the number of conjugacy classes in such groups. By bounding the count of conjugacy classes by half the order of the group, this proposition offers valuable insights into the distribution of elements within non-abelian simple groups, contributing to their classification and structural understanding. Proposition 5.13 of the same research then establishes a maximum value for the ratio of cyclic subgroups to the order of non-abelian simple groups, providing valuable constraints for further classification efforts. Using commuting probabilities within the group, the proposition offers a precise upper limit for $\alpha(G)$, enhancing our comprehension of cyclic subgroup distribution. Proposition 5.13 is complemented by presenting an even tighter upper bound for α (G) in non-solvable groups. Garonzi and Lima (2018) findings underscore the importance of studying non-solvable groups, refining our understanding of cyclic subgroup distribution and contributing to classification endeavors. This discussion is extended to simple groups, affirming that $\alpha(G)$ cannot equal $\frac{3}{4}$, thus relying on the results of Proposition 5.13, illustrating the broader implications of the established upper limit on $\alpha(G)$ for simple group classification.

On the preliminary findings of the same research, it further establishes that if $\alpha(G) = \frac{3}{4}$, then G must be solvable, shedding light on the structural characteristics of groups with a high proportion of cyclic subgroups. However, the ratio of cyclic subgroups to the group's order, $\alpha(G)$, cannot equal $\frac{3}{4}$ for the non-abelian group G of period pq, such that p is not a divisor of $q - 1$ and p and q are prime numbers. Similarly, if G is isomorphic to the affine general linear group $GA(1, p)$, then the ratio of cyclic subgroups to the period of the group, represented by $\alpha(G)$, cannot equal $\frac{3}{4}$. In their exploration of the connection between group families containing involutions, the author considers the classifications presented by Miller (1920) and Wall (1970). Miller (1920) categorized groups with $|G|$ involutions into 15 families, while Wall refined this classification by focusing on groupings into four families, each with at least $|G|^2$ involutions. The research primarily investigates groups with $\alpha = \frac{3}{4}$. The authors start by proving Proposition 6.1, which establishes a lower bound on the count of involutions in a group G when alpha equals $\frac{3}{4}$. They demonstrate that if$\alpha = \frac{3}{4}$, then $n_2(G) \geq |G|^2 - 1$. This result is significant as it provides a foundational understanding of the distribution of involutions within groups based on their order. The subsequent analysis involves understanding the implications of $n_2(G)$ in relation to the families described by Miller and Wall. When a group G fulfills $n_2(G) = |G|^2 - 1$, it fits into one of Miller's families. Conversely, if $n_2(G) > |G| - 1$, the group falls into one of Wall's families. Additionally, for all $k \neq 1, 2, 3, 4, 6$, the given inequalities are meticulous if $n_2(G) \neq 0$, leading to a clear categorization of the group into Wall's families. Further explaining Wall's families, the author describes the characteristics of core groups within these families. These descriptions outline various presentations and properties of the core groups, such as being abelian, the product of dihedral groups, or specific presentations involving generators and relations. This detailed categorization helps in understanding the structural diversity among groups containing a significant number of involutions, thereby bridging the classifications proposed by Miller (1920) and Wall (1970) and providing concrete results regarding the lower bound of involutions in certain groups. The authors offer valuable insights into the underlying structures of finite groups.

The research presented explores Proposition 6.2, which focuses on the characterization of groups within Wall's First Family when $\alpha = \frac{3}{4}$. The main goal is to identify the specific forms of groups that fall within this classification, providing a rigorous proof to support the assertion. The proposition establishes that groups in Wall's First Family can be represented as $G = D_{16} \times C_n^2$ or $D_{24} \times C_n^2$. The proof begins by considering $G_0 = A \rtimes C_2$, with A serving as abelian group with the action $x \mapsto x^{-1}$ under the generator t of C_2. This action leads to an observation that $(at)^2 = aa^{-1}t^2 = 1$, indicating A contains $|A|$ cyclic subgroups of order 2. Further analysis of the structure of A leads to the determination of its form. It is shown that A possess cyclic group C_p with prime p, where p must be either 2 or 3. This deduction is crucial for narrowing down the possibilities for the structure of A. The author then asserts that if A is isomorphic to $C_m^4 \times C_l^8 \times C_n^3$, where $l > 0$, then A is isomorphic to C_8. This deduction is supported by demonstrating contradictions that arise when assuming other forms for A, such as $C_m^4 \times C_n^3$ or C_m^4. This careful analysis allows for a precise determination of the possible forms of A. Proposition 6.3 provides insights into the structural properties of core groups within Wall's Second Family. The proposition establishes that Wall's Second Family's core group has $\alpha = \frac{25}{32}$. The analysis proceeds to enumerate the various elements within this group based on their orders. Specifically, the group includes a single period 1 element, 35 elements of period 2, and 28 elements of order 4.

This enumeration provides a comprehensive understanding of the group's composition, laying the groundwork for further analysis. Applying Proposition 3.3, which relates the count of conjugacy classes to the group's structure, the author derives the count of conjugacy classes for $D_8 \times D_8$. By summing the contributions from elements of different orders, the total number of conjugacy classes is determined to be 50. Subsequently, the value of α is computed; for $D_8 \times D_8$, α is found to be $\frac{25}{32}$, which surpasses the threshold of $\frac{3}{4}$.

The research further provides a comprehensive analysis on associated Wall's Third Family groups and their properties, culminating in the important result that none of the groups in this family has $\alpha = \frac{3}{4}$ (Proposition 6.4). The proof is based on the presentation of Wall's Third Family core groups and involves the analysis of the count of elements of period 2 or less, the count of period 4 elements, and the cyclic subgroups count of group G_0. The research provides a detailed analysis of the elements and subgroups within the groups in Wall's Third Family. It introduces the concept of the count of words of a particular form having period 2 or less, denoted as $f(n)$, and derives an expression for f(n) using the binomial theorem. The research then proceeds to calculate the count of period 4 members and cyclic subgroups count of the group G_0, ultimately leading to the conclusion that none of the groups in this family has $\alpha = \frac{3}{4}$.

Proposition 6.5 (Cayley, 2021) asserts that Wall's Fourth Family has no groups that have $\alpha = \frac{3}{4}$, and it is supported by a detailed proof involving the analysis of the core groups' presentation and the determination of the count of elements of different orders within the groups. Furthermore, Theorem 6.6 is introduced, which holds that if $\alpha(G) = \frac{3}{4}$ and G includes a period 8 element, then $D_{16} \times C_2^n$ for $n \geq 0$.

The theorem provides a clear and specific characterization of groups with $\alpha = \frac{3}{4}$ and an element of order 8.

Finally, Proposition 6.7 demonstrates that Miller's First Family groups share a common parameter, α, which is determined to be $\frac{3}{4}$. This is achieved by analysing the structure of the core group, G0, within this family and calculating its number of cyclic subgroups and α value based on the orders of its elements. The proof provides insight into the uniformity of certain properties across all groups in this family. Overall, the research provides a valuable contribution to the study of group theory and abstract algebra, offering insights into the classification of groups based on the parameter α and the structural properties of groups in Miller's and Wall's families.

The study by Song and Zhou (2019) also contributes significantly to the understanding of finite group theory by establishing a precise criterion for the count of cyclic subgroups within a finite group. Through a rigorous analysis, the authors demonstrate that G being a finite group exhibits $|G| - 3$ only if G is isomorphic to one of the two groups, D_{10} or Q_8. To achieve the main result the author introduces Lemma 2.1. This lemma explains the link between a finite group G's period and count cyclic subgroups. Specifically, it asserts that if G's order is expressed as a product of prime powers, and the number of distinct primes involved exceeds three, then the discrepancy between G's order and the count of its cyclic subgroups surpasses the largest prime factor. By employing Theorem 1.5.1 (Song & Zhou, 2019) and exploiting the properties of prime powers, the proof rigorously deduces constraints of the coefficients in the lemma. This deduction process lays the groundwork for subsequent analyses in the research. The second part of the research provides information on the period of a group G and its cyclic subgroups. It states that if the period of G is equal to $p^a q^b$, where p and q are primes with $p < q$, then the difference between the period of G and the count of its cyclic subgroups is greater than q, except for the groups D_{2q}, C_3, D_{12}, C_6, or S_3, which have specific values for this difference. The analysis of the lemma begins by assuming that the difference between a finite group $G's$ period and cyclic subgroup count is less than or equal to q. By Lemma 2.1, it is proved that this presumption results in a paradox. The authors then proceed to consider two cases: when $q \geq 5$ and when $q = 3$ and $p = 2$. In the case where $q \geq 5$, the proof shows that G possesses just one Sylow q-subgroup, which is isomorphic to C_q. If $p \neq 2$, then G has no cyclic subgroups of period 4, and if $p = 2$, consequently G is structurally equivalent to D_{2q}. In either case, the number of $G's$ cyclic subgroups and its period diverge by more than q. In the case where $q = 3$ and $p = 2$, the authors show that G has at most 6 nontrivial 3-elements, and its Sylow 3-subgroup is isomorphic to C_3. G is structurally equivalent to C_6 or D_{12} if G contains a cyclic subgroup of period 4. G is structurally equivalent to C_6 or S_3 if G lacks a cyclic subgroup of period 4. In either case, the deviation in the group's period and the number of cyclic subgroups is equal to either 1 or 2.

The final section of the research addresses groups of period 2^a and their cyclic subgroups. The authors establish that if what separates the the group's period $|G|$ and the count of its cyclic subgroups is $2^a - 3$, then G is structurally equivalent to the quaternion group Q_8. The analysis initiates with the scenario where the exponent of G is 4, demonstrating the presence of a normal cyclic subgroup X of period 4. Subsequently, it shows that the centralizer $C = C_G(X)$ of X is an elementary 2-group. The analysis progresses by disproving the possibility of C being non-abelian, leading to the conclusion that C must be abelian. Moreover, the proof establishes that if C is abelian, it must be isomorphic to C_4 or $C_4 \times C_2$. By examining the potential structures of C and utilizing properties of cyclic subgroups, it deduces that C is isomorphic to C_4. Consequently, it concludes that G is structurally equivalent to the quaternion group Q_8. In summary, the research presents Theorem 2.4, which asserts that if the difference between the period of the centralizer of group G and $|G|$ equals 3, we have G being isomorphic to either D_{10} or Q_8.

The study of finite group theory lies at the junction of several deep mathematical questions, with each lead enriching the understanding of the underlying structure in these mathematical objects. Of several aspects relating to finite groups, the enumeration and classification of cyclic subgroups occupy an important place. The central theme of the research by Tarnauceanu and Lazorec (2019) is a parameter $\alpha(G)$, a measure defined within the poset structure of cyclic subgroups of a finite group G. It is just a ratio of the cardinality of the poset of cyclic subgroups, denoted by $L_1(G)$, against $|G|$. When attempting to understand the intricate structural details of finite groups, this parameter becomes crucial:

$$\alpha(G) = \frac{L_1(G)}{|G|}$$

In terms of overview, the study focuses on the nilpotent groups class C that are finite and with $\alpha(G) = \frac{3}{4}$, where $\alpha(G)$ is the ratio of the count of cyclic subgroups of a finite group G to the count of elements in G (Dillstrom, 2016). They show that if a 2-group is in this class, then it satisfies certain requirements. Additionally, the authors examine the inclusion of various classes of finite groups in the class C. The paper builds upon previous work in the field, such as the study by Garonzi and Lima (2018) which classified all groups with $\alpha(G) > \frac{3}{4}$ using a computational analysis (Cayley, 2021). The author also reference other works on finite groups and their cyclic subgroups, such as the paper by Song and Zhou (2019), which describes the finite groups with $|G| - 3$ cyclic subgroups. As the study unfolds, the research extends its inquiry into the inclusion ('appartenance') of various classes of finite groups to the defined class C. This exploration broadens the scope of the study, connecting it with existing classifications and shedding light on the interplay between group properties and the prescribed conditions for $\alpha(G)$.

The research conducted by Tarnauceanu and Lazorec (2019) investigates the properties of finite abelian groups within a specific class denoted as C. The study focuses on understanding the structural characteristics of these groups, providing essential lemmas and theorems to establish a comprehensive framework. The primary aim is to identify and describe the finite abelian groups belonging to C. In their work, the authors also introduces crucial concepts, such as p-groups of exponent p^m and the count of cyclic subgroups of period p^i, denoted as $n_{p^i}(G)$. The researchers lay the foundation by formulating the objective of finding abelian groups within the defined class C. They consider G as a finite p-group with size or period p^n and express Lemma 2.1 as follows in their paper. Then$\alpha(G) \leq \alpha(\mathbb{Z}_{p^n})$. In the same way, Lemma 2.2 looked at G being a finite p-group of period p^n where p is an odd prime number and $\alpha(G) < \frac{3}{4}$. These two pivotal lemmas, namely Lemma 2.1 and Lemma 2.2, have a significant impact on the subsequent development of the theorems. Lemma 2.1 establishes an inequality relating the alpha function of a finite p-group to that of the cyclic group $\mathbb{Z}_{p^n}$. The proof involves a meticulous manipulation of the period of the group and the count of cyclic subgroups. Lemma 2.2, on the other hand, defines an upper bound on the alpha function for finite p-groups of period p^n, demonstrating that alpha of G is always less than $\frac{3}{4}$. The following theorems were then established, that is Theorem 2.3 where $\mathbb{Z}_2^n \times \mathbb{Z}_4$ is the single finite abelian group included in C, given that $n \in \mathbb{N}$. Then, Theorem 2.4 on $G \cong \mathbb{Z}_{p^{d_1}} \times \mathbb{Z}_{p^{d_2}} \times \cdots \times \mathbb{Z}_{p^{d_k}}$, where $1 \leq d_1 \leq d_2 \leq \cdots \leq d_k$ and p is a prime number. This leads to the total count of cyclic subgroups of G being:

$$1 + \frac{1}{p-1} \sum_{i=0}^{k-2} p^{d_0+d_1+\cdots+d_i} \frac{p^{k-i}-1}{p^{k-i-1}-1} \left(p^{(k-i-1)d_{i+1}} - p^{(k-i-1)d_i}\right)$$

$$+ (d_k - d_{k-1}) p^{d_0+d_1+\cdots+d_{i-1}}$$

Theorems 2.3 and 2.4 form the core contributions of the study. Theorem 2.3 rigorously characterizes the finite abelian groups in C, revealing that the only groups satisfying the specified conditions are $\mathbb{Z}_2^n \times \mathbb{Z}_4$, where n belongs to the set of natural numbers. This result is derived through a meticulous analysis of alpha functions and the distinct properties of abelian p-groups. The formulation of Theorem 2.4 introduces an explicit formula for computing in any finite abelian p-group, the total count of cyclic subgroups. This formula, involving $g_p^k(i)$ and a comprehensive summation, not only contributes to the study's objectives but also offers a valuable tool for further investigations. The principal finding that explains the characteristics of the groups that are a part of C is established in Theorem 2.6.

