UNIVERSITY OF CAPE COAST

QUARTIC RANK TRANSMUTATION OF GAMMA–TYPE DISTRIBUTIONS: CHARACTERISTICS AND ESTIMATION

JONES ASANTE MANU

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BY

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Thesis submitted to the Department of Statistics of the School of Physical Sciences, College of Agriculture and Natural Sciences, University of Cape Coast, in partial fulfilment of the requirements for the award of Doctor of Philosophy degree in Statistics

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DECLARATION

Candidate's Declaration

I hereby declare that this thesis is the result of my own original research and that no part of it has been presented for another degree in this university or elsewhere.

Candidate's Signature .... Date ....

Name: Jones Asante Manu

Supervisors' Declaration

We hereby declare that the preparation and presentation of the thesis were supervised in accordance with the guidelines on supervision of thesis laid down by the University of Cape Coast.

Supervisor's Signature .... Date ....

Name: Professor Nathaniel Howard

Supervisor's Signature .... Date ....

Name: Professor Bismark Kwao Nkansah

ABSTRACT

Rank transmutation maps have emerged as one of the adopted methods for proposing new probability distributions. This study used the quartic rank transmutation to introduce three new probability distributions: the Quartic Transmuted Exponential Distribution, Quartic Transmuted Lindley Distribution, and Quartic Transmuted Rayleigh Distribution. The construction of these distributions involves meticulously examining mathematical concepts, encompassing probability density functions, survival functions, moments, entropies, and order statistics. Visual aids, including cumulative distribution functions, probability density functions, and hazard rate functions, enhance the comprehension of distribution characteristics. A comprehensive simulation study underscores a consistent trend: a reduction in bias for maximum likelihood estimation and refinement in standard errors with increasing sample size. The practical applicability of these newly proposed distributions was demonstrated using real-world datasets. The quartic transmuted exponential distribution was effectively employed to model the lifetime of 50 devices, referencing data from Aarset's study in 1987. Similarly, the quartic transmuted Lindley distribution was adeptly applied to remission times (measured in months) of 128 bladder cancer patients. Finally, the quartic transmuted Rayleigh distribution was successfully utilized to analyze a dataset comprising 72 instances of exceedance from the Wheaton River flood data near Carcoss in Yukon Territory, Canada. Evaluation criteria such as log-likelihood, AIC, AICc, and BIC affirm the superior flexibility and performance of the proposed distributions. This research significantly contributes to distribution theory, offering innovative methods to enhance distribution adaptability in diverse applications.

KEY WORDS

Exponential distribution

Lindley distribution

Quartic transmuted exponential distribution

Quartic transmuted Lindley distribution

Quartic transmuted Rayleigh distribution

Rayleigh distribution

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DEDICATION

To my family

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LIST OF ACRONYMS/ABBREVIATIONS

AIC	Akaike Information Criterion
AICc	corrected Akaike Information Criterion
BIC	Bayesian Information Criterion
CTED	Cubic Transmuted Exponential Distribution
CTLD	Cubic Transmuted Lindley Distribution
CTRD	Cubic Transmuted Rayleigh Distribution
TED	Transmuted Exponential Distribution
TLD	Transmuted Lindley Distribution
TRD	Transmuted Rayleigh Distribution

CHAPTER ONE

INTRODUCTION

This chapter provides a comprehensive introduction to the thesis, outlining the motivation and background for the study. It includes the problem statement, which highlights the gaps and challenges addressed by the research. The chapter also details the study's objectives, setting the stage for the subsequent chapters. Key elements such as the significance and the scope of the study are presented. Finally, an overview of the thesis structure is provided to guide the reader through the journey of the research. Each chapter is briefly described, giving insights into how the thesis unfolds to address the stated objectives and achieve the research aims.

Background to the Study

Statistics, as a discipline within mathematical science, encompasses the process of collecting, organizing, analyzing, and drawing well-founded conclusions about a population. This field consists of two primary branches: descriptive statistics and inferential statistics. Descriptive statistics involves the systematic summarization and depiction of key characteristics of data, utilizing various techniques such as means, percentages, graphs, and sums. These methods facilitate the extraction of meaningful insights and patterns from the data, enabling researchers to grasp its essential features and properties. On the other hand, inferential statistics aims to make informed judgments and decisions about a population by utilizing information obtained from a sample. By drawing inferences from the sample data, statisticians strive to generalize their findings and make conclusions that hold true for the broader population. While both descriptive and inferential statistics play crucial roles, statisticians often emphasize inferential statistics, prioritizing the process of making accurate inferences over the descriptive aspects. Consequently, in recent years, the significance and emphasis placed on inferential statistics have grown substantially within the field of statistics.

At the core of statistical inference lies a vast toolbox of statistical methods and distributions employed by statisticians. These tools are essential for making scientific developments and drawing conclusions amidst inherent uncertainty and variability. By harnessing these statistical methods and distributions, researchers gain the means to explore and understand the complexities of real-world phenomena. These methods provide systematic frameworks for analyzing data, identifying patterns, quantifying relationships, and uncovering hidden insights. They offer robust approaches to investigate research questions, validate hypotheses, and make well-informed decisions based on empirical evidence. They provide a fundamental basis for quantifying uncertainty, estimating probabilities, and performing statistical tests. By fitting observed data to appropriate distributions, researchers gain valuable insights into the underlying mechanisms and dynamics governing real-world phenomena.

Consequently, given the wide applicability and utility of statistical methods and distributions, their usage have become pervasive across various scientific disciplines. From medicine to economics, sociology to environmental studies, researchers in virtually all fields rely on statistical tools for data analysis and interpretation. This widespread adoption underscores the importance of statistical inference in advancing knowledge and facilitating evidence-based decision-making in the pursuit of scientific understanding.

Recognizing the profound significance of statistical methods and distributions, extensive efforts are devoted to the development of comprehensive collections of classical probability distributions, as well as the associated statistical methodologies. These classical distributions have been used in modelling statistical data in diverse domains within the applied and social sciences. However, the successful utilization of these distributions needs a thorough understanding of the underlying assumptions and a familiarity with the real-life phenomena to which specific distributions can be applied. For example, in fields like reliability engineering, statistical methods and distributions play a vital role in modelling the failure rate of machines. By utilizing these tools, researchers can predict potential malfunctions, optimize maintenance strategies, and enhance the overall reliability of systems. Environmental science heavily relies on statistical methods and distributions to investigate and quantify environmental pollution. These tools enable scientists to analyse pollutant concentrations, identify sources of pollution, and assess the impact on ecosystems and human health. In the realm of medical science, statistical methods and distributions are instrumental in examining the survival time of patients after surgery. Through these analyses, researchers can assess the effectiveness of different treatments, identify relevant risk factors, and estimate patient prognosis. Actuarial science makes extensive use of statistical methods and distributions to model insurance loss data. This enables insurers to evaluate risks, determine appropriate premiums, and establish reserves that align with potential liabilities.

The aforementioned examples merely scratch the surface of the diverse applications of statistical methods and distributions across various scientific fields. By leveraging these tools, researchers gain valuable insights into complex phenomena, make informed predictions, and derive meaningful conclusions from empirical data. Consequently, a comprehensive understanding of the assumptions underlying statistical distributions and their appropriate usage is crucial for their effective application in practical contexts. This understanding fosters advancements in scientific research, facilitates evidence-based decision-making, and ultimately contributes to the progress of knowledge across disciplines.

Despite the usefulness of classical probability distributions, the intricate nature of certain research often gives rise to data sets that are challenging to model using these distributions. This is because classical distributions may not always offer a satisfactory fit to such complex data. As Allison (1995) suggests, many classical probability distributions may not be adequate for analysing intricate phenomena, leading to results that are more of an approximation than a true representation of reality. Consequently, researchers in recent years have been actively exploring various methods to extend, generalize, or modify existing classical distributions. Researchers are also endeavouring to develop entirely new statistical distributions that exhibit greater flexibility in modelling real-life data. The primary objective is to create distributional models that can more accurately capture the nuances of complex phenomena, leading to improved goodness of fit. By extending, generalizing, or modifying existing classical distributions, researchers aim to address the limitations posed by traditional approaches and cater to the intricacies of real-life data. This, in turn, enhances the accuracy and reliability of statistical analyses and facilitates a more comprehensive understanding of intricate research areas.

For this reason, numerous generalization or transformation methods have been proposed in the existing literature. These methods, put forward by researchers such as Pearson (1895), Johnson (1949), Tukey (1960), Eugene, Lee, and Famoye (2002), Zografos and Balakrishnan (2009), and AAlzaatreh, Lee and Famoye (2013) aim to extend or transform existing distributions or develop entirely new distributions. According to Alzaatreh et al. (2013), the development of new methods and distributions serves several essential purposes. First, it involves creating skewness in distributions that are otherwise symmetrical. This is crucial for capturing asymmetry and tail behaviour in real data sets. Second, the focus is on developing heavy-tailed distributions that can effectively model a wide range of real data sets with varying degrees of tail behaviour. Third, the objective is to define special distributions that encompass various types of hazard rate functions, allowing for the modelling of different failure patterns and durations. Fourth, the aim is to generate distributions with different types of skewness, enabling the representation of diverse forms of asymmetry in data.

Finally, the goal is to develop distributions that consistently offer better fits to the underlying data compared to other generated distributions. That is, these new generalized, extended, or developed distributions are proposed with the aim of generating distributions with non-monotonic failure rates, heavy tails, flexibility in application, and workable distribution.

Problem Statement

Several approaches to developing new probability distributions are discussed in the literature (Johnson (1949), Tukey (1960), Eugene et al. (2002), Zografos and Balakrishnan (2009), and Alzaatreh et al. (2013)). However, rank transmutation maps have recently emerged as a well-situated approach for generating new statistical distributions. Shaw and Buckley (2009) introduced the concept of transmuting probability distributions, proposing the quadratic rank transmutation map (QRTM) distributions. They utilized these distributions to extend non-Gaussian distributions by incorporating additional parameters into their distributional functions. Several authors (Aryal & Tsokos (2011), Elbatal & Aryal (2013), Merovci (2013a)) have used the QRTM method to obtain new distributions. However, according to Granzotto et al. (2017), QRTM distributions capture the intricacies of unimodal datasets, but real-life data often possesses greater complexity, such as multimodal or bimodal characteristics. Consequently, there are instances where the QRTM may not be suitable for fitting such intricate distributions. In essence, the quadratic transmutation method captures the quadratic patterns within data, thus imposing a limitation on its applicability.

To extend the QRTM to model bimodal (multi-modal) data, Granzotto, Louzada and Balakrishnan (2017) introduced a new family of transmuted distributions called the cubic rank transmutation map (CRTM). Several authors (Rahman, AL-Zahrani & Shahbaz, 2018; Celik, 2018; Bhatti, Hamedani, Najibi & Ahmad, 2020) have applied the CRTM to obtain new probability distributions. However, there are some situations where one needs to model some complex data that both the quadratic and the cubic ranks transmutation distributions cannot offer a reasonable fit to the data. In such circumstances, there is a need to develop a more suitable method or distribution to model the complexity of the data. According to Granzatto et al. (2017), when data is bimodal, a higher function can be thought of as a good fit for the data. The authors further stated that for each order added to the transmutation map, a new parameter is added to the model. That is, by increasing the order, the model becomes more flexible than the previous order.

Few researchers have tried to generalize the rank-transmuted map. For instance, Riffi (2019), studied higher-rank transmuted families of distribution. Further, Ali and Athar (2021) studied the generalized rank-mapped transmuted distribution. In this thesis, an attempt is made to propose the quartic transmuted distributions of gamma–type probability distributions and deduce some of their statistical properties. The proposed distributions will be used to model complex data arising in financial, environmental, and other areas of life.

Motivation for the Study

The motivation behind this study is the transmuted distributions' ability to offer a high level of flexibility and suitability for fitting data. Though both the QRTM and CRTM have been studied in the literature, they at a point fail to fit more complex data reasonably well. Hence, the researcher is motivated to study a higher degree of the rank transmutation map to provide more flexibility and tractability to data. Thus, this study aims at developing a new model from existing distributions using the quartic transmuted rank map. Higher transmuted distributions offer a more flexible approach to modelling complex data.

Objective of the Study

The primary objective of this thesis is to develop more flexible probability distributions over the classical probability distributions. The specific objectives are to:

1. To propose an extension of the gamma–type probability distributions using the quartic rank transmutation map.
2. To derive the statistical characteristics of the proposed distributions (moments, moment-generating functions, order statistics, quantile functions, etc.).
3. To find parameter estimates of the proposed new distributions.
4. To demonstrate the applications of the new distributions using simulation and real data sets.

Significance of the Study

1. The development of a new statistical distribution will expand the range of tools available to statisticians, providing greater flexibility in analysing data across various fields.
2. The new distribution will improve the accuracy and robustness of statistical analysis.
3. The new distribution will offer a better fit for non-standard or unusual datasets.
4. The development of a new statistical distribution will stimulate innovation in statistical theory and methodology, leading to new techniques and algorithms that can improve statistical inference and decision-making.

5. It will encourage interdisciplinary collaborations and the adoption of new statistical techniques by researchers from diverse fields, leading to a better understanding of complex phenomena and real-world applications.

Organization of the Thesis

Chapter One presents the background of the study, statement of problem, motivation, objectives, and significance of the study. Chapter Two describes generators (approaches to developing, modifying, or extending probability distributions). The chapter also comprehensively reviews the quadratic and cubic rank transmuted distributions. Chapter Three presents the derivation of the formula for quartic rank transmuted distributions, and definitions of some basic statistical terms and describes gamma-type distributions. Chapter Four deals with new classes of probability distribution called the quartic transmuted distributions. Various statistical properties are explored. Estimation of the parameters of the family is performed through maximum likelihood estimation. A simulation study is conducted to estimate the model parameters of the quartic transmuted gamma-type distributions. Applications of the quartic transmuted gamma-type distributions are demonstrated. Chapter Five is concerned with the conclusions and possible further extensions for this research work.

CHAPTER TWO

LITERATURE REVIEW

Introduction

The exploration and development of statistical distributions have long been integral components of the field of statistics. The comprehensive review analyses conducted by Kotz and Vicari (2005) sheds light on the fundamental beginnings of statistical distribution development. The extensive work of Pearson (1895) played a pivotal role in establishing the underpinning for a multitude of methodologies aimed at generating diverse families of statistical distributions.

Lee, Famoye and Alzaatreh (2013) provided a comprehensive and thorough overview of methods developed before 1980, along with an in-depth examination of those introduced after the 1980s. This literature review adopts their comprehensive framework to examine the evolution of these methods and their applications to various datasets. By delving into this timeline, we aim to present a comprehensive overview of the evolution of techniques for constructing statistical distributions. This investigation underscores the dynamic nature of the field and its continual growth, providing insights into the progress made in the development of these fundamental tools for statistical analysis. Additionally, the chapter delves into the examination of quadratic and cubic transmuted distributions, along with an overview of the diverse research endeavours conducted on these distributions.

Methods of Developing Statistical Distributions Before 1980

Before 1980, various methodologies were employed to generate families of statistical distributions. These methodologies encompassed techniques based on differential equations, transformation (translation), and quantiles.

The methods of Differential Equations

According to Lee et al. (2013), the first method for generating statistical distributions is accredited to Pearson (1895). The phenomenal work of Pearson (1895) detailed an approach for generating statistical distributions using differential equations. These families of distributions were developed to model non-systematic datasets. The Pearson probability distribution is defined by its pdf $y = p(x)$, which fulfils a differential equation expressed as follows:

$$\frac{1}{q}\frac{d}{dx}\left[q(x)\right] = -\frac{\beta + x}{\mu_0 + \mu_1 x + \mu_2 x^2}$$

Here, β, μ_0, μ_1, and μ_2 are parameters that influence the form of the distribution $y = p(x)$. The solution to this equation varies according to the roots of the polynomial $\mu_0 + \mu_1 x + \mu_2 x^2 = 0$, meaning that distinct types of distributions arise from different solutions to that equation. Originally proposed by Pearson in 1895, there are four primary distribution types: Type I to Type IV, with the normal distribution later categorized as Type V. Subsequent refinements led to a redefinition of Type V and the introduction of Type VI. Types VII through XII were established as specific cases, and Johnson et al. (1994) provided an in-depth examination of various distributions derived from the Pearson family.

Similar to the Pearson system of distributions, Burr (1942) introduced an alternative system of distributions that comprises twelve types of cdfs, each resulting in a wide range of density shapes. These distributions are derived by considering cdfs that satisfy a specific differential equation, the solution of which is provided by:

$$dQ = Q(1 - Q)g(x)\, dx$$

where Q lies within the interval $[0, 1]$ and $g(x) \geq 0$ and is defined on the variable x. Most of the proposed distributions by Burr (1942) are unimodal. Numerous studies have investigated the characterization and extensions of distributions derived from Pearson systems of distribution and Burr systems of distribution. For example, the generalization of Pearson's differential equation proposed by Dunning and Hanson (1977) can be expressed as:

$$\frac{d}{dx}\left[p(x)\right] = \frac{\left(\lambda_0 + \lambda_1 x + \lambda_2 x^2 + \cdots + \lambda_n x^n\right) p(x)}{\left(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_r x^r\right)}$$

where $n > 0$ and $r > 0$. Several characterizations and generalizations of Pearson's system have since been developed, leading to the creation of new families of distributions (Chaudhry & Ahmed, 1993; Kibria & Shakil, 2011).

Methods of Transformation (Translation)

The method of generating family of distributions proposed by Johnson (1949) has become the milestone of this method. Johnson (1949) introduced a technique for generating distributions through a normalization transformation described by the equation:

$$Z = \lambda + \gamma f\left(\frac{X - \zeta}{\alpha}\right)$$

In this equation, $\lambda, \gamma, \zeta,$ and α are parameters, with $f(.)$ representing the translation function and Z being the Z-score variable. This approach is known as the transformation or translation method, and Johnson (1949) notably assumed that both γ and α are positive without losing generality. The author also proposed three specific transformation functions, which are:
The first transformation defined for this method is given by:

$$Z = \lambda + \gamma \ln\left(\frac{X - \zeta}{\alpha}\right), \qquad \text{for } X > \zeta$$

This transformation covers the lognormal family. The second transformation is denoted by:

$$Z = \lambda + \gamma ln\left(\frac{Y-\zeta}{\zeta+\alpha-Y}\right), \ \zeta \leq Y \leq \zeta+\alpha$$

This transformation is the bounded of family distributions. The distribution can be bounded on the lower end, the upper end or both ends. This family covers gamma, beta, and many other distributions. The third transformation is defined by:

$$Z = \lambda + \gamma \ln\left\{\left(\frac{X-\zeta}{\alpha}\right) + \sqrt{\left[\left(\frac{X-\zeta}{\alpha}\right)^2 + 1\right]}\right\}, \quad -\infty < X < \infty$$
$$= \lambda + \gamma \sinh^{-1}\left(\frac{X-\zeta}{\alpha}\right)$$

This transformation is the unbounded family of distributions, and they cover the distribution, normal distribution, and many other distributions.

As a special case, the Birnbaum–Saunders (1969) distribution was obtained from the Johnson (1949) families of distributions. The Birnbaum–Saunders distribution, named after its developers, is a probability distribution commonly used in reliability engineering and material science. It is particularly suited for modelling material fatigue and lifetime data. This distribution has found applications in various fields, especially when dealing with failure and survival analysis. It offers a flexible framework for modelling lifetimes, making it a valuable tool in statistical analysis. Johnson's systems have been widely employed in the literature to generate several generalized Birnbaum–Saunders distributions, such as the location–scale Birnbaum–Saunders family, the non-central Birnbaum–Saunders distribution, and the four-parameter generalized Birnbaum–Saunders distribution (Athayde, Azevedo, Leiva & Sanhueza, 2012).

As an extension, Tukey (1977) used the transformation method to construct skew and heavy–trailed distributions and called these systems of distributions as Tukey's GH-Family. This family is expressed by:

$$Z = W \times T^{\alpha}(W), \alpha \in \mathbb{R}$$

For $w \geq 0$, the function $T(w)$ is characterized by being positive, symmetric, and strictly increasing. This distribution was extended and formalized by Martinez and Iglewicz (1984) and Hoaglin (1985) and has demonstrated its effectiveness in various fields, including medicine, finance, and environmental studies (Fischer, 2010). Various forms of the $T(w)$ function have been extensively analyzed and defined in the literature. A widely recognized instance is Tukey's GH-transformation, defined as:

$$Z_{m,n}(W) = \alpha + \beta(e^{mW} - 1)\frac{e^{\left(\frac{nW^2}{2}\right)}}{m}$$

The parameters α is the location parameter and β is the scale parameter. The parameters m and n are associated with skewness and kurtosis, respectively. The $Z_{m,0}$ represents the m-distribution, which features skewness without kurtosis, whereas the $Z_{0,n}$ denotes the n-distribution, which displays fat or thin tails without skewness. According to Headrick, Kowalchuk and Sheng (2008), the GH-distribution has two main limitations: first, the corresponding pdf and cdf are not explicitly defined due to the lack of well-defined shape parameters; second, fitting a GH-distribution to real datasets is challenging because estimating the parameters m and n is difficult.

Hoaglin (1985) suggested fitting the GH-distribution by approximating it with a chi-square distribution having six degrees of freedom. Subsequent research has concentrated on estimating the parameters m and n or developing approximations for the parametric pdf and cdf to improve estimation techniques for the GH-distribution in practical applications. For instance, Headrick et al. (2008) derived the moments of the GH-distribution as functions of

(m, n). Additionally, there has been significant work on extending and generalizing the GH-distribution. Fischer (2010) reviewed various generalizations and introduced new extensions, while Rayner and MacGillivray (2002) proposed an alternative G-transformation method known as K-transformation. Furthermore, Fischer and Klein (2004) investigated the J-transformation, suggesting a general transformation within Tukey's GH-distribution.

The Method of Quantile Function

This method is accredited to the works of Hastings et al. (1947) and Tukey (1960). The focus was on the development of the lambda distribution. However, Ramberg and Schmeiser (1972, 1974) expanded the scope of this distribution and introduced it as the generalized lambda distributions (GLDs). This class of distributions is characterized by its percentile function and is expressed as follows:

$$\begin{aligned} Q(u) &= Q(u; \alpha_1, \alpha_2, \alpha_3, \alpha_4) \\ &= \alpha_1 + \frac{u^{\alpha_3} - (1-u)^{\alpha_4}}{\alpha_2}, \quad \text{where } 0 \leq u \leq 1 \end{aligned}$$

The parameters α_1 and α_2 represent the location and scale parameters, while α_3 and α_4 control the skewness and kurtosis, respectively. The corresponding pdf is of the form:

$$f(x) = \frac{\alpha_2}{\alpha_3 u^{\alpha_3 - 1} + \alpha_4 (1-u)^{\alpha_4 - 1}}, \quad \text{with } x = Q(u)$$

For a valid pdf, it is required that the condition $\alpha_3 u^{\alpha_3 - 1} + \alpha_4 (1-u)^{\alpha_4 - 1}$ must have the same sign throughout the interval $[0, 1]$ and must be consistently positive or negative. Freimer, Kollia, Mudholkar and Lin (1988) explored both the similarities and differences between Pearson's system and the generalized lambda distribution. The authors noted that Pearson's system does not include the logistic distribution, whereas the generalized lambda distribution does not cover all possible values of skewness and kurtosis. Karian, Dudewicz and Mcdonald (1996), and Karian and Dudewicz (2000) developed an extended GLDs. The extended GLDs entails both GLD and the generalized beta distribution and is defined by:

$$f(x) = \begin{cases} \dfrac{(x-a_1)a_3(a_1+a_2-x)^{a_4}}{B(a_3+1,a_4+1)a_2^{(a_3+a_4+1)}}, & \text{for } a_1 \leq x \leq a_1+a_2 \\ 0, & \text{otherwise} \end{cases}$$

where $B(.,.)$ is the complete beta function. Several studies have explored extensions and generalizations of this method. For instance, Karian and Dudewicz (2000) provided a thorough discussion on both the generalized lambda distribution and its extended form. Additionally, Tuner and Pruitt (1978), Morgenthaler and Tukey (2000), and Jones (2002) have investigated quantiles associated with Tukey's lambda distribution.

While the traditional techniques for generating families of probability distributions through differential equations, transformations, and quantiles were established before 1980, they still remain actively utilized and have garnered increasing attention. New probability distributions, along with their statistical characteristics, are continually emerging in the literature based on these methodologies and their practical applications. However, as the need for distributions that are both flexible and straightforward for practical applications persisted, researchers encountered challenges in creating entirely new distributions. Consequently, different approaches emerged after 1980. These approaches entailed integrating fundamental distributions to construct more elaborate and adaptable distributions.

Methods of Developing Statistical Distributions after 1980

This section provides an overview of advancements in the methods used to generate probability distributions since 1980.