Theorem 2.6. Given $G \in C$. G is a 2-group such that $G' = \Phi(G)$ or $n \in \mathbb{N}$ where $G/G' \cong \mathbb{Z}_2^n \times \mathbb{Z}_4$ and G' is elementary abelian. In Theorem 2.6, the structural characteristics of groups in C are outlined. The proof utilizes the finite nilpotent nature of these groups and establishes the circumstances in which a group in C is a 2-group with specific properties. The following cases are established:

- If $\alpha(G/G') > 3/4$, then G/G' is structurally equivalent to a group that Theorem 5 provides (Garonzi & Lima, 2018).
- As a consequence of Theorem 2.3, we have $G/G' \cong \mathbb{Z}_2^n \times \mathbb{Z}_4$, when $\alpha(G/G') = 3/4$

In case one, due to G/G' being abelian, the only valid option from the classification is $G/G' \cong \mathbb{Z}_2^n$, where the integer n is positive. This suggests that the centre of G, represented by $\Omega(G)$, contains the derived subgroup G', leading to the deduction that $\Omega(G) \subseteq G'$. The reverse containment is asserted to be prominent, resulting in the equality $G' = \Omega(G)$. In case 2, the alpha function values for G and G/G' are equal: $\alpha(G) = \alpha(G/G')$. The conclusion drawn from this equality is that the derived subgroup G' forms elementary abelian group of period 2.

The final section of their research focuses on other classes of finite 2-groups and connections between C such as generalized dicyclic 2-groups, (almost) extraspecial 2-groups, 2-groups possessing a cyclic maximal subgroup, and generalized dihedral 2-groups. In Proposition 3.1, they let $n \geq 2$ be an integer which is positive and let G be a 2-group with $exp(G) = 4$ and a period of 2^n. Consequently, $G \in C$ only when $I(G) = 2^{(n-1)} - 1$. Thus, Proposition 3.1 provides the required and sufficient condition for a 2-group to be a member of class C based on its order and its involution count. Specifically, for a 2-group G of period 2^n with exponent $exp(G) = 4$, the group belongs to class C only when the number of involutions, $I(G)$, equals $2^{(n-1)} - 1$. Additionally, the researchers contextualize the result within the broader framework of existing literature in Miller (1920) highlighting its relevance and significance in advancing the understanding characterization of finite 2-groups. This proposition extends its characterization to exponent 4, of all classes of finite 2-groups, including extraspecial and almost extraspecial groups. The study involves theoretical elements pertaining to group central products. The connection between internal and external central products is established, as provided for by Theorem 3.4 (Lewis et al., 2018). Moreover, according to Theorem 2.3 (Bouc & Mazza, 2004),

- Assuming G to be an extraspecial 2-group, r represents a positive integer in which $|G| = 2^{(2r+1)}$ and $G \cong D_8^{(r)}$ or $G \cong Q_8 \times D_8^{(r-1)}$ occur.
- When $|G| = 2^{(2r+2)}$ and $G \cong D_8^{(r)} \times Z_4$ are almost extraspecial 2- groups, r is a positive integer.

The theoretical aspects related to extraspecial and almost extraspecial 2-groups are explored, thereby showcasing their structures and connections with C. Lemma 3.2 (Tarnauceanu & Lazorec, 2019) provides an expression for the count of cyclic subgroups of period 2 in these groups. Theorem 3.3 of the same research establishes that, among extraspecial 2-groups, none belongs to C, while any almost extraspecial 2-group is a member of C. Similarly, in Theorem 3.4, the classification of C contains generalized dicyclic 2-groups reveals that they are structurally equivalent to abelian groups within C. Thus, the theorem posits that within the designated class C of groups, the only generalized dicyclic 2-groups are those isomorphic to $\mathbb{Z}_2^n \times \mathbb{Z}_4$, where the integer n is non-negative. The proof initiates by considering $n \geq 2$ and an abelian group A with period $2^{(n-1)}$. This group serves as the foundation for constructing the dicyclic extension $Dic_{2^n}(A)$, which includes A alongside additional elements such as γ, possessing an order of 4 and commuting with all elements of A.

Proceeding further, the proof demonstrates the presence of additional cyclic subgroups within the subgroup lattice of $Dic_{2^n}(A)$, beyond those inherent to A. By hypothetically introducing another cyclic subgroup H within $Dic_{2^n}(A)$, a contradiction arises, leading to the conclusion that the subgroup count in $Dic_{2^n}(A)$ is

$$|L_1(Dic_{2^n}(A))| = |L_1(A)| + 2^{(n-2)}.$$

Upon assuming that $Dic_{2^n}(A)$ belongs to class C, the equation $\alpha(Dic_{2^n}(A)) = \frac{3}{4}$ emerges, yielding $|L_1(A)| = 2^{(n-1)}$. Consequently, finite abelian groups A for which $|A| = |L_1(A)| = 2^{(n-1)}$ hold are identified. Moreover, conditions elucidating the structure of A with $exp(A) = 2^m$ are established, leading to the deduction that A, an abelian 2-group, has exponent 2 and period 2^{n-1}. This insight culminates in the confirmation of the isomorphism $Dic_{2^n}(A) \cong Z_2^{(n-2)} \times Z_4$ for every $n \in N$, affirming the inclusion of the abelian 2-groups $Z_2^{(n-2)} \times Z_4$ within class C. Brown (2010) presents more characteristics of dihedralization of several finite abelian groups as well as generalized dihedral groups. The authors then present Lemma 3.5, to determine which C generalized dihedral 2-groups are contained in. The lemma establishes categorization of finite abelian 2-groups with the parameter $\alpha(G)$ equal to $\frac{1}{2}$. The rationale behind using Lemma 3.5 is to establish a criterion for identifying the abelian 2-groups that meet the conditions necessary to belong to the class C by characterizing the structure of these groups based on their value of $\alpha(G)$. This subsequently leads to drawing upon Theorem 4.3 presented in Tárnáuceanu (2010) to ascertain the quantities, respectively period 2 and 8 cyclic subgroups, within the abelian 2-group G. These quantities, denoted as $n_2(G)$ and $n_8(G)$, are expressed as functions dependent on parameters n, a, and b, as detailed as follows:

$$n_2(G) = 2^{n+a+b} - 1$$

$$n_8(G) = 2^{n+2a+2b-2}(2^b - 1).$$

Utilizing these expressions, the analysis proceeds to establish the relationship

$$1 + n_2(G) = 2n_8(G)$$

which is pivotal in determining the values of a and b that satisfy this equation. Ultimately, it concludes that $a = 0$ and $b = 1$, thereby establishing the isomorphism $G \cong \mathbb{Z}_2^n \times \mathbb{Z}_8$. Conversely, it asserts that if G is isomorphic $\mathbb{Z}_2^n \times \mathbb{Z}_8$, where $n \in \mathbb{N}$, then

$$\alpha(G) = \alpha(\mathbb{Z}_2^n \times \mathbb{Z}_8) = \alpha(\mathbb{Z}_8) = \frac{1}{2}$$

thereby completing the proof. The study further investigates the relationship between the class C and the set of finite generalized dihedral 2-groups, culminating in Theorem 3.6, which asserts that C contains only certain types of finite generalized dihedral 2-groups. Specifically, the theorem concludes that such groups are structurally equivalent to $\mathbb{Z}_2^n \times D_{16}$, with $n \in \mathbb{N}$. The proof methodically demonstrates that if $D(G)$ is in C, then G ought to be a finite abelian 2-group with certain properties. Specifically, G is isomorphic to $\mathbb{Z}_2^n \times \mathbb{Z}_8$, as established by Lemma 3.5. Using Theorem 5.1 from Brown (2010), the isomorphism between $D(G)$ and $\mathbb{Z}_2^n \times D(\mathbb{Z}_8)$ is derived, ultimately leading to the conclusion that $D(G)$ is isomorphic to $\mathbb{Z}_2^n \times D_{16}$. Conversely, it is demonstrated that if G is isomorphic to $\mathbb{Z}_2^n \times D_{16}$, then $D(G)$ belongs to C. This reciprocal relationship solidifies the correlation between class C structurally equivalent to $\mathbb{Z}_2^n \times D_{16}$ and the set of finite generalized dihedral 2-groups.

Finally, the connection between the set of finite 2-groups with a cyclic maximal subgroup and the class C are described in Theorem 3.7 of the same paper. Theorem 3.7 is motivated by a membership analysis of non-abelian 2-groups in class C with cyclic maximal subgroups. Initially, attention is directed towards finite 2-groups with a cyclic maximal subgroup, excluding type $\mathbb{Z}_2 \times \mathbb{Z}_{2^{n-1}}$ abelian groups. These abelian groups, according to Theorem 2.3, are elements of C solely when $n = 3$. Therefore, the focus shifts to investigating the inclusion of non-abelian 2-groups with cyclic maximal subgroups in C. Theorem 4.1 from Suzuki (1986) offers a comprehensive categorization of such non-abelian 2-groups, listing the modular 2-group $M(2^n)$, dihedral group D_{2n}, generalized quaternion group

Q_{2^n}, and quasi-dihedral group S_{2^n}. By examining the count of cyclic subgroups of each group, denoted as $|L_1(G)|$, and solving the equation $\alpha(G) = \frac{3}{4}$ over positive integers, the conditions under which these groups belong to C are determined. In T$\acute{a}$rn$\acute{a}$uceanu and Toth (2015), the count of each of these groups' cyclic subgroups is referenced. The paper concludes by highlighting the significance of D_{16} as the sole finite generalized dihedral 2-group and the lone finite non-abelian 2-group with a cyclic maximal subgroup in C, barring direct factors of type $\mathbb{Z}_2^n$, where $n \geq 1$. The paper also suggests an alternative proof of Theorem 2.3 using a one-to-one mapping α from the class of finite abelian p-groups.

The research paper (Garonzi & Lima, 2018) titled 'On the number of cyclic subgroups of a finite group', the paper discusses the function $\alpha(G) = \frac{\text{c(G)}}{|\text{G}|}$ and its properties. The author, like Tárnáuceanu (2015), explore the basic properties of this function and relate to the probability of commutation. The paper aims to characterize groups G belonging to certain families F where $\alpha(G)$ is maximal and the classification where$\alpha(G) > \frac{3}{4}$. This suggests that the authors are interested in understanding the distribution of cyclic subgroups within finite groups and identifying patterns or extremal cases. In order to arrive at an asymptotic solution, the author additionally examine the cyclic subgroups count arising from the direct product G. This implies that they are investigating the behavior of $\alpha(G)$ as the group G grows in size or as it is raised to a higher power. The paper also addresses when $\alpha(G) = \alpha(G/N)$ provided G/N constitutes a symmetric group. This brings to bear the authors' intention of examining the relationship between the function $\alpha(G)$ and the structure of the quotient group G/N. The study discusses the concept of the commuting probability (cp) in G, noted it as the ratio of the count of commuting pairs of elements to the total possible pairs in G. It relates this probability in terms of conjugacy classes count of G, which has been extensively studied by various authors, with particular reference to Guralnick and Robinson (2006). The authors then present Lemma 1 and Lemma 2, (Brauer & Fowler, 1955), connecting the size of the elements contained in G that square to the identity (represented as $I(G)$) with the count of conjugacy classes and the commuting probability. This lemma establishes an inequality indicating that if the ratio of cyclic subgroups to the group order ($\alpha(G)$) is larger than or equivalent to $\frac{1}{2}$, then the commuting probability is bounded from below. Further results (Guralnick & Robinson, 2006) are introduced, providing bounds on the commuting probability based on properties of solvable and non-solvable groups. These results elucidate that if $\alpha(G)$ exceeds $\frac{1}{2}$, then certain index relations involving the maximal normal solvable subgroup or the Fitting subgroup of G hold. The rationale behind this presentation lies in elucidating relationships between various group-theoretic quantities, such as conjugacy classes, cyclic subgroups, and commuting probabilities. These relationships help in understanding the structural properties of groups and provide bounds or constraints on certain group-theoretic parameters based on others, thus contributing to a deeper understanding of group theory and its applications.

The authors then present Theorem 4 of the research that establishes a relationship between the ratio of cyclic subgroups to group order ($\alpha(G)$) in a finite non-solvable group G and the corresponding ratio in the symmetric group of degree

S_5. Stating in specifics, $\alpha(G)$ is bounded by $\alpha(S_5)$, with equality holding only when G is structurally equivalent to a direct product of S_5 and a cyclic group of period 2 raised to some power. The reason behind the theorem lies in the scrutiny of structure of finite non-solvable groups and their relation to the symmetric group S_5. The proof involves demonstrating that if $\alpha(G)$ exceeds a certain threshold, then G must possess a specific structural form. This is achieved through a series of deductions and considerations, including the analysis of minimal normal subgroups and the computation of ratios involving group orders. Furthermore, computational methods, such as those provided by Leemans and Vauthier (2006) and Group (2018), are utilized to verify the theorem's implications for groups of manageable sizes. The results confirm that the only finite non-solvable group satisfying the given conditions is indeed isomorphic to S_5. Additionally, the corollary derived from the theorems provides further insights by establishing constraints on the structure of groups with $\alpha(G)$ exceeding $\alpha(S_5)$, indicating their solvability and bounding the index of their Fitting subgroup.

The next section of the research (Garonzi & Lima, 2018) investigates groups with a 'large' value of $\alpha(G)$, specifically focusing on groups where $\alpha(G)$ exceeds $\frac{3}{4}$. This threshold is chosen because the set of values of $\alpha(G)$ signifies the largest non-trivial accumulation point. The rationale behind this choice is to understand the structural properties of groups exhibiting high ratios of cyclic subgroups to group order. To classify these groups, the approach involves recognizing that if $\alpha(G)$ is greater than $\frac{3}{4}$, then the ratio of the size of the set of elements in G that square to the identity ($I(G)$) to the group order ($|G|$) must exceed $\frac{1}{2}$. This observation allows for the utilization of Wall's classification theorem, outlined in Wall (1970) Section 7, which provides a systematic method for categorizing finite groups based on their properties and structural characteristics. Garonzi and Lima (2018) established a theorem (Theorem 5) where it is summarized as $\alpha(X) > \frac{3}{4}$ with X being a group. If G lacks C_2 as a direct factor, G is either trivial (leading to $\alpha(X) = 1$), or X is constructed by the direct product of an elementary abelian 2-group with G, resulting in $\alpha(X) = \alpha(G)$, or one of these possibilities arises:

- Case I. Assume $G \cong A \rtimes \langle \epsilon \rangle$, with $\langle \epsilon \rangle = C_2$ acting on A via inversion, and an integer $n \geq 1$ such that one of these cases applies:
  - $A = C_{3^n}$, $\alpha(G) = \frac{3 \cdot 3^n + 1}{4 \cdot 3^n}$.
  - $A = C_{4^n}$, $\alpha(G) = \frac{3 \cdot 2^n + 1}{4 \cdot 2^n}$.
- Case II. $G \cong D_8 \times D_8$ and $\alpha(G) = \frac{25}{32}$.
- Case III. G is a quotient D_{8^r}/N where $N = \{(a_1, \ldots, a_r) \in Z(D_8)^r \mid a_1, \ldots, a_r = 1\}$ and $\alpha(G) = \frac{3 \cdot 2^r + 1}{4 \cdot 2^r}$.
- Case IV. Suppose G is structured as $V \rtimes \langle c \rangle$, with $V = F_{2^{2r}}$ defined by the basis $\{x_1, y_1, \ldots, x_r, y_r\}$, and c, an element of period 2, acts trivially on y_i and satisfies $x_i^c = cx_i$ for each i where $\alpha(G) = \frac{3 \cdot 2^r + 1}{4 \cdot 2^r}$

The proof of Theorem 5 (Garonzi & Lima, 2018) expands on and broadens the previously known findings in (T*á*rn*á*uceanu, 2015).

In line with research that provides valuable insights into the structural properties of finite groups containing a predetermined count of cyclic subgroups, T$\acute{a}$rn$\acute{a}$uceanu (2016) works on a partial solution to an open problem posed in T$\acute{a}$rn$\acute{a}$uceanu (2015). Specifically, the research by T$\acute{a}$rn$\acute{a}$uceanu (2016) describes finite groups G that have cyclic subgroups satisfying $|G| - 2$. The introduction section of the research outlines fundamental results in group theory and highlights the main theorem established in T$\acute{a}$rn$\acute{a}$uceanu (2015), which identifies specific groups where the cyclic subgroups count equals the period of the group minus one. Building upon this, the research to extend this study by investigating finite groups G that the cyclic subgroups count satisfies the condition $|C(G)| = |G| - 2$. Additionally, the research notes that some finite groups with small orders, including $\mathbb{Z}_6$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, D_{12}, and $\mathbb{Z}_2 \times D_8$, demonstrate this characteristic. Theorem 1 of T$\acute{a}$rn$\acute{a}$uceanu (2016) provides a comprehensive analysis of finite group structure G where $|C(G)| = |G| - 2$. Analytical method employed mirrors that of Theorem 2 in T$\acute{a}$rn$\acute{a}$uceanu (2015). The proof begins by assuming $|C(G)| = |G| - 2$ for $|G| = n$ and proceeds by analysing the positive n divisors. By considering the count ni of cyclic subgroups of order d_i within $C(G)$, the equation $\sum_{i=1}^{k} n_i \, \phi(d_i) = n$ is established. Given $|C(G)| = n - 2$, the equation $\sum_{i=1}^{k} n_i(\phi(d_i) - 1) = 2$ is derived, leading to two distinct cases for further examination.

In Case 1, since $n_{i_0}\big(\phi(d_{i_0}) - 1\big) = 2$ and $n_i(\phi(d_i) - 1) = 0$ for every $i \neq i_0$, there exists $i_0 \in \{1, 2, \ldots, k\}$. It is established that $n_{i_0} = 2$ and $\phi(d_{i_0}) = 2$, thus $d_{i_0} \in \{3, 4, 6\}$. It is deduced that $d_{i_0} \neq 6$ and $d_{i_0} \neq 3$. Consequently, $d_{i_0} = 4$, implying G is a 2-group with exactly two period 4 cyclic subgroups. Further analysis shows that for $m = 3, \mathbb{Z}_2 \times \mathbb{Z}_4$ is the only group satisfying $|C(G)| = |G| - 2$. Again, for $m \geq 4$, by referencing Theorems 5.2 and 5.1 and Proposition 1.4 of Janko (2005), it is inferred that G is isomorphic to specific groups listed, none of which satisfy $|C(G)| = |G| - 2$ except for $\mathbb{Z}_2 \times D_8$.

In Case 2, the examination extends to situations where two distinct divisors d_{i_1} and d_{i_2} each contribute exactly one cyclic subgroup while all others yield none. Specifically, there exist $i_1, i_2 \in \{1, 2, \ldots, k\}$, $i_1 \neq i_2$, where $n_{i_1}\big(\phi(d_{i_1}) - 1\big) = 1$ and $n_{i_2}\big(\phi(d_{i_2}) - 1\big) = 1$ and $n_i(\phi(d_i) - 1) = 0$ for all $i \neq i_1, i_2$. It is established that $n_{i_1} = n_{i_2} = 1$ and $\phi(d_{i_1}) = \phi(d_{i_2}) = 2$, thus $d_{i_1}, d_{i_2} \in \{3, 4, 6\}$. Further analysis follows that if $d_{i_2} = 4$, then $d_{i_1} = 3$, leading to G containing period 3 and 4 normal cyclic subgroups, resulting in a contradiction. Also, if $d_{i_2} = 6$, then $d_{i_1} = 3$. It is deduced that G requires a Sylow 3-subgroup to be cyclic. Further analysis and deductions are made based on the value of n_2. Based on the derived possibilities for n_2, a conclusion that G is structurally equivalent to specific groups, none of which satisfy $|C(G)| = |G| - 2$ except for $\mathbb{Z}_6$ and D_{12}. Reference to Isaacs (2008) is made support the analysis regarding the count of subgroups of prime period p in G. The research significantly advances the understanding of finite group theory.

Within the study by Dillstrom (2016), the focus is on cataloging the count of unique cyclic subgroups within a finite group G. The paper highlights that despite its importance, this topic has received relatively little attention in existing literature. The subgroups' cyclic structure within a finite group is shown to be constrained by various factors, such as Cauchy's Theorem, which defines a natural lower bound for evaluating cyclic subgroups count. The research contributes by establishing characterizations for different values cyclic subgroups count ($|C(G)| = k$) ranging from 1 to 4. It also delves into the specific cases of dihedral groups and elementary abelian p-groups. Furthermore, the study shows alternative approach for situations where the count of cyclic subgroups is the same as the group's order minus a constant k, with k being 0 or 1. Of particular note is the characterization of the case where $k = 2$, which addresses an open problem previously posed by T$\acute{a}$rn$\acute{a}$uceanu (2016). In Remark 4.3, Dillstrom (2016) referenced T$\acute{a}$rn$\acute{a}$uceanu (2016) work, indicating period n of G with $d_1, d_2, \ldots, d_k$ being $n's$ positive divisors. For each $i \in \{1, 2, \ldots, k\}$, set $n_i = |\{H \in C(G) | |H| = d_i\}|$, then $\sum_{i=1}^{k} n_i\, \phi(d_i) = n$ and $\phi(x)$ denote Euler's phi function. T˘arn˘auceanu (2015). However, the paper also acknowledges the complexity of the $k = 2$ problem, suggesting that further investigation may require a more robust approach than the one outlined in Remark 4.3 of T$\acute{a}$rnauce$\acute{a}$nu's article, which provides a criterion for determining which cases to consider when the count of cyclic subgroups is relatively large compared to the group order. In pursuit of the overarching goals outlined in Dillstrom (2016) research, the author initiates the investigation by examining the subgroup structure of finite groups, with a specific emphasis on groups of period less than or equal to 10. The author utilizes these smaller groups as a foundational framework for analysing the subgroup compositions of larger groups within related families.

The presentation includes a tabulated summary, as illustrated in Table 2, detailing the distribution of small order groups up to isomorphism. Table 2 provides a concise overview, delineating the number of groups and their corresponding isomorphism types for each order. Noteworthy observations arise, highlighting the prevalence of cyclic, dihedral, and direct product groups within this constrained domain, while also identifying the quaternion group Q_8 as a notable exception. Through, these initial observations, Dillstrom (2016) sets forth clear objectives for the investigation. The primary aim is to characterize the count of cyclic subgroups for groups of varying types and orders, particularly focusing on those exhibiting a limited count of cyclic subgroups, up to four. By conducting manual calculations, the study records the cardinality of cyclic subgroups ($|C(G)|$) for each of these small-order groups, elucidating notable arithmetic relationships between these values and the order of the groups ($|G|$). These observations are succinctly captured and presented in Table 1, which serves as a comprehensive record of the calculated values and their corresponding notes.

In furthering the discourse initiated by Dillstrom (2016) on finite groups that have many cyclic subgroups relative to their size, the research presents an alternative proof aimed at characterizing finite groups G satisfying the condition $|C(G)| = |G| - 1$. This endeavor builds upon existing frameworks, particularly drawing from T$\acute{a}$rn$\acute{a}$uceanu (2016) methodology up to a certain juncture in the argumentative process. The author also provides alternative proofs to characterize other finite groups as proved by other authors. The author specifically worked on a finite group having the characterization $|C(G)| = |G| - 1$ being true when G belongs to C_3, C_4, S_3, or D_8 (see workings in Dillstrom (2016)). Again, solution to the subsequent groups: $C_4 \times C_2$, $C_2 \times D_8$, C_6, or D_{12} as having $|C(G)| = |G| - 2$ were provided (see proof in Dillstrom (2016)). Dillstrom (2016) research not only expands the existing body of knowledge within finite group theory but also fosters a spirit of intellectual inquiry and innovation. This endeavor underscores the dynamic and collaborative nature of mathematical inquiry, wherein researchers build upon each other's work to collectively advance the frontiers of knowledge. Through the exploration of divergent proof techniques and analytical approaches, the study contributes to the ongoing discourse surrounding subgroup structure characterization in finite groups, enriching the theoretical landscape with nuanced perspectives and novel methodologies.

CHAPTER THREE

METHODOLOGY

Introduction

In this chapter we describe and elaborate on the methods, theorems and procedures employed in analysing and obtaining the results of the conjecture and other complex open problem concerning finite groups and their properties. The main research objective revolves around determining the structural characteristics and intrinsic properties of finite groups exhibiting specific alpha invariants of $\frac{3}{4}$ and discerning their nilpotency conditions. This chapter expounds upon the systematic approach applied to unveil solutions and findings that are explained in subsequent chapters.

To establish the requisite theoretical groundwork for objective one, we begin with a comprehensive grasp of important concepts and theorems underpinning our research analysis. The following foundational concepts are instrumental in our study:

Subgroup Count

To ensure that we list all the subgroup within group G, the subsequent systematic approach is utilized:

- Group Order: We start by considering subgroups with periods that are factors of the group's order.
- Trivial Subgroups: These are always subgroups of any group.
- Cyclic Subgroups: We test if any elements belonging to the group can generate cyclic subgroups.

- Pairs of Elements: We explore subgroups generated by pairs of elements.
- Normal Subgroups: Determine if any of the subgroups are normal subgroups, which may lead to additional subgroups known as quotient groups.
- Sylow Subgroups: We also employ Sylow theory to analyze subgroups of specific orders and determine if there are additional subgroups.
- Group theory Software: To confirm our steps, we employ group theory software, GAP (Groups, Algorithms, and Programming), to calculate subgroups automatically.

Characterizing non-nilpotent groups with specific alpha value

The first objective of our research aims to solve an open problem regarding the requirements for G, finite-ordered, with a specific structure having $\alpha(G) = \frac{3}{4}$. The problem is investigated through a structured approach, beginning with the construction of the general form of such a group, then with an exploration of its subgroup structure, particularly focusing on cyclic subgroups and the implications for nilpotency.

Constructing the General Form of *G*: We start by considering a finite group G that possesses $\alpha(G) = \frac{3}{4}$. By applying Theorem 33, which asserts that subgroup's order must divide $|G|$, we derive the relationship between cyclic subgroups count $C(G)$ and $|G|$:

$$\frac{4}{3} \cdot C(G) = |G|$$

Given that $|G|$ is finite, it emerges that $\frac{4}{3} \cdot C(G)$ must be an integer. Consequently,

$C(G)$ must be divisible by 3.

Analysing G's structure: Explore the composition of group G under the condition that $C(G)$, the count of cyclic subgroups including the trivial subgroup, is a multiple of 3. Assume $|G|$ has p as a prime divisor. By Theorem 35, an element p being the period must exist in G which consequently implies, cyclic subgroup of period p exists. Therefore, the count of cyclic subgroups $C(G)$ must be at least $p+1$, considering each cyclic subgroup contributing one subgroup, including the trivial one. Extending this reasoning gives the expression:

$$1 + (2 + 1) + (3 + 1) + \cdots + (p + 1)$$

This summation simplifies to a quadratic expression, which, by extension, must satisfy:

$$\frac{p^2 + 3p + 2}{2} = 3k$$

where k is an integer. This condition restricts p to being either an odd prime or 2, leading to the general form:

$$|G| = 2^a \times 3^b \times p_1 \times p_2 \times \cdots \times p_k$$

with non-negative integers a and b and $p_1, p_2, \cdots, p_k$ distinct odd primes.

Verifying Subgroup Structure: The factors $2^a, 3^b, p_1, p_2, \cdots, p_k$ represent the possible orders of subgroups of G. By Theorem 36, G has Sylow subgroups $P, Q, R, S_2, \ldots, S_k$ corresponding to the orders $2^a, 3^b, p_1, p_2, \cdots, p_k$ respectively. To confirm that any H subgroup of G has period $2^a \times 3^b \times p_1 \times p_2 \times \cdots \times p_k$, we apply the direct product theorem, demonstrating that:

$$|H| = |P| \times |Q| \times |R| \times |S_2| \times \cdots \times |S_k|$$

Since each Sylow p-subgroup's order divides $|G|$, Consequently $|H|$ divides the group's order, and H has the desired structure, thereby confirming isomorphism to the direct product of the Sylow subgroups.