The Methods of Generating Skewed Distributions

This method involves combining two symmetric distributions to generate a skewed distribution. It was first introduced by Azzalini (1985) and referred to as the skew normal family of distributions. The skew normal family is given by: If Y is a random variable with a pdf symmetric about 0 and U has the cdf $H(.)$ that is absolutely continuous, with $H^{/}(.)$ also being symmetric, then, for any real number $k \in \mathbb{R}$,

$$0.5 = P(Y - kX < 0) = E_Z\left[P(U < kz\,|Z = z\right] = \int_{-\infty}^{\infty} f_0(z)H(kz)du$$

Consequently, $2f_0(z)H(kz), -\infty < z < \infty$ represent a valid pdf. When Z follows a standard normal distribution $N(0,1)$, the random variable Z_k is said to follow a skew-normal distribution $SN(k)$ with skewness parameter k if its pdf is given by

$$f_{Z_k}(z;k) = 2\Phi(z)\lambda(kz), \; z \in \mathbb{R}, \; k \in \mathbb{R}$$

where $\Phi(z)$ and $\lambda(z)$ are $N(0,1)$ pdf and cdf respectively. The distribution of $(SN(K))$ is characterized by a single parameter k. However, location and scale parameters can be added using the translation $Y = a + bX$. Many alternatives of the skew normal distribution and its extensions have been greatly proposed and studied. Pearson (1895) provided a comprehensive summary of the developments, characterizations, and generalizations of $(SN(K))$. As an extension and generalization, Azzalini (1986), pointed out that the family $(SN(K))$ can only produce tails thinner than the normal ones, and proposed a broader class of densities of the form:

$$2g(z;\varpi)G(kz)$$

where $g(.)$ and $G(.)$ are the pdf and cdf of a symmetric random variable. Different definitions lead to different families of skewed distributions. Azzalini (1986), defined $g(.)$ as:

$$g(x;\varpi) = C_{\varpi} exp\left(-\frac{|x|^{\varpi}}{\varpi}\right)$$

where $\varpi > 0$ and $C_{\varpi} = (2\varpi^{\frac{1}{\varpi}}\Gamma(\frac{1}{\varpi}))^{-1}$. It can be seen that $g(x, 2)$ is N(0,1), $g(x, 1)$ is the Laplace, and $g(x, \varpi)$ converges to uniform (-1,1) as ϖ goes to infinity. It is easy to see that one can apply different symmetric distribution from $2g(z;\varpi)G(kz)$ to generate different skewed distributions. Various extensions of $SN(k)$ have been developed in the literature. Some extensions are based on normal distribution, while other generalizations are based on other symmetric densities. Some extensions based on normal distribution were given by Arnold, Beaver, Groeneveld and Meeker (1993).

$$f(x;k_0,k_1) = \phi(x)\Phi(k_0 + k_1x)\Phi[k_0^1 + k_1^2] \quad \text{with } k_0, k_1 \in \mathbb{R}$$

and

$$f(x;k_0,k_1,\mu,\sigma) = \frac{\varphi(\frac{x-\mu}{\sigma})\Phi(k_0 + k\frac{x-\mu}{\sigma})}{\Phi(\frac{k_0}{1+k_1^2})}$$

which includes location and scale parameters. Pewsey (2000) proposed the wrapped skew normal distribution on the circle. Let X be $SN(k)$ and $Y = g + hX$. Define the random variable $Q = Y(Mod2\pi)$. The density of Q is given by

$$f(Q;g,h,k) = \frac{2}{h}\sum_{n=-\infty}^{\infty}\varphi\left(\frac{Q+2\pi n-g}{h}\right)\Phi\left[k\left(\frac{Q+2\pi n-g}{h}\right)\right], 0 \leq Q \leq 2\pi$$

Arellano-Valle, Gómez, and Quintana, (2004) pointed out that as the skewing parameter tends to infinity, the $SN(k)$ distribution behaves like a half–normal distribution. To mitigate such issue, they proposed the following extensions:

$$f(x;k_0,k_1) = 2\varphi(x)\Phi\left(\frac{k_1 x}{\sqrt{1+k_2x^2}}\right), \text{with } \; k_1 \in \mathbb{R}, k_2 \geq 0.$$

An extension of the distribution developed by Arellano-Valle et al. (2004) using the marginal density from bivariate normal density function was proposed by Arnold et al. (2007) and was defined by:

$$f(x;k_1,k_2) \propto \varphi(x)\Phi\left(\frac{k_1 + k_2 x}{\sqrt{1+(k_1+k_2x)^2}}\right), \text{with } \; k_1, k_2 \in \mathbb{R}.$$

According to Choudhury and Matin (2011), the $SN(k)$ distribution and the extensions by Arellano–Valle et al. (2004) do not characterize the kurtosis of a distribution well and therefore proposed a distribution that characterize skewness and kurtosis.

$$f(x;k_1,k_2,k_3) = 2\varphi(x)\Phi\left(\frac{k_1 x}{\sqrt{1+k_2x^2+k_3x^4}}\right), \text{with } \; k_1 \in \mathbb{R}, k_2, k_3 \geq 0.$$

Balakrishnan (2002) also proposed an extension based on normal distribution which has the connection to normal order statistics and is defined by:

$$f_n(x;k) = \frac{\varphi(x)[\Phi(kx)]^n}{C_n(k)}$$

where n is positive integer and $C_n(k) = \frac{1}{\pi}\int_{-\infty}^{\infty}[\Phi(kx)]^n \phi(x)\,dx$. When n is odd, $C_n(k)$ can be expressed:

$$C_{2n+1}(k) = \sum_{i=1}^{2n+1}(-1)^{i+1}\binom{2n+1}{i}\frac{1}{2i} \times C_{2n+1-i}(k), \quad n = 0,1,2,\ldots, k \in \mathbb{R}$$

According to Balakrishnan (2002), when $k = 1$, $f_n(x, 1)$ becomes the density of the largest order statistic in a sample of size $(n+1)$ from the standard normal distribution. Gupta and Gupta (2004), Sharafi and Behboodian (2008), and Yadegari, Gerami and Khaledi, (2008) have all studied some extensions, details statistical properties and applications of this generalizations version have been investigated. Generalizations of the $SN(k)$ distribution to skewed symmetric distributions have been studied extensively in the literature by Azzalini (2005) using the following definition.

$$f(x) = 2f_0(x)G[w(x)], \quad x \in \mathbb{R}$$

where f_0 is a pdf symmetric about 0, G is a cdf such that $G^{/}$is a pdf symmetric about 0, and $w(.)$ is any odd function. There are still some works in the literature that generate families of skew symmetric distributions. The generalization proposed by Azzalini (2005) included skewed distributions with more flexible tails for the generated skewed distributions using a non–normal distribution $g(.)$. Arnold and Beaver (2000, 2002) proposed using two different independent, non–normal symmetric distributions to generate "non–normal skewed distributions". Let Ψ_1 and Ψ_1 be the pdf and cdf of X and let Ψ_2 and Ψ_2 be the pdf and cdf of Y. The random variables X and Y are independent and symmetric. The conditional density of X given that $(k_0 + k_1X > Y)$ is of the form:

$$f(x; k_0, k_1) = \frac{\psi_1(x)\psi_2(k_0 + k_1x)}{P(k_0 + k_1X > Y)}$$

Another extension is the skew scale mixtures of normal distributions (SSMN) proposed by Ferreira et al., (2011). The SSMN distribution uses the mixture of normal distribution as f_0 and normal as $G(.)$. Nadarajah and Kotz ((2006), (2007a), (2007b)) published three papers to study many specific skewed distributions by applying different pdfs f_0 and different cdf G. Nadarajah and Kotz (2006) provided a list of skewed distributions and defined the moments, and characteristic functions. For each of the skewed distributions, the same group G was applied including Student's t, Cauchy, Laplace, logistic, and uniform distributions. Further, Nadarajah and Kotz ((2007a), (2007b)) defined and presented hazard functions, central and non-central moments of many skewed distributions by taking both f_0 and G for the same random variable. Several authors have proposed and developed some alternatives for generating skewed distributions for symmetric random variables (Fernandez & Steel, 1998; Mudholkar & Hutson, 2000; Arnold, 2004; Abtahi et al., 2012; Chang & Genton, 2004; Ferreira & Steel, 2006). It should be noted that the skewed symmetric methods described here are methods that introduce a skewing mechanism into symmetric distributions to generate skew families of distributions. The skewness of the distributions often is directly connected to the skewing factor, which is useful for capturing the magnitude of skewness.

Beta–Generated Method

The first work produced in this category was the paper by Eugene et al. (2002). Eugene et al. (2002) used the beta distribution as the generator to develop the beta–distributions. The cdf of the beta–class random variable Y is defined by:

$$F(y) = \int_0^{G(y)} b(s)\, ds$$

where $b(s)$ is the pdf of the beta random variable and $G(y)$ is the cdf of any random variable. The pdf corresponding to the beta–class distribution is given by:

$$f(y) = \frac{1}{B(a,b)} g(y) G^{a-1}(y) \left(1 - G(y)\right)^{b-1}$$

According to Eugene et al., (2002) and Jones (2004), this family of distributions can be considered as a generalization of distributions of order statistics for the random variable Y with cdf, $G(y)$. When a and bare integers, the pdf is the a^{th} order statistics of the random sample of size $(a + b - 1)$. It is of interest to know that the beta-generated family was developed by generating distributions with more parameters using beta distribution as the generator. The new developed distributions add more parameters for fitting different types of shapes. Hence, the skewness is not directly defined by a specific parameter; instead, it is the combination of all shape parameters that play the role of measuring skewness. The beta generated method provides an easy way to generate new distributions. Let $G(y)$ be the parent distribution, the beta distribution the generator, and the resulting beta generated distribution in a "B-G distribution". Aside generating new distributions, some existing generalized distributions can also be generated using the pdf by properly defining $G(y)$ (McDonald, 1984; Jones, 2004; Jones & Faddy, 2003; Sepanski & Kong, 2008).

The statistical properties of each of the B–G distribution that are commonly studied are: moments, modes, limiting behaviour, hazard functions, distribution shapes, measure of entropy, mean, and median deviations. However, the moments of Cauchy distribution do not exist and thus, makes the Cauchy distribution rarely used for modelling real data. In the context of generalizations and extensions, the beta generated method can straightforwardly be extended by replacing beta distribution using any distribution defined on a finite [a, b]. For instance, any generalized form of the beta distribution such as the generalized three parameter beta distributions proposed by McDonald (1984), or the Johnson's family of bounded distributions (S_B) can be used as the generator by normalizing the domain to [0, 1]. Jones (2009) and Cordeiro et al. (2011) extended the beta–class family by replacing the beta distribution using the Kumaraswamy (1980) distribution. The Kumaraswamy (1980) distribution is defined as:

$$f(y) = aby^{a-1}\left(1-y^{a}\right)^{b-1}, \quad \text{with} \quad G(y) = 1-\left[1-F(y)^{a}\right]^{b}, \quad y \in (0,1).$$

The pdf of the Kumaraswamy generated distribution (Kw–G) is a family of distribution defined by:

$$g(y) = abf(y)F^{a-1}(x)\left(1 - F^{a}(x)\right)^{b-1}$$

Cordeiro et al., (2011) andNadarajah, Cordeiro and Ortega (2012) have studied some general properties of the Kw-G family of distribution. Various Kw-G distributions have been studied in the literature including: Kw-normal distribution by Cordeiro et al., (2011), Kw-gamma distribution by Cordeiro et al. (2011), Kw-inverse Gaussian distribution by Cordeiro et al. (2011), Kw-Weibull distribution by Cordeiro et al., (2011) and Cordeiro et al., (2010), Kw-generalized gamma by de Pascoa et al., (2011).

Alexandra et at., (2012) introduced generalization of the beta generated family by using the generalized beta of type I, McDonald (1984) as the generator. Further, Zografos and Balakrishnan (2009) also extended the beta generated family by using the generalized gamma density as the generator. This generalization was called generalized gamma-generated family (GGG). The generalized gamma has the pdf:

$$P_X(x; a, b, \gamma) = \frac{\gamma}{b^{a\gamma}\Gamma(a)} x^{a\gamma-1} exp[-(x/b)^{\gamma}]$$

The pioneering work by Eugene et al. (2002) laid the foundation for the study of various Beta-generated distributions in the literature. Subsequent research has explored a wide range of these distributions, including the Beta-normal (Eugene et al., 2002; Famoye et al., 2004; Gupta & Nadarajah, 2004), Beta-Gumbel (Nadarajah & Kotz, 2004), Beta-Frechet (Nadarajah & Gupta, 2004; Barreto & Cordeiro, 2011), Beta-exponential (Nadarajah & Kotz, 2005), Beta-Weibull (Famoye & Lee, 2005; Lee et al., 2007; Cordeiro et al., 2011).

Method of Adding Parameters

This method is about adding parameter(s) to a parent (existing) distribution to generate a generalized or an extended distribution and is a very common approach for developing more flexible statistical distributions. The literature contains many generalized distributions like generalized gamma, generalized beta, generalized Pareto, generalized Weibull, etc. Johnson, Kotz and Balakrishnan (1994) have a section on generalization and many of these distributions were obtained by adding one or two parameters to the parent distribution to create its generalized version. It should be noted that, there is a clear distinction between the method adding parameters and the methods of combination. The method of adding parameters is to add extra parameter(s) to an existing distribution; while the methods of combination combine two existing distributions to form new distributions. Consequently, the methods of combination are able to generate larger families of distributions. Conversely, the method of adding parameters can be applied to any family of distributions generated through the methods of combination to generate the 'exponentiated' version of the new family, and vice versa. However, according to Johnson et al., (1994), fitting real world data using more than four parameters are often not practical.

The Exponentiated Method

The pioneering work on the exponentiated method is given in Mudholkar and Srivastava (1993) was on the exponentiated Weibull family defined by:

$$F(y) = \left[1 - \exp\left(-(ay)^b\right)\right]^\gamma, \quad y > 0, \ a, b, \gamma > 0$$

When we put $\gamma = 1$, the cdf of the Weibull distribution is obtained. In general, the exponentiated method can be described as follows: Let $G(y)$ and $G_a(y)$ be the cdfs of random variable Y and the exponentiated Y. Then,

$$G_\alpha(y) = [G(y)]^\alpha, \quad \alpha > 0$$

Gupta, Gupta and Gupta (1998) gave a systematic treatment of $G_a(y) = [G(y)]^a, \quad a > 0$. Gupta and Kundu (2001) studied exponentiated exponential family. Nadarajah and Kotz (2006) studied a list of exponentiate X distributions including exponentiated exponential, gamma, Weibull, Gumbel, and Frechet distributions. Gupta and Kundu (2007) gave a review of the development of generalized exponential distributions. Nadarajah et al., (2012) gave a thorough survey of the exponentiated Weibull distributions.

In addition to the exponentiated method, there have been different alternatives to generate different types of flexible families of distributions using different functions of $G(y)$. Gera (1997) defined $G_a(y)$ as follows:

$$G_a(y) = e^{-ay^\gamma} G(y), \quad a, \gamma > 0$$

Cancho and Bolfarine (2001) also defined $G_E(y)$ as:

$$G_E(y) = a - a[1 - G(y)], \quad a \in (0, 1)$$

This method can be considered as a mixture method. The Kumaraswamy generated family can also be defined as an exponentiated method by adding one exponent to F and the other to $(1 - F^a)$. Furthermore, Marshall and Olkin (1997) proposed a method of adding an extra parameter from a lifetime distribution perspective. The authors applied their method to extend exponential and

Weibull distributions. The survival function of this family is defined by:

$$\begin{aligned} G(y) &= 1 - \frac{\lambda S(y)}{1-(1-\lambda)S(y)} \\ &= 1 - \frac{\lambda S(y)}{F(y)+\lambda S(y)} \end{aligned}$$

$$S(x) = 1 - F(y), \in \mathbb{R}, \lambda > 0$$

Some new distributions have been proposed by taking the base distribution to the extended Weibull (Ghitany, Al-Hussaini & Al-Jarallah, 2005), normal (García, Gómez-Déniz & Vázquez-Polo, 2010), and Birnhaum–Saunders distributions (2013). For a thorough discussion and review of life distributions, one may refer to the book by Marshall and Olkin (2007).

The Transformed–Transformer Method $(T-X)$ Family

The beta-generated distributions are generated using distributions with support between 0 and 1 as the generator. The limitation of using a generator between 0 and 1 raises an interesting question, "can other distributions with different supports as a generator be used to derive different classes of distributions?" Alzaatreh et al., (2013) introduced a general method that allows for using continuous pdf as the generator. Let X be a random variable with pdf $g(x)$ and cdf $G(x)$. Also, let T be a continuous random variable with the pdf $r(t)$, defined on [a, b]. The cdf of a new family of distribution is defined by:

$$G_{T-Y}(x) = \int_a^{W[G(x)]} r(t)dt = R[W(x)],$$

where $W[G(x)]$ satisfies the following conditions

$$\left.\begin{array}{l} W[G(x)] \in [a, b], \\ W[G(x)] \text{ is differentiable and monotonically non-decreasing}, \\ W[G(x)] \to a \text{ as } x \to -\infty, \\ W[G(x)] \to b \text{ as } x \to \infty \end{array}\right\}$$

The corresponding pdf is:

$$g(y) = -\frac{d}{dx}W[G(x)]\, r[W[G(x)]]$$

This family of distribution is named as "transformed – transformer" class of family. The new pdf $g(x)$ is the pdf transformed from the random variable T through the transformer random variable X. Different $W[G(x)]$ will define different new family of $T - X$ distributions. The definition of depends on the support of the random variable T. Alzaatreh et al. (2013) defined the $T - X$ family for several different $W[G(x)]$ functions under different supports of T and studied the family when $W[G(x)] = -log(1 - F(x))$ in some details.

The $T - X$ Family when $W(G(x))$ is defined as $-log(1 - F(x))$

By $W[G(x)] = -\log(1 - F(x))$, the corresponding T-X family has the cdf and pdf, respectively, as:

$$G(x) = R[-\log(1 - F(x))] = R[H_f(x)]$$

$$g(x) = \frac{f(x)}{1 - F(x)} r[1 - \log(1 - F(x))] = b_f(x) r[H_f(x)], \quad x > 0$$

where $b_f(x)$ and $H_f(x)$ are the hazard and cumulative hazard functions of the random variable X with PDF $f(x)$.

A list of subfamilies of this T-X family includes: γ-X, Exponential-X, Beta-Exponential-X, Exponentiated-Exponential-X, Half-Normal-X, Levy-X, Log-Logistic-X, Rayleigh-X, Type-II Gumbel-X, Lomax-X, Inverted Beta-X, Inverse Gaussian-X, and Weibull-X. Some subfamilies of the $T-X$ family based on $W[G(x)] = -log(1-F(x))$ are:

Gamma-X Family

When $T \sim \text{Gamma}(a,b)$, the resulting gamma-family is defined as:

$$g(x) = \frac{1}{\Gamma(a)b^a} f(x) \left[-\log(1-F(x))\right]^{a-1} (1-F(x))^{\frac{1}{b}-1}$$

The upper record value distribution is a special case of the gamma-normal distribution, and Tracy's generalized gamma distribution is a special case of the gamma-gamma Weibull distribution.

Weibull-X Family

When $T \sim \text{Weibull}(v,\gamma)$, the resulting Weibull-X family is defined as:

$$g(x) = \frac{v}{b}\frac{f(x)}{1-F(x)} \left\{\frac{-\log(1-F(x))}{b}\right\}^{v-1} \exp\left\{-\left(\frac{-\log(1-F(x))}{b}\right)^{v}\right\}$$

When $v = 1$, the Weibull-X family reduces to the $\text{Exp}(1-F(x))$ distributions. The Type II generalized logistic distribution is a special case of the Weibull-logistic distribution. The Weibull-Pareto was studied in detail in Roy (2004).

Beta-Exponential-X Family

When $T \sim \text{Beta-Exponential}(a,b,v)$, the PDF of the Beta-Exponential-X family is:

$$g(x) = \frac{v}{B(a,b)} f(x)(1-F(x))^{vb-1} \left[1-(1-F(x))^{v}\right]^{a-1}$$

where $B(a,b)$ is the beta function.

Special cases include:

- When $b = 1$ and $v = 1$, $g(x)$ reduces to the Exp(F) distributions.
- When $a = 1$, $g(x)$ reduces to the Exp$(1 - F)$ distributions.
- When $b = 1$ and $v = 1$, $g(x)$ also reduces to the exponentiated–exponential-X family with odf:

$$f(x)\left[1-(1-F(x))^{v}\right]^{a-1}(1-F(x))^{v-1}.$$

Composite Methods

The term composite distribution discussed here is different from the commonly used compound distribution or mixture distribution. Cooray and Ananda (2005) proposed the composite method by combining two distributions in the following way: Let X be a random variable with pdf: The piecewise function $f(x)$ is defined as:

$$f(x)=\begin{cases} cf_1(x), & \text{if } x\in(0,\theta) \\ cf_2(x), & \text{if } x\in(\theta,\infty) \end{cases}$$

where c is the normalizing constant, and f_1 and f_2 are PDFs with positive support. The unknown θ is determined so that the newly formed PDF $f(x)$ is continuous and differentiable at θ. The continuity and differentiability are achieved by imposing the following constraints $f_1(\theta) = f_2(\theta)$ and $f_1'(\theta) = f_2'(\theta)$

According to Cooray and Ananda (2005), the rationale for the composite distribution is that f_1 models a large portion of the data well but quickly fades to zero, thus fitting poorly a portion of the tail. On the other hand, f_2 fits the tail portion well but fits the other portion poorly. By combining two distributions, where one fits the portion below a given threshold and the other fits the portion larger than the threshold, the composite distribution was proposed.

Various new composite distributions have been developed in the literature, including: composite-Burr (Nadarajah & Bakar, 2012), composite exponential (Cooray & Ananda, 2005; Teodorescu & Vernic, 2009), Composite Weibull-Pareto (Preda & Ciumara, 2006), composite lognormal (Scollnik, 2007; Preda & Ciumara, 2006; Pigeon & Denuit, 2011), truncated composite lognormal-Pareto (Teodorescu, 2010), truncated composite Weibull-Pareto (Teodorescu & Panaitescu, 2009), composite lognormal-lognormal (Cooray, Gunasekera & Ananda, 2010), composite inverse Weibull-Weibull (Cooray et al., 2010), composite log-Gauss-Pareto (Eliazar & Cohen, 2012).

Quadratic Rank Transmutation Map

Shaw and Buckley (2009) developed a new family of probability distributions using the quadratic rank transmutation map and called it the transmuted distributions. The cdf of the quadratic transmuted distribution is defined as:

$$G(x) = (1+\lambda)F(x) - \lambda F^2(x)$$

where $F(x)$ is the cumulative distribution function of the baseline distribution, and $|x| \leq 1$. By putting $\lambda = 0$, we notice that the baseline cumulative (cdf) is the same as the original pdf.

Proof

Proving the quadratic transmutation map is intuitively simple. Let X_1 and X_2 be independent and identically distributed non-negative random variable with cdf $F(x)$. Then, we have

$$Y \stackrel{\text{def}}{=} \begin{cases} \min(X_1, X_2) & \text{with probability } \alpha \\ \max(X_1, X_2) & \text{with probability } 1-\alpha \end{cases}$$

where $0 \leq \alpha \leq 1$. The distribution of Y is clearly

$$G_Y(x) = \alpha \Pr[\min(X_1, X_2) \leq x] + (1-\alpha) \Pr[\max(X_1, X_2) \leq x]$$

We know that $G_{\text{min}}(x) = 1 - [1 - F(x)]$ and $G_{\text{max}}(x) = [F(x)]^n$. Hence,

$$G_Y(x) = \alpha \left[1 - (1 - F(x))^2\right] + (1-\alpha)F^2(x)$$

Expanding, we have

$$\begin{aligned}
G_Y(x) &= \alpha \left[1 - \left(1 - 2F(x) + F^2(x)\right)\right] + F^2(x) - \alpha F^2(x) \\
&= \alpha \left[1 - 1 + 2F(x) - F^2(x)\right] + F^2(x) - \alpha F^2(x) \\
&= 2\alpha F(x) - \alpha F^2(x) + F^2(x) - \alpha F^2(x) \\
&= 2\alpha F(x) + (1 - 2\alpha)F^2(x)
\end{aligned}$$

When we let $2\alpha = \lambda$, we obtain the well-known quadratic transmuted distribution. The corresponding pdf is given by

$$g(x) = f(x)[1 + \lambda - 2\lambda F(x)]$$

Several authors have applied the quadratic transmuted to develop various distributions following the idea of Shaw and Buckley (2009). The work of Aryal and Tsokos (2009, 2011) stands as a pioneering contribution in introducing the quadratic transmuted method. Their research unveiled a novel approach to transmutation and presented a set of transmuted probability distributions that demonstrated enhanced distributional flexibility, particularly beneficial in the realms of environmental and reliability analysis. Building on this foundation, Bourguignon et al. (2016) and Das (2015) further delved into the quadratic transmuted family of distributions. Their studies provided valuable insights into the general characteristics and outcomes associated with this transmuted family of distributions, contributing to the broader understanding and application of transmutation methods in statistical modelling and analysis. Tahir and Cordeiro (2016) have detailed a list for quadratic transmuted distributions. The research works done on the quadratic transmuted family of distributions is evidently enriched enough in the literature and is also speedily convalescing. We now discuss the cubic transmuted families of distributions.

Cubic Rank Transmutation Map

Granzotto et al., (2017) developed a cubic transmuted family of the form:

$$G(x) = \lambda_1 F(x) + (\lambda_2 - \lambda_1)F^2(x) + (1 - \lambda_2)F^3(x)$$

where $\lambda_1 \in [0, 1]$ and $\lambda_2 \in [-1, 1]$.

Proof:

Let $X_1, X_2, \text{and} X_3$ be independent and identically random variables distributed with cdf $F(x)$. We know that:

$X_{1:3} = \min\{X_1, X_2, X_3\}, X_{2:3} =$ the 2nd smallest of (X_1, X_2, X_3),

$X_{3:3} = \max\{X_1, X_2, X_3\}$

and let

$$Y \overset{\text{def}}{=} \begin{cases} X_{1:3}, & \text{with probability } \alpha_1, \\ X_{2:3}, & \text{with probability } \alpha_2, \\ X_{3:3}, & \text{with probability } \alpha_3, \end{cases}$$

where

$\sum_{i=1}^{3} \alpha_i = 1 \implies \alpha_3 = 1 - \alpha_1 - \alpha_2$

The distribution of $G_Y(x)$ is clearly given by

$$\begin{aligned}
G_Y(x) &= \alpha_1 \Pr\left[\min(X_1, X_2, X_3) \le x\right] + \alpha_2 \Pr\left[X_{2:3} \le x\right] \\
&\quad + (1-\alpha_1-\alpha_2) \Pr\left[\max(X_1, X_2, X_3) \le x\right] \\
&= \alpha_1\left[1-(1-F(x))^3\right] + \alpha_2 \left\{ \sum_{i=1}^{3} \binom{3}{i} F(x)^i [1-F(x)]^{3-i} \right\} \\
&\quad + (1-\alpha_1-\alpha_2)[F(x)]^3 \\
&= 3\alpha_1 F(x) - 3\alpha_1 F(x)^2 + \alpha_1 F(x)^3 + 3\alpha_2 F(x)^2 - 2\alpha_2 F(x)^3 \\
&\quad + [F(x)]^3 - \alpha_1 [F(x)]^3 - \alpha_2 [F(x)]^3 \\
&= 3\alpha_1 F(x) + 3(\alpha_2 - \alpha_1) F(x)^2 + (1-3\alpha_2) F(x)^3
\end{aligned}$$

When we let $\lambda_1 = 3\alpha_1$ and $\lambda_2 = 3\alpha_2$, we obtain the cubic rank transmuted distribution given by Granzotto et al., (2017). The corresponding pdf is also given by:

$$g(x) = f(x)[\lambda_1 + 2(\lambda_2 - \lambda_1)F(x) + 3(1-\lambda_2)F^2(x)$$

Numerous researchers have explored various forms of cubic rank transmuted distributions. Rahman et al. (2018a, 2018c, 2019b) introduced three novel families of cubic rank transmuted distributions. AL-Kadim (2018) also examined a generalized transmuted distribution family, which was found to be a specific case of the generalized transmuted families proposed by Rahman et al. (2018a, 2018c). Ali and Athar (2021) presented a generalized rank-mapped transmuted distribution method for generating continuous distribution families, and they studied the n^{th} degree generalized transmutation map derived from continuous distribution families. Aslam et al. (2018) analyzed another cubic transmuted-G family of distributions and investigated its associated statistical properties. Several cubic rank transmuted distributions have been introduced in the literature, including the cubic transmuted Weibull (Granzotto et al., 2017; AL-Kadim & Mohammed, 2017), cubic transmuted log-logistic (Granzotto et al., 2017), cubic transmuted exponential (Rahman, Al-Zahrani & Shahbaz, 2018b; Rahman, Al-Zahrani & Shahbaz 2018c), cubic transmuted Pareto (Rahman et al., 2018b; Ansari & Eledum, 2018), cubic transmuted Frechet (Celik, 2018), cubic transmuted Gumbel (Celik, 2018), cubic transmuted Gompertz (Celik, 2018), cubic transmuted Burr III-Pareto (Bhatti et al., 2019), cubic transmuted uniform (Rahman et al., 2019b), and cubic transmuted Gompertz–Makeham (Hamdam & Riffi, 2020) distributions. The cubic rank transmuted distributions have been shown to be more adaptable in modelling complex and bimodal data compared to quadratic transmuted distributions.

Chapter Summary

In this chapter, various rank transmutation map methodologies for developing and modifying probability distributions were comprehensively examined, encompassing periods both preceding and following the 1980s. The exploration delved into the historical timeline of these methodologies, offering insights into the dynamic evolution of techniques for constructing statistical distributions. The investigation highlighted the continual growth of the field, emphasizing the progress made in developing fundamental tools for statistical analysis. Furthermore, the chapter undertook an examination of quadratic and cubic transmuted distributions, providing an overview of the diverse research endeavors conducted on these distributions.

CHAPTER THREE

METHODOLOGY

Introduction

This chapter presents definitions of some probability concepts and terminologies in statistics.

Some Statistics Terminologies

In this section, we outline several key properties of probability distributions that will be referenced throughout this thesis. These definitions are based on works by Hogg, McKean and Craig (2009), Ofosu and Hesse (2011), and Howard and Nkansah (2016). The focus of this thesis is primarily on continuous random variables.

Sample Space

Let Ω be a sample space with an associated probability function $f(x)$, and let X be a continuous random variable defined on Ω. For any real number x, the cdf of X, denoted by $F(x)$, represents the probability that the values of X on the real line are less than or equal to x. Explicitly, this is expressed as:

$$F(x) = P(s \in \Omega \mid X(s) \leq x).$$

For a continuous random variable x, the function $F(x)$ is non-decreasing and satisfies the conditions that it equals 0 for values less than or equal to the beginning of Ω and rises to 1 at the end and beyond Ω. This implies that for all x, the range of $F(x)$ is $0 \leq F(x) \leq 1$, with the probability $P(X = y) = 0$ for any specific value y. Consequently, $F(x)$ is continuous, which allows for the existence of an inverse function.

Continuous Random Variable

Let X be a continuous random variable with probability density function (pdf) f. Then, f is defined by the following relationship:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(y)\, dy \quad \text{for all } x.$$

Expected Value of a Random Variable

Let X represent any random variable with a pdf or probability mass function (pmf) denoted as $f(x)$. The mean or expected value of X, often denoted by μ or $E(X)$, is defined as follows:

1. For the continuous case:

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

2. For the discrete case:

$$\mu = E(X) = \sum_{\text{all } i} x_i f(x_i) \quad \text{exists.}$$

Moment Generating Functions

The r^{th} moment of a random variable X with pdf $f(x)$ represents the expected value of X raised to the power of r. This is defined as:

$$E\left(X^r\right) = \begin{cases} \sum_x x^r f(x), & \text{if } X \text{ is discrete} \\ \int_R x^r f(x)\, dx, & \text{if } X \text{ is continuous} \end{cases}$$

Let X be a random variable with cumulative distribution function $F(x)$ and probability density function $f(x)$. Assume there exists a positive constant h such that the expected value $E\left(e^{tX}\right)$ exists for all $t \in (-h, h)$. The function of

t given by $E\left(e^{tX}\right)$ is known as the moment generating function (MGF) of X in a neighbourhood of zero. The MGF, denoted by $M_X(t)$, is expressed as:

$$M_X(t) = E\left(e^{tX}\right),$$

with specific forms:

$$M_X(t) = \begin{cases} \sum_{x=0}^{\infty} e^{tx} f(x), & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} e^{tx} f(x)\, dx, & \text{if } X \text{ is continuous} \end{cases}$$

Clearly, we can see how the MGF generates moments.

Order Statistics

Consider a random sample $X_1, \ldots, X_n$ of size n drawn from a continuous distribution with probability density function $f(x)$ and cumulative distribution function $F(x)$. Let $X_{(1)}$ denote the smallest value in the sample $(X_1, \ldots, X_n)$, $X_{(2)}$ the second smallest, and so on, up to $X_{(n)}$, which is the largest. In other words, $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$ represent the ordered values of $X_1, X_2, \ldots, X_n$ when arranged in increasing order. The variable $X_{(k)}$ for $k = 1, 2, \ldots, n$, is referred to as the k^{th} order statistic of the sample.

The cumulative distribution function (CDF) of $X_{(n)}$, the maximum of n observations, is given by:

$$\begin{aligned} G_{X_{(n)}}(x) &= P\left[X_{(n)} \leq x\right] \\ &= P\left(X_1 \leq x, X_2 \leq x, \ldots, X_n \leq x\right) \\ &= \prod_{i=1}^{n} P\left(X_i \leq x\right) \quad \text{(by independence)} \\ &= \{F(x)\}^n. \end{aligned}$$

Differentiating with respect to x, the pdf of $X_{(n)}$ is:

$$g_{X_{(n)}}(x) = n[F(x)]^{n-1} f(x).$$

The cdf of $X_{(1)}$, the minimum of n observations, is given by:

$$\begin{aligned}
G_{X_{(1)}}(x) &= P\left[X_{(1)} \leq x\right] \\
&= 1 - P(X_1 > x, X_2 > x, \ldots, X_n > x) \\
&= 1 - \prod_{i=1}^{n} P\left(X_i > x\right) \quad \text{(by independence)} \\
&= 1 - \prod_{i=1}^{n} \left[1 - P\left(X_i \leq x\right)\right] \\
&= 1 - [1 - F(x)]^n.
\end{aligned}$$

Differentiating with respect to x, the PDF of $X_{(1)}$ is:

$$g_{X_{(1)}}(x) = n[1 - F(x)]^{n-1} f(x).$$

In general, the cdf of $X_{(k)}$, the k^{th} order statistic, is:

$$G_{X_{(k)}}(x) = P\left[X_{(k)} \leq x\right] = \sum_{j=k}^{n} \binom{n}{j} [F(x)]^j [1-F(x)]^{n-j} = I_{F(x)}(k, n-k+1),$$

where $I_{F(x)}(k, n-k+1)$ denotes the incomplete beta function.

The corresponding pdf of $X_{(k)}$ is:

$$g_{X_{(k)}}(x) = \frac{d}{dx}\left[G_{X_{(k)}}(x)\right] = \frac{n!}{(k-1)!(n-k)!}[F(x)]^{k-1}[1 - F(x)]^{n-k} f(x).$$

Entropy Measure

The concept of entropy, first introduced by Clausius (1865), has since found applications across a range of disciplines, including classical thermodynamics, statistical science, and information theory. Shannon (1948) extended the notion of entropy to characterize uncertainty and missing information in telecommunication signals, leading to the development of information theory. This interpretation, which was suggested by Von Neumann in 1932, has become the prevailing understanding of entropy. The definition of entropy varies across fields: in information theory, it quantifies the amount of information contained in a signal, while in statistics, it measures the uncertainty associated with the probability distribution of a random variable. Higher entropy values reflect greater uncertainty in the data. This section also covers three specific types of entropy: Shannon's entropy, as defined by Shannon (1948); Renyi's entropy, introduced by Renyi (1961); and q-entropy, defined by Ullah (1996).

Shannon Entropy

Consider a non-negative continuous random variable Y with probability density function $g(y)$. The Shannon entropy of Y, denoted by $H(g)$, is defined as:

$$H(g) = E[-\log g(Y)] = -\int_{-\infty}^{\infty} g(y) \log g(y)\, dy$$

This is commonly referred to as differential entropy or continuous entropy.

Renyi Entropy

Consider a non-negative continuous random variable Y with its probability density function denoted as $g(y)$. The Renyi entropy of order β for Y, symbolized as $H_\beta(g)$, is defined by:

$$H_\beta(g) = \frac{1}{1-\beta} \log \left(\int_{-\infty}^{\infty} g(y)^\beta\, dy \right), \quad \text{for } \beta > 0 \text{ and } \beta \neq 1$$

The Shannon entropy, denoted as $H(Y)$, can be obtained as the limit of Renyi entropy as β approaches 1:

$$H(Y) = \lim_{\beta \to 1} H_\beta(Y) = -\int_{-\infty}^{\infty} \log g(y)\, dy,$$

assuming that both integrals are well-defined.

Q-Entropy

Let Y be a non-negative continuous random variable with density function $g(y)$. The r-entropy of Y, denoted as $J_{H(r)}$, is defined as:

$$J_{H(r)} = \frac{1}{r-1}\left(1 - \int_{0}^{\infty} g(y)^r\, dy\right),$$

where $r > 0 \text{and} r \neq 1$.

Some Basic Survival Quantities

Probability distributions have certain basic properties with regards to its reliability or survival features by means of some functions. The commonly used basic quantities are survival functions (reliability function), failure rate function (hazard rate function) and the residual mean functions. Theoretically, these functions are related such that, an existence of one can be used to obtain or derive the others. In this thesis, the basic quantities of survival functions along with their corresponding symbolizations and definitions are exploited. Though these functions are not squarely best, there are still some reasons that are of interest for studying all these functions. These definitions can be found in Klein and Moeschberger (2003).

Survival Function (Reliability Function)

The survival function is a fundamental measure used to describe time-to-event scenarios. It represents the probability that an individual will survive beyond a specified time x, or equivalently, that the event will occur after time x. Formally, the survival function $S(X)$ is defined as:

$$S(X) = P(X > x)$$

In contexts involving the failure of equipment or products, this survival function is often referred to as the reliability function $R(x)$.

The survival function is a key metric used to describe time-to-event situations. It denotes the probability that an individual will endure beyond a specified time t, or equivalently, that the event occurs after time t. It is formally defined as:

$$S(T) = P(T > t)$$

In the context of equipment failures or product lifespan, the survival function $S(t)$ is often termed the reliability function $R(t)$.

For a continuous random variable T, $S(t)$ is a strictly decreasing and continuous function. For a continuous random variable T, the survival function $S(t)$ is the complement of the cumulative distribution function $G(t)$, given by:

$$S(t) = 1 - G(t) \quad \text{where} \quad G(t) = P(T \leq t).$$

Moreover, the survival function $S(t)$ can also be expressed as the integral of the probability density function (pdf) $g(t)$. That is:

$$S(t) = P(T > t) = \int_t^{\infty} g(u)\, du$$

Thus,

$$g(t) = -\frac{d}{dt}S(t)$$

It is noteworthy that $g(t)\,dt$ can be seen as the "approximate" probability of the event occurring at time t, and $g(t)$ is a non-negative function with the integral over its range summing to one.

The Hazard Rate Function (or the Failure Rate)

A fundamental concept in survival analysis is the hazard rate function, which finds applications in various domains. In reliability analysis, it is known as the conditional failure rate; in demography, it is referred to as the force of mortality; in stochastic processes, it is called the intensity function; in epidemiology, it is termed the age-specific failure rate; and in economics, it is known as the inverse of Mill's ratio. Commonly, it is simply referred to as the hazard rate. The hazard rate function is mathematically defined as:

$$\lambda(t) = \lim_{\Delta t \to 0} \frac{P(t \leq T < t + \Delta t \mid T \geq t)}{\Delta t}$$

For a continuous random variable T, the hazard rate function can be expressed as:

$$\lambda(t) = \frac{g(t)}{S(t)} = -\frac{d}{dt}\ln[S(t)]$$

Another related quantity is the cumulative hazard function $\Lambda(t)$, defined by:

$$\Lambda(t) = \int_0^t \lambda(u)\,du = -\ln[S(t)]$$

Thus, for continuous survival times, the survival function $S(t)$ can be written as:

$$S(t) = \exp\left(-\Lambda(t)\right) = \exp\left(-\int_0^t \lambda(u)\,du\right)$$

The Mean Residual Life Function

A key measure in survival analysis is the mean residual lifetime at a specific time t. For individuals who have reached time t, this measure quantifies their anticipated remaining duration. It is defined as:

$$\text{ERL}(t) = \mathbb{E}(T - t \mid T > t)$$

For a continuous random variable T, the expected remaining lifetime can be formulated as:

$$\text{ERL}(t) = \frac{\int_t^\infty (u-t) f(u)\,du}{S(t)} = \frac{\int_t^\infty S(u)\,du}{S(t)} = \frac{1}{1-F(t)} \int_t^\infty (1 - F(u))\,du$$

where $f(t)$ denotes the probability density function (pdf) and $S(t)$ represents the survival function.

Furthermore, the mean of T is expressed as:

$$\mu = \mathbb{E}(T) = \int_0^\infty u f(u)\,du = \int_0^\infty S(u)\,du$$

The variance of T in terms of the survival function is given by:

$$\text{Var}(T) = 2\int_0^t u S(u)\,du - \left[\int_0^\infty S(u)\,du\right]^2$$

The p-th quantile of the distribution of T, also known as the $100p$-th percentile, is the smallest value t_p such that:

$$S(t_p) \leq 1 - p \quad \text{or} \quad t_p = \inf\{u \mid S(u) \leq 1 - p\}$$

For a continuous random variable T, the p-th quantile can be determined by solving:

$$S(t_p) = 1 - p$$

Parameter Estimation Techniques

Estimators offer the foundation for the practical discussion of statistical inference. The problem of estimation, as it shall be studied in this thesis, is loosely expressed as: Consider a random variable X representing some characteristic of the elements in a population, where its density or mass function is denoted as $f(x; \theta)$. This function depends on an unknown parameter vector $\theta = (\theta_1, \theta_2, \ldots, \theta_r)$, which has r components. The parameter space, denoted by Ω, defines the permissible range for θ. The objective is to identify a statistic $t(X)$ such that it provides an approximate value of θ. This statistic $t(X)$ is referred to as an estimator of θ, and the value $t(\theta)$ is known as the estimate of θ. Estimation involves using the information from a sample to infer characteristics of the population from which the sample originates. Various methods are employed to estimate unknown parameters when fitting probability distributions.

Method of moments

A traditional approach to parameter estimation is the method of moments, which was first introduced by Karl Pearson (1894). This method involves specifying the true distribution by expressing the population moments as functions of the parameter of interest. These theoretical moments are then equated to the corresponding sample moments. The number of parameters to be estimated dictates the number of equations needed. Solving these equations yields the estimates for the parameters of interest.

Consider estimating unknown parameters $\theta = (\theta_1, \theta_2, \ldots, \theta_r)$ that characterize the distribution of a random variable Z with probability density function (pdf) or probability mass function (pmf) $f_z(z;\theta)$. Suppose the first r moments of the population can be expressed as functions of the parameters θ as follows:

$$
\begin{aligned}
\mu_1 &= \mathbb{E}[Z] = \lambda_1(\theta_1, \theta_2, \ldots, \theta_r), \\
\mu_2 &= \mathbb{E}[Z^2] = \lambda_2(\theta_1, \theta_2, \ldots, \theta_r), \\
&\vdots \\
\mu_r &= \mathbb{E}[Z^r] = \lambda_r(\theta_1, \theta_2, \ldots, \theta_r).
\end{aligned}
$$

Given a sample of size n with values $z_1, z_2, \ldots, z_n$, let $\hat{\mu}_j = \frac{1}{n}\sum_{i=1}^{n} z_i^j$ denote the j^{th} sample moment, which estimates μ_j. The method of moments estimator for $\theta = (\theta_1, \theta_2, \ldots, \theta_r)$, denoted by $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_r)$, is determined by solving (if possible) the following equations:

$$\hat{\mu}_1 = \lambda_1(\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_r),$$

$$\hat{\mu}_2 = \lambda_2(\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_r),$$

$$\vdots$$

$$\hat{\mu}_r = \lambda_r(\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_r).$$

The method of moments estimators $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_r)$ rely on the strong law of large numbers, which asserts that for a random sample $X_1, X_2, \ldots, X_n$ from the distribution of X,

$$\frac{1}{n}\sum_{i=1}^{n} X_i^k \xrightarrow{a.s.} \mathbb{E}(X^k).$$

Thus, if the k^{th} population moment is finite, the k^{th} sample moment converges almost surely to the k^{th} population moment (Ofosu & Hesse, 2011).

Method of Maximum Likelihood

The method of maximum likelihood is a widely used estimation technique due to its ease of application and the desirable statistical properties of its estimators. Originally developed by C.F. Gauss, this method was formalized as a general estimation approach by Fisher in 1920. The core principle of this method involves finding the parameter values that maximize a likelihood function.

Suppose X is a random variable with probability density function (pdf) or probability mass function (pmf) $f(x;\phi)$, where $\phi = (\phi_1, \phi_2, \ldots, \phi_r)$ represents the unknown parameters. Let $x_1, x_2, \ldots, x_n$ denote the observed values from a sample of size n of X. The likelihood function for this sample is defined as:

$$L(\phi_1, \phi_2, \ldots, \phi_r) = L(\phi) = \prod_{i=1}^{n} f(x_i; \phi_1, \phi_2, \ldots, \phi_r) = \prod_{i=1}^{n} f(x_i; \phi),$$

where $\phi = (\phi_1, \phi_2, \ldots, \phi_r)$.
This likelihood function represents the joint probability of the observed sample, viewed as a function of the unknown parameters, with the observed values of the random variables treated as fixed.

Two key points simplify the process of working with the likelihood function. First, for independent events, the likelihood function is typically the product of individual probabilities. Second, it is often convenient to work with the logarithm of the likelihood function. Let $L(\phi)$ denote the likelihood function, then:

$$\ell(\phi) = \log L(\phi).$$

To estimate the parameters ϕ, we select the value $\hat{\phi}$ that maximizes $L(\phi)$, subject to $\phi \in \Psi$, where Ψ is the parameter space. Since the maxima of $L(\phi)$ and $\ell(\phi)$ occur at the same parameter values, we typically maximize $\ell(\phi) = \log L(\phi)$ with respect to $\phi = (\phi_1, \phi_2, \ldots, \phi_r)$. Therefore, under certain conditions, the maximum likelihood estimates $\phi = (\phi_1, \phi_2, \ldots, \phi_r)$ are obtained by solving the r likelihood equations:

$$\frac{\partial}{\partial \phi_i} \ell(\phi = (\phi_1, \phi_2, \ldots, \phi_r)) = 0, \quad i = 1, 2, \ldots, r.$$

It turns out that this method of estimation has many desirable properties. The main justification depends on asymptotic properties, which hold as $n \to \infty$, but there are some very useful small sample properties as well.

Bayesian Estimation Method

Bayesian analysis is a statistical approach that integrates two sources of information to infer about an unknown parameter. Specifically, Bayesian analysis combines prior knowledge about a parameter with evidence from sample data to guide the inference process. Initially, a prior probability distribution for the parameter is specified. As new data is collected, Bayes' theorem is applied to update the prior distribution to a posterior probability distribution. This posterior distribution is then used to make statistical inferences about the parameter. Unlike frequentist statistics, Bayesian methods are predictive, allowing for the calculation of the conditional probability distribution of future observations based on the sample data.

Mathematically, let X be an iid random variable with a sample of size n (i.e., $x_1, \ldots, x_n$), where the distribution and density functions are $F(X \mid \theta)$ and $f(x \mid \theta)$, respectively, with θ as a vector of parameters. Since the true value of θ is unknown, θ is treated as a random variable with a prior distribution $F(\theta)$ and density $f(\theta)$. The parameter space is denoted by Θ. Suppose we are interested in r observations with a sample size r from a population with pdf $f(X \mid \theta)$. In this case, θ is assumed to be a random variable with pdf $p(\theta)$. The joint density of the observations given θ is:

$$p(\theta \mid x_1, \ldots, x_r) = \prod_{i=1}^{r} f(x_i \mid \theta) = L(x_1, \ldots, x_r \mid \theta),$$

where $L(x_1, \ldots, x_r \mid \theta)$ denotes the likelihood function.

The marginal density of $(x_1, \ldots, x_r)$ is given by:

$$p(x_1, \ldots, x_r) = \int_{\Theta} p(x_1, \ldots, x_r \mid \theta)\, p(\theta)\, d\theta.$$

The posterior density of θ given the data $(x_1, \ldots, x_r)$ is:

$$\begin{aligned} p(\theta \mid x_1, \ldots, x_r) &= \frac{p(x_1, \ldots, x_r \mid \theta)\, p(\theta)}{p(x_1, \ldots, x_r)} \\ &= \frac{L(x_1, \ldots, x_r \mid \theta)\, p(\theta)}{\int_{\Theta} L(x_1, \ldots, x_r \mid \theta)\, p(\theta)\, d\theta}. \end{aligned}$$

Prior to finding $(x_1, \ldots, x_r)$, the deviations in θ were denoted by $P(\theta)$ (identified as the prior distribution on θ). However, after data $(x_1, \ldots, x_r)$ was obtained, based on the new information, the deviations in θ are denoted by $P(\theta \mid x_1, \ldots, x_r)$ (the posterior distribution of θ). That is, prior to the experiment is denoted by $P(\theta)$ and the same after the experiment is denoted by $P(\theta \mid x_1, \ldots, x_r)$ and this process is a straightforward application of Bayes' theorem. As soon as the posterior distribution is estimated, it becomes the key focus of the study.

Information Criteria

Increasing the number of parameters in a model generally enhances the fit, leading to a higher likelihood, regardless of the significance of the additional parameters. However, when comparing non-nested models, the likelihood ratio test (LRT) may not be suitable, necessitating the use of alternative methods. Information criteria facilitate comparisons in such cases. The most commonly used information criteria include the Akaike Information Criterion (AIC), the Corrected Akaike Information Criterion (AICc), and the Bayesian Information Criterion (BIC).

Akaike Information Criterion

The Akaike Information Criterion (AIC) was introduced by Akaike (1973) and further elaborated in Akaike (1974). It is a widely used tool for model selection among researchers. To apply the AIC, one begins with a set of candidate models that are considered appropriate for the data at hand. The AIC statistic is defined as:

$$\text{AIC} = -2 \log L(\hat{\theta}) + 2p$$

where p represents the number of parameters estimated in the model. The model with the lowest AIC value is considered the best fit for the dataset. One of the strengths of the AIC is its capability to penalize models with a large number of parameters. While the AIC provides reliable model selection for large samples, it may be biased in small samples. To address this issue, the Corrected Akaike Information Criterion (AICc) was introduced by Sugiura (1978). Hurvich and Tsai (1995) demonstrated that the AICc offers improved model selection even for small sample sizes and is preferred when the model includes a large number of parameters. The AICc statistic is given by:

$$\text{AICc} = \text{AIC} + \frac{2p(p+1)}{n-p-1}$$

where n is the sample size.

Bayesian Information Criterion

The Bayesian Information Criterion (BIC), also known as the Schwarz Information Criterion (SIC), was introduced by Schwarz (1978). The BIC is derived from approximating the Bayes factor under the assumption of independent and identically distributed data. The test statistic for the BIC is defined as:

$$\text{BIC} = -2\log L(\hat{\theta}) + q\log(n)$$

where n represents the sample size and q is the number of estimated parameters. The term $\log L(\hat{\theta})$ denotes the natural logarithm of the likelihood function.

The BIC is particularly effective at penalizing models with a large number of parameters compared to the AIC and AICc, and it performs well in both large and small sample sizes. Therefore, it is advisable to use the BIC in conjunction with the AIC and AICc when selecting the best model from competing candidates. As with the AIC, the model with the lowest BIC value is considered the most appropriate.

Chapter Summary

In this chapter, various probability concepts and terminologies in statistics were defined, providing a foundational understanding of key principles. The discussion extended to cover topics such as maximum likelihood estimations, shedding light on the essential methods for estimating parameters in statistical models. Additionally, the chapter explored information criteria, including AIC (Akaike Information Criterion), AICc (corrected Akaike Information Criterion), and BIC (Bayesian Information Criterion). These criteria were discussed in the context of model selection and evaluation, offering valuable insights into the decision-making process when comparing different statistical models.