Nilpotency: We conclude the first part of the proof by exploring the non-nilpotency of G using Theorem 45. Assuming, by contradiction, that G is nilpotent, we argue that G would have a lower central series with abelian factor groups. However, the presence of a minimal normal subgroup PQ, which is abelian and a product of Sylow subgroups, leads to a contradiction. This demonstrates that G, given its specific structure, cannot be nilpotent.

The second part of the proof utilizes a series of logical deductions based on established group theoretical principles, Sylow's theorems, and properties of nilpotent groups to analyze the given group structure. We start with the group $G \cong D_{24} \times C_{2^n}$. The researcher's primary goal is to determine the parameter $\alpha(G)$ and identify specific instances when G simplifies to either D_{24} or $D_{24} \times C_{2^n}$. The calculation of $\alpha(G)$ follows the equation:

$$\alpha(G) = \frac{C(D_{24}) \cdot C(C_{2^n})}{|D_{24}| \cdot |C_{2^n}|}$$

This simplification implies that n must be either 0 or 1, leading to G being either D_{24} or $D_{24} \times C_{2^n}$. Next, we apply Lemma 4 concerning the dihedral group D_{2n}, where we are able to establish the subgroup $C_n = \langle r \rangle$ is period n cyclic because $r^n = e$ implies $|r| = n$. Thus, C_n is a cyclic subgroup of D_{2n}. We also employ Lemma 5, which facilitates the process of creating subgroups inside a group's direct product. Now, by the implication on non-nilpotent subgroups, we employ Theorem 46 and use the proof by contradiction, assuming that G is nilpotent and subgroup H

is not, leading to a contradiction through the properties of the lower central series and commutator subgroups, thereby proving that a non-nilpotent subgroup implies the non-nilpotency of the group. By extension, D_{24} is within the group $D_{24} \times C_{2^n}$ and, utilizing Lemma 4, we note that D_{24} contains subgroup that is cyclic of period 12 generated by a rotation element, and so

$$H = \langle r, e \rangle \times C_{2^n} \leq G$$

By Lemma 5, it emerges $H \leq G$ and, applying Theorem 46, H contains a non-nilpotent subgroup (period 12 cyclic subgroup), H is not nilpotent. Consequently, G cannot be nilpotent. The remainder of the technique is structured to cater for two corollaries, that is, Corollary 1 and Corollary 2. By Theorem 47, we establish that given $\alpha(G) = 0.75$ and the formula for $\alpha(D_{2n})$, it demonstrates that n must be even. Then by Lemma 6 and Lemma 7, we identify that $\alpha(D_{24} \times C_2) = \frac{3}{4}$ and also D_{16} and D_{24} as the only dihedral groups satisfying the given α value. We finally apply the induction method to prove Theorem 49, thus by verifying the base cases and utilizing inductive reasoning for $n = k + 1$, we demonstrate that G maintains $\alpha(G) = 0.75$ and conforms to the structures $G \cong D_{24} \times C_{2^n}$ or $G \cong D_{16} \times C_{2^n}$, for all $n = 0,1$. This completes the proof.

Non-Trivial Centre Analysis

In objective two, we establish that G has a non-trivial centre by embarking upon a rigorous proof technique through analysing the centralizers, conjugacy classes, and commutativity within G. To investigate the group's, G, structure and establish the non-triviality of its centre, we utilize the concept of group actions.

Specifically, we consider the conjugation operation of G acting upon its own elements. This action is expressed below:

Action Definition: The action $g \cdot x = gxg^{-1}$ is defined for every element $g \in G$ and any $x \in G$.

Conjugacy Classes and Orbits: The conjugacy class of x in G, under this action, is described as the orbit of x. The conjugacy class $Orb(x)$ can be expressed as:

$$Conjugacy\ Class\ (Orbit): Orb(x) = \{gxg^{-1}: g \in G\}.$$

The size of this orbit is a crucial factor in our analysis.

Utilizing the Orbit-Stabilizer Theorem: To relate the period of the conjugacy class to the centralizer of x, we apply Proposition 12, which is the Orbit-Stabilizer Theorem.

Orbit-Stabilizer Theorem:

$$|Orb(x)| = \frac{|G|}{|C_G(x)|}$$

where the cardinality of the centralizer of $x \in G$ is $|C_G(x)|$

Calculation of Conjugacy Class Size: To determine the size of the conjugacy class of x, we use the Orbit-Stabilizer Theorem.

Total Size of *G*: Given that $G \cong D_{24} \times C_{2^n}$, total size of G is computed as:

$$|G| = |D_{24}| \cdot |C_{2^n}|.$$

Centralizer Components: The centralizer of $x \in G$ is factored into the centralizers within $D_{24}$and C_{2^n}. Thus, the conjugacy class's size:

$$|C_G(x)| = |C_{D_{24}}(x_{D_{24}})| \cdot |C_{C_{2^n}}\ (x_{C_{2^n}}\ )|.$$

In conclusion, we achieve that since $|C_{D_{24}}(x_{D_{24}})| < |D_{24}|$, it consequently yields $|Orb(x)| > 1$, indicating that conjugacy class period is greater than 1. This result shows that x has a non-trivial conjugacy class, this conflict with the premise that $x \notin Z(G)$. Therefore, x must lie in $Z(G)$, implies that G possesses a meaningful non-trivial centre.

Subgroup Normality and Characteristics Analysis

In objective three, we demonstrate the structured approach to put forth that a subgroup of $G \cong D_{24} \times C_{2^n}$ is both preserved by conjugation and automorphism, thereby establishing that it is both normal and characteristic. We specifically aim to demonstrate that a non-trivial subgroup $Z(G)$ of G possesses these properties. We employ a combination of computational and theoretical techniques to establish this result. Group (2021) is utilized to compute and analyze the elements and subgroups of D_{24}. The elements or generators $\{f_1, f_2, f_3, f_4\}$ and subgroups of D_{24} were generated, providing a comprehensive list of subgroups, which was cross-referenced with the theoretical findings. Next, the subgroups of D_{24} were identified theoretically, highlighting subgroups spanned by various combinations of R and S. We then establish that a non-trivial subgroup $H = \{(R_6, e)\} \leq Z(G) = \{(e,e), (e,c), (R_6, e), (R_6, c)\}$ is used for further analysis. Next, we show that $H = \{(R_6, e)\}$ is normal in G. For any element $g \in G$, where $g = (x, y)$ with $x \in D_{24}$ and $y \in C_2^n$, we demonstrate that gHg^{-1} is done by examining two scenarios depending on whether y is the identity element or not. In both cases, we show that conjugating H by g leaves H invariant, proving its normality. Finally, we establish that H is characteristic by showing that it is invariant under any automorphism φ

of G. Since $D_{24} \times C_2^n$, any automorphism φ can be decomposed into automorphisms φ_1 of D_{24} and φ_2 of C_2^n. Given that H is a subgroup of the centre of G and commutes with every member of G, we derive that $\varphi(H) = H$ for any automorphism φ, thereby proving H is characteristic. In all, we combine computational tools and theoretical analysis to prove that a subgroup that is not trivial H of $D_{24} \times C_2^n$ is both normal and characteristic. This methodology ensures a comprehensive and robust proof, blending computational verification with classical group theory to provide a clear demonstration of the subgroup's invariance under conjugation and automorphism.

Normal subgroups and isomorphism classes

Another critical aspect this research looks, is the identification and analysis of normal subgroups in objective four. Thus, we delve into the normal subgroups of a group G and examine their structural properties up to isomorphism, with the aim to uncover the internal symmetries and ultimately demonstrate that G is not a simple group but has non-trivial normal subgroups, making it solvable. To achieve our objective, we employ a systematic approach that includes analysing the structure of G, identifying normal subgroups in D_{24} and C_2 and their structural properties. We start by using the properties of D_{24} to find its normal subgroups. By employing Theorem 33 and Euler's totient function, we determine the possible orders of elements in D_{24}. We list the elements and their orders and use Theorem 25 to classify elements into conjugacy classes. The class equation for D_{24} helps us understand the distribution of these elements. Next, we classify the subgroups of D_{24} based on their orders. We identify subgroups of orders 1, 2, 3, 4, 6, 8, 12, and

24, and verify their normality by using Proposition 7, Proposition 4 and Proposition 8. This step ensures that we correctly identify all normal subgroups of D_{24}. With the normal subgroups of D_{24} identified, we move on to $D_{24} \times C_2$. The cyclic group C_2 has only two subgroups: the group itself {e, a} and its trivial subgroup {e}, where a is the non-identity element. By pairing each normal subgroup of D_{24} with these subgroups of C_2, we form the normal subgroups of G. We then analyze their structures to determine their isomorphism types. This systematic approach not only provides insight into the composition of G but also highlights the rich internal symmetry and solvable nature of the group.

Characterizing cyclic subgroup count

In objective five, we describe and characterize finite groups G for which cyclic subgroups count satisfies the condition $|C(G)| = |G| - 6$. Our approach involves a thorough and detailed analysis of the structural properties of these groups, building on previous classification techniques by Tárnáuceanu (2016) and Song and Zhou (2019) and the introduction of new technique. By understanding their methodologies and results, we establish a strong foundation for our research, enabling us to adapt and extend their approaches to our specific problem. Thus, our methodology uniquely applies a detailed analysis of possible divisors and specific group orders to identify groups meeting the condition. We utilize the notion of Equation (7) and Proposition 2 to establish the general structure of the group G, thus;

$$|G| = p_1^n \cdot p_2^m \cdot p_3^r$$

where $m \leq 3, n \leq 2, r \leq 1, p_1 < p_2 < p_3$ and $p_1, p_2, \ldots, p_k$ are distinct primes.

The Sylow theorems are utilized to analyze the structure and count of Sylow p-subgroups. This helps us establish conditions for the existence and uniqueness of these subgroups and also deduce the possible subgroup structures and their implications for the group's overall structure.

In our first approach, we classify finite p-groups of order p^k for $1 \leq k \leq 3$, ensuring that they satisfy the characterization $|C(G)| = |G| - 6$. The approach first uses the Sylow theorems, showing that G contains a nontrivial centre $Z(G)$. If G is cyclic, its structure is determined by the divisors of p^k. For non-cyclic groups, the methodology explores group extensions, determining whether G can be decomposed into smaller cyclic subgroups through a split extension. If the extension does not split, G remains indecomposable but cyclic. This technical process rigorously addresses each case of k, ensuring the correct count of cyclic subgroups.

In our next approach, we explore the finite groups structure of composite orders of the factorization $|G| = p^k q^l$, where p and q are distinct primes. We begin by identifying a normal subgroup N with $|N| = p^k$, establishing a short exact sequence that represents G as an extension of N by G/N. Our analysis, as done previously, focuses on the different cases based on the orders of G, utilizing Sylow theorems to determine the existence and count of cyclic subgroups. We then investigate whether the extension splits, which leads us to discern between a direct product or a semi-direct product structure. In our exploration, we pay special attention to the role of automorphisms of N and how their non-trivial actions can introduce non-abelian characteristics into the group structure. By examining these aspects for groups with orders $|G| = p^k q^l$, under the constraints $k, l \leq 3$ and $gcd(k, l) = 1$, we reveal critical insights in the group structure that conforms with $|G| - 6$ cyclic subgroups.

We Conclude by considering Proposition 3 and Theorem 22 to establish that certain groups satisfy $|C(G)| = |G| - 6$. This case-by-case approach involves detailed proofs and logical deductions that finally ensure that our findings are accurate and valid.

Chapter Summary

This chapter elaborate the approach and methods employed in achieving research objectives. The techniques employed in proving the conjecture are first examined, followed by non-trivial centre analysis, subgroup normality and characteristics analysis, then finally on characterizing cyclic subgroup count.

CHAPTER FOUR

RESULTS AND DISCUSSION

Introduction

This chapter presents the key findings of the research, taking into account fundamental theorems and results that have motivated this research in the area of group theory.

Characterizing finite groups with specific alpha invariant value

Motivated by the work of Cayley (2021), we solve the open problem and present conditions under which the open problem holds. Using $\alpha(G) = \frac{3}{4}$, we first construct the generic form of the finite group G.

Open Problem: Let group G be finite. If $\alpha(G) = \frac{3}{4}$ and G is non-nilpotent, then $G \cong D_{24} \times C_{2^n}$.

We first establish the solution to the initial part of the open problem, which sets the stage for proving the subsequent result.

Lemma 1

Let G be finite-ordered with $\alpha(G) = \frac{3}{4}$. Then $|G|$ is structured as $2^a \cdot 3^b \cdot p_1 \cdot p_2 \cdots p_k$, with a and b as non-negative integers, and $p_1, p_2, \cdots, p_k$ as distinct odd primes.

Proof. From $\alpha(G) = \frac{3}{4}$, Theorem 33 suggests that subgroup's order must divide $|G|$. Thus, we derive that the count of cyclic subgroups, $C(G)$, must divide $|G|$. Now, Equation (1) yields

$$|G| = \frac{4}{3} \times C(G) \quad (2)$$

Since $|G|$ is finite, left-hand side of Eq. (2) must be a positive integer, indicating that $C(G) \equiv 0\ (mod\ 3)$. We then analyze the structure of G given that $C(G)$, including the trivial subgroup, is a multiple of 3. Suppose one of $|G|'s$ prime divisors is p, Theorem 35 guarantees the existence of a cyclic subgroup of period p in G. Therefore, the count of cyclic subgroups $C(G)$ must be at least $p + 1$, such that each cyclic subgroup of period dividing p contributes one subgroup of which the trivial subgroup is included. Thus, by extension we have

$$1 + (2 + 1) + (3 + 1) + \cdots + (p + 1). \quad (3)$$

Eq. (3) is a multiple of 3, it must be of the form $3k$, with k being an integer:

$$\frac{p^2+3p+2}{2} = 3k \quad (4)$$

$$p^2 + 3p + 2 = 6k \quad (5)$$

Since $6k$ is divisible by 2, the left-hand side of Eq. (5) must also be divisible by 2. This implies that p must be an odd prime or only the even prime 2, as otherwise, Eq. (5) would be odd, causing a contradiction. Therefore, if the count of cyclic subgroups of G is a multiple of 3, and p is an odd prime or only the even prime 2 dividing $|G|$, we can write that $|G|$ must be of the form:

$$2^a \cdot 3^b \cdot p_1 \cdot p_2 \cdots p_k \quad (6)$$

We further show that the factors $2^a, 3^b, p_1, p_2, \cdots, p_k$ represent the potential subgroup orders of the finite group G. By Theorem 36, G has a Sylow 2-subgroup P of order 2^a, a Sylow 3-subgroup Q of order 3^b, and a Sylow p_1-subgroup R of order p_1. For each odd prime pi with $2 \leq i \leq k$, G also has a Sylow p_i-subgroup S_i of period p_i. We show that any subgroup H of G has period $2^a \cdot 3^b \cdot p_1 \cdot p_2 \cdots p_k$, by considering the direct product theorem, which asserts that $|H|$ is the product of the periods of its factors. Therefore, it suffices to demonstrate that $|P| = 2^a$, $|Q| = 3^b$, $|R| = p_1$, and $|S_i| = p_i$ for all $2 \leq i \leq k$. Since $H \leq G$, its order must divide $|G|$. Theorem 33 ensures that each Sylow p-subgroup divides $|G|$. Thus, $|P|$ divides 2^a, |Q| divides 3^b, |R| divides p_1, and $|S_i|$ divides p_i for all $2 \leq i \leq k$. But each S_i is ap_i-group and p_i's are distinct, so they commute with each other and with P, Q, and R, so the order of H is the product of the orders of its factors, $|H| = 2^a \cdot 3^b \cdot p_1 \cdot p_2 \cdots p_k$. Hence, G has a subgroup H with the desired structural equivalent to the direct product of $P, Q, R, S_2, \ldots, S_k$.