CHAPTER FOUR

RESULTS AND DISCUSSIONS

Introduction

In this chapter, we have introduced three new classes of probability distributions. These probability distributions have been developed using the quartic transmuted distribution as a baseline distribution. Specifically, the quartic transmuted exponential distribution, quartic transmuted Rayleigh distribution, and quartic transmuted inverse exponential distribution are the new classes of distributions that have been developed. Additionally, we have thoroughly examined and discussed some structural properties like moments, moment-generating function, and reliability measures that are associated with the newly developed distributions. To estimate the parameters for these distributions, we have also employed the widely recognized method of maximum likelihood estimation. By utilizing this approach, we can determine the most probable values for the parameters based on the available data. To effectively compare these distributions with others in the literature, we have utilized a range of statistical measures. These measures include the log-likelihood, AIC, AICc, and the BIC. These metrics provide valuable insights into the performance and suitability of the distributions when compared to alternative options.

Gamma-Type Distributions

Gamma–type distributions entail all probability distributions that are members of the generalized gamma family. They include the classical gamma, generalized gamma, log-gamma, inverse (Vinci) gamma, exponential, inverse exponential, Gompertz, Rayleigh, Lindley, and Weibull distributions. In literature, the applications of these distributions are substantial in the applied sciences. For this thesis, the focus will be on three of these distributions defined in this chapter.

Quartic Rank Transmutation Map

In this section, we propose a quartic ranking transmutation map, along the lines of transmutation described earlier in chapter two.

Theorem

Let X_1, X_2, X_3, and X_4 be independent and identically distributed random variables with cdf $F(x)$. Then, the quartic rank transmutation map is given by:

$$G(x) = F(x)\Big[4\lambda_1 + 6(\lambda_2 - \lambda_1)F(x) + 4(\lambda_1 - 2\lambda_2 + \lambda_3)F(x)^2 + (1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)F(x)^3\Big]$$

Proof

Let X_1, X_2, X_3, and X_4 be independent and identically distributed random variables with cdf $F(x)$. From order statistics, we know that:

$$X_{1:4} = \min(X_1, X_2, X_3, X_4)$$

$$X_{2:4} = \text{the } 2^{\text{nd}} \text{ smallest of } (X_1, X_2, X_3, X_4)$$

$$X_{3:4} = \text{the } 3^{\text{rd}} \text{ smallest of } (X_1, X_2, X_3, X_4)$$

$$X_{4:4} = \max(X_1, X_2, X_3, X_4)$$

The cdf of these order statistics are given as follows:

$$F(X_{1:4}) = [1 - (1 - F(x))]^4$$

$$F(X_{2:4}) = \sum_{i=2}^{4} \binom{4}{i} F(x)^i [1 - F(x)]^{4-i}$$

$$F(X_{3:4}) = \sum_{i=3}^{4} \binom{4}{i} F(x)^i [1 - F(x)]^{4-i}$$

$$F(X_{4:4}) = [F(x)]^4$$

Now, let

$$G_Y(x) = \begin{cases} Y \stackrel{\text{d}}{=} X_{1:4}, & \text{with probability } \lambda_1 \\ Y \stackrel{\text{d}}{=} X_{2:4}, & \text{with probability } \lambda_2 \\ Y \stackrel{\text{d}}{=} X_{3:4}, & \text{with probability } \lambda_3 \\ Y \stackrel{\text{d}}{=} X_{4:4}, & \text{with probability } \lambda_4 \end{cases}$$

It follows that

$$\sum_{i=1}^{4} \lambda_i = 1 \implies \lambda_4 = 1 - \lambda_1 - \lambda_2 - \lambda_3$$

Then, the distribution of $G_Y(x)$ is given by

$$\begin{aligned} G_Y(x) &= \lambda_1 P(X_{1:4}) + \lambda_2 P(X_{2:4}) + \lambda_3 P(X_{3:4}) + \lambda_4 P(X_{4:4}) \\ &= \lambda_1 [1 - (1 - F(x))]^4 + \lambda_2 \sum_{i=2}^{4} \binom{4}{i} F(x)^i [1 - F(x)]^{4-i} \\ &+ \lambda_3 \sum_{i=3}^{4} \binom{4}{i} F(x)^i [1 - F(x)]^{4-i} + \lambda_4 [F(x)]^4 \end{aligned}$$

Expanding, we have

$$\begin{aligned} G(x) = {} & 4\lambda_1 F(x) - 6\lambda_1 F(x)^2 + 4\lambda_1 F(x)^3 - \lambda_1 F(x)^4 + 6\lambda_2 F(x)^2 \\ & - 8\lambda_2 F(x)^3 + 3\lambda_2 F(x)^4 + 4\lambda_3 F(x)^3 - 3\lambda_3 F(x)^4 + F(x)^4 \\ & - \lambda_1 F(x)^4 - \lambda_2 F(x)^4 - \lambda_3 F(x)^4 \end{aligned}$$

Now, simplifying, we obtain

$$\begin{aligned} G(x) = F(x)\Big[&4\lambda_1 + 6(\lambda_2 - \lambda_1)F(x) + 4(\lambda_1 - 2\lambda_2 + \lambda_3)F(x)^2 \\ &+ (1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)F(x)^3\Big] \end{aligned} \tag{1}$$

The corresponding pdf of the quartic rank transmutation map is given by:

$$\begin{aligned} g(x) = f(x)\Big[&4\lambda_1 + 12(\lambda_2 - \lambda_1)F(x) + 12(\lambda_1 - 2\lambda_2 + \lambda_3)[F(x)]^2 \\ &+ 4(1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)[F(x)]^3\Big] \end{aligned} \tag{2}$$

Quartic Transmuted Exponential Distribution

In this section, we present a generalization of the exponential distribution, known as the quartic transmuted exponential distribution (QTED). The QTED is derived based on the quartic rank transmutation map, providing a versatile extension of the traditional exponential distribution. To provide context, we first give a brief description of the exponential distribution, outlining its key characteristics and properties. Subsequently, we introduce the pdf and cdf of the quartic transmuted exponential distribution. These mathematical expressions reveal how the QTED differs from the standard exponential distribution and offer valuable insights into its behaviour. In addition to its mathematical properties, we explore some statistical properties of the quartic transmuted exponential distribution, such as moments, variance, and skewness. These statistical measures contribute to a comprehensive understanding of the QTED's shape and central tendencies. Furthermore, we delve into estimation procedures for the QTED, including maximum likelihood estimation, which allows us to obtain the best estimates for the distribution's parameters based on observed data. Finally, we investigate practical applications of the quartic transmuted exponential distribution in various fields. By examining real-world scenarios where the

QTED finds utility, we demonstrate its relevance and potential advantages over the standard exponential distribution in modelling and analysing various phenomena. Throughout this section, we aim to provide a thorough exploration of the quartic transmuted exponential distribution, illustrating its theoretical foundations, statistical properties, estimation techniques, and practical applicability. This comprehensive analysis contributes to a broader understanding of the QTED and its potential impact in various domains of research and practice.

Exponential Distribution

The exponential distribution is a fundamental continuous probability distribution widely employed in the realms of probability theory and statistics. Its versatile applicability extends to diverse scenarios, notably describing inter-arrival times in homogenous Poisson processes, where events occur at a constant average rate. For instance, it finds utility in representing the time intervals between specific events, such as mutations on a DNA strand or road kills on a particular road. In the context of queuing theory, the exponential distribution serves as a favored model for the service times of agents within a system. This modelling approach proves valuable in assessing performance and optimizing queueing systems for various applications. Moreover, exponential distribution plays a pivotal role in reliability theory and engineering due to its memoryless property. This property makes it an ideal candidate for modelling the constant hazard rate portion of the well-known bathtub curve, which is frequently employed in reliability analysis to assess the reliability and failure rates of systems over time. Advancements in the understanding and application of the exponential distribution have been made by esteemed researchers, including Balakrishnan (1995). Their contributions have refined and expanded the practical and theoretical aspects of the exponential distribution in various fields, fostering deeper insights and broadening its scope of applicability. As a result, the exponential distribution continues to be an invaluable tool for researchers and practitioners in diverse scientific and engineering disciplines.

A random variable, X , is said to have an exponential distribution if it has the following pdf.

$$f(x;\theta) = \begin{cases} \theta e^{-\theta x}, & \text{for } x > 0 \\ 0, & \text{for } x \leq 0 \end{cases} \tag{3}$$

The corresponding cdf is given as:

$$F(x;\theta) = \begin{cases} 1 - e^{-\theta x}, & \text{for } x \geq 0 \\ 0, & \text{for } x < 0 \end{cases} \tag{4}$$

Derivation and Characteristics of the QTED

A random variable X is said to have a QTED if its cdf, using Equations (1) and (4) is given as follows:

$$\begin{aligned} G(x) = \left(1 - e^{-\theta x}\right)\Big[&4\lambda_1 + 6(\lambda_2 - \lambda_1)\left(1 - e^{-\theta x}\right) \\ &+ 4(\lambda_1 - 2\lambda_2 + \lambda_3)\left(1 - e^{-\theta x}\right)^2 \\ &+ (1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)\left(1 - e^{-\theta x}\right)^3\Big] \end{aligned} \tag{5}$$

and the corresponding pdf using equations (2), (3), and (4) becomes

$$\begin{aligned} g(x) = \theta e^{-\theta x}\Big[&4\lambda_1 + 12(\lambda_2 - \lambda_1)\left(1 - e^{-\theta x}\right) \\ &+ 12(\lambda_1 - 2\lambda_2 + \lambda_3)\left(1 - e^{-\theta x}\right)^2 \\ &+ 4(1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)\left(1 - e^{-\theta x}\right)^3\Big] \end{aligned} \tag{6}$$

where $\theta > 0$ and $\lambda_i \in [0, 1]$.

Proposition 1

Let X be a QTED random variable. Then $g(x)$ is a valid pdf of X if and only if:

1. $g(x) \geq 0$ for all x

2. $\int_0^\infty g(x)\,dx = 1$

Proof:

It can be seen that $g(x) \geq 0$ for all x. Also,

$$\int_0^\infty g(x)\,dx = \int_0^\infty \theta e^{-\theta x}\Bigg[4\lambda_1 + 12(\lambda_2 - \lambda_1)(1 - e^{-\theta x}) + 12(\lambda_1 - 2\lambda_2 + \lambda_3)(1 - e^{-\theta x})^2 + 4(1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)(1 - e^{-\theta x})^3\Bigg]dx$$

Now, since each term under the integration sign is integrable with respect to x, we integrate term by term. That is,

$$g(x) = \theta e^{-\theta x}\Bigg[4\lambda_1 + 12(\lambda_2 - \lambda_1)\left(1 - e^{-\theta x}\right) + 12(\lambda_1 - 2\lambda_2 + \lambda_3)\left(1 - e^{-\theta x}\right)^2 + 4(1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)\left(1 - e^{-\theta x}\right)^3\Bigg]$$

Now, holding the constants and integrating the expressions under the integral sign, we have

For the first term:

$$\int_0^\infty e^{-\theta x}\,dx$$

$u = -\theta x \rightarrow du = -\theta dx \Rightarrow dx = -\frac{du}{\theta}$

Now,

$$\int_0^\infty e^{-\theta x}\,dx = -\int_0^\infty e^u\,du = -e^u$$

Putting $u = -\theta x$ back, we obtain

$$\int_0^\infty e^{-\theta x}\,dx = \left[-e^{-\theta x}\right]_0^\infty = 1$$

For the second term:

$$\int_0^\infty e^{-\theta x}(1-e^{-\theta x})\,dx$$

Let $u = 1 - e^{-\theta x} \to du = \theta e^{-\theta x}\,dx$

Now,

$$\int_0^\infty e^{-\theta x}(1-e^{-\theta x})\,dx = -\int_0^\infty u\,du = \frac{u^2}{2}$$

Putting $u = 1 - e^{-\theta x}$ back, we obtain

$$\int_0^\infty e^{-\theta x}(1-e^{-\theta x})\,dx = \left[\frac{(1-e^{-\theta x})^2}{2}\right]_0^\infty = \frac{1}{2}$$

For the third term:

$$\int_0^\infty e^{-\theta x}(1-e^{-\theta x})\,dx$$

Let $u = 1 - e^{-\theta x} \to du = \theta e^{-\theta x}\,dx$

Now,

$$\int_0^\infty e^{-\theta x}(1-e^{-\theta x})\,dx = -\int_0^\infty u\,du = \frac{u^2}{2}$$

Putting $u = 1 - e^{-\theta x}$ back, we obtain

$$\int_0^\infty e^{-\theta x}(1-e^{-\theta x})\,dx = \left[\frac{(1-e^{-\theta x})^2}{2}\right]_0^\infty = \frac{1}{2}$$

For the fourth term:

$$\int_0^\infty e^{-\theta x}(1-e^{-\theta x})^3\,dx$$

Let $u = 1 - e^{-\theta x} \to du = \theta e^{-\theta x}\,dx$

Now,

$$\int_0^\infty e^{-\theta x}(1-e^{-\theta x})^3\,dx = -\int_0^\infty u^3\,du = \frac{u^4}{4}$$

Putting $u = 1 - e^{-\theta x}$ back, we obtain

$$\int_0^\infty e^{-\theta x}(1-e^{-\theta x})^3\,dx = \left[\frac{(1-e^{-\theta x})^4}{4}\right]_0^\infty = \frac{1}{4}$$

Now putting all the constants and the integrated values back into $g(x)$, we have

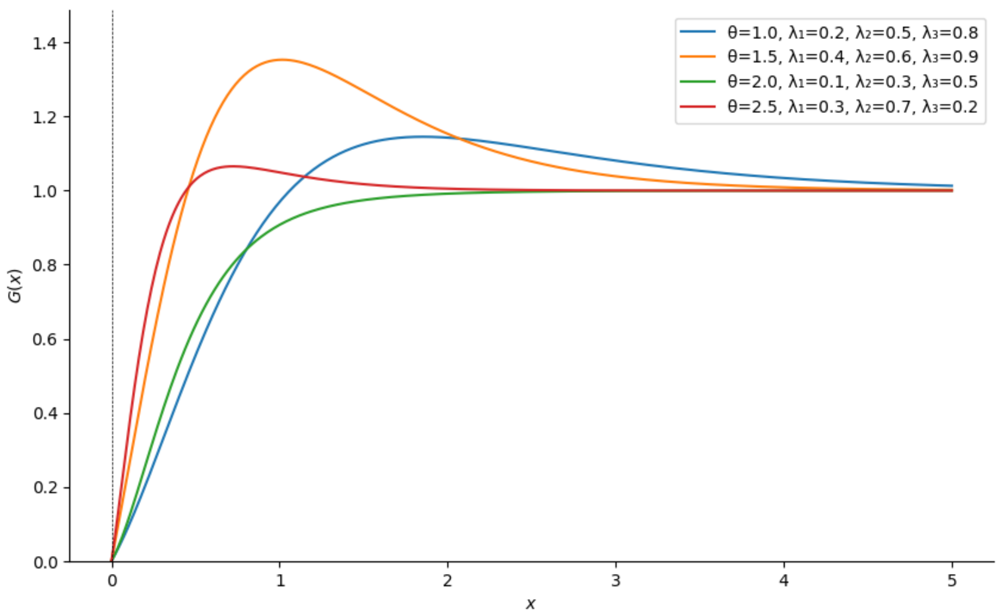
$$\int_0^\infty g(x) = 4\lambda_1 \cdot 1 + 12(\lambda_2 - \lambda_1)\cdot\frac{1}{2} + 12(\lambda_1 - 2\lambda_2 + \lambda_3)\cdot\frac{1}{3} + 4(1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)\cdot\frac{1}{4}$$

Expanding the brackets, we have

$$\int_0^\infty g(x) = 4\lambda_1 + 6\lambda_2 - 6\lambda_1 + 4\lambda_1 - 8\lambda_2 + 4\lambda_3 + 1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3 = \underline{1}\square$$

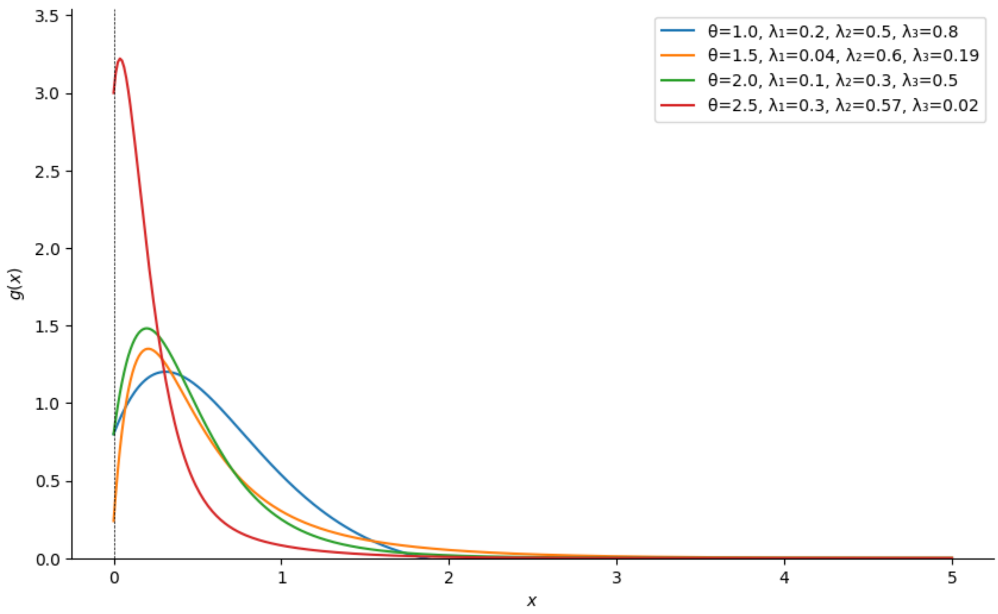
Graphical representations are presented for both the cdf and the pdf of the QTED. The figures presented below depict the graphical representations of the cdf, $G(x)$, and pdf, $g(x)$, respectively.

Figure 1: CDF Plot of QTED



Source: Author, 2023

Figure 2: PDF Plot of QTED



Source: Author, 2023

Figure 1 depicts cdf plot of the QTED for various parameter values. The ascending nature of the plot signifies an increasing distribution, highlighting the progressive accumulation of probability as the random variable X advances. This upward trend in the cdf further reinforces the understanding that the QTED distribution exhibits a positively skewed nature, emphasizing its ability to model data with increasing probability as values of X rise. Figure 2 presents the probability density function (pdf) plot of the QTED across different parameter values. The observed shape of the plot indicates a positively skewed distribution. The variation in parameters contributes to the distinctive forms of the pdf, illustrating the flexibility of the QTED in capturing different skewness patterns. This graphical representation offers valuable insights into the characteristics of the distribution under diverse parameter settings, aiding in the interpretation and understanding of its behavior.

Survival Functions of the QTED

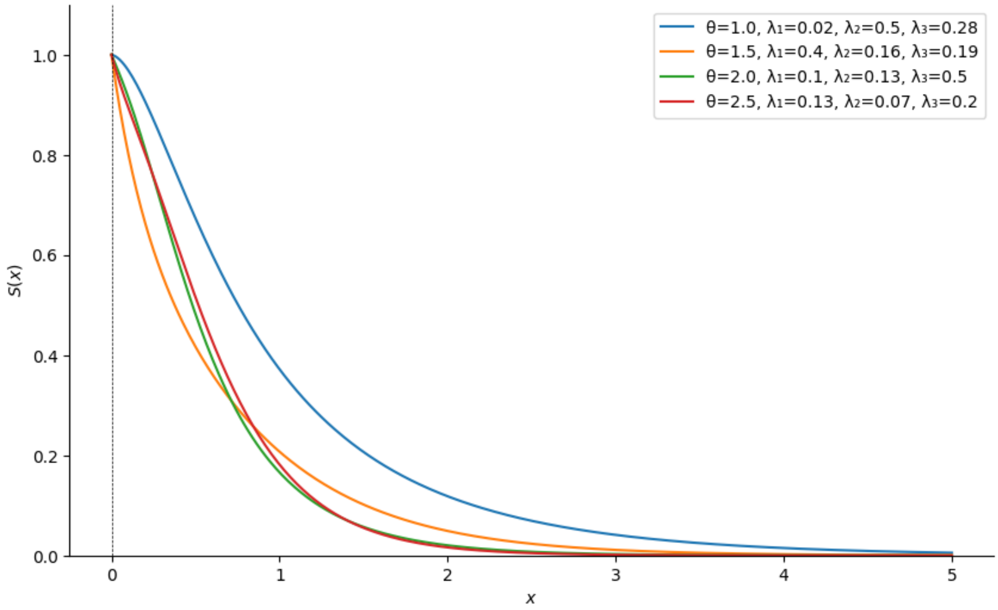
In this section, we examine several reliability-related functions for the QTED. These functions offer important insights into the behaviour and characteristics of the distribution. Specifically, we will investigate the following functions for QTED: (Reliability Function (Survival Function), Hazard Rate Function,Reversed Hazard Function, Cumulative Hazard Function, and Odds Function)

Reliability Analysis

The reliability function, also known as the survival function, represents the probability that a random variable X from the QTED distribution exceeds a given value x. It is mathematically obtained as the complement of the cdf. Let X be a random variable with cdf $G(x)$. Then the reliability (survival) function $S(x)$ is given by:

$$\begin{aligned} S(x) &= 1 - G(x) \\ &= 1 - \left\{(1-e^{-\theta x})\left[4\lambda_1 + 6(\lambda_2 - \lambda_1)(1-e^{-\theta x})\right.\right. \\ &\quad + 4(\lambda_1 - 2\lambda_2 + \lambda_3)(1-e^{-\theta x})^2 \\ &\quad \left.\left.+(1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)(1-e^{-\theta x})^3\right]\right\} \end{aligned}$$

Figure 3: Survival Plot of QTED



Source: Author, 2023

The survival plots in Figure 3 illustrate the survival function for the QTED under different parameter configurations. These plots showcase the probability of survival beyond various time points, providing insights into the distribution's tail behaviour. The declining nature of the survival curves indicates a decrease in survival probability over time, offering valuable information about the distribution's characteristics in terms of reliability and failure patterns. The variations in the plots demonstrate the impact of different parameter values on the survival function, emphasizing the flexibility of the QTED in capturing diverse survival behaviours in real-world applications.

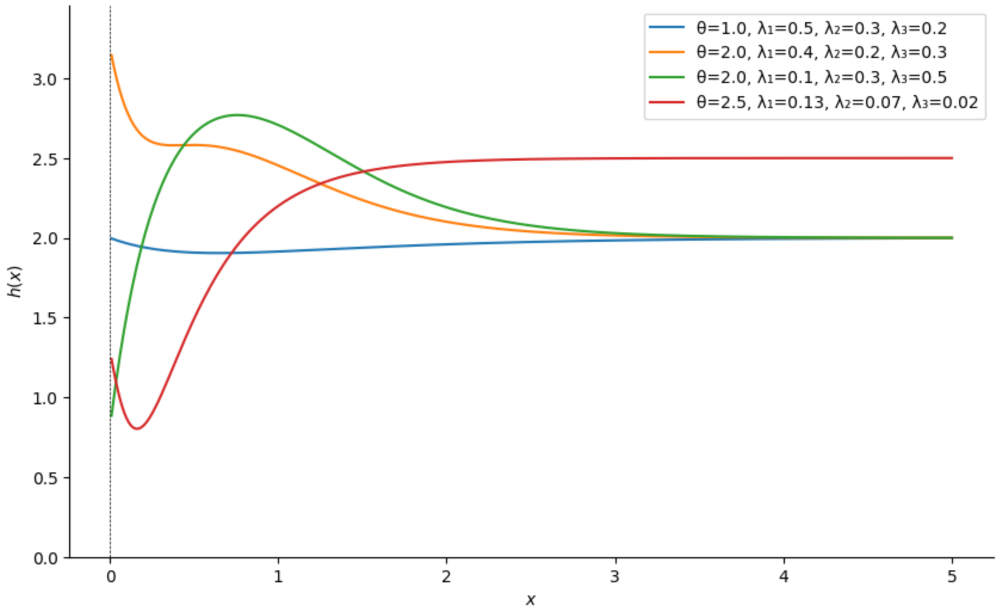
Hazard Function

The hazard rate function, denoted by $h(x)$, describes the instantaneous failure rate of a QTED random variable X at time x. Mathematically, the hazard function is defined as the ratio of the probability of an event occurring in a small-time interval to the width of that interval, given that the individual or object has survived up to that time. That is, it is a ratio of the pdf $g(x)$ to the reliability (survival) function $R(x)$. Thus, the hazard rate for the QTED is given by:

$$\begin{aligned} h(x) &= \frac{g(x)}{1 - G(x)} \\ &= \frac{\theta e^{-\theta x}\left[4\lambda_1 + 12\kappa(\lambda,\theta) + 12\Psi(\lambda,\theta) + 4\Upsilon(\lambda,\theta)\right]}{1 - \left\{(1 - e^{-\theta x})\left[4\lambda_1 + 6\kappa(\lambda,\theta) + 4\Psi(\lambda,\theta) + \Upsilon(\lambda,\theta)\right]\right\}} \end{aligned}$$

where $\kappa(\lambda,\theta) = (\lambda_2 - \lambda_1)(1 - e^{-\theta x})$, $\Psi(\lambda,\theta) = (\lambda_1 - 2\lambda_2 + \lambda_3)(1 - e^{-\theta x})^2$ and $\Upsilon(\lambda,\theta) = (1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)(1 - e^{-\theta x})^3$

Figure 4: Hazard Plot of QTED



Source: Author, 2023

Figure 4 depicts the hazard function QTED across varying parameter settings. The hazard function, a crucial measure in reliability analysis, represents the instantaneous failure rate at any given time. These plots reveal how the hazard rate changes over time, offering insights into the distribution's risk profile. The varying shapes of the hazard curves highlight the sensitivity of the QTED to different parameter values, showcasing its adaptability in modelling complex failure patterns.

Cumulative Hazard Function

The cumulative hazard function, denoted by $H(t)$, gives the cumulative failure rate up to time t for the QTED random variable X. Mathematically, the cumulative hazard function is defined as the integral of the hazard function from the beginning of the observation period (often denoted as time 0) up to a given time t). The cumulative hazard function of the QTED is therefore defined as:

$$\begin{aligned} H(x) &= \int_0^t h(u)\,du = -\ln\left[R(x)\right] = -\ln\left[1 - G(x)\right] \\ &= -\ln\left\{1 - \left[\left(1 - e^{-\theta x}\right)\left(4\lambda_1 + 6(\lambda_2 - \lambda_1)(1 - e^{-\theta x}\right)\right.\right. \\ &\quad + 4(\lambda_1 - 2\lambda_2 + \lambda_3)(1 - e^{-\theta x})^2 \\ &\quad \left.\left. + (1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)(1 - e^{-\theta x})^3)\right]\right\} \end{aligned}$$

Reserved Hazard Rate

The reversed hazard function, denoted by $r(x)$, represents the probability that a QTED random variable X fails exactly at time t, given that it has survived up to $.t$ The reserved hazard rate is defined as the ratio of the pdf to the cdf of a random variable. Mathematically, RHR, $r(x)$, is defined as:

$$r(x) = \frac{\theta e^{-\theta x}\left[A + 12(B)\left(1 - e^{-\theta x}\right) + 12(C)\left(1 - e^{-\theta x}\right)^2 + 4(D)\left(1 - e^{-\theta x}\right)^3\right]}{e^{-\theta x}\left[A + B\left(1 - e^{-\theta x}\right) + C\left(1 - e^{-\theta x}\right)^2 + D\left(1 - e^{-\theta x}\right)^3\right]}$$

where $A = 4\lambda_1$, $B = 6(\lambda_2 - \lambda_1)$, $C = 4(\lambda_1 - 2\lambda_2 + \lambda_3)$, and $D = 1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3$

Odds Ratio

The odds function, denoted by $O(x)$, characterizes the odds of failure to the odds of survival for the QTED random variable X at time t. It is defined as the ratio of the hazard rate function to the reversed hazard function and is given by:

$$OR = \frac{\left(1-e^{-\theta x}\right)\left[A+B\left(1-e^{-\theta x}\right)+C\left(1-e^{-\theta x}\right)^2+D\left(1-e^{-\theta x}\right)^3\right]}{e^{-\theta x}\left[A+B\left(1-e^{-\theta x}\right)+C\left(1-e^{-\theta x}\right)^2+D\left(1-e^{-\theta x}\right)^3\right]}$$

where $A = 4\lambda_1$, $B = 6(\lambda_2 - \lambda_1)$, $C = 4(\lambda_1 - 2\lambda_2 + \lambda_3)$, and $D = 1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3$

Moment-Based Measures

This section entails some of the moment properties of the QTED. Understanding these properties is crucial for analyzing the distribution's behavior and characteristics. Specifically, we will focus on calculating and interpreting the first few moments, including the mean, variance, and higher-order moments. These moments provide insights into the central tendency, dispersion, and shape of the distribution. Additionally, we will explore the moment generating function (MGF), which is a powerful tool for deriving moments and studying the distribution's behaviour in detail. By examining these properties, we gain a deeper understanding of the QTED's statistical properties and its applications in various fields.

The Moment Generating Function

Theorem 2 Let X be a QTED random variable with pdf $g(x)$ as defined in Equation (4.6). Then, the MGF, defined as $M_x(t)$ is given as

$$M_x(t) = -\frac{4\theta\left(6t\theta^2\lambda_3 + \left(9t\theta^2 - 3t^2\theta\right)\lambda_2 + \left(11t\theta^2 - 6t^2\theta + t^3\right)\lambda_1 - 6\theta^3\right)}{(\theta - t)(2\theta - t)(3\theta - t)(4\theta - t)}$$

where $4\theta - t > 0,\ \ 2\theta - t > 0,\ \ 3\theta - t > 0,\ \ \theta - t > 0.$

Proof

$$M_X(t) = \int_0^\infty e^{tx}\theta e^{-\theta x}\left[4\lambda_1 + 12(\lambda_2-\lambda_1)\left(1-e^{-\theta x}\right)\right.$$
$$+\,12(\lambda_1-2\lambda_2+\lambda_3)\left(1-e^{-\theta x}\right)^2$$
$$\left.+4(1-2\lambda_1+2\lambda_2-4\lambda_3)\left(1-e^{-\theta x}\right)^3\right]dx$$

$$= \int_0^\infty \lambda e^{-(\theta-t)x}\left[4\lambda_1 + 12(\lambda_2-\lambda_1)\left(1-e^{-\lambda x}\right)\right.$$
$$+\,12(\pi_1-2\pi_2+\pi_3)\left(1-e^{-\lambda x}\right)^2$$
$$\left.+4(1-2\lambda_1+2\lambda_2-4\lambda_3)\left(1-e^{-\theta x}\right)^3\right]dx$$

$$= \lambda\left[4\lambda_1\int_0^\infty e^{-(\theta-t)x}dx + 12(\lambda_2-\lambda_1)\int_0^\infty e^{-(\theta-t)x}\left(1-e^{-\theta x}\right)dx\right.$$
$$+\,12(\lambda_1-2\lambda_2+\lambda_3)\int_0^\infty e^{-(\theta-t)x}\left(1-e^{-\theta x}\right)^2dx$$
$$\left.+4(1-2\lambda_1+2\lambda_2-4\lambda_3)\int_0^\infty e^{-(\theta-t)x}\left(1-e^{-\theta x}\right)^3dx\right]$$

$$= -\frac{4\theta\left(6t\theta^2\lambda_3 + \left(9t\theta^2-3t^2\theta\right)\lambda_2 + \left(11t\theta^2-6t^2\theta+t^3\right)\lambda_1 - 6\theta^3\right)}{(\theta-t)(2\theta-t)(3\theta-t)(4\theta-t)}$$

Theorem 3

Let X be a QTED random variable with pdf $g(x)$ as defined in Equation (4.6). The rth moment of the QTED can be obtained using the relation

$$E(X^r) = \mu_r = \int_{-\infty}^\infty x^r g(x)dx \tag{7}$$

Putting Equation (4.6) into Equation (4.7), we have

$$E(X^r) = \int_{-\infty}^{\infty} x^r \Bigg\{ \theta e^{-\theta x} \times \Big[4\lambda_1 + 12(\lambda_2 - \lambda_1)(1 - e^{-\theta x}) + 12(\lambda_1 - 2\lambda_2 + \lambda_3)(1 - e^{-\theta x})^2 + 4(1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)(1 - e^{-\theta x})^3 \Big] \Bigg\} dx$$

By integrating term by term, we have

$$\begin{aligned} E(X^r) =& 4\lambda_1\theta \int_0^{\infty} x^r e^{-\theta x}\, dx + 12\theta(\lambda_2 - \lambda_1) \int_0^{\infty} x^r e^{-\theta x}(1 - e^{-\theta x})\, dx \\ &+ 12\theta(\lambda_1 - 2\lambda_2 + \lambda_3) \int_0^{\infty} x^r e^{-\theta x}(1 - e^{-\theta x})^2\, dx \\ &+ 4\theta(1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3) \int_0^{\infty} x^r e^{-\theta x}(1 - e^{-\theta x})^3\, dx \end{aligned}$$

This can be reduced to

$$\begin{aligned} E(X^r) = \frac{\Gamma(r+1)}{\theta^r} \Bigg[& \frac{(-4\lambda_3 + 2\lambda_2 - 2\lambda_1 + 1)}{4^r} + \frac{4(3\lambda_3 + \lambda_1 - 1)}{3^r} \\ + \quad & 4(\lambda_3 + \lambda_2 + \lambda_1 - 1) - 3(2\lambda_3 + \lambda_2 + \lambda_1 - 1)2^{1-r} \Bigg] \end{aligned}$$

Therefore, the first four moments are obtained by setting $r = 1, 2, 3, 4$ into $E(X^r)$.

$$E(X) = -\frac{12\lambda_3 + 18\lambda_2 + 22\lambda_1 - 25}{12\theta}$$

$$E(X^2) = -\frac{300\lambda_3 + 378\lambda_2 + 406\lambda_1 - 415}{72\theta^2}$$

$$E(X^3) = -\frac{4980\lambda_3 + 5670\lambda_2 + 5818\lambda_1 - 5845}{288\theta^3}$$

$$E(X^4) = -\frac{70140\lambda_3 + 75330\lambda_2 + 76030\lambda_1 - 76111}{864\theta^4}$$

Hence, the variance, $Var(X)$ is obtained as

$$\text{Var}(X) = \left\{ -\frac{300\lambda_3 + 378\lambda_2 + 406\lambda_1 - 415}{72\theta^2} - \left[-\frac{12\lambda_3 + 18\lambda_2 + 22\lambda_1 - 25}{12\theta} \right]^2 \right\}$$

The standard deviation is: $\sigma = \sqrt{\text{Var}(X)}$

The coefficient of variation is: CV $= \frac{\sigma}{E(X)}$

$$\text{Skewness} = \frac{E(X^3) - 3E(X)E(X^2) + 2(E(X))^3}{\sigma^3}$$

$$= \frac{-\frac{4980\lambda_3+5670\lambda_2+5818\lambda_1-5845}{288\theta^3} - 3E(X)E(X^2) + 2E(X)^3}{\sigma^3}$$

$$\text{Kurtosis} = \frac{E(X^4) - 4E(X)E(X^3) + 6(E(X))^2E(X^2) - 3(E(X))^4}{\sigma^4}$$

$$= \frac{-\frac{70140\lambda_3+75330\lambda_2+76030\lambda_1-76111}{864\theta^4} - 4E(X)E(X^3) + 6E(X)^2E(X^2) - 3E(X)^4}{\sigma^4}$$

We present the numerical results of the QTED parameters. By varying the parameters θ, λ_1, λ_2, and λ_3, specifically: $\lambda_1 \in \{0.1, 0.2\}$ $\lambda_2 \in \{0.2, 0.3\}$, $\lambda_3 \in \{0.1, 0.2\}$, and $\theta \in \{1.0, 2.0\}$. we have computed the mean, variance, standard deviation, coefficient of variation, skewness, and kurtosis. These statistical measures provide a comprehensive understanding of the distribution's characteristics under different parameter settings. The calculations were performed using Python software, specifically utilizing the 'numpy' and 'scipy.stats' libraries for statistical computations. Table 1 summarizes the results for selected parameter combinations.

Table 1: Statistical Measures for Different Parameter Sets

λ_1	λ_2	λ_3	θ	E(X)	Variance	Std Dev	Skewness	Kurtosis
0.1	0.2	0.1	1.0	1.5	1.4833	1.2179	1.4161	6.0441
0.1	0.2	0.1	2.0	0.75	0.3708	0.6090	1.4161	6.0441
0.1	0.2	0.2	1.0	1.4	1.3567	1.1648	1.5423	6.6143
0.1	0.2	0.2	2.0	0.7	0.3392	0.5824	1.5423	6.6143
0.1	0.3	0.1	1.0	1.35	1.3858	1.1772	1.5732	6.6035
0.1	0.3	0.1	2.0	0.675	0.3465	0.5886	1.5732	6.6035
0.1	0.3	0.2	1.0	1.25	1.2292	1.1087	1.7229	7.4097
0.1	0.3	0.2	2.0	0.625	0.3073	0.5543	1.7229	7.4097
0.2	0.2	0.1	1.0	1.317	1.4358	1.1983	1.5310	6.3655
0.2	0.2	0.1	2.0	0.658	0.3590	0.5991	1.5310	6.3655
0.2	0.2	0.2	1.0	1.217	1.2725	1.1281	1.6813	7.1514
0.2	0.2	0.2	2.0	0.608	0.3181	0.5640	1.6813	7.1514
0.2	0.3	0.1	1.0	1.167	1.2833	1.1328	1.7470	7.3056
0.2	0.3	0.1	2.0	0.583	0.3208	0.5664	1.7470	7.3056
0.2	0.3	0.2	1.0	1.067	1.0900	1.0440	1.9233	8.4384
0.2	0.3	0.2	2.0	0.533	0.2725	0.5220	1.9233	8.4384

Source: Author, 2023

The table provides a summary of statistical measures computed for various combinations of parameters λ_1, λ_2, λ_3, and θ. The expected value $E(X)$ ranges from 0.533 to 1.5, showing how the central tendency varies with different parameter settings. Variance, which measures the dispersion of the data, is relatively small across the table, with a maximum value of 1.4833, indicating that the outcomes are generally close to the expected values. The standard deviation, derived from the variance, lies between 0.2725 and 1.2179, further highlighting the variability in the distribution. Skewness values are positive throughout, suggesting a right-skewed distribution where the tail extends toward higher values. This skewness is more pronounced for higher parameter values. Kurtosis values, which are all above 6, indicate distributions with heavy tails and sharp peaks, implying the potential presence of outliers. Overall, the table illustrates the significant impact of parameter variations on these statistical properties, offering useful insights for modeling and further analysis.

Order Statistics and Quantile Function

Order Statistics

Let $X_1, X_2, ..., X_n$ be a random sample from QTED with pdf $g(x)$ and $X_{(1)}, X_{(2)}, ..., X_{(n)}$ the order statistics of the sample, where $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$, $X_{(1)} = \min(X_1, X_2, ..., X_n)$ and $X_{(n)} = \max(X_1, X_2, ..., X_n)$.
Then the pdf of the kth order statistic, $X_{(k)}$ is given by

$$f_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!}[G(x)]^{(k-1)}[1-G(x)]^{(n-k)} \times g(x)$$

Substituting $G(x)$ and $g(x)$, we have

$$\begin{aligned} f_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!} &\times \Bigg\{\theta e^{-\theta x}\Big[4\lambda_1 + 12(\lambda_2 - \lambda_1)(1-e^{-\theta x}) \\ &+ 12(\lambda_1 - 2\lambda_2 + \lambda_3)(1-e^{-\theta x})^2 + 4(1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)(1-e^{-\theta x})^3\Big]\Bigg\} \\ &\times \Bigg\{(1-e^{-\theta x})\Big[4\lambda_1 + 6(\lambda_2 - \lambda_1)(1-e^{-\theta x}) \\ &+ 4(\lambda_1 - 2\lambda_2 + \lambda_3)(1-e^{-\theta x})^2 + (1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)(1-e^{-\theta x})^3\Big]\Bigg\}^{(k-1)} \\ &\times \Bigg\{1 - (1-e^{-\theta x})\Big[4\lambda_1 + 6(\lambda_2 - \lambda_1)(1-e^{-\theta x}) + 4(\lambda_1 - 2\lambda_2 + \lambda_3)(1-e^{-\theta x})^2 \\ &+ (1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)(1-e^{-\theta x})^3\Big]\Bigg\}^{(n-k)} \end{aligned}$$

Therefore, by putting $k = 1$, the distribution of the minimum order statistic for the QTED is given by

$$f_{1:n}(x) = n \times \theta e^{-\theta x}\Bigg\{\Big[4\lambda_1 + 12(\lambda_2 - \lambda_1)(1 - e^{-\theta x}) + 12(\lambda_1 - 2\lambda_2 + \lambda_3)(1 - e^{-\theta x})^2 + 4(1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)(1 - e^{-\theta x})^3\Big]\Bigg\} \times \Bigg\{1 - (1 - e^{-\theta x})\Big[4\lambda_1 + 6(\lambda_2 - \lambda_1)(1 - e^{-\theta x}) + 4(\lambda_1 - 2\lambda_2 + \lambda_3)(1 - e^{-\theta x})^2 + (1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)(1 - e^{-\theta x})^3\Big]\Bigg\}^{(n-1)}$$

Similarly, by putting $k = n$, the maximum order statistic distribution for the QTED can be expressed as

$$f_{n:n}(x) = n\theta e^{-\theta x} \times \Bigg\{\Big[4\lambda_1 + 12(\lambda_2 - \lambda_1)(1 - e^{-\theta x}) + 12(\lambda_1 - 2\lambda_2 + \lambda_3)(1 - e^{-\theta x})^2 + 4(1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)(1 - e^{-\theta x})^3\Big]\Bigg\} \times \Bigg\{(1 - e^{-\theta x})\Big[4\lambda_1 + 6(\lambda_2 - \lambda_1)(1 - e^{-\theta x}) + 4(\lambda_1 - 2\lambda_2 + \lambda_3)(1 - e^{-\theta x})^2 + (1 - \lambda_1 + 2\lambda_2 - 4\lambda_3)(1 - e^{-\theta x})^3\Big]\Bigg\}^{(n-1)}$$

Quantile Function

Given the cdf in Equation (4.5), to derive the quantile function, substitute $G(x) = u$:

$$u = \left(1 - e^{-\theta x}\right)\Big[4\lambda_1 + 6(\lambda_2 - \lambda_1)\left(1 - e^{-\theta x}\right) + 4(\lambda_1 - 2\lambda_2 + \lambda_3)\left(1 - e^{-\theta x}\right)^2 + (1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)\left(1 - e^{-\theta x}\right)^3\Big]$$

Let $y = 1 - e^{-\theta x}$, then the equation simplifies to:

$$u = y\Big[4\lambda_1 + 6(\lambda_2 - \lambda_1)y + 4(\lambda_1 - 2\lambda_2 + \lambda_3)y^2 + (1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)y^3\Big]$$

This is a cubic equation in y:

$$u = y \cdot P(y)$$

where

$$P(y) = 4\lambda_1 + 6(\lambda_2 - \lambda_1)y + 4(\lambda_1 - 2\lambda_2 + \lambda_3)y^2 + (1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)y^3$$

The equation becomes:

$$y = \frac{u}{P(y)}$$

Numerical methods are typically employed to solve this equation for y given u. Once y is determined, we find x using:

$$x = -\frac{1}{\theta}\ln(1-y)$$

Thus, the quantile function $x(u)$ is:

$$x(u) = -\frac{1}{\theta}\ln\left(1 - y(u)\right)$$

where $y(u)$ is obtained by solving the cubic equation $u = y \cdot P(y)$ numerically. Quantiles at different probabilities p represent the values x below which a certain proportion of the data falls. For the given distribution, quantiles were calculated at specific probability levels (0.1, 0.2, ..., 0.9), providing insight into the distribution's shape and behaviour.

Table 2: Calculated Quantiles at Specific Probabilities

Probability (p)	Quantile (x)
0.1	0.052946
0.2	0.112707
0.3	0.181073
0.4	0.260650
0.5	0.355443
0.6	0.472110
0.7	0.623038
0.8	0.835821
0.9	1.197801

Source: Author, 2023

These quantiles offer a detailed view of how the distribution behaves across its range. For instance, the quantile at $p = 0.5$ (also known as the median) is $x = 0.355443$, meaning that 50% of the data is expected to fall below this value. Similarly, the quantile at $p = 0.1$ is $x = 0.052946$, indicating that 10% of the data will fall below this threshold. The increasing values of x as p increases indicate the distribution's cumulative nature, where higher probabilities correspond to larger quantiles. This progression highlights the distribution's skewness and spread, contributing to a better understanding of its characteristics. For example, the relatively larger gap between the quantiles at $p = 0.8$ and $p = 0.9$ suggests a longer right tail, implying a possible skew in the distribution. This analysis of quantiles is critical for modelling and predicting the behaviour of datasets within this distribution framework. By understanding where the majority of the data lies and how the distribution spreads, more informed decisions can be made regarding the application of this distribution in various contexts.

Entropy

Statistical entropy is a measure of uncertainty in a distribution function. We will discuss in this section the Renyi's, Shannon and Tsallis entropies.

The Shannon entropy

Shannon entropy (Shannon, 1948) is a measure of the uncertainty or information content in a probability distribution. It quantifies the average amount of information produced by a stochastic source of data. The Shannon entropy (Shannon, 1948), $H(g)$, is defined by:

$$
\begin{aligned}
H(g) &= -\int_0^\infty g(x)\ln g(x)\,dx \\
&= -\int_0^\infty \theta e^{-\theta x}\Big[4\lambda_1 + 12(\lambda_2-\lambda_1)(1-e^{-\theta x}) \\
&\quad + 12(\lambda_1 - 2\lambda_2 + \lambda_3)(1-e^{-\theta x})^2 \\
&\quad + 4(1-2\lambda_1+2\lambda_2-4\lambda_3)(1-e^{-\theta x})^3\Big] \\
&\quad \times \ln\left(\theta e^{-\theta x}\left[4\lambda_1 + 12(\lambda_2-\lambda_1)(1-e^{-\theta x})\right.\right. \\
&\quad + 12(\lambda_1 - 2\lambda_2 + \lambda_3)(1-e^{-\theta x})^2 \\
&\quad \left.\left. +4(1-2\lambda_1+2\lambda_2-4\lambda_3)(1-e^{-\theta x})^3\right]\right)\,dx
\end{aligned}
$$

Renyi entropy

The Renyi entropy $H_\alpha(g)$ of order α is given by:

$$
\begin{aligned}
H_\alpha(g) &= \frac{1}{1-\alpha}\ln\left(\int_0^\infty g(x)^\alpha\,dx\right) \\
&= \frac{1}{1-\alpha}\ln\left(\theta^\alpha\int_0^\infty e^{-\alpha\theta x}\left[4\lambda_1 + 12(\lambda_2-\lambda_1)(1-e^{-\theta x})\right.\right. \\
&\quad +12(\lambda_1 - 2\lambda_2 + \lambda_3)(1-e^{-\theta x})^2 \\
&\quad \left. + 4(1-2\lambda_1+2\lambda_2-4\lambda_3)(1-e^{-\theta x})^{3\alpha}\,dx\right)
\end{aligned}
$$

Tsallis entropy

The Tsallis entropy $S_q(g)$ of order q is given by:

$$
\begin{aligned}
S_q(g) &= \frac{1}{q-1}\left(1-\int_0^{\infty} g(x)^q\,dx\right)\\
&= \frac{1}{q-1}\Bigg(1-\theta^q\int_0^{\infty} e^{-q\theta x}\left[4\lambda_1+12(\lambda_2-\lambda_1)(1-e^{-\theta x})\right.\\
&\quad +12(\lambda_1-2\lambda_2+\lambda_3)(1-e^{-\theta x})^2\\
&\quad \left.+\,4(1-2\lambda_1+2\lambda_2-4\lambda_3)(1-e^{-\theta x})^{3q}\,dx\right)
\end{aligned}
$$

Maximum Likelihood Estimation of Parameters of the QTED

The likelihood function for a sample of n observations $x_1, x_2, ..., x_n$ assumed to be taken from QTED with pdf $g(x)$ is given by

$$L(x_1, x_2, ..., x_n|\theta, \lambda_1, \lambda_2, \lambda_3) = \prod_{i=1}^{n} g(x_i; \theta, \lambda_1, \lambda_2, \lambda_3)$$

Let $L(x_1, x_2, ..., x_n|\theta, \lambda_1, \lambda_2, \lambda_3)$ be denoted as $L(X|\theta, \Lambda)$

$$
\begin{aligned}
L(X|\theta,\Lambda) = \prod_{i=1}^{n}\Bigg\{\theta e^{-\theta x_i}\times\Bigg[&4\lambda_1+12(\lambda_2-\lambda_1)(1-e^{-\theta x_i})\\
&+12(\lambda_1-2\lambda_2+\lambda_3)(1-e^{-\theta x_i})^2\\
&+4(1-2\lambda_1+2\lambda_2-4\lambda_3)(1-e^{-\theta x_i})^3\Bigg]\Bigg\}
\end{aligned}
$$

The log-likelihood functions are $l_i = log(L_i)$ and the likelihood equations are

$$
\begin{aligned}
l(x_1, x_2, \ldots, x_n|\theta, \lambda_1, \lambda_2, \lambda_3) &= \sum_{i=1}^{n} \ln[g(x)] \\
&= \sum_{i=1}^{n} \Big[\ln\theta - \theta x_i + \ln\big(4\lambda_1 + 12(\lambda_2 - \lambda_1)(1 - e^{-\theta x_i}) \\
&\quad +12(\lambda_1 - 2\lambda_2 + \lambda_3)(1 - e^{-\theta x_i})^2 \\
&\quad +4(1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)(1 - e^{-\theta x_i})^3\big)\Big]
\end{aligned}
$$

Taking the partial derivative of the log-likelihood function with respect to θ, λ_1, λ_2, and λ_3 and setting to zero gives

$$
\frac{\partial l_i}{\partial \theta} = 0, \frac{\partial l_i}{\partial \lambda_1} = 0, \frac{\partial l_i}{\partial \lambda_2} = 0, \text{ and } \frac{\partial l_i}{\partial \lambda_3} = 0
$$

The equations may be solved numerically to obtain the maximum likelihood estimates (MLEs). The python software is used to estimate the parameters based on available data.

We compute the second partial derivatives to form the Hessian matrix H:

$$
H = \begin{bmatrix}
\frac{\partial^2 \ell}{\partial \theta^2} & \frac{\partial^2 \ell}{\partial \theta \partial \lambda_1} & \frac{\partial^2 \ell}{\partial \theta \partial \lambda_2} & \frac{\partial^2 \ell}{\partial \theta \partial \lambda_3} \\
\frac{\partial^2 \ell}{\partial \lambda_1 \partial \theta} & \frac{\partial^2 \ell}{\partial \lambda_1^2} & \frac{\partial^2 \ell}{\partial \lambda_1 \partial \lambda_2} & \frac{\partial^2 \ell}{\partial \lambda_1 \partial \lambda_3} \\
\frac{\partial^2 \ell}{\partial \lambda_2 \partial \theta} & \frac{\partial^2 \ell}{\partial \lambda_2 \partial \lambda_1} & \frac{\partial^2 \ell}{\partial \lambda_2^2} & \frac{\partial^2 \ell}{\partial \lambda_2 \partial \lambda_3} \\
\frac{\partial^2 \ell}{\partial \lambda_3 \partial \theta} & \frac{\partial^2 \ell}{\partial \lambda_3 \partial \lambda_1} & \frac{\partial^2 \ell}{\partial \lambda_3 \partial \lambda_2} & \frac{\partial^2 \ell}{\partial \lambda_3^2}
\end{bmatrix}
$$

Computing the inverse of the Hessian matrix evaluated at the MLEs to obtain the covariance matrix:

$$
\text{Cov}(\hat{\theta}, \hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3) = -H^{-1}(\hat{\theta}, \hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3)
$$

The diagonal elements of this covariance matrix provide the variances of the MLEs. The standard deviations are the square roots of these variances.