Lemma 2

Sylow p-subgroups of G are cyclic if the period is the maximal p-power dividing $|G|$.

Proof. Suppose H is a Sylow p-subgroup of G with order p^m, noting that the highest power of p dividing $|G|$ is p^m. By Theorem 36, such a subgroup exists. Since $|H| = p^m$, H is a p-group. By the Fundamental Theorem of Finite Abelian Groups, every finite p-group is cyclic. Hence, H is cyclic.

Lemma 3

Let G be a group and let H be the unique Sylow p-subgroup of G for a prime p. Then, H is a normal subgroup of G.

Proof. By Theorem 39, $n_p \equiv 1 (mod\ p)$ and n_p divides $|G|/p^k$, where p^k is $p's$ highest power that divides $|G|$. Since H is the only Sylow p-subgroup, $n_p = 1$. Thus, $|H\ = p^k$, and by Lemma 2, H is cyclic. Since cyclic subgroups are always normal in their containing group, H is normal in G.

Theorem 45. *Given G to be finite with $|G| = 2^a \cdot 3^b \cdot p_1 \cdot p_2 \cdots p_k$, where $a, b > 0$ and $p_1, p_2 \cdots, p_k$ are distinct odd primes. Then G is non-nilpotent, and $G \cong D_{24} \times C_{2^n}$.*

Proof. Suppose by contradiction that G is nilpotent. According to the notion of nilpotency, G must have a series of normal subgroups

$$\{e\} = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$$

such that each quotient G_{i+1}/G_i is abelian. Let P and Q denote Sylow 2-subgroup and Sylow 3-subgroup of G, respectively. Consider the smallest index i such that both P and Q are contained in G_i. As P and Q are normal in G_i, the product PQ is a subgroup of G_i. By the uniqueness of Sylow subgroups (Lemma 3), PQ is the unique Sylow subgroup of its order, and thus $P \cap Q = \{e\}$. Consequently, $PQ \cong P \times Q$, meaning PQ is abelian. Now, consider any odd prime divisor p of $|G|$. Since p does not divide the order of PQ, the quotient G/PQ is abelian. However, in a nilpotent group, all factors in the series must be abelian. This contradicts the fact that PQ is a minimal normal subgroup of G that is abelian. Therefore, G cannot be nilpotent.

Next, we demonstrate that G is structurally equivalent to $D_{24} \times C_{2^n}$. Since $|G| = 2^a \cdot 3^b \cdot p_1 \cdot p_2 \cdots p_k$, Theorem 36 guarantees the existence of Sylow subgroups for each prime divisor of $|G|$. For each p_i, let H_i denote the corresponding Sylow p_i-subgroup of G. By Theorem 39, the count n_i of Sylow p_i-subgroups satisfies $n_i \equiv 1$ mod pi and divides $|G|/p_i^{m_i}$. Since $p_1 \cdot p_2 \cdots p_k$ are distinct primes, the numbers n_i are pairwise coprime. The product of all n_i equals the order of the group, implying $n_i = 1$ for all i. Hence, each Sylow subgroup is unique and normal in G. Moreover, each Sylow subgroup H_i is cyclic, and their intersections are trivial. Thus, the product of the Sylow subgroups forms a direct product:

$$G \cong H_2 \times H_3 \times \cdots \times H_k,$$

where each H_i is isomorphic to $C_{p_i^{m_i}}$, the cyclic group of order $p_i^{m_i}$. Since G is the direct product of cyclic groups, we have:

$$G \cong C_{2^a} \times C_{3^b} \times C_{p_1^{m_1}} \times \cdots \times C_{p_k^{m_k}}.$$

Finally, we observe the structure of G shares the same prime factorization as $D_{24} \times C_{2^n}$. Therefore, we conclude that:

$$G \cong P \times Q \times R \times S_2 \times \ldots \times S_k \cong C_{2^a} \times C_{3^b} \times C_{p_1} \times C_{p_2} \times \cdots \times C_{p_k} \cong D_{24} \times C_{2^n}$$

With the core structure defined, we now advance to demonstrate the second part of the open problem.

Lemma 4

Let D_{2n} be of period $2n$, with $n \geq 3$. There exists C_n of D_{2n} generated by a rotation element of size n.

Proof. By definition, $D_{2n} = \langle r, s \mid r^n = s^2 = e, srs = r^{-1} \rangle$. We suppose that, subgroup $C_n = \langle r \rangle$. Since $r^n = e \Rightarrow |r| = n$, C_n is cyclic of size n. Hence, $C_n \subset D_{2n}$.

Lemma 5

For groups G_1 and G_2, it follows that $H_1 \times H_2 \leq G_1 \times G_2$ if $H_1 \leq G_1$ and $H_2 \leq G_2$.

Proof. The subgroup $H_1 \times H_2$ is specified as the collection of ordered pairs (h_1, h_2) with $h_1 \in H_1$ and $h_2 \in H_2$. The group operations are defined component-wise, making $H_1 \times H_2$ a subgroup of $G_1 \times G_2$.

Theorem 46. *Consider G as a group and $H \leq G \cong D_{24} \times C_{2^n}$. If H is a non-nilpotent subgroup, then G is non-nilpotent and $\alpha(G) = \frac{3}{4}$.*

Proof. Suppose, by contradiction, that H is a non-nilpotent subgroup and G is a nilpotent group. Since H is not nilpotent, existence is established for a positive whole number k for which $H_k \neq \{e\}$, and H_k is the k-th term in the lower central series of H. Let $x \in H_k$ such that $x \neq e$. Now consider the element x in the group G. Since $H \leq G$, and $x \in H_k \subseteq H$, this implies $x \in G$. By notion, lower central series of a group G exhibit $G = G_1 \geq G_2 \geq \ldots$, where we define G_{k+1} as $[G_k, G]$ for each $k \geq 1$ and $[G_k, G]$ represents the commutator subgroup of G_k with G. Since $x \in G$, we have the commutator $[x, G]$ being the set of all commutators of the form $[x, g] = xgx^{-1}g^{-1}$ for $g \in G$. By iterating this commutator process k times, we obtain the kth lower central commutator, denoted as $[x, G]_k = [x, [G, \ldots, G]]$, with k nested commutators. Since G is assumed to be nilpotent, there must exist an n such that $([x, G]_k)^n = \{e\}$. However, this implies that $x^n = ([x, G]_k)^n = \{e\}$, which contradicts the choice of x not being the identity element. Therefore, if H is not nilpotent, then G cannot be Nilpotent. In the reverse direction, we consider D_{24} in $G \cong D_{24} \times C_{2^n}$. By Lemma 4, D_{24} incorporates a cyclic subgroup of the specified period 12 generated by a rotation element. So, we can write $H = \langle r, e \rangle \times C_{2^n} \leq G$. Thus, Lemma 5 gives us $H \leq G$. Applying Theorem 46, and noting H contains a non-nilpotent subgroup (the cyclic subgroup of order 12),we formalize that G is non-nilpotent since G contains non-nilpotent H.

Building on this insight, we proceed, given $G \cong D_{24} \times C_{2^n}$, to establish the constraint on the n copies, thus;

$$\alpha(G) = \frac{C(D_{24}).C(C_{2^n})}{|D_{24}|.|C_{2^n}|} = \frac{3.(n+1)}{4.2^n} = \frac{3}{4}$$

where $n = 0$ or 1 such that $G = D_{24}$ or $G = D_{24} \times C_2$, respectively. Thus, the conjecture holds true, as demonstrated by the preceding arguments.

From the established result, we immediately derive the following corollaries:

Corollary 1

When $n = 0$, then $G \cong D_{24} \times C_{2^0} \cong D_{24}$ such that $\alpha(G) = \frac{3}{4}$ 3 and G is not nilpotent.

Corollary 2

When $n = 1$, then $G \cong D_{24} \times C_2$ such that $\alpha(G) = \frac{3}{4}$ and G is not nilpotent.

To support the validity of Corollaries 1 and 2, it is necessary to examine the preceding lemmas and theorems, which furnish the requisite theoretical framework.

Theorem 47. *Let $\alpha(G) = \frac{3}{4}$ and G be a dihedral group D_{2n}, then n must be an even integer.*

Proof. It is observed from Theorem 23, $\alpha(D_{2n}) = \frac{n+\tau(n)}{2n}$, and $G = D_{2n}$. Since $\alpha(D_{2n}) = 0.75$, we have

$$\frac{n+\tau(n)}{2n} = 075$$

By subtracting n from both sides, we arrive at the equation $0.5n = \tau\,(n)$, signifying that the count of divisors of G is half of n. To explore the possible values for n, we first examine its parity. If n is even, then $\tau\,(n)$ is a positive integer. Conversely, if n is odd, $\tau\,(n)$ fails to be an integer, leading to a contradiction. Therefore, n must be an even integer. Consequently, for the dihedral group $G = D_{2n}$, takes values from the set $\{2, 4, 6, \dots, k\}$, where k is a positive even integer.

Lemma 6

The proportion of cyclic subgroups in the dihedral group D_{16} satisfies $\alpha(D_{16}) = \frac{3}{4}$.

Proof. To establish this result, Theorem 23 is utilized. Substituting the parameters for D_{16}, we obtain

$$\alpha(D16) = \frac{3}{4}.$$

Hence, the claim holds.

Lemma 7

The proportion of cyclic subgroups in the dihedral group D_{24} satisfies $\alpha(D_{24}) = \frac{3}{4}$

Proof. Similarly, from Theorem 23 $\alpha(D_{24}) = \frac{3}{4}$. This follows from an explicit enumeration of count of cyclic subgroups to groups order. From Lemma 6 and Lemma 7, we can similarly write $\alpha(D_{24} \times C_2) = \alpha(D_{24}) \times \alpha(C_2) = \frac{3}{4} \times 1 = \frac{3}{4}$.

Theorem 48. *The only dihedral groups with* $\alpha(G) = 0.75$ *are* D_{16} *or* D_{24}.

Proof. Inspired by the logic presented in Cayley (2021) proof, we present a similarly structured argument to demonstrate the result for Theorem 48. From Theorem 23, we observe

$$\alpha(G) = \frac{\tau(n) + n}{2n} = \frac{3}{4} \Longrightarrow \tau(n) = \frac{n}{2}$$

Since $\tau\ (n)$ count positive divisors of n, we can determine an upper limit on the count of divisors that n can have. Therefore

$$\tau\ (n) = \frac{n}{2} \leq 2\sqrt{n} \Longrightarrow n(n - 16) \leq 0 \Longrightarrow 0 \leq n \leq 16$$

But by Theorem 47, $n = 2, 4, 6, 8, 10, 12$, or 16. Where only $n = 8$ or 12 satisfies $\alpha(G)$, hence $G = D_{24}$ or D_{16}.

From Theorem 46 and Theorem 48, we establish our final theorem.

Theorem 49. *Given a finite group* G *and* $\alpha(G) = \frac{3}{4}$. $G \cong D_{24} \times C_{2^n}$ *or* $G \cong D_{16} \times C_{2^n}$, *for all* $n = 0,1$.

Proof. Base case: For $n = 0,1$:

$$G \cong D_{24} \times C_{2^n} \cong D_{24} \times C_1 \cong D_{24}$$

or

$$G \cong D_{16} \times C_{2^n} \cong D_{16} \times C_{2^0} \cong D_{24} \times C_1 \cong D_{16}$$

Suppose $z \in \{16, 24\}$.

Inductive step: Suppose $n = k$ and $\alpha(G) = 0.75$ and $G \cong D_z \times C_{2^k}$ holds a certain positive integer k. Proving the validity of the assertion for $n = k + 1$ is prerequisite. Thus, for $n = k + 1$, we have:

$$G \cong D_z \times C_{2^{k+1}}.$$

Rewrite this as:

$$G \cong \left(D_z \times C_{2^k}\right) \times C_2.$$

We can deduce that $D_z \times C_{2^{2k}}$ is isomorphic to $\acute{G} = G/C_2$, where $\acute{G} = D_z \times C_{2^{2k}}$. Using the assumption that $\alpha(\acute{G}) = 0.75$, we have:

$$\alpha(\acute{G}) = \alpha(D_z \times C_{2^k}) = 0.75.$$

Now, we consider the action of C_2 on G. Since C_2 belongs to the order 2 cyclic group, it has only two elements, say $\{e, a\}$. We consider two issues:

Case 1: C_2 acts trivially on $\acute{G}$. If C_2 acts trivially on $\acute{G}$, then every element of $\acute{G}$ is fixed by the action of C_2. This implies:

$$G = \acute{G} \times C_2$$

Since $\alpha(\acute{G}) = 0.75$ and C_2 acts trivially on $\acute{G}$, we have:

$$\alpha(G) = \alpha(\acute{G} \times C2) = \alpha(\acute{G}) \times \alpha(C2) = 0.75 \times 1 = 0.75.$$

Therefore, the claim is true for $n = k + 1$.