Simulation Process and Random Sample Generation

Given the cdf $G(x)$ and pdf $g(x)$, we can derive the simulation process for random sample generation as follows:

Step 1: Inverse Transform Sampling

To generate random samples, we use the inverse transform sampling method. The goal is to find the inverse of the CDF, $G^{-1}(u)$, where u is a random variable uniformly distributed on $[0, 1]$.

Step 2: Numerical Solution

Given the complexity of $G(x)$, it may not be possible to obtain an explicit form for $G^{-1}(u)$. Therefore, we use numerical methods to solve the equation:

$$G(x) = u$$

for x, where u is a random number drawn from a uniform distribution, $u \sim \text{Uniform}(0, 1)$.

Step 3: Random Sample Generation Process

The steps to generate random samples are as follows:

1. Generate a random number u from a uniform distribution $u \sim \text{Uniform}(0, 1)$.
2. Numerically solve the equation $G(x) = u$ for x using a root-finding method such as the Newton-Raphson method.
3. The solution x is a random sample from the distribution.
4. Repeat the process to generate the desired number of random samples.

Reproducibility Package

To ensure the reproducibility of the random sample generation, we use the following packages:

- **numpy**: For generating uniform random samples.
- **scipy**: For numerical methods such as root finding.
- **matplotlib**: For plotting the generated samples.

Simulation Study and Application

A simulation study is conducted by considering samples of sizes 50, 100, 150, 200, 300, 500, and 800 from the QTED. A total of 1000 random samples are generated for each setup with the parameters fixed as $\theta = 1.5,\ \lambda_1 = 0.5,\ \lambda_2 = 0.6,$ and $\lambda_3 = 0.2.$ Table 3 presents the mean of the true value, estimates, bias, and standard error (SE) of the model parameters. From the table, it can be observed that the estimated values of the parameters $\theta, \lambda_1, \lambda_2, \lambda_3$ tend to be close to the true values for larger sample sizes.

Table 3: Parameter Estimates of the QTED by Simulation

Sample size	True value	Estimate	—Bias—	SE
50	$\theta = 1.5$	1.2084	0.2916	0.0412
	$\lambda_1 = 0.5$	0.3148	0.1852	0.0262
	$\lambda_2 = 0.6$	0.4327	0.1673	0.0234
	$\lambda_3 = 0.2$	0.0934	0.1066	0.0151
100	$\theta = 1.5$	1.3172	0.1828	0.0183
	$\lambda_1 = 0.5$	0.3736	0.1264	0.0126
	$\lambda_2 = 0.6$	0.4651	0.1349	0.0135
	$\lambda_3 = 0.2$	0.1018	0.0982	0.0098
150	$\theta = 1.5$	1.3382	0.1618	0.0132
	$\lambda_1 = 0.5$	0.3865	0.1135	0.0093
	$\lambda_2 = 0.6$	0.4651	0.1349	0.0110
	$\lambda_3 = 0.2$	0.1018	0.0982	0.0080
200	$\theta = 1.5$	1.4382	0.0618	0.0044
	$\lambda_1 = 0.5$	0.4144	0.0856	0.0061
	$\lambda_2 = 0.6$	0.5164	0.0836	0.0059
	$\lambda_3 = 0.2$	0.1168	0.0832	0.0058
300	$\theta = 1.5$	1.5382	0.0382	0.0022
	$\lambda_1 = 0.5$	0.4365	0.0635	0.0037
	$\lambda_2 = 0.6$	0.5656	0.0344	0.0020
	$\lambda_3 = 0.2$	0.1450	0.0550	0.0031
500	$\theta = 1.5$	1.4823	0.0177	0.0007
	$\lambda_1 = 0.5$	0.5214	0.0214	0.0009
	$\lambda_2 = 0.6$	0.6252	0.0252	0.0011
	$\lambda_3 = 0.2$	0.1837	0.0163	0.0007
800	$\theta = 1.5$	1.5138	0.0138	0.0005
	$\lambda_1 = 0.5$	0.5140	0.0140	0.0005
	$\lambda_2 = 0.6$	0.6179	0.0179	0.0006
	$\lambda_3 = 0.2$	0.1889	0.0111	0.0004

Source: Author, 2023

Overall, the results indicate that increasing the sample size improves the accuracy and precision of parameter estimation. Larger sample sizes tend to yield parameter estimates with lower biases, MSE, and variances, resulting in more reliable and robust inference.

Life Test Data

In this section, the QTED is used to fit a real-life data set. The data set (given in Table 3) contains the times to failure of 50 devices put on life test at time 0 (in weeks). The data was extracted from Aarset (1987).

Table 4: Lifetimes of 50 Devices

0.1	0.2	1	1	1	1	1	2	3	6
7	11	12	18	18	18	18	18	21	32
36	40	45	46	47	50	55	60	63	63
67	67	67	67	72	75	79	82	82	83
84	84	84	85	85	85	85	85	86	86

Source: Aarset 1987

The Table 5 presents a summary of the data's descriptive statistics. These statistics offer valuable insights into the distribution and central tendencies of the data, allowing us to understand its characteristics better.

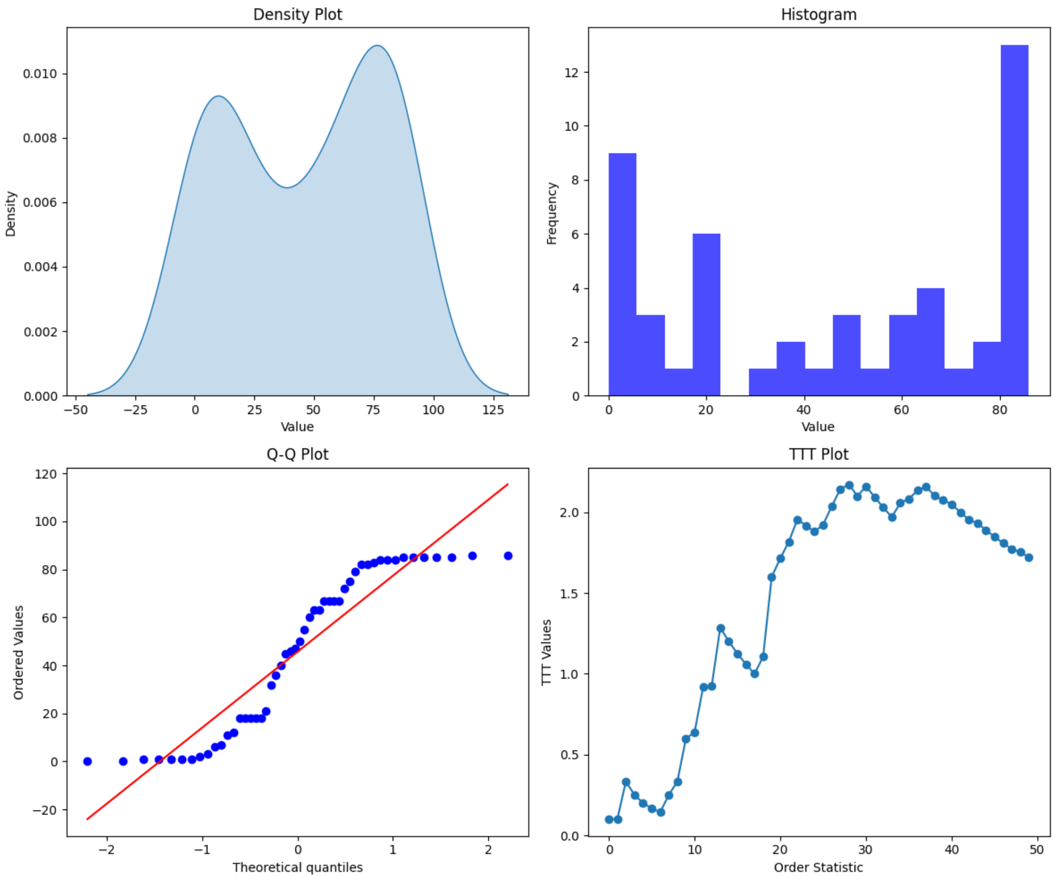
Table 5: Descriptive statistics of the Lifetime Data

Min	1^{st} Qu.	Median	Mean	3^{rd} Qu.	Max	SDv.	Skew	Kurt
0.1	13.50	48.50	45.69	81.25	86	32.84	-0.138	1.414

Source: Author, 2023

The obtained skewness value is less than 0, indicating that the distribution is skewed to the left (negative skewness). Again, the obtained kurtosis value indicates that the distribution is platykurtic. We now plot some statistical plots for the lifetime data. These plots provide a comprehensive view of the device lifetime data. Analysing these plots together aids in understanding the characteristics and behaviour of the dataset.

Figure 5: Some Statistical Plots for the Lifetime Data



Source: Author, 2023

The density curve shows the estimated probability density function of the data, while the histogram shows the frequency distribution of the data. The Q-Q plot compares the quantiles of the data against the quantiles of a theoretical normal distribution, which helps assess whether the data follows a normal distribution. The TTT plot is useful for assessing the shape of the hazard function in reliability analysis and survival analysis. The diagonal line represents the case of a constant hazard rate. This combination of plots provides a comprehensive visual summary of the data's distribution and characteristics. The goodness of fit of the QTED is compared with the following distributions:

- The exponential distribution given in Equation (4.3)
- Cubic Transmuted Exponential (CTED) (Rahman, Shahbaz, & Al-Zahrani, 2019) which given as

$$f(x) = \frac{1}{\theta}e^{-\frac{x}{\theta}}\left[1 + \lambda_1 + 3\lambda_2(1 - e^{-\frac{x}{\theta}})^2\right] \tag{8}$$

  where $\lambda_1, \lambda_2 \in [-1, 1]$, $\theta \in (0, \infty]$, such that $-2 \leq \lambda_1 + \lambda_2 \leq 1$ and $x \in (0, \infty)$

- Transmuted Exponential Distribution (TED) (Owoloko, Oguntunde, & Adejumo, 2015) which is given as

$$f(x) = \frac{1}{\theta}\exp\left(-\frac{x}{\theta}\right)\left[1 - \lambda + 2(\lambda)\exp\left(-\frac{x}{\theta}\right)\right] \tag{9}$$

  where $x > 0,\ \theta > 0$, and $|\lambda \leq 1|$

Table 6 presents the estimates for the parameters of four different probability distributions: QTED, CTED, TED, and the exponential distribution. The table provides the estimated values of the parameters for each distribution, which are obtained from the MLE based on the available data.

Table 6: MLEs of Selected Distributions

Distribution	Parameter	Estimate
QTED	$\theta, \lambda_1, \lambda_2, \lambda_3$	0.2395, 0.5548, 0.6741, 0.2124
CTED	$\theta, \lambda_1, \lambda_2$	33.77, 0.064, -0.27
TED	θ, λ	41.157, -0.243
Exponential	θ	27.36

Source: Author, 2023

Table 7 shows the results of LogLik, AIC, AICc, and BIC of the fitted distributions.

Table 7: Selection Criteria Values for Selected Models

Distribution	-LogLik	AIC	AICc	BIC
QTED	233.671	475.341	476.230	482.989
CTED	236.018	478.036	478.557	483.772
TED	240.677	485.355	485.610	489.179
Exponential	257.780	517.560	517.861	522.440

Source: Author, 2023

$$\mathbf{H} = \begin{bmatrix} 4.16862043 \times 10^{4} & 1.38588970 \times 10^{3} & 7.24375546 \times 10^{2} & 4.56496575 \times 10^{2} \\ 1.38588970 \times 10^{3} & 4.61137026 \times 10^{1} & 2.40584587 \times 10^{1} & 1.53179793 \times 10^{1} \\ 7.24375546 \times 10^{2} & 2.40584587 \times 10^{1} & 1.36251652 \times 10^{1} & 7.96595304 \times 10^{0} \\ 4.56496575 \times 10^{2} & 1.53179793 \times 10^{1} & 7.96595304 \times 10^{0} & 6.00144782 \times 10^{0} \end{bmatrix}$$

Standard errors: [0.24066153 7.45317572 1.00356678 1.45547812]

Distance Measures between QTED and the Exponential Distribution

Given the pdf $g(x)$ and the exponential distribution, the following distance measures are computed:

1. Kullback-Leibler (KL) Divergence

The KL divergence between QTED and the exponential distribution $f(x)$ is given by:

$$D_{\text{KL}}(g\|f) = \int_0^\infty g(x) \log\left(\frac{g(x)}{f(x)}\right) dx$$

The computed KL divergence between QTED and the exponential distribution is:

$$D_{\text{KL}}(g\|f) = 0.006383982906484991$$

2. Hellinger Distance

The Hellinger distance between QTED and the exponential distribution $f(x)$ is given by:

$$H(g, f) = \frac{1}{\sqrt{2}} \left(\int_0^\infty \left(\sqrt{g(x)} - \sqrt{f(x)} \right)^2 dx \right)^{\frac{1}{2}}$$

The computed Hellinger distance between QTED and the exponential distribution is:

$$H(g, f) = 0.1802174075559232$$

3. Total Variation (TV) Distance

The Total Variation distance between QTED and the exponential distribution $f(x)$ is given by:

$$D_{\text{TV}}(g, f) = \frac{1}{2} \int_0^\infty |g(x) - f(x)| \, dx$$

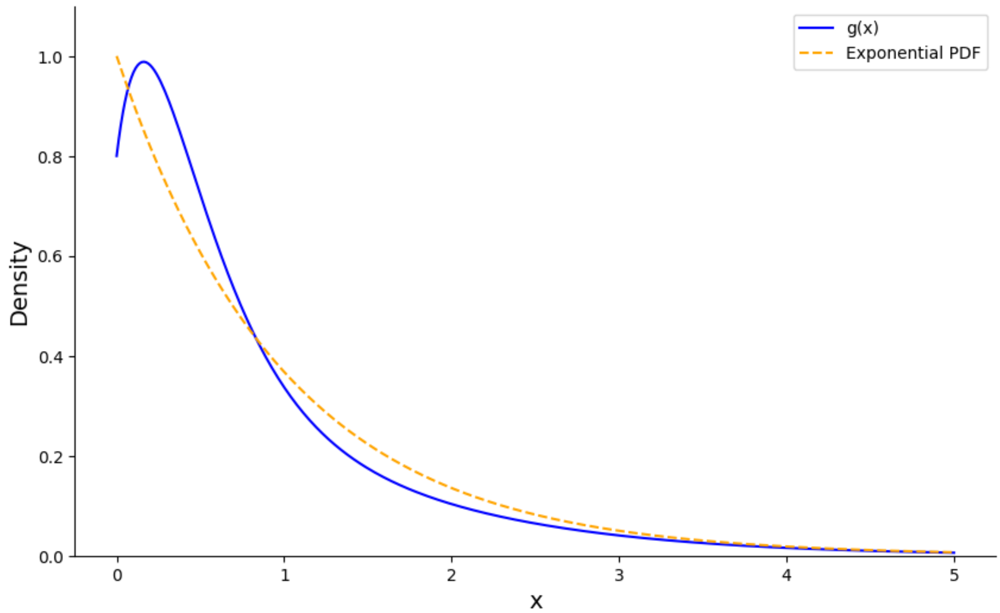
The computed Total Variation distance between QTED and the exponential distribution is:

$$D_{\text{TV}}(g, f) = 0.9559515047728921$$

The small KL divergence and moderate Hellinger distance suggest that QTED is quite similar to the exponential distribution in key areas, particularly where the distributions have significant probability mass. However, the high

Total Variation distance indicates that there are regions where QTED deviates significantly from the exponential distribution, contributing to the overall difference. This mixed picture shows that while QTED and the exponential distribution share some similarities, they are not the same, and certain features of QTED create larger discrepancies when viewed across the entire range.

Figure 6: QTED and Exponential Comparison



Source: Author, 2023

Quartic Transmuted Lindley Distribution

In this section, we introduce the quartic transmuted Lindley distribution (QTLD), which serves as a generalized version of the Lindley distribution. The QTLD is derived by employing the quartic rank transmutation map, offering a flexible extension of the conventional Lindley distribution. To establish context, we initially present a concise overview of the Lindley distribution, outlining its fundamental characteristics and properties. Subsequently, we proceed to introduce the pdf and cdf of the QTLD. These mathematical expressions shed light on the distinctions between the QTLD and the standard Lindley distribution, providing valuable insights into its behaviour. Additionally, we explore various statistical properties of the QTLD, such as moments, variance, and skewness. These statistical measures contribute to a comprehensive understanding of the QTLD's shape and central tendencies.

Furthermore, we delve into the estimation procedures for the QTLD, with a particular focus on MLE. This approach enables us to obtain optimal parameter estimates for the distribution, based on observed data. By applying this estimation technique, we can better model and analyze various phenomena using the QTLD. Throughout this section, our goal is to offer a thorough exploration of the QTLD, encompassing its theoretical foundations, statistical properties, estimation techniques, and practical applications. By examining real-world dataset where the QTLD finds usefulness, we aim to demonstrate its relevance and potential advantages over the standard Lindley distribution in diverse fields. This comprehensive analysis aims to contribute to a broader understanding of the QTLD and its potential impact in various domains of research and practice.

The Lindley Distribution

The Lindley distribution holds significant convenience in characterizing the lifetimes of processes and devices across diverse domains, encompassing fields like biology, engineering, and medicine. Moreover, it has proven efficacy in the modelling of mortality studies. Introduced by Lindley in 1958, this one-parameter probability distribution—known as the Lindley distribution—has emerged as an alternative to existing statistical probability distributions. However, it is important to note that the Lindley distribution exhibits limitations in certain contexts. It is less suited for modelling datasets with extensive right tails or tails that converge rapidly toward zero. Various extensions have been proposed in the literature to address these limitations and enhance the Lindley distribution's applicability. Sankaran (1970) introduced the discrete Poisson-Lindley distribution among the earliest extensions. This extension was tailored for scenarios involving errors in copying groups of random digits and accidents involving women working with high explosive shells.

Ahsan-ul-Haq (2022) introduced a generalized Lindley distribution, demonstrating improved hazard rate properties compared to distributions like gamma, lognormal, and Weibull. Shanker et al. (2019) expanded the framework with a two-parameter Lindley distribution, which found application in survival time data for guinea pigs infected with virulent tubercle bacilli and in data related to waiting times for bank customers. Numerous other extensions have been proposed to cater to specific data characteristics and requirements. These include the extended Lindley Poisson distribution (Pararai et al., 2015), truncated Lindley distribution (Zaninetti, 2019), double Lindley distribution (Kumar & Jose, 2018), and the Zografos Balakrishnan Power Lindley Distribution (Khokhar & Shahid, 2020), among several others. Collectively, these extensions contribute to the versatility and adaptability of the Lindley distribution framework, enabling it to accommodate a broader range of data patterns and applications across various domains. The pdf of the Lindley distribution is as given as:

$$f(x) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}, x > 0, \theta > 0 \tag{10}$$

The corresponding cdf is also given by

$$F(x) = 1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}, x > 0, \theta > 0 \tag{11}$$

Derivation and Characteristics of the QTLD

Given the baseline distribution with cdf $G(x)$ in Equation (1), and using Equation (11), the cdf of the QTLD is given by

$$\begin{aligned} G(x) =& 1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\left[4\lambda_1 + 6(\lambda_2-\lambda_1)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)\right. \\ &+ 4(\lambda_1-2\lambda_2+\lambda_3)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^2 \\ &\left.+ (1-2\lambda_1+2\lambda_2-4\lambda_3)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^3\right] \end{aligned} \tag{12}$$

The corresponding pdf using (2), (10), and (11) is given as

$$\begin{aligned} g(x) =& \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\left[4\lambda_1 + 12(\lambda_2-\lambda_1)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)\right. \\ &+ 12(\lambda_1-2\lambda_2+\lambda_3)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^2 \\ &\left.+ 4(1-2\lambda_1+2\lambda_2-4\lambda_3)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^3\right] \end{aligned} \tag{13}$$

where $x > 0, \theta > 0, \lambda_i \in [0,1]$

Proposition 2

Let X be a QTED random variable. Then $g(x)$ is a valid pdf of X if and only if:

1. $g(x) \geq 0$ for all x
2. $\int_0^\infty g(x)\,dx = 1$

Proof:

It can be seen that $g(x) \geq 0$ for all x. Also,

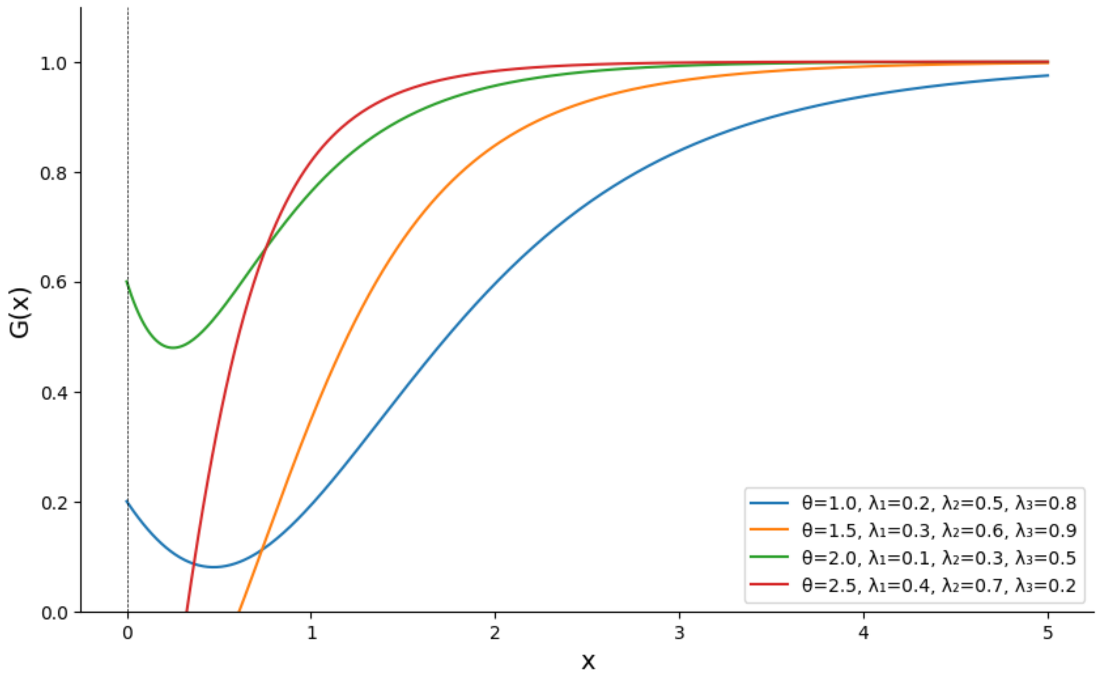
$$\int_0^\infty g(x)\,dx = \int_0^\infty \frac{\theta^2(1+x)}{\theta+1}e^{-\theta x}\left\{\begin{array}{l} 4\lambda_1 + 12(\lambda_2-\lambda_1)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right) + \\ 12(\lambda_1-2\lambda_2+\lambda_3)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^2 + \\ 4(1-2\lambda_1+2\lambda_2-4\lambda_3)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^3 \end{array}\right\}dx$$

Now since each term under the integration sign is integrable with respect to x , we do the integration term by term. That is,

$$\begin{aligned}
\int_0^\infty g(x)\,dx = 4\lambda_1 \int_0^\infty \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\,dx \\
+ 12(\lambda_2-\lambda_1)\int_0^\infty \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\left[1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right]dx \\
+ 12(\lambda_1-2\lambda_2+\lambda_3)\int_0^\infty \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\left[1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right]^2 dx \\
+ 4(1-2\lambda_1+2\lambda_2-4\lambda_3)\int_0^\infty \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\left[1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right]^3 dx \\
= 4\lambda_1\cdot(1) + \frac{1}{2}\cdot[12(\lambda_2-\lambda_1)] + 13\cdot[12(\lambda_1-2\lambda_2+\lambda_3)] \\
+ 14\cdot[4(1-2\lambda_1+2\lambda_2-4\lambda_3)] \\
= 4\lambda_1 + 6\lambda_2 - 6\lambda_1 + 4\lambda_1 - 8\lambda_2 + 4\lambda_3 + 1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3 \\
= \underline{1}
\end{aligned}$$

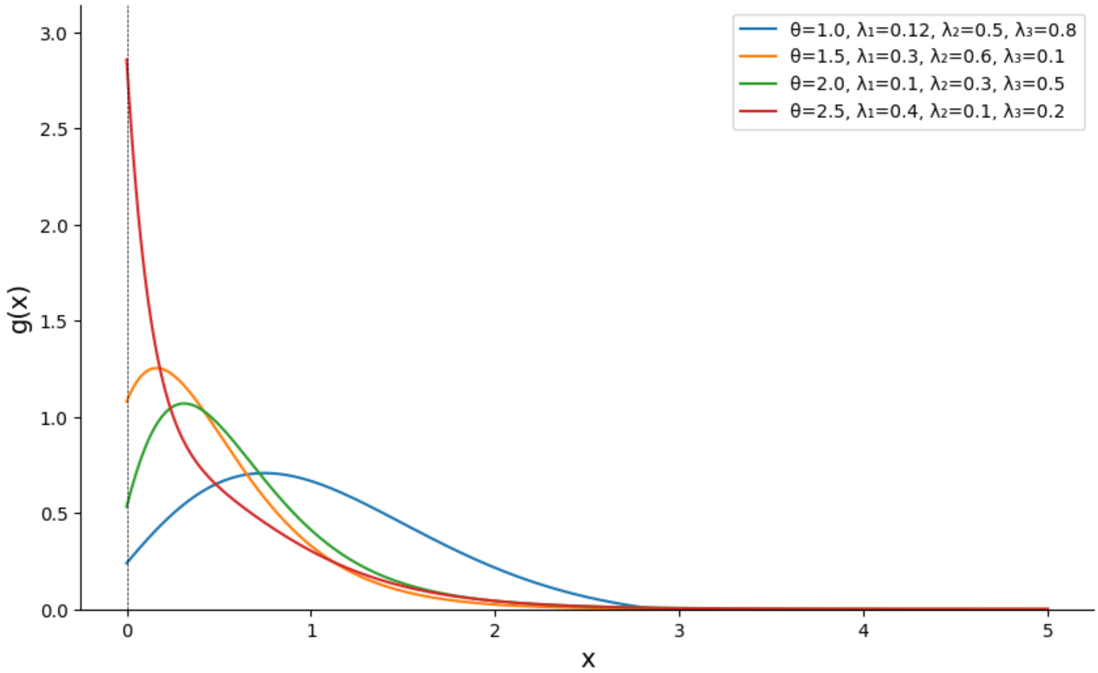
We present graphical representations for both the cdf, $G(x)$ and the pdf $g(x)$ of the QTED. These figures presented below depict the graphical representations of the cdf, $G(x)$ and pdf, $g(x)$, respectively.