Case 2: C_2 acts non-trivially on $\acute{G}$. If C_2 acts non-trivially on $\acute{G}$, then there is an existence of $a \in C_2$ with $a \neq e$. In this case, we can write:

$$G = (\acute{G} \times C_2)/\langle(e, a)\rangle,$$

where $\langle(e, a)\rangle$ is the subgroup generated by (e, a). But (e, a) has order 2, and its action on $\acute{G}$ is non-trivial. However, since (e, a) has order 2, its square is:

$$(e, a) \cdot (e, a) = (e, a \cdot a) = (e, e) = e.$$

Thus, the element (e, a) is self-inverse, and its action on any group is trivial, regardless of the specific action on $\acute{G}$. Hence, we can conclude that the action of (e, a) on G is indeed trivial, regardless of how it acts on $\acute{G}$. Then,

$$G = (\acute{G} \times C_2)/\langle(e, a)\rangle \cong \acute{G} \times C_2.$$

Implying, $G \cong \acute{G} \times C_2$, which leads to

$$\alpha(G) = \alpha(\acute{G} \times C2) = \alpha(\acute{G}) \times \alpha(C2) = 0.75 \times 1 = 0.75.$$

Therefore, $n = k + 1$ is valid, which completes the proof.

Orbit-Stabilizer Theorem and Centralizers in Direct Products

In the field of group theory, analysing the structure and characteristics of groups is essential to various mathematical disciplines and applications. A key aspect of group theory deals with the centre of a group, the presence of a non-trivial centre, especially in non-nilpotent groups, significantly influences the group's overall structure and dynamics. In objective two consider the concept of group actions and apply the Orbit-Stabilizer Theorem to analyse the centralizers in the direct products of $G \cong D_{24} \times C_{2^n}$. This technique provides a more general and theoretically grounded framework for analyzing the structure of G and the behavior of its conjugacy classes.

Theorem 50. *Let $G \cong D_{24} \times C_{2^n}$ when $\alpha(G) = \frac{3}{4}$ and G is not nilpotent. Then G has a non-trivial centre.*

Proof. Suppose $x \in G$ is not in $Z(G) = D_{24} \times C_{2^n}$. Accordingly, there is some $y \in G$ such that $xy \neq yx$. We examine the action of G on itself by conjugation, defined by $g \cdot x = gxg^{-1}$ for all $g \in G$. The orbit of x under this action is precisely the conjugacy class of x, denoted by $Orb(x) = \{gxg^{-1} : g \in G\}$. From Proposition 12, period of orbit is given by

$$|Orb(x)| = \frac{|G|}{|C_G(x)|}$$

where $C_G(x)$ is the centralizer of x in G. Since G is the direct product of D_{24} and C_{2^n}, we have

$$|G| = |D_{24}| \cdot |C_{2^n}|$$

and can write

$$|Orb(x)| = \frac{|D_{24}| \cdot |C_{2^n}|}{|C_G(x)|} = \frac{|D_{24}| \cdot |C_{2^n}|}{\left|C_{D_{24}}(x_{D_{24}})\right| \cdot \left|C_{C_{2^n}}(x_{C_{2^n}})\right|}.$$

Now, because C_{2^n} is abelian, $C_{C_{2^n}}(x_{C_{2^n}}) = C_{2^n}$, so

$$\left|C_{C_{2^n}}(x_{C_{2^n}})\right| = |C_{2^n}|,$$

and hence

$$|Orb(x)| = \frac{|D_{24}|}{\left|C_{D_{24}}(x_{D_{24}})\right|}.$$

Since D_{24} is non-abelian, $C_{D_{24}}(x_{D_{24}})$ must be a proper subgroup of D_{24}, so

$$\left|C_{D_{24}}(x_{D_{24}})\right| < |D_{24}|$$

and thus $|Orb(x)| > 1$. This implies that x has a non-trivial orbit, which contradicts the assumption that $x \notin Z(G)$. Therefore, x must lie in $Z(G)$, and the centre is non-trivial.

Invariance of Subgroups under Conjugation and Automorphism

In objective 3, we seek to prove that the group $G's$ subgroup is unaffected by conjugation and automorphism by showing that the subgroup of G is respectively normal and characteristic. We specifically show that a subgroup $Z(G)$ of G has this property and is non-trivial.

Theorem 51. *Given $G \cong D_{24} \times C_{2^n}$ when $\alpha(G) = \frac{3}{4}$ and G is not nilpotent, it follows that G possesses a non-trivial subgroup which is both normal and characteristic.*

Proof. To support the analysis of this theorem, we utilized Group (2021) to compute and examine the elements and subgroups of D_{24}. Using the Order and Subgroups commands, we generated the elements and subgroups of D_{24}, resulting in four generators $\{f_1, f_2, f_3, f_4\}$ and 34 subgroups. This computational approach provides a concrete foundation for the theoretical results employed in the proof.

$D_{24} = \{e, f_1, f_2, f_3, f_4, f_1 * f_2, f_1 * f_3, f_1 * f_4, f_2 * f_3, f_2 * f_4, f_3 * f_4, f_4^2, f_1 * f_2 * f_3, f_1 * f_2 * f_4, f_1 * f_3 * f_4, f_1 * f_4^2, f_2 * f_3 * f_4, f_2 * f_4^2, f_3 * f_4^2, f_1 * f_2 * f_3 * f_4, f_1 * f_2 * f_4^2, f_1 * f_3 * f_4^2, f_2 * f_3 * f_4^2, f_1 * f_2 * f_3 * f_4^2\}$

Subgroups of D24: $Group([\ ])$, $[Group([f_3 * f_4])$, $Group([f_1])$, $Group([f_1 * f_3])$, $Group([f_1 * f_4])$, $Group([f_1 * f_4^2])$, $Group([f_1 * f_2])$, $Group([f_1 f_3 * f_4^2])$

$Group([f_1 * f_3 * f_4])$, $Group([f_1 * f_2 * f_3])$, $Group([f_1 * f_2 * f_4])$, $Group([f_4, f_3])$, $Group([f_1 * f_2 * f_3 * f_4])$, $Group([f_1 * f_2 * f_4^2])$, $Group([f_1 * f_2 * f_3f_4^2])$ $Group([f_4])$, $Group([f_1, f_3 * f_4])$, $Group([f_1 * f_4, f_3 * f_4])$, $Group([f_1 * f_3, f_4])$ $Group([f_1 * f_4^2, f_3 * f_4])$, $Group([f_2 * f_4^2, f_3 * f_4])$, $Group([f_1 * f_2 * f_4, f_3 * f_4])$ $Group([f_1 * f_2, f_3 * f_4])$, $Group([f_1 * f_2 * f_4^2, f_3 * f_4])$, $Group([f_1 * f_2 * f_3, f_4])$ $Group([f_1 * f_4, f_2 * f_4^2, f_3 * f_4])$, $Group([f_4, f_3, f_1 * f_2])$, $Group([f_4, f_3, f_1, f_2])]$ $Group([f_4, f_3, f_1])$, $Group([f_1 * f_4^2, f_2 * f_4^2, f_3 * f_4])$, $Group([f_4, f_1 * f_2])$, $Group([f_1, f_2 * f_4^2, f_3 * f_4])$, $Group([f_4, f_3, f_2])$, $Group([f_4, f_1])$

Theoretically using rotations (R) and reflections (S), D_{24} or $D_{2(12)}$ has a group presentation given as: $D_{2(12)} = \langle R, S: R^{12} = S^2 = e, RS = SR^{-1}\rangle$. Thus, $D_{2(12)} = \{e, R, R^2, R^3, \dots, R^{11}, S, SR, SR^2, SR^3, \dots, SR^{11}\}$. From Theorem 24 there are 34 subgroups of D_{24}. Therefore, we proceed to generate these subgroups with two generators: subgroup spanned by R: $\{e, R, R^2, R^3, R^4, R^5, R^6, R^7, R^8, R^9, R^{10}, R^{11}\}$, subgroup spanned by R^2: $\{e, R^2, R^4, R^6, R^8, R^{10}\}$, subgroup spanned by R^3: $\{e, R^3, R^6, R^9\}$, subgroup spanned by R^4: $\{e, R^4, R^8\}$, subgroup spanned by R^6: $\{e, R^6\}$, subgroup spanned by R^{12} or the trivial subgroup: $\{e\}$, subgroup spanned by R and S or the whole group: D_{24}, subgroup spanned by R^2 and S: $\{e, R^2, R^4, R^6, R^8, R^{10}, S, R^2S, R^4S, R^6S, R^8S, R^{10}S\}$, subgroup spanned by R^2 and RS: $\{e, R^2, R^4, R^6, R^8, R^{10}, RS, R^3S, R^5S, R^7S, R^9S, R^{11}S\}$, subgroup spanned by R^3 and S: $\{e, R^3, R^6, R^9, S, SR^3, SR^6, SR^9\}$, subgroup spanned by R^3 and RS: $\{e, R^3, R^6, R^9, RS, R^4S, R^7S, SR^{10}\}$, subgroup spanned by R^3 and R^2S: $\{e, R^3, R^6, R^9, R^2S, R^5S, R^8S, SR^{11}\}$, subgroup spanned by R^4 and S: $\{e, R^4, R^8, S, R^4S, R^8S\}$, subgroup spanned by R^4 and RS: $\{e, R^4, R^8, RS, R^5S, R^9S\}$, subgroup spanned by R^4 and R^2S: $\{e, R^4, R^8, R^2S, R^6S, R^{10}S\}$, subgroup spanned by R^4 and R^3S: $\{e, R^4, R^8, R^3S, R^7S, R^{11}S\}$, subgroup spanned by R^6 and S: $\{e, R^6, S, R^6S\}$, subgroup spanned by R^6 and RS: $\{e, R^6, RS, R^7S\}$, subgroup spanned by R^6 and R^2S: $\{e, R^6, R^2S, R^8S\}$, spanned by R^6 and R^3S: $\{e, R^6, R^3S, R^9S\}$, subgroup spanned by R^6 and R^4S: $\{e, R^6, R^4S, R^{10}S\}$, subgroup spanned by R^6 and R^5S: $\{e, R^6, R^5S, R^{11}S\}$, $\{S, e\}$, $\{RS, e\}$, $\{R^2S, e\}$, $\{R^3S, e\}$, $\{R^4S, e\}$, $\{R^5S, e\}$, $\{R^6S, e\}$, $\{R^7S, e\}$, $\{R^8S, e\}$, $\{R^9S, e\}$, $\{R^{10}S, e\}$, $\{R^{11}S, e\}$.

From Theorem 50, we established that G has a non-trivial centre, and $Z(G)$ is the centre of G. Now let $H \leq Z(G)$ where H is non-trivial. This is possible since $Z(G)$ is non-trivial. To show that H is normal, we demonstrate that for every element $g \in G$, $gHg^{-1} = H$. Since H is a subgroup of the centre of G, it commutes with all elements of G. Thus, for any $g \in G$, $gh = hg$ for $h \in H$. Now, consider gHg^{-1}, where $g \in G$. It is

evident that:

$$gHg^{-1} = \{ghg^{-1} : h \in H\}.$$

Since H commutes with all elements of G, we can rewrite this as:

$$gHg^{-1} = \{hgg^{-1} : h \in H\}.$$

But g and g^{-1} cancel out, leading to:

$$gHg^{-1} = \{h \mid h \in H\} = H.$$

Thus, $H \lhd G$. From Theorem 20, $Z(D_{24}) = Z(D_{2(12)}) = \{e, R^6\}$. Now, let's consider C_2. The subgroups of $C_2 = \{e, c\}$ are the trivial subgroup $\{e\}$ and the whole group C_2. Since C_2 is abelian, its centre is itself: $Z(C_2) = C_2 = \{e, c\}$. Therefore, we can write:

$$D_{24} \times C_2 = \{(e,e),(e,c),(R,e),(R,c),(R^2,e),(R^2,c),(R^3,e),(R^3,c) \ ...,$$
$$(SR^{11},e),(SR^{11},c)\}$$

This leads to:

$$Z(D_{24} \times C_2) = \{(e,e),(e,c),(R^6,e),(R^6,c)\}.$$

Thus, $H = \{(R^6,e)\}$ is a non-trivial subgroup of $Z(D_{24} \times C_2)$, and since $H = \{(R^6,e)\}$ is the centre of D_{24} and also a subgroup of the centre of G, we can say H is likewise a centre of G. Now, suppose $g = (x,y) \in G$, with $x \in D_{24}$ and $y \in C_2$. We need to show that $gHg^{-1} = H$. We examine two scenarios based on the value of y.

Case 1: If $y = e$ (the identity element of C_2), then $g = (x,e)$. In this case,

$$gHg^{-1} = (x,e)H(x,e)^{-1}.$$

But $(x,e)^{-1} = (x^{-1},e)$, so

$$gHg^{-1} = \{(x,e)(R^6,e)(x^{-1},e) : (R^6,e) \in H\}$$
$$= \{(xR^6x^{-1},e) : (R^6,e) \in H\}$$
$$= \{(R^6,e)\} = H$$

Therefore, $gHg^{-1} = H$ is satisfied in this case.

Case 2: If $y = c$ (a non-identity element of C_2), then $g = (x,c)$. Similarly,

$$gHg^{-1} = (x,c)H(x,c)^{-1}.$$

But $(x,c)^{-1} = (x^{-1},c^{-1})$ where $c \cdot c = e$ in C_2, implying that c is its own inverse. Now,

$$gHg^{-1} = \{(x,c)(R^6,e)(x^{-1},c^{-1}) : (R^6,e) \in H\},$$

where $(x,c) \cdot (R^6,e) = (xR^6,c)$ and $(xR^6,c) \cdot (x^{-1},c^{-1}) = (R^6,e) = H$. Thus, for both cases, we have shown that $gHg^{-1} = H$, confirming that the centre $H =$

$\{R^6, e\}$ is indeed invariant under the action of any element g in the group G. Hence, $H \lhd G$.

We then prove that H is characteristic, thus we need to demonstrate that for every automorphism φ of G, $\varphi(H) = H$. Let an automorphism of G be φ. Since $G \cong D_{24} \times C_{2^n}$, φ is of the form $\varphi = \varphi_1 \times \varphi_2$, where φ_1 is an automorphism of D_{24}, and φ_2 is an automorphism of C_{2^n}. Since $H \leq Z(G)$, it commutes with all elements of G. Therefore, for any automorphism φ_1 of D_{24} and φ_2 of C_{2^n}, we have: $\varphi_1(H) = H$, and $\varphi_2(H) = H$. Therefore, $\varphi(H) = \varphi_1(H) \times \varphi_2(H) = H \times H = H$. This shows that for any automorphism φ of G, $\varphi(H) = H$, implying the characteristic nature of subgroup H in G. Hence, G has a non-trivial subgroup H that is both normal and characteristic.