Figure 7: CDF Plot of QTLD



Source: Author, 2023

Figure 8: PDF Plot of QTLD



Source: Author, 2023

Figures 7 and 8 show the cdf plot and the pdf plot of the QTLD for various parameter values. The rising nature of the cdf plot suggests an increasing distribution, underlining the progressive accumulation of probability as the random variable X advances. This rising trend in the cdf further buttresses the understanding that the QTLD distribution exhibits a positively skewed nature, emphasizing its ability to model data with increasing probability as values of X rise. The pdf plot displays QTLD across different parameter values. The observed shape of the plot indicates a positively skewed distribution. The variation in parameters contributes to the distinctive forms of the pdf, illustrating the flexibility of the QTLD in capturing different skewness patterns. This graphical representation offers valuable insights into the characteristics of the distribution under diverse parameter settings, aiding in the interpretation and understanding of its behaviour.

Basic Survival Quantities of the QTLD

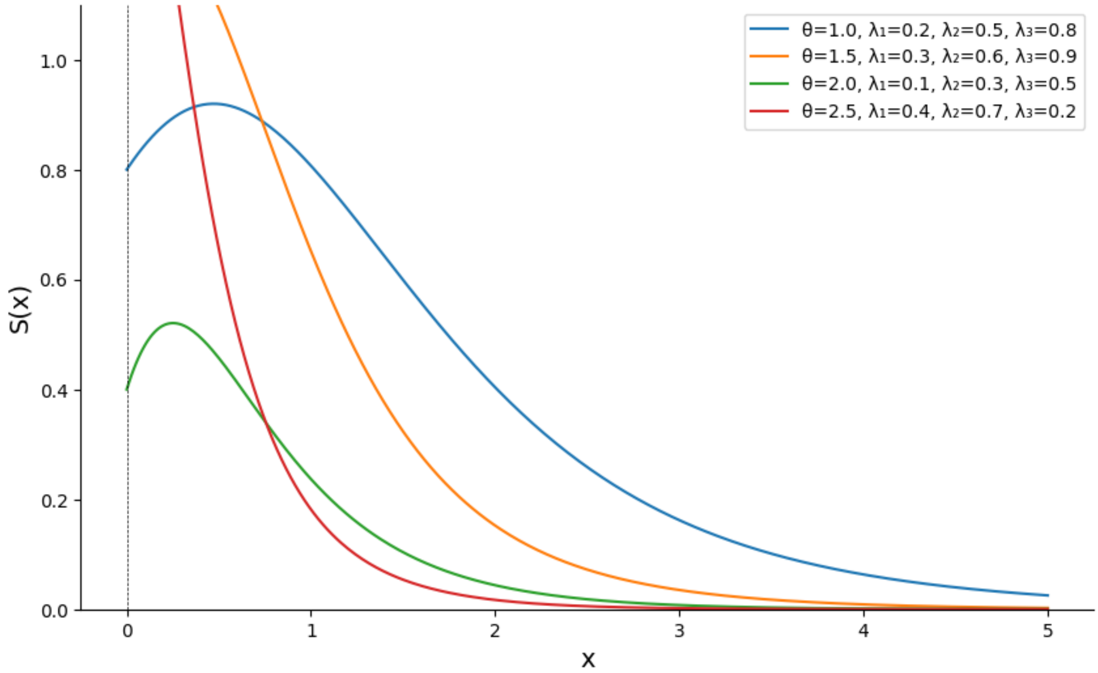
Here, we present the reliability, hazard rate, reversed hazard, cumulative hazard, and odds functions for the QTLD. These functions play a crucial role in reliability engineering, survival analysis, and risk assessment of systems.

Reliability Analysis

In this section, we present the reliability, hazard rate, reversed hazard, cumulative hazard and odds functions for the QTLD. The reliability function is obtained mathematically as the complement of $G(x)$. Let be a QTLD with pdf, $g(x)$ and cdf, $G(x)$. Then the reliability function of X is defined as:

$$
\begin{aligned}
R(x) &= 1 - G(x) \\
&= 1 - \left\{ \frac{\theta + 1 + \theta x}{\theta + 1} e^{-\theta x} \left[4\lambda_1 + 6(\lambda_2 - \lambda_1) \left(1 - \frac{\theta + 1 + \theta x}{\theta + 1} e^{-\theta x} \right) \right. \right. \\
&\quad + 4(\lambda_1 - 2\lambda_2 + \lambda_3) \left(1 - \frac{\theta + 1 + \theta x}{\theta + 1} e^{-\theta x} \right)^2 \\
&\quad \left. \left. + (1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3) \left(1 - \frac{\theta + 1 + \theta x}{\theta + 1} e^{-\theta x} \right)^3 \right] \right\}
\end{aligned}
$$

Figure 9: Survival Plot of QTLD



Source: Author, 2023

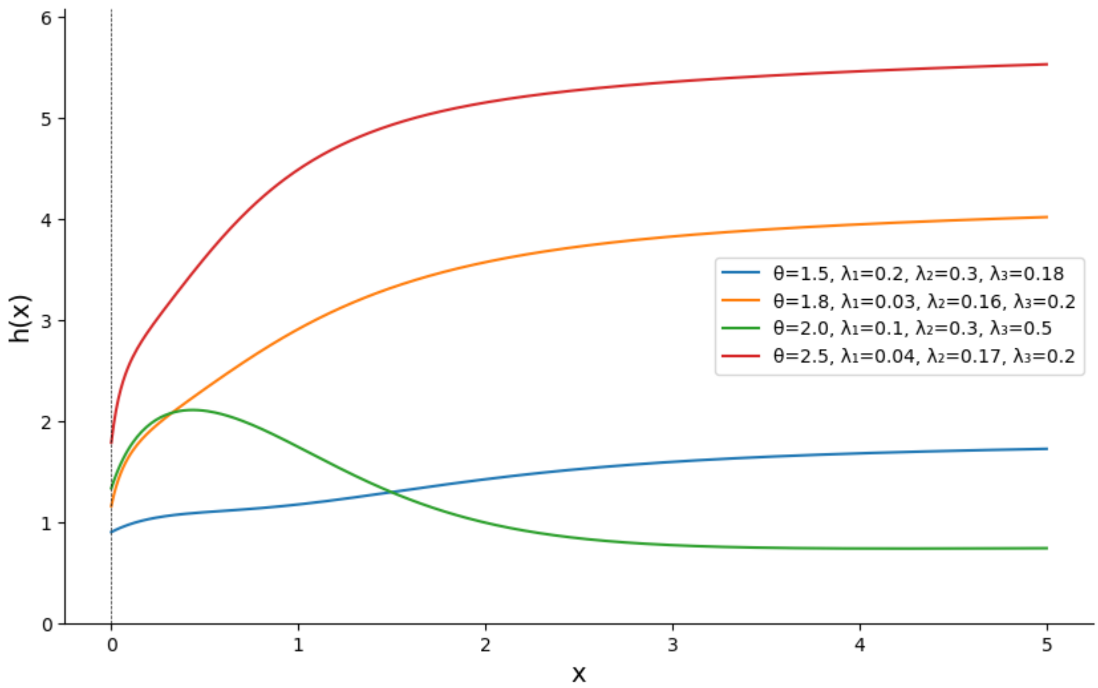
The plot in Figure 9 illustrate the survival function for the QTLD plotted under different parameter values. These plots showcase the probability of survival beyond various time points, providing insights into the distribution's tail behaviour. The declining nature of the survival curves indicates a decrease in survival probability over time, offering valuable information about the distribution's characteristics in terms of reliability and failure patterns. The variations in the plots demonstrate the impact of different parameter values on the survival function, emphasizing the flexibility of the QTLD in capturing diverse survival behaviours in real-world applications.

Hazard Function

The hazard rate function is obtained mathematically as the ratio of the pdf, $g(x)$ to the reliability function, $R(x)$. Thus, the hazard function for the QTLD is defined as:

$$h(x) = \frac{\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\left\{\begin{aligned}&4\lambda_1 + 12(\lambda_2-\lambda_1)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)\\&+12(\lambda_1-2\lambda_2+\lambda_3)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^2\\&+4(1-2\lambda_1+2\lambda_2-4\lambda_3)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^3\end{aligned}\right\}}{\left\{\begin{aligned}&1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\left[4\lambda_1+6(\lambda_2-\lambda_1)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)\right.\\&+4(\lambda_1-2\lambda_2+\lambda_3)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^2\\&\left.+(1-2\lambda_1+2\lambda_2-4\lambda_3)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^3\right]\end{aligned}\right\}}$$

Figure 10: Hazard Plot of QTLD



Source: Author, 2023

Cumulative Hazard

The cumulative hazard function of the QTLD is defined as:

$$H(x) = -\ln\left[1 - G(x)\right]$$
$$= -\ln\left(1 - \left[1 - \frac{\theta + 1 + \theta x}{\theta + 1}e^{-\theta x}\right]\left\{\begin{aligned}&4\lambda_1 + 6(\lambda_2 - \lambda_1)(1 - A)\\&+ 4(\lambda_1 - 2\lambda_2 + \lambda_3)(1 - A)^2\\&+ (1 - A)^3\end{aligned}\right\}\right)$$

where

$$A = \left(1 - \frac{\theta + 1 + \theta x}{\theta + 1}e^{-\theta x}\right)$$

Reversed Hazard

The reversed hazard rate is defined as the ratio of the pdf to the cdf. Thus, the reversed hazard rate is given as:

$$\begin{aligned}H(x) = &\frac{\theta^2}{\theta + 1}(1 + x)e^{-\theta x}\left[4\lambda_1 + 12(\lambda_2 - \lambda_1)\left(1 - \frac{\theta + 1 + \theta x}{\theta + 1}e^{-\theta x}\right)\right.\\&+ 12(\lambda_1 - 2\lambda_2 + \lambda_3)\left(1 - \frac{\theta + 1 + \theta x}{\theta + 1}e^{-\theta x}\right)^2\\&\left.+ 4(1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)\left(1 - \frac{\theta + 1 + \theta x}{\theta + 1}e^{-\theta x}\right)^3\right] /\\&\left[1 - \frac{\theta + 1 + \theta x}{\theta + 1}e^{-\theta x}\left(4\lambda_1 + 6(\lambda_2 - \lambda_1)\left(1 - \frac{\theta + 1 + \theta x}{\theta + 1}e^{-\theta x}\right)\right.\right.\\&+ 4(\lambda_1 - 2\lambda_2 + \lambda_3)\left(1 - \frac{\theta + 1 + \theta x}{\theta + 1}e^{-\theta x}\right)^2\\&+ (1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)\left(1 - \frac{\theta + 1 + \theta x}{\theta + 1}e^{-\theta x}\right)^3\end{aligned}$$

The Odds Function

The odds function of the quartic transmuted Lindley distribution is given as:

$$O(x) = \frac{\begin{aligned}&\left[4\lambda_1 + 6(\lambda_2 - \lambda_1)\left(1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)\right.\\ &+4(\lambda_1 - 2\lambda_2 + \lambda_3)\left(1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^2\\ &\left.+(1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)\left(1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^3\right]\end{aligned}}{\begin{aligned}&\left[1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right] \times\\ &\left[4\lambda_1 + 6(\lambda_2 - \lambda_1)\left(1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)\right.\\ &+4(\lambda_1 - 2\lambda_2 + \lambda_3)\left(1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^2\\ &\left.+(1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)\left(1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^3\right]\end{aligned}}$$

Moments and Moments Generating Function

In this section, we present the moment and moment generating functions for the QTLD.

Moments Generating Function

Let X be a QTLD random variable with pdf $g(x)$ as defined in equation (4.13). Thus, the MGF, defined by $M_x(t)$ is given as:

$$M_X(t) = \int_0^\infty e^{tx}\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\begin{bmatrix}4\lambda_1\\ +12(\lambda_2 - \lambda_1)(1-\Omega)\\ +12(\lambda_1 - 2\lambda_2 + \lambda_3)(1-\Omega)^2\\ +4(1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)(1-\Omega)^3\end{bmatrix}dx.$$

$$= \frac{\theta^2}{\theta+1} \begin{bmatrix} 4\lambda_1 \int_0^\infty e^{tx}(1+x)e^{-\theta x}\, dx \\ + 12(\lambda_2 - \lambda_1) \int_0^\infty e^{tx}(1+x)e^{-\theta x}\, (1-\Omega)\, dx \\ + 12(\lambda_1 - 2\lambda_2 + \lambda_3) \int_0^\infty e^{tx}(1+x)e^{-\theta x}\, (1-\Omega)^2\, dx \\ + 4(1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3) \int_0^\infty e^{tx}(1+x)e^{-\theta x}\, (1-\Omega)^3\, dx \end{bmatrix}$$

where

$$\Omega = \left(1 - \frac{\theta+1+\theta x}{\theta+1} e^{-\theta x}\right)$$

Thus, $E(e^{tX}) = a + b + c + d$ where

$$a = \frac{4\lambda_1\theta^2}{\theta+1} \int_0^\infty e^{tx}(1+x)e^{-\theta x} dx = \frac{4\lambda_1\theta^2}{\theta+1} \int_0^\infty (x+1)e^{tx-\theta x} dx$$

Integrating by parts,

$$\begin{aligned} &\text{Let } f = x+1 \Rightarrow f' = 1 \\ &g' = e^{tx-\theta x} \Rightarrow \frac{e^{tx-\theta x}}{t-\theta} \\ &= \frac{(x+1)e^{tx-\theta x}}{t-\theta} - \int_0^\infty \frac{e^{tx-\theta x}}{t-\theta} dx \\ &= \frac{4\lambda_1\theta^2}{\theta+1} \left[-\frac{t-\theta-1}{(t-\theta)^2} \right] \end{aligned}$$

$$b = \frac{12\,(\lambda_2-\lambda_1)\,\theta^2}{\theta+1} \int_0^\infty e^{tx}(1+x)e^{-\theta x} \left[1 - \frac{\theta+1+\theta x}{\theta+1} e^{-\theta x}\right] dx$$

$$= \frac{12\,(\lambda_2-\lambda_1)\,\theta^2}{\theta+1} \begin{bmatrix} -\frac{-\theta-1}{\theta+1} \int_0^\infty x e^{(t-\theta)x}\, dx - \frac{-\theta-1}{\theta+1} \int_0^\infty e^{(t-\theta)x}\, dx \\ -\frac{\theta}{\theta+1} \int_0^\infty x^2 e^{(t-2\theta)x}\, dx - \frac{2\theta+1}{\theta+1} \int_0^\infty x e^{(t-\theta)x}\, dx \\ -\int_0^\infty e^{(t-2\theta)x}\, dx \end{bmatrix}$$

$$= \frac{12\,(\lambda_2-\lambda_1)\,\theta^2}{\theta+1} \left[-\left(\frac{1}{t-\theta} - \frac{-\theta-1}{(\theta+1)(t-\theta)^2} + \frac{2\theta+1}{(\theta+1)(t-2\theta)^2} - \frac{1}{t-2\theta} - \frac{2\theta}{(\theta+1)(t-2\theta)^2} \right) \right]$$

$$
\begin{aligned}
c &= \frac{12\left(\lambda_1 - 2\lambda_2 + \lambda_3\right)\theta^2}{\theta+1} \int_0^\infty e^{tx}(1+x)e^{-\theta x}\left[1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right]^2 dx \\
&= \frac{12\left(\lambda_1 - 2\lambda_2 + \lambda_3\right)\theta^2}{\theta+1}\left[\int_0^\infty (x+1)\left[1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right]^2 e^{(t-\theta)x}\, dx\right] \\
&= \frac{12\left(\lambda_1 - 2\lambda_2 + \lambda_3\right)\theta^2}{\theta+1}\left[\int_0^\infty \frac{x\left\{(-\theta-1)e^{-\theta x} + \theta x + \theta + 1\right\}^2 e^{(t-3\theta)x}}{(\theta+1)^2}\, dx\right. \\
&\quad \left. + \frac{\left\{(-\theta-1)e^{-\theta x} + \theta x + \theta + 1\right\}^2 e^{(t-3\theta)x}}{(\theta+1)^2}\, dx\right] \\
&= \frac{12\left(\lambda_1 - 2\lambda_2 + \lambda_3\right)\theta^2}{\theta+1}\left\{\frac{1}{(\theta+1)^2}\int_0^\infty x\left\{(-\theta-1)e^{-\theta x} + \theta x + \theta + 1\right\}^2 e^{(t-3\theta)x}\, dx\right. \\
&\quad \left. + \frac{1}{(\theta+1)^2}\int_0^\infty \left\{(-\theta-1)e^{-\theta x} + \theta x + \theta + 1\right\}^2 e^{(t-3\theta)x}\, dx\right\} \\
&= \frac{12\left(\lambda_1 - 2\lambda_2 + \lambda_3\right)\theta^2}{\theta+1}\left\{-\left[\begin{array}{l}
\frac{(-\theta-1)^2}{(\theta+1)^2(t-\theta)} - \frac{(-\theta-1)^2}{(\theta+1)^2(t-\theta)} + \frac{-2\theta(-\theta-1)}{(\theta+1)^2(t-2\theta)^2} \\
+\frac{2(-\theta-1)}{(\theta+1)(t-2\theta)} - \frac{2(-\theta-1)}{(\theta+1)(t-2\theta)^2} + \frac{4\theta(-\theta-1)}{(\theta+1)^2(t-2\theta)^3} \\
+\frac{2\theta^2}{(\theta+1)^2(t-3\theta)^3} - \frac{2\theta}{(\theta+1)^2(t-2\theta)^3} + \frac{1}{t-3\theta} - \frac{1}{(t-3\theta)^2} \\
+\frac{4\theta}{(\theta+1)(t-3\theta)^3} - \frac{6\theta^2}{(\theta+1)^2(t-3\theta)^4}
\end{array}\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
d &= \frac{4(1-2\lambda_1+2\lambda_2-4\lambda_3)\theta^2}{\theta+1}\int_0^\infty e^{tx}(1+x)e^{-\theta x}\left[1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right]^3 dx \\
&= \frac{4(1-2\lambda_1+2\lambda_2-4\lambda_3)\theta^2}{\theta+1}\left[\int_0^\infty (x+1)\left[1 - \frac{\theta+1+\theta x}{\theta+1}\right]^3 e^{tx-\theta x}\right] dx \\
&= \frac{4(1-2\lambda_1+2\lambda_2-4\lambda_3)\theta^2}{\theta+1}\int_0^\infty -x\left[(-\theta-1)\,e^{\theta x} + \theta x + \theta + 1\right]^3 e^{tx-4\theta x}(\theta+1)^3\, dx \\
&= \frac{4(1-2\lambda_1+2\lambda_2-4\lambda_3)\theta^2}{\theta+1}\left[-\frac{1}{(\theta+1)^3}\int_0^\infty x\left((-\theta-1)e^{\theta x} + \theta x + \theta + 1\right]^3 e^{tx-4\theta x}\, dx\right] \\
&\quad - \frac{1}{(\theta+1)^3}\int_0^\infty \left((-\theta-1)e^{\theta x} + \theta x + \theta + 1\right)^3 e^{tx-4\theta x}\, dx
\end{aligned}
$$

Integrating the terms in the integral signs, we obtain d as

$$
\begin{aligned}
d = \frac{4(1-2\lambda_1+2\lambda_2-4\lambda_3)\theta^2}{\theta+1}&\left[-\left\{\frac{(-\theta-1)^3}{(\theta+1)^3(t-\theta)} + \frac{(-\theta-1)^3}{(\theta+1)^3(t-\theta)^2} + \frac{3\theta(-\theta-1)^2}{(\theta+1)^3(t-2\theta)^2}\right.\right. \\
&- \frac{3(-\theta-1)^2}{(\theta+1)^2(t-2\theta)} + \frac{3(-\theta-1)}{(\theta+1)^2(t-3\theta)^2} + \frac{6\theta(-\theta-1)^2}{(\theta+1)^3(t-2\theta)^3} + \frac{6\theta(-\theta-1)}{(\theta+1)^2(t-3\theta)^2} \\
&- \frac{3(-\theta-1)}{(\theta+1)^2(t-3\theta)} + \frac{3(-\theta-1)}{(\theta+1)(t-3\theta)^3} - \frac{12\theta^3(-\theta-1)}{(\theta+1)^3(t-3\theta)^4} + \frac{18\theta(-\theta-1)}{(\theta+1)^3(t-3\theta)^4} \\
&+ \frac{6\theta^3}{(\theta+1)^3(t-4\theta)^4} - \frac{6\theta^2}{(\theta+1)^2(t-4\theta)^3} + \frac{3\theta}{(\theta+1)(t-4\theta)^3} + \frac{18\theta^2}{(\theta+1)^2(t-4\theta)^4} \\
&\left.\left. + \frac{24\theta^3}{(\theta+1)(t-4\theta)^5}\right\}\right].
\end{aligned}
$$

Theorem

Let X be a QTLD random variable with pdf $g(x)$ as defined in Equation (4.13).

Then the rth moment of X, $E(X^r)$, is given as

$$E(X^r) = \int_0^\infty x^r \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x} \begin{bmatrix} 4\lambda_1 \\ +12(\lambda_2-\lambda_1)(1-A) \\ +12(\lambda_1-2\lambda_2+\lambda_3)(1-A)^2 \\ +4(1-2\lambda_1+2\lambda_2-4\lambda_3)(1-A)^3 \end{bmatrix} dx.$$

$$E(X^r) = \frac{\theta^2}{\theta+1} \begin{bmatrix} 4\lambda_1 \int_0^\infty x^r(1+x)e^{-\theta x}\,dx \\ +12(\lambda_2-\lambda_1) \int_0^\infty x^r(1+x)e^{-\theta x}(1-A)\,dx \\ +12(\lambda_1-2\lambda_2+\lambda_3) \int_0^\infty x^r(1+x)e^{-\theta x}(1-A)^2\,dx \\ +4(1-2\lambda_1+2\lambda_2-4\lambda_3) \int_0^\infty x^r(1+x)e^{-\theta x}(1-A)^3\,dx \end{bmatrix}$$

where

$$A = \left(1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)$$

Integrating term by term, we have

$$E(X^r) = p + q + m + n$$

where

$$\begin{aligned} p &= \frac{4\lambda_1\theta^2}{\theta+1}\int_0^\infty x^r(1+x)e^{-\theta x}\,dx \\ &= \frac{4\lambda_1\theta^2}{\theta+1}\int_0^\infty (x^r+x^{r+1})e^{-\theta x}\,dx \\ &= \frac{4\lambda_1\theta^2}{\theta+1}\left[\int_0^\infty x^r e^{-\theta x}\,dx + \int_0^\infty x^{r+1}e^{-\theta x}\,dx\right] \\ &= \frac{4\lambda_1\theta^2}{\theta+1}\left[\theta^{-r-2}\Gamma(r+2) + \theta^{-r-1}\Gamma(r+1)\right]. \end{aligned}$$

$$
\begin{aligned}
q &= \frac{12(\lambda_2-\lambda_1)\theta^2}{\theta+1}\int_0^\infty x^r(1+x)e^{-\theta x}\left[1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right]dx \\
&= \frac{12(\lambda_2-\lambda_1)\theta^2}{\theta+1}\left[-\frac{1}{\theta+1}\int_0^\infty x^r(x+1)e^{-\theta x}\left[(\theta x+\theta+1)e^{-\theta x}-(\theta+1)\right]dx\right].
\end{aligned}
$$