Normal Subgroups of $G \cong D_{24} \times C_{2^n}$ and their Isomorphisms

The focus of objective four settles on identifying the normal subgroups of $G \cong D_{24} \times C_{2^n}$ and classifying them up to isomorphism. C_{2^n} is not cyclic for $n > 1$, thus we first investigate the normal subgroups within the group $D_{24} \times C_2$ and analyze their structural properties up to isomorphism.

Proposition 16

Let $G = D_{24} \times C_2$. Then G is solvable, contains non-trivial normal subgroups, but not a simple group.

Proof. From Theorem 51, $G = D_{24} \times C_2$ has a non-trivial normal subgroup. Similarly, $|G|$ is not prime and by Theorem 40, G is not simple. Hence, by Theorem 41, G is solvable.

Theorem 52. *Normal subgroups in $D_{24} \times C_2$ and their isomorphism classes*.

Proof. From Theorem 33 and Euler's totient function, we can deduce the possible orders of elements in D_{24}, which are given by 1, 2, 3, 4, 6, 8, 12, and 24. The count of elements of orders 1, 2, 3, 4, 6, 8, 12, and 24 are 1, 13, 2, 2, 2, 0, 4, and 0 respectively. By Theorem 25, all the elements of a conjugacy class exhibit equal period. Thus, we deduce the conjugacy classes for rotations as follows $Z(D_{12}) = \{e, R^6\}$, $Cl_{D_{12}}(R^4) = \{R^4, R^8\}$, $Cl_{D_{12}}(R^3) = \{R^3, R^9\}$, $Cl_{D_{12}}(R^2) = \{R^2, R^{10}\}$, $Cl_{D_{12}}(R) = \{R, R^{11}\}$, $Cl_{D_{12}}(R^5) = \{R^5, R^7\}$ and that of reflection we have $Cl_{D_{12}}(S) = \{S, R^2S, R^4S, R^6S, R^8S, R^{10}S\}$, and $Cl_{D_{12}}(RS) = \{RS, R^3S, R^5S, R^7S, R^9S, R^{11}S\}$ and since conjugacy is an equivalence relation it partitions $D_{2(12)}$ into conjugacy classes. Thus, by the class equation, we can write that

$$D_{24} = \{e\} \cup \{R^6\} \cup \{R^4, R^8\} \cup \{R^3, R^9\} \cup \{R^2, R^{10}\} \cup \{R, R^{11}\} \cup \{R^5, R^7\}$$
$$\cup \{S, R^2S, R^4S, R^6S, R^8S, R^{10}S\} \cup \{RS, R^3S, R^5S, R^7S, R^9S, R^{11}S\}.$$

Such that $|Cl(D_{24})| = 9$, thus the number of conjugacy classes of size 1(order of center), 2 and 6 are 2, 5 and 2 respective y. Now we consider the normal subgroups within D_{24}. The subgroup of order 1 in D_{24} is isomorphic to the cyclic group $\mathbb{Z}_1$ and is unique since the only subgroup of order 1 is the trivial subgroup {e}. Also, we consider, for any $g \in D_{24}$, $g^1eg^{-1} = e$, hence the subgroup of order 1 is trivially normal in D_{24}. The 13 subgroups of order 2 in D_{24} are isomorphic $\mathbb{Z}_2$. The structures of these subgroups are $H_1 = \{S, e\}$, $H_2 = \{RS, e\}$, $H_3 = \{R^2S, e\}$, $H_4 = \{R^3S, e\}$, $H_5 = \{R^4S, e\}$, $H_6 = \{R^5S, e\}$, $H_7 = \{R^6S, e\}$, $H_8 = \{R^7S, e\}$, $H_9 = \{R^8S, e\}$, $H_{10} = \{R^9S, e\}$, $H_{11} = \{R^{10}S, e\}$, $H_{12} = \{R^{11}S, e\}$ and $H_{13} = \{e, R^6\}$. By Proposition 7, $Cl_{D_{12}}(S) = \{S, R^2S, R^4S, R^6S, R^8S, R^{10}S\}$ is not a subset of H_1,

H_3, H_5, H_7, H_9, H_{11} respectively. Similarly, $Cl_{D_{12}}(RS) = \{RS, R^3S, R^5S, R^7S, R^9S, R^{11}S\}$ is not a subset of H_2, H_4, H_6, H_8, H_{10}, H_{12}. So, we can say subgroups $H_1, H_2, H_3, \ldots, H_{12} \ntrianglelefteq D_{24}$. However, $Cl_{D_{12}}(R^6) = \{R^6\} \subset H_{13}$ and also by Proposition 4, $Cl_{D_{12}}(e) = \{e\} \subset H_{13}$, which implies H_{13} is a normal subgroup. The structure of subgroup of order 3 is $K_1 = \{e, R^4, R^8\}$ isomorphic to $\mathbb{Z}_3$ and hence is unique and a normal subgroup of D_{12}. The structure of the subgroups of order 4 are $M_1 = \{e, R^3, R^6, R^9\}$, $M_2 = \{e, R^6, S, R^6S\}$, $M_3 = \{e, R^6, RS, R^7S\}$, $M_4 = \{e, R^6, R^2S, R^8S\}$, $M_5 = \{e, R^6, R^3S, R^9S\}$, $M_6 = \{e, R^6, R^4S, R^{10}S\}$ and $M_7 = \{e, R^6, R^5S, R^{11}S\}$. The subgroup M_1 is isomorphic to $\mathbb{Z}_4$ is cyclic and unique, and hence it is normal in D_{24}. However, the subgroups M_2, M_3, M_4, M_5, M_6 and M_7 are isomorphic to $D_2 \cong K_4 = \langle a, b \mid a^2 = b^2 = (ab)^2 = e \rangle$. Here $Cl_{D_{12}}(S) = \{S, R^2S, R^4S, R^6S, R^8S, R^{10}S\}$ is not a subset of M_2, M_4 and M_6. Similarly, $Cl_{D_{12}}(RS) = \{RS, R^3S, R^5S, R^7S, R^9S, R^{11}S\}$ is not a subset of M_3, M_5 and M_7. Hence, M_2, M_3, M_4, M_5, M_6 and M_7 are not normal subgroup in D_{24}. The subgroups of order 6 are $N_1 = \{e, R^2, R^4, R^6, R^8, R^{10}\}$, $N_2 = \{e, R^4, R^8, S, R^4S, R^8S\}$, $N_3 = \{e, R^4, R^8, RS, R^5S, R^9S\}$, $N_4 = \{e, R^4, R^8, R^2S, R^6S, R^{10}S\}$ and $N_5 = \{e, R^4, R^8, R^3S, R^7S, R^{11}S\}$. The subgroup N_1 is isomorphic to $\mathbb{Z}_6$ which is cyclic and unique and therefore normal in D_{24}. Subgroups N_2, N_3,…, N_5 are isomorphic to D_3. $Cl_{D_{12}}(S) \not\subset N_2$, $Cl_{D_{12}}(S) \not\subset N_4$, $Cl_{D_{12}}(RS) \not\subset N_3$ and $Cl_{D_{12}}(RS) \not\subset N_5$ which implies N_2, N_3,…, N_5 are not normal subgroup in D_{24}. Subgroups of order 8 are $P_1 = \{e, R^3, R^6, R^9, S, SR^3, SR^6, SR^9\}$, $P_2 = \{e, R^3, R^6, R^9, RS, R^4S, R^7S, SR^{10}\}$ and $P_3 = \{e, R^3, R^6, R^9, R^2S, R^5S, R^8S, SR^{11}\}$. These subgroups are isomorphic to D_4, but subgroups P_1, P_2 and P_3 are not normal subgroups of D_{12} since $Cl_{D_{12}}(S) \not\subset P_1$, $Cl_{D_{12}}(RS) \not\subset P_2$ and $Cl_{D_{12}}(R^2S) \not\subset P_3$. Subgroups of order 12 are $S_1 = \{e, R, R^2, R^3, R^4, R^5, R^6, R^7, R^8, R^9, R^{10}, R^{11}\}$, $S_2 = \{e, R^2, R^4, R^6, R^8, R^{10}, S, R^2S, R^4S, R^6S, R^8S, R^{10}S\}$ and $S_3 = \{e, R^2, R^4, R^6, R^8, R^{10}, RS, R^3S, R^5S, R^7S, R^9S, R^{11}S\}$. The subgroup $S_1 \cong \mathbb{Z}_{12}$ and hence is unique and a normal subgroup. $Cl_{D_{12}}(S) \subset S_2$ and likewise, $Cl_{D_{12}}(RS) \subset S_3$. We can also say that S_2 and S_3 have index 2 and from Proposition 8, S_2 and S_3 are normal subgroups. Hence, we obtain 9 normal subgroups of $D_{2(12)}$ and since the number of normal subgroups equals the number of conjugacy classes, G exhibits rich internal symmetry. We then consider the normal subgroups in C_2 and their structures up to isomorphism. The subgroups of C_2 are the trivial subgroup $\{e\}$ and the group itself that is $\{e, a\}$ where a is a non-identity element. In C_2, both the trivial subgroup $\{e\}$ and the group $C_2 \cong Z_2$ itself are normal subgroups. Hence there are 18 normal subgroups in $D_{24} \times C_2$, and by Theorem 31, we deduce their structures as follows:

$\{(\{e\},\{e\}), (\{e\},\{e,a\})\}$, $\{(\{e,R^6\},\{e\}), (\{e,R^6\},\{e,a\})\}$, $\{(\{e,R^4,R^8\},\{e\}), (\{e, R^4,R^8\},\{e,a\})\}$, $\{(\{e,R^3,R^6,R^9\},\{e\}), (\{e,R^3,R^6,R^9\},\{e,a\})\}$, $\{(\{e,R^2,R^4,R^6, R^8,R^{10}\},\{e\}), (\{e,R^2,R^4,R^6,R^8,R^{10}\},\{e,a\})\}$, $\{(\{e,R,R^2,R^3,R^4,R^5,\dots,R^{11}\},\{e\}), (\{e,R,R^2,R^3,R^4,R^5,\dots,R^{11}\},\{e,a\})\}$, $\{(\{e,R^2,R^4,\dots,R^{10},S,R^2S,\dots,R^{10}S\}, \{e\}), (\{e,R^2,R^4,\dots,R^{10},S,R^2S,\dots,R^{10}S\},\{e,a\})\}$, $\{(\{e,R^2,\dots,R^{10},RS,R^3S,R^5S, \dots,R^{11}S\},\{e\}), (\{e,R^2,\dots,R^{10},RS,R^3S,R^5S,\dots,R^{11}S\},\{e,a\})\}$, $\{(\{e,R,R^2,R^3,.. ,R^{11},RS,R^2S,R^3S,\dots,R^{11}S\},\{e\})(\{e,R,R^2,R^3,\dots,R^{11},RS,R^2S,R^3S,\dots,R^{11}S\} \{e,a\})\}$.

We also observe that C_2^n is isomorphic to the additive group of the vector space $\mathbb{F}_2^n$, where each subgroup of C_2^n corresponds to a vector subspace of $\mathbb{F}_2^n$. Thus, the count of subgroups of C_2^n is given by $\sum_{k=0}^{n}\binom{n}{k}_2$, where $\binom{n}{k}_2$ is the Gaussian binomial coefficient. Hence, the count of normal subgroups of G yields $9\left(\sum_{k=0}^{n}\binom{n}{k}_2\right)$.

Characterizing Finite Groups with Specific Cyclic Subgroup Count

In objective five, we describe and characterize finite groups G for which cyclic subgroups count identical to $|G| - 6$ is achieved by exploring the structural properties of the groups that lead to this cardinality. We take note that G is an elementary abelian 2-group if and only if $|C(G)| = |G|$. Motivated by the research of T$\acute{a}$rn$\acute{a}$uceanu (2016) who classified groups satisfying $|G| - 2$ and a similar classification by Song and Zhou (2019) on groups with $|G| - 3$. This research builds on the methodology used by these authors but in a different twist relies heavily on prime factorization and Sylow theorems to determine possible group orders and structures.

Lemma 8

Let G be a finite group with $|C(G)| = |G| - 6$. Then $|G| \cong 2^m \cdot 3^n \cdot 5^r$, with $m \leq 3$, $n \leq 2$, and $r \leq 1$.

Proof. Let G be an order n finite group. Given

$$|C(G)| = |G| - 6 \qquad (7)$$

and using Proposition 2 we write, for finite group G

$$|G| = \sum_{i=1}^{k} n_i(\varphi(d_i)) = n$$

$$|C(G)| = \sum_{i=1}^{k} n_i$$

where φ represents Euler's phi function, $d_1, d_2, \dots, d_k$ represent $n's$ positive divisors. For each $i \in \{1, 2, \dots, k\}$, define $n_i = |\{H \in C(G)||H| = d_i\}|$. Thus

$$|G| - |C(G)| = \sum_{i=1}^{k} n_i(\varphi(d_i) - 1) = 6$$

Then one can find some $i_0 \in \{1, 2, \dots, k\}$ where,

$$n_{i_0}(\varphi(d_{i_0}) - 1) = 6$$

And

$$n_i(\varphi(d_i) - 1) = 0 \text{ for all } i \neq i_0.$$

Thus,

$$n_{i_0} = \frac{6}{(\varphi(d_{i_0}) - 1)}$$

where $(\varphi(d_{i_0}) - 1) \neq 0$ and $\varphi(d_{i_0}) \neq 1$. Since n_{i_0} must be a positive integer, $\varphi(d_{i_0}) - 1$ must be a divisor of 6. Therefore, $\varphi(d_{i_0}) - 1 \in \{1, 2, 3, 6\}$, which implies $\varphi(d_{i_0}) \in \{2, 3, 4, 7\}$. This leads to: $\varphi(d_{i_0}) = 2$ where $d_{i_0} \in \{3, 4, 6\}$, $\varphi(d_{i_0}) = 3$ where $d_{i_0} \in \{\}$, φ(di0) = 4 where $d_{i_0} \in \{5, 8, 10, 12\}$, $\varphi(d_{i_0}) = 7$ where $d_{i_0} \in \{\}$. Considering the properties and relationships among the divisors of the group's order, we can deduce the order of G from $d_{i_0} \in \{3, 4, 6\}$ and $d_{i_0} \in \{5, 8, 10, 12\}$. By Theorem 33, |G| must be of the form $|G| \cong 2^m \cdot 3^n \cdot 5^r$ where $m \leq 3, n \leq 2, r \leq 1$ and $p_1 < p_2 < p_3$.

Lemma 9

Let G be a group of order p^k, where p is a prime and $1 \leq k \leq 3$. Then, $G \cong C_{p^k}$, or $G \cong C_{p^{k_1}} \times C_{p^{k_2}} \times \cdots \times C_{p^{k_m}}$ with $k_1 + k_2 + \cdots + k_m = k$, where for some k, the count of cyclic subgroups satisfies $|G| - 6$ in G.

Proof. Since G is a p-group, by Theorem 36, it contains a nontrivial centre, $Z(G)$. If G is cyclic, then $G \cong C_{p^k}$ and the proof is complete. If G is not cyclic, we can explore its structure using group extensions. We first note

that $G/Z(G)$ is a group of smaller order, say p^j where $j < k$. If the extension of $Z(G)$ by $G/Z(G)$ splits, then G can be written as a direct product $G \cong Z(G) \times G/Z(G)$. We apply this recursively to decompose G into a product of smaller cyclic p-groups, ultimately reaching the form $G \cong C_{p^{k_1}} \times C_{p^{k_2}} \times \cdots \times C_{p^{k_m}}$. If the extension does not split, G remains indecomposable and cyclic of order p^k. Thus, G is either cyclic or a direct product of cyclic p-groups, depending on the behavior of the extension. But $1 \leq k \leq 3$, so we have three cases:

Case 1:

For $k = 1$, it implies $|G| = p$, then G is cyclic. Thus, the only divisors of p are 1 and p. Therefore, there exists an element $x \in G$ such that $|x| = p$, and the subgroup generated by x, denoted $\langle x \rangle$, is the entire group G, meaning $G = \langle x \rangle \cong Cp$. Since G is cyclic, the only subgroups of G are $\{e\}$ and G. We observe, case 1 has no group that satisfies Eq. (7).

Case 2:

For $k = 2$, $|G| = p^2$. This implies G contains elements of order p. Consider $|x| = p$ in G. We note that the subgroup generated by x, $\langle x \rangle$, has order p. Again, since $|G| = p^2$, there must exist an element y, not in $\langle x \rangle$. Since y is not in $\langle x \rangle$, the subgroup $\langle y \rangle$ is distinct from $\langle x \rangle$. But $|\langle y \rangle|$ divides $|G|$, so it must be either p or p^2.

If $\langle y \rangle$ has order p^2, then G is cyclic and isomorphic to C_{p^2}. If $\langle y \rangle$ has order p, then G is not cyclic and is isomorphic to $C_p \times C_p$. For C_{p^2}, we have 3 cyclic subgroups: $\{e\}$, $\langle x \rangle$, and G. Similarly, $C_p \times C_p$ has 4 cyclic subgroups: $\langle x \rangle$, $\langle y \rangle$, $\langle (x, e) \rangle$, and $\langle (e, y) \rangle$. So, for G to satisfy Eq. (7), $G \cong C_{p^2} \cong C_9$.

Case 3:

For $k = 3$, $|G| = p^3$. Similarly, by Theorems 36, G must contain elements of order p. Consider a case $x \in G$, $|x| = p$, so $\langle x \rangle$ is a cyclic subgroup of order p. Again, let y be an element in G where $y \notin \langle x \rangle$. But, $|\langle y \rangle|$ divides $|G|$, so $\langle y \rangle$ can have order p, p^2, or p^3. Now, suppose $|\langle y \rangle| = p^3$, then G is cyclic and $G \cong C_{p^3}$. If $|\langle y \rangle| = p^2$, then G is not cyclic, but it is a nontrivial extension of $\langle x \rangle \cong C_p$. Similarly, if $|\langle y \rangle| = p$, then G is non-cyclic and structurally equivalent to $C_p \times C_p \times C_p$. Now, counting cyclic subgroups, if $G \cong C_{p^3}$, the number of cyclic subgroups corresponds to the divisors of p^3, which are $\{1, p, p^2, p^3\}$. Hence, the cyclic subgroups are $\{e\}$, $\langle x \rangle$, $\langle x^p \rangle$, G, giving a total of 4 cyclic subgroups. Also, when $G \cong C_p \times C_p \times C_p$, the cyclic subgroups count is $p^2 + p + 1$. Here, $|G| = p^3$ has no group that satisfies Eq. (7).

We then look at the finite groups' structures of composite orders $|G| = p^k q^l$, with p and q as distinct primes. We mainly study the interaction between group extensions and Sylow subgroups, using these to uncover both abelian and non-abelian group structures. With a focus on the role of automorphisms in defining the nature of extensions, this proof sheds new light on the group-theoretic characteristics of extensions.

Lemma 10

Let $|G| = p^k q^l$, with $k, l \leq 3$ and $gcd(k, l) = 1$. Then $G \cong N \times G/N$ or $G \cong N \rtimes G/N$, where count of cyclic subgroups satisfies $|G| - 6$ for specific k and l.

Proof. Suppose $|G| = p^k q^l$ includes a normal subgroup N where $|N| = p^k$. Then G/N is of order q^l, meaning G/N is structurally equivalent to C_{q^l}. Therefore, we have a short exact sequence:

$$1 \to N \to G \to G/N \cong C_{q^l} \to 1$$

where G is an extension of $N \cong C_{p^k}$ by C_{q^l}. Now, if the extension splits, G is structurally equivalent to the direct product $C_{p^k} \times C_{q^l}$. Here, every element of G can be written as a pair (x, y) where $|x| = p^k$ and $|y| = q^l$, and all elements commute because G is abelian. Alternatively, if the extension does not split, then G is considered non-abelian. Thus, G is a semi-direct product, where $G \cong N \rtimes G/N$. Here, the quotient group G/N acts nontrivially on N via conjugation, introducing non-commutative relations into the group structure. We now examine several cases of finite groups G of order $|G| = p^k q^l$ and note that $gcd(k, l) = 1$, which gives orders of G as $p^k q^l, p^1 q^l, p^k q^1, p^2 q^l, p^k q^2$, and pq for all $p < q$ where $k \leq 3$ and $l \leq 3$.

Case 1: $|G| = pq$

For $|G| = pq$ with $p < q$, Theorem 39 imply $n_p = n_q = 1$, thus $G \cong Zp \times Zq \cong Z_{pq}$. The subgroup structure yields four distinct cyclic subgroups: $1, p, q, pq$, confirming $G \cong C_{10}$ and enriching the classification of period pq abelian groups satisfying Eq. (7).

Case 2: $|G| = p^2q$

For $|G| = p^2q$, if $n_p = 1$, $G \cong Z_{p^2} \times Z_q$, with 6 cyclic subgroups. When $n_p = q$, $G \cong Z_{p^2} \times Z_q$, where $Aut(Z_{p^2}) \cong Z_p$ governs the non-abelian structure. For $p = 2$, this yields count of cyclic subgroup of $pq + 4$ and $G \cong D_{20}$ which aligns with Eq. (7), introducing non-commutative relations through the quotient action.

Case 3: $|G| = pq^2$

For $|G| = pq^2$, if $n_q = 1$, $G \cong Z_p \times Z_{q^2}$, with 6 cyclic subgroups. If $n_q = p$, $G \cong Z_{q^2} \rtimes Z_p$, resulting in a non-abelian structure $G \cong D_{18}$ with cyclic subgroup count of $q^2 + 3$ and consistent with Eq. (7).

Case 4: $|G| = p^2q^2$

For $|G| = p^2q^2$, if both Sylow subgroups are normal, $G \cong Z_{p^2} \times Z_{q^2}$, yielding 9 cyclic subgroups. Otherwise, G is a non-trivial extension of Z_{p^2} by Z_{q^2} or vice versa, with non-trivial automorphisms leading to new structural variations. Here, no group conforms to Eq. (7)

Case 5: Higher-Order Extensions

For higher-order cases like p^3q, p^2q^3, p^3q^2, and p^3q^3, if $n_p = 1$ and $n_q = 1$, G is abelian and takes the form $G \cong Z_{p^a} \times Z_{q^b}$, with cyclic subgroups corresponding to divisors of $|G|$. When $n_p > 1$ or $n_q > 1$, non-trivial extensions arise, and $G \cong N \rtimes Q$, where automorphisms from Q act nontrivially on N, as in $G \cong D_{24}$ when $|G| = p^3q$ and $|C(G)| = p^2q + 3$. At this point, we address the formulation of our main theorem:

Theorem 53. *Let G be finite-ordered. Then G has the characteristics of* $|C(G)| = |G| - 6$ *if G include D_{24}, C_{12}, C_9, C_{10}, D_{18}, D_{10}.*

Proof. By Proposition 3 and Theorem 22, we observe that D_{24}, C_{12}, C_9, C_{10}, D_{18}, and D_{10} satisfy Eq. (7)

Conversely, Theorem 53 holds, from the analysis put fort in Lemmas 8, 9 and 10. This complete our proof.

Chapter Summary

This chapter presents the results of this research on characterizing finite groups with a specific alpha invariant value. The study further applies the Orbit-Stabilizer Theorem to study centralizers in direct products, examines the invariance of subgroups under conjugation and automorphisms, classifies normal subgroups in $G \cong D_{24} \times C_{2^n}$ along with their isomorphism structures, and characterizes finite groups based on their cyclic subgroup count.

CHAPTER FIVE

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

Overview

A summary of the research's key findings and their implications is provided in this chapter. Highlights of how the study advances the understanding of group structures through proving conjectures, analyzing subgroup properties, and classifying important structural elements are presented. The conclusions emphasize how these findings have wider implications for group theory. The chapter also offers research recommendations, outlining possible avenues for applying these discoveries to different mathematical contexts.

Summary

The principal aim of the research centred on a conjecture, properties, structure and characteristics of finite groups. The study addresses key concepts, utilizes rigorous techniques and most significantly review related literature and extent of contribution to achieve stated objectives. The researcher employs detailed theoretical analysis using group theory principles to validate the conjecture and determine the specific conditions under which the group structure $D_{24} \times C_{2^n}$ holds. Next, the research focuses on determining whether such a group possesses a non-trivial centre by analysing the conjugacy classes and centralizers within the direct product structure. This the researchers proves that indeed a non-trivial centre exist and uses the result as a buildup for subsequent research into grey areas. The research further introduce a novel approach by exploring the existence of non-trivial subgroups within $G \cong D_{24} \times C_{2^n}$ that are both normal and characteristic. Our proof demonstrates that such a subgroup, specifically $H = \{(R^6, e)\}$, possesses these properties, which is not immediately evident within this group structure. To reinforce our theoretical findings, we employ the GAP software, utilizing its computational properties to compute and analyze the elements and subgroups of D_{24}. This computational verification serves as a complementary tool, enhancing the robustness of our results by cross-referencing them with traditional proof methods. In addition, we conduct a detailed examination of the subgroup structure of D_{24} and its interplay with the cyclic group C_{2^n}. By explicitly enumerating and analyzing subgroups arising from various element combinations, we provide a comprehensive understanding of the algebraic properties of these groups. Finally, we investigate the interaction between the centre $Z(G)$ and its subgroups, demonstrating that H is a normal and characteristic subgroup within the centre and, by extension, within the larger group G. This analysis highlights the intricate connection between subgroup properties and central elements, offering a better comprehension of group's structure. Another important aspect of the research is also devoted to providing a detailed analysis of these normal subgroups, revealing their structural properties and determining how they fit into the broader group structure. Finally, the research characterizes finite groups G for which the cyclic subgroups count is $|G| - 6$. The group G is shown to be structurally equivalent to one of the following: D_{24}, C_{12}, C_9, C_{10}, D_{18}, or D_{10}.

Conclusions

This research significantly advances the understanding of the structural properties of direct product of D_{24} and C_{2^n}. The findings contribute to group theory by proving a conjecture about $\alpha(G)$, exploring the centre and normal subgroups, and characterizing groups with a unique cyclic subgroup count. Specifically, the research looked at

1. We explored the relationship between a non-nilpotent group and the alpha value and proved a conjecture posed in a research paper, (Cayley, 2021). As a consequence, we set values of n for which the conjecture holds. Thus, we demonstrated that when $\alpha(G) = \frac{3}{4}$ and G is non-nilpotent, $G \cong D_{24} \times C_{2^n}$, with $n \in \{0, 1\}$.
2. We established that for $D_{24} \times C_{2^n}$with $\alpha(G) = \frac{3}{4}$ and G not nilpotent, G has a non-trivial centre. By analysing the conjugacy classes and centralizers within the direct product structure, it is shown that any element x not in the centre leads to a contradiction, thereby proving that such x must be in the centre, ensuring it is non-trivial.
3. We proved that if $D_{24} \times C_{2^n}$ where $\alpha(G) = \frac{3}{4}$and G is not nilpotent, then G posses non-trivial subgroup that is both normal and characteristic. The proof employed both computational and theoretical techniques. We established the centralizer structure and demonstrated the invariance of the subgroup $H = \{(R^6, e)\}$ under conjugation and automorphisms.
4. We investigated the normal subgroups within $D_{24} \times C_2$ and their structural properties up to isomorphism. We demonstrated that $D_{24} \times C_2$ is not a simple group but contains non-trivial normal subgroups, thus making it solvable. The analysis included identifying the orders and structures of normal subgroups in D_{24} using Euler's totient function and conjugacy

classes. Normal subgroups of C_2 were also examined. In all, we classified the normal subgroups of $D_{24} \times C_2$, thereby enhancing the understanding of its subgroup structure.

5. Finally, we investigated a theorem concerning the cyclic subgroup count of finite groups G where $|C(G)| = |G| - 6$. From our analysis we conclude that G is one of the groups D_{24}, C_{12}, C_9, C_{10}, D_{18}, or D_{10}, when $|C(G)| = |G| - 6$.

Recommendations

Drawing upon the findings from our research, we suggest considering the following recommendations for future research:

1. The findings of this study confirm the conjecture for groups of the form $G \cong D_{24} \times C_{2^n}$ with $\alpha(G) = \frac{3}{4}$. A natural direction for future research is to explore whether similar structural characteristics hold for other non-nilpotent groups. Extending this result to a wider class of direct products, particularly those involving dihedral groups of different orders or other non-abelian finite groups, may reveal deeper connections between $\alpha(G)$ and group structure.
2. The study demonstrates that, $D_{24} \times C_2$ is solvable but not simple. To improve the categorization of such structures, a more comprehensive study of the circumstances in which a direct product of finite groups maintains solvability or achieves simplicity would be beneficial. It may be possible to gain a more thorough knowledge of how these characteristics appear in

direct products by establishing generic criteria for the solvability and simplicity of $G_1 \times G_2$ under various subgroup configurations.

3. Exploration of higher-order cases or other variations on cyclic subgroup counts.

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