Expanding and applying linearity, we have

$$
\begin{aligned}
q &= \frac{12(\lambda_2-\lambda_1)\theta^2}{\theta+1}\left[(-\theta-1)\int_0^\infty x^{r+1}e^{-\theta x}\,dx+(-\theta-1)\int_0^\infty x^r e^{-\theta x}\,dx+\theta\int_0^\infty x^{r+2}e^{-2\theta x}\,dx\right. \\
&\quad \left.+(2\theta+1)\int_0^\infty x^{r+1}e^{-2\theta x}\,dx+(\theta+1)\int_0^\infty x^r e^{-2\theta x}\,dx\right] \\
&= \frac{12(\lambda_2-\lambda_1)\theta^2}{\theta+1}\left[\frac{\Gamma(r+3)}{\theta^{r+3}}+\frac{2(2\theta+1)\Gamma(r+2)}{\theta^{r+2}}-\frac{2(\theta+1)\Gamma(r+2)}{\theta^{r+2}}\right. \\
&\quad \left.-\frac{(2r+1)\Gamma(r+1)}{\theta^{r+1}}+\frac{2r+1\theta\Gamma(r+1)}{\theta^{r+1}}\right]\frac{1}{\theta+1}.
\end{aligned}
$$

$$
\begin{aligned}
m &= \frac{12(\lambda_1-2\lambda_2+\lambda_3)\theta^2}{\theta+1}\int_0^\infty x^r(1+x)e^{-\theta x}\left[1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right]^2 dx \\
&= \frac{12(\lambda_1-2\lambda_2+\lambda_3)\theta^2}{\theta+1}\left[\frac{(-\theta-1)^2\theta^2}{(\theta+1)^3}\int_0^\infty x^r e^{-\theta x}\,dx+\frac{2(-\theta-1)\theta^3}{(\theta+1)^3}\int_0^\infty x^{r+1}e^{-2\theta x}\,dx\right. \\
&\quad +\frac{2(-\theta-1)\theta^2}{(\theta+1)^3}\int_0^\infty x^r e^{-2\theta x}\,dx+\frac{\theta^4}{(\theta+1)^3}\int_0^\infty x^{r+2}e^{-3\theta x}\,dx \\
&\quad \left.+\frac{2\theta^3}{(\theta+1)^2}\int_0^\infty x^{r+1}e^{-3\theta x}\,dx+\frac{\theta^2}{(\theta+1)}\int_0^\infty x^r e^{-3\theta x}\,dx\right] \\
&= \frac{12(\lambda_1-2\lambda_2+\lambda_3)\theta^2}{\theta+1}\Big[(-3\theta-2)\theta^{-r-3}\Gamma(r+3)-3\theta^{-r-4}\Gamma(r+4)(\theta+1)^2 \\
&\quad +2\theta^{-r-2}\Gamma(r+3)+(-3\theta-1)\theta^{-r-2}\Gamma(r+2)+(2\theta+1)\theta^{-r-2}\Gamma(r+2) \\
&\quad -\theta^{-r-2}\Gamma(r+2)-3\theta^{-r-1}\Gamma(r+1)+\theta^{-r-1}\Gamma(r+1)+2r\theta^{-r-1}\Gamma(r+1)\Big].
\end{aligned}
$$

$$n = \frac{4(1-2\lambda_1+2\lambda_2-4\lambda_3)\theta^2}{\theta+1}\int_0^\infty x^r(1+x)e^{-\theta x}\left[1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right]^3 dx$$

$$= \frac{4(1-2\lambda_1+2\lambda_2-4\lambda_3)\theta^2}{\theta+1}\Bigg[-\frac{1}{(\theta+1)^3}\Bigg\{4\Gamma(r+5)-(5\theta+4)\Gamma(r+4)(\theta+1)^{-3}$$

$$+\Big[-3\theta-4\Gamma(r+4)+3(2\theta+1)\Gamma(r+3)-(3\theta+2)\Gamma(r+2)(\theta+1)^{-2}\Big]$$

$$+\Big[3\cdot 2^{-(r+3)}\theta^{-r-2}\Gamma(r+3)+(4\theta+1)\Gamma(r+2)$$

$$-(3\theta+1)\theta^{-r-2}\Gamma(r+2)+3(2\theta+1)\theta^{-r-2}\Gamma(r+2)\Big]$$

$$-\Big[(4-r-1)\theta^{-r-1}\Gamma(r+1)-\theta^{-r-1}\Gamma(r+1)\Big]$$

$$+\Big[-\theta^{-r-1}\Gamma(r+1)-\theta^{-r-1}\Gamma(r+1)\Big]\Bigg\}\Bigg]$$

Therefore, the first four moments are obtained by setting $r = 1, 2, 3, 4$ into $E(X^r)$.

$$E(X) = -\frac{1}{3456\theta(\theta+1)^4}\Bigg[(3456\theta^4+17280\theta^3+27360\theta^2+18480\theta+4620)\lambda_3$$

$$+(5184\theta^4+25920\theta^3+42700\theta^2+29160\theta+7290)\lambda_2$$

$$+(6336\theta^4+31680\theta^3+53616\theta^2+37912\theta+9478)\lambda_1$$

$$-(7200\theta^4+36000\theta^3+62040\theta^2+45580\theta+12259)\Bigg]$$

$$E(X^2) = -\frac{1}{20736\theta^2(\theta+1)^4}\Bigg[(864000\theta^4+5184000\theta^3+9374400\theta^2+6988800\theta+188580)\lambda_3$$

$$+(1088640\theta^4+6531840\theta^3+1224720\theta^2+9331200\theta+255150)\lambda_2$$

$$+(1169280\theta^4+7015680\theta^3+1335024\theta^2+10416640\theta+288850)\lambda_1$$

$$-(1195200\theta^4+7171200\theta^3+1371960\theta^2+1083136\theta+307561)\Bigg]$$

$$E(X^3) = -\frac{1}{829440^3(\theta+1)^4}\Big[(14342240\theta^4 + 100396800\theta^3 + 200188880\theta^2 + 160305560\theta + 4573380)\lambda_3$$
$$+ (16329600\theta^4 + 114307200\theta^3 + 234446400\theta^2 + 191872800\theta + 5562270)\lambda_2$$
$$+ (16755840\theta^4 + 117290880\theta^3 + 242389440\theta^2 + 200984800\theta + 5891170)\lambda_1$$
$$- (16833600\theta^4 + 117835200\theta^3 + 243905760\theta^2 + 202987120\theta + 5997361)\Big]$$

$$E(X^4) = -\frac{1}{2488320\theta^4(\theta+1)^4}\Big[(202003200\theta^4 + 1616025600\theta^3 + 3464092800\theta^2 + 2920444800\theta + 86731260)\lambda_3$$
$$+ (216950400\theta^4 + 1735603200\theta^3 + 3796632000\theta^2 + 3257755200\theta + 98130690)\lambda_2$$
$$+ (218966400\theta^4 + 1751731200\theta^3 + 3845582240\theta^2 + 331444800\theta + 100709630)\lambda_1$$
$$- (219199680\theta^4 + 1753597440\theta^3 + 3851531040\theta^2 + 3330426080\theta + 101252735)\Big]$$

Hence,

Variance

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

Standard Deviation

$$\sigma = \sqrt{\text{Var}(X)}$$

Coefficient of Variation

$$\text{CV} = \frac{\sigma}{E(X)}\%$$

Skewness

$$\gamma_1 = \frac{E\left[(X - E(X))^3\right]}{[\text{Var}(X)]^{3/2}}$$

$$\text{Skewness} = \frac{E(X^3) - 3E(X)E(X^2) + 2[E(X)]^3}{[\text{Var}(X)]^{3/2}}$$

Kurtosis

$$\gamma_2 = \frac{E\left[(X - E(X))^4\right]}{[\text{Var}(X)]^2} - 3$$

$$\text{Kurtosis} = \frac{E(X^4) - 4E(X)E(X^3) + 6[E(X)]^2E(X^2) - 3[E(X)]^4}{[\text{Var}(X)]^2} - 3$$

Statistical measures and numerical results of the QTLD parameters are presented by varying the parameters θ, λ_1, λ_2, and λ_3, specifically: $\lambda_1 \in \{0.1, 0.2\}$ $\lambda_2 \in \{0.02, 0.03\}$, $\lambda_3 \in \{0.01, 0.02\}$, and $\theta \in \{0.5, 1.5\}$. The mean, variance, standard deviation, coefficient of variation, skewness, and kurtosis are presented. These statistical measures provide a comprehensive understanding of the distribution's characteristics under different parameter settings. The calculations were performed using Python software, specifically utilizing the 'numpy' and 'scipy.stats' libraries for statistical computations. Table 8 summarizes the results for selected parameter combinations.

Table 8: Statistical Measures for Different Parameter Sets

θ	λ_1	λ_2	λ_3	$E(X)$	Variance	Std Dev	Coeff Var	Skewness	Kurtosis
0.5	0.1	0.02	0.01	5.7084	10.3514	3.2174	56.36	1.0053	2.8755
1.5	0.1	0.02	0.01	1.6724	0.7326	0.8559	51.18	4.4606	-2.0294
0.5	0.1	0.02	0.02	5.6820	10.3308	3.2142	56.57	1.0122	2.8938
1.5	0.1	0.02	0.02	1.6640	0.7302	0.8545	51.35	4.4752	-1.9362
0.5	0.1	0.03	0.01	5.6671	10.3722	3.2206	56.83	1.0214	2.8376
1.5	0.1	0.03	0.01	1.6595	0.7090	0.8420	50.74	4.9043	-2.6406
0.5	0.1	0.03	0.02	5.6407	10.3495	3.2171	57.03	1.0286	2.8578
1.5	0.1	0.03	0.02	1.6511	0.7064	0.8404	50.90	4.9226	-2.5387
0.5	0.2	0.02	0.01	5.1803	11.1316	3.3364	64.41	1.0727	3.7875
1.5	0.2	0.02	0.01	1.5114	0.5240	0.7239	47.89	10.6871	-6.8231
0.5	0.2	0.02	0.02	5.1539	11.0832	3.3291	64.59	1.0834	3.8358
1.5	0.2	0.02	0.02	1.5030	0.5189	0.7203	47.93	10.8136	-6.6276
0.5	0.2	0.03	0.01	5.1390	11.1089	3.3330	64.86	1.0957	3.8005
1.5	0.2	0.03	0.01	1.4985	0.4962	0.7044	47.01	11.9116	-8.1310
0.5	0.2	0.03	0.02	5.1126	11.0582	3.3254	65.04	1.1067	3.8513
1.5	0.2	0.03	0.02	1.4901	0.4909	0.7006	47.02	12.0687	-7.9299

Source: Author, 2023

The statistical measures reveal a significant dependency on the parameters θ, λ_1, λ_2, and λ_3. The expected values $E(X)$ are notably higher for lower values of θ (e.g., 0.5) compared to higher values (e.g., 1.5), indicating that increasing θ shifts the distribution toward lower values. Variance and standard deviation are larger for lower θ values, suggesting a more dispersed distribution that becomes more concentrated as θ increases, with variance dropping from around 10.3 to 0.49. The coefficient of variation is relatively high for $\theta = 0.5$ (approximately $56.36\% - 65.04\%$), indicating considerable variability relative to the mean, which decreases to $47.01\% - 51.35\%$ for $\theta = 1.5$. Skewness values are positive for $\theta = 0.5$, suggesting a right-skewed distribution, while the skewness for $\theta = 1.5$ increases significantly, indicating a more pronounced right tail. The kurtosis values indicate a distribution that is more peaked than normal for lower θ values (ranging from about 2.8376 to 3.8513), while for $\theta = 1.5$, kurtosis values drop below 3 (down to -8.1310), suggesting a lighter tail.

Order Statistics and Quantile Function

Order Statistics

Given the expressions for $G(x)$ in Equation (4.12) and $g(x)$ in Equation (4.13), the kth moment of the kth order statistic $X_{(k)}$ in a sample of size n is given by:

$$
\begin{aligned}
f_{X_{(k)}}(x) &= \frac{n!}{(k-1)!(n-k)!}\left[G(x)\right]^{k-1}\left[1-G(x)\right]^{n-k} g(x) \\
&= \frac{n!}{(k-1)!(n-k)!}\left\{1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\left[4\lambda_1+6(\lambda_2-\lambda_1)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)\right.\right. \\
&\quad \left.\left.+4(\lambda_1-2\lambda_2+\lambda_3)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^2+(1-2\lambda_1+2\lambda_2-4\lambda_3)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^3\right]\right\}^{k-1} \\
&\quad \times\left\{\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\left[4\lambda_1+6(\lambda_2-\lambda_1)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)\right.\right. \\
&\quad \left.\left.+4(\lambda_1-2\lambda_2+\lambda_3)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^2+(1-2\lambda_1+2\lambda_2-4\lambda_3)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^3\right]\right\}^{n-k} \\
&\quad \times\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\left[4\lambda_1+12(\lambda_2-\lambda_1)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)\right. \\
&\quad \left.+12(\lambda_1-2\lambda_2+\lambda_3)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^2+4(1-2\lambda_1+2\lambda_2-4\lambda_3)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^3\right]
\end{aligned}
$$

Therefore, for $k = 1$, the distribution of the minimum order statistics is given by

$$
\begin{aligned}
f_{X_{(1)}}(x) &= n[1-G(x)]^{n-1}g(x)\\
&= n\left\{\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\left[4\lambda_1+6(\lambda_2-\lambda_1)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)\right.\right.\\
&\left.\left.+4(\lambda_1-2\lambda_2+\lambda_3)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^2+(1-2\lambda_1+2\lambda_2-4\lambda_3)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^3\right]\right\}^{n-1}\\
&\times\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\left[4\lambda_1+12(\lambda_2-\lambda_1)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)\right.\\
&\left.+12(\lambda_1-2\lambda_2+\lambda_3)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^2+4(1-2\lambda_1+2\lambda_2-4\lambda_3)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^3\right]
\end{aligned}
$$

Similarly, for $k = n$, the minimum order statistics distribution for the QTLD can be expressed as

$$
\begin{aligned}
f_{X_{(n)}}(x) &= n[G(x)]^{n-1}g(x)\\
&= n\left\{1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\left[4\lambda_1+6(\lambda_2-\lambda_1)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)\right.\right.\\
&\left.\left.+4(\lambda_1-2\lambda_2+\lambda_3)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^2+(1-2\lambda_1+2\lambda_2-4\lambda_3)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^3\right]\right\}^{n-1}\\
&\times\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\left[4\lambda_1+12(\lambda_2-\lambda_1)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)\right.\\
&\left.+12(\lambda_1-2\lambda_2+\lambda_3)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^2+4(1-2\lambda_1+2\lambda_2-4\lambda_3)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^3\right]
\end{aligned}
$$

Quantile Function

Given the cdf in Equation (4.12), we want to find the quantile function, which is the value of x that satisfies $G(x) = p$ for a given probability p. Setting $G(x) = p$, we have

$$
\begin{aligned}
p = 1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}&\left[4\lambda_1+6(\lambda_2-\lambda_1)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)\right.\\
&+4(\lambda_1-2\lambda_2+\lambda_3)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^2\\
&\left.+(1-2\lambda_1+2\lambda_2-4\lambda_3)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^3\right].
\end{aligned}
$$

To solve for x, we generally use numerical methods because the equation is too complex to solve algebraically for x explicitly. Hence, the quantile function

$x = G^{-1}(p)$ needs to be computed numerically for given values of p. Thus, numerical methods are required to compute the quantile values for specific probabilities p. The quantile function, derived from the cdf $G(x)$, is crucial for understanding the distribution's behaviour at specific probability levels. The quantiles corresponding to probabilities $p = 0.1$ through $p = 0.9$ are as follows:

Table 9: Quantile Values for Selected Probabilities

Probability p	Quantile x
0.1	0.924822
0.2	1.055912
0.3	1.204032
0.4	1.374640
0.5	1.576242
0.6	1.823172
0.7	2.142498
0.8	2.595420
0.9	3.378140

Source: Author, 2023

The calculated quantile values for selected probabilities provide insight into the distribution's spread and central tendency. Thus, these values illustrate the distribution's behaviour across different levels of the cumulative probability.

Entropy Measures for QTLD

Shannon, Tsallis and Renyi entropies measure for QTLD are discussed.

Shannon entropy

$$
\begin{aligned}
H(g) = -\int_{-\infty}^{\infty} \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\Bigg[4\lambda_1 + 12(\lambda_2-\lambda_1)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)\\
+ 12(\lambda_1 - 2\lambda_2 + \lambda_3)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^2\\
+ 4(1-2\lambda_1+2\lambda_2-4\lambda_3)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^3\Bigg]\\
\times \log\Bigg(\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\Bigg[4\lambda_1 + 12(\lambda_2-\lambda_1)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)\\
+12(\lambda_1 - 2\lambda_2 + \lambda_3)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^2\\
+4(1-2\lambda_1+2\lambda_2-4\lambda_3)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^3\Bigg]\Bigg)\, dx
\end{aligned}
$$

Simplifying this expression further requires additional approximations or numerical evaluations.

Tsallis entropy

The Tsallis entropy for the QTLD random variable with pdf $g(x)$ is defined as

$$
\begin{aligned}
S_q(g) = \frac{1}{q-1}\Bigg(1-\int_{-\infty}^{\infty}\Bigg[\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\Bigg[4\lambda_1 + 12(\lambda_2-\lambda_1)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)\\
+12(\lambda_1 - 2\lambda_2 + \lambda_3)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^2\\
+4(1-2\lambda_1+2\lambda_2-4\lambda_3)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^3\Bigg]\Bigg]^q dx
\end{aligned}
$$

Similarly, simplifying this expression further requires additional approximations or numerical evaluations.

Renyi entropy

The Renyi's entropy for the QTLD rand0m variable with pdf $g(x)$ is defined as

$$H_q(g) = \frac{1}{1-q}\log\left(\int_{-\infty}^{\infty}\left[\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\left[4\lambda_1 + 12(\lambda_2-\lambda_1)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)\right.\right.\right.$$
$$+12(\lambda_1-2\lambda_2+\lambda_3)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^2$$
$$\left.\left.+4(1-2\lambda_1+2\lambda_2-4\lambda_3)\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)^3\right]\right]^q dx$$

Likelihood Function

The likelihood function $L(\theta, \lambda_1, \lambda_2, \lambda_3)$ for a sample $\{x_1, x_2, \ldots, x_n\}$ is given by:

$$L(\theta, \lambda_1, \lambda_2, \lambda_3) = \prod_{i=1}^{n} g(x_i; \theta, \lambda_1, \lambda_2, \lambda_3)$$

Taking the natural logarithm of the likelihood function:

$$\ell(\theta, \lambda_1, \lambda_2, \lambda_3) = \sum_{i=1}^{n} \log g(x_i; \theta, \lambda_1, \lambda_2, \lambda_3)$$

To find the MLE, set the first-order partial derivatives of the log-likelihood with respect to θ, λ_1, λ_2, and λ_3 to zero:

$$\frac{\partial \ell(\theta, \lambda_1, \lambda_2, \lambda_3)}{\partial \theta} = 0,$$

$$\frac{\partial \ell(\theta, \lambda_1, \lambda_2, \lambda_3)}{\partial \lambda_1} = 0,$$

$$\frac{\partial \ell(\theta, \lambda_1, \lambda_2, \lambda_3)}{\partial \lambda_2} = 0,$$

$$\frac{\partial \ell(\theta, \lambda_1, \lambda_2, \lambda_3)}{\partial \lambda_3} = 0$$

These equations yield the MLEs $\hat{\theta}$, $\hat{\lambda}_1$, $\hat{\lambda}_2$, $\hat{\lambda}_3$.

The Hessian matrix $\mathbf{H}$ is the matrix of second-order partial derivatives of the log-likelihood function:

$$\mathbf{H} = \begin{bmatrix} \frac{\partial^2 \ell}{\partial \theta^2} & \frac{\partial^2 \ell}{\partial \theta \partial \lambda_1} & \frac{\partial^2 \ell}{\partial \theta \partial \lambda_2} & \frac{\partial^2 \ell}{\partial \theta \partial \lambda_3} \\ \frac{\partial^2 \ell}{\partial \lambda_1 \partial \theta} & \frac{\partial^2 \ell}{\partial \lambda_1^2} & \frac{\partial^2 \ell}{\partial \lambda_1 \partial \lambda_2} & \frac{\partial^2 \ell}{\partial \lambda_1 \partial \lambda_3} \\ \frac{\partial^2 \ell}{\partial \lambda_2 \partial \theta} & \frac{\partial^2 \ell}{\partial \lambda_2 \partial \lambda_1} & \frac{\partial^2 \ell}{\partial \lambda_2^2} & \frac{\partial^2 \ell}{\partial \lambda_2 \partial \lambda_3} \\ \frac{\partial^2 \ell}{\partial \lambda_3 \partial \theta} & \frac{\partial^2 \ell}{\partial \lambda_3 \partial \lambda_1} & \frac{\partial^2 \ell}{\partial \lambda_3 \partial \lambda_2} & \frac{\partial^2 \ell}{\partial \lambda_3^2} \end{bmatrix}$$

The standard errors of the MLEs are obtained from the inverse of the negative Hessian matrix:

$$\text{Var}(\hat{\theta}, \hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3) = -\mathbf{H}^{-1}$$

The standard error for each parameter is the square root of the diagonal elements of $-\mathbf{H}^{-1}$:

$$\text{SE}(\hat{\theta}) = \sqrt{[-\mathbf{H}^{-1}]_{11}},$$

$$\text{SE}(\hat{\lambda}_1) = \sqrt{[-\mathbf{H}^{-1}]_{22}}$$

$$\text{SE}(\hat{\lambda}_2) = \sqrt{[-\mathbf{H}^{-1}]_{33}},$$

$$\text{SE}(\hat{\lambda}_3) = \sqrt{[-\mathbf{H}^{-1}]_{44}}$$

Simulation Process and Random Sample Generation

Given the cdf $G(x)$ and pdf $g(x)$, we can derive the simulation process for random sample generation as follows:

Step 1: Inverse Transform Sampling

To generate random samples, we use the inverse transform sampling method. The goal is to find the inverse of the CDF, $G^{-1}(u)$, where u is a random variable uniformly distributed on $[0, 1]$.

Step 2: Numerical Solution

Given the complexity of $G(x)$, it may not be possible to obtain an explicit form for $G^{-1}(u)$. Therefore, we use numerical methods to solve the equation:

$$G(x) = u$$

for x, where u is a random number drawn from a uniform distribution, $u \sim \text{Uniform}(0, 1)$.

Step 3: Random Sample Generation Process

The steps to generate random samples are as follows:

1. Generate a random number u from a uniform distribution $u \sim \text{Uniform}(0, 1)$.
2. Numerically solve the equation $G(x) = u$ for x using a root-finding method such as the Newton-Raphson method.
3. The solution x is a random sample from the distribution.
4. Repeat the process to generate the desired number of random samples.

Reproducibility Package

To ensure the reproducibility of the random sample generation, we use the following packages:

- **numpy**: For generating uniform random samples.
- **scipy**: For numerical methods such as root finding.
- **matplotlib**: For plotting the generated samples.

Simulation and Application

In this section, we present the results of a simulation study designed to assess the properties of the proposed estimation procedure. Moreover, we illustrate the usefulness of the proposed models with real data set.

Simulation

A simulation study was performed using sample sizes of 50, 100, 150, 200, 300, 500, and 800 drawn from the QTLD. For each configuration, 1,000 random samples were generated with the parameters set to $\theta = 2.5, \lambda_1 = 0.8, \lambda_2 = 0.4$, and$\lambda_3 = 0.25$. Table 10 presents the means of the estimates, absolute bias, MSE as well as the standard error for the model parameters.

The table presents the estimates of various parameters (θ, λ_1, λ_2, and λ_3) obtained from simulations across different sample sizes (50, 100, 150, 200, 300, 500, and 800). For each sample size, the mean estimates, absolute bias, and standard error are reported. Key observations include the consistency of estimates, where the mean estimates for all parameters remain relatively stable across sample sizes, indicating that the estimation method is robust. The absolute bias for θ shows a higher value compared to the λ parameters, suggesting that while the estimates for λ_1, λ_2, and λ_3 are closer to their true values, the estimate for θ exhibits greater deviation. Additionally, the standard errors for all parameters decrease as the sample size increases, demonstrating that larger samples yield more precise estimates. The standard errors remain notably low, indicating that the parameter estimates are reliable. Furthermore, the estimates for λ_1, λ_2, and λ_3 show minimal variability across sample sizes, suggesting that these parameters may be less sensitive to changes in the sample size compared to θ.

Table 10: Estimates of Parameters for Different Sample Sizes

Sample Size	Parameter	Mean Estimate	Absolute Bias	Standard Error
50	θ	0.4001	2.0999	0.0018
	λ_1	0.3201	0.4799	0.0014
	λ_2	0.1601	0.2399	0.0007
100	θ	0.4023	2.0977	0.0013
	λ_1	0.3219	0.4781	0.0010
	λ_2	0.1609	0.2391	0.0005
	λ_3	0.1006	0.1494	0.0003
150	θ	0.4011	2.0989	0.0011
	λ_1	0.3209	0.4791	0.0009
	λ_2	0.1604	0.2396	0.0004
	λ_3	0.1003	0.1497	0.0003
200	θ	0.4001	2.0999	0.0009
	λ_1	0.3201	0.4799	0.0007
	λ_2	0.1600	0.2400	0.0004
	λ_3	0.1000	0.1500	0.0002
300	θ	0.4006	2.0994	0.0007
	λ_1	0.3205	0.4795	0.0006
	λ_2	0.1602	0.2398	0.0003
	λ_3	0.1001	0.1499	0.0002
500	θ	0.4001	2.0999	0.0006
	λ_1	0.3201	0.4799	0.0005
	λ_2	0.1600	0.2400	0.0002
	λ_3	0.1000	0.1500	0.0001
800	θ	0.4005	2.0995	0.0004
	λ_1	0.3204	0.4796	0.0004
	λ_2	0.1602	0.2398	0.0002
	λ_3	0.1001	0.1499	0.0001

Source: Author, 2023

Application

In this section, the QTLD is used to fit a real-life data set. This data set consists of remission times (in months) of a random sample of 128 bladder cancer. The data was observed and reported by Lee and Wang (2003). Other authors who studied this data include Sakthivel, Rajitha & Dhivakar (2020) and Merovci (2013).

To provide a comprehensive overview of the remission times for the 128 bladder cancer patients, we present the descriptive statistics for this dataset.

Table 11: Remission Times of 128 Bladder Cancer Patients

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63	0.20	2.23
3.52	4.98	6.97	9.02	13.29	0.40	2.26	3.57	5.06	7.09
9.22	13.80	25.74	0.50	2.46	3.64	5.09	7.26	9.47	14.24
25.82	0.51	2.54	3.70	5.17	7.28	9.74	14.76	26.31	0.81
2.62	3.82	5.32	7.32	10.06	14.77	32.15	2.64	3.88	5.32
7.39	10.34	14.83	34.26	0.90	2.69	4.18	5.34	7.59	10.66
15.96	36.66	1.05	2.69	4.23	5.41	7.62	10.75	16.62	43.01
1.19	2.75	4.26	5.41	7.63	17.12	46.12	1.26	2.83	4.33
7.66	11.25	17.14	79.05	1.35	2.87	5.62	7.87	11.64	17.36
1.40	3.02	4.34	5.71	7.93	11.79	18.10	1.46	4.40	5.85
8.26	11.98	19.13	1.76	3.25	4.50	6.25	8.37	12.02	2.02
3.31	4.51	6.54	8.53	12.03	20.28	2.02	3.36	6.76	12.07
21.73	2.07	3.36	6.93	8.65	12.63	22.69	5.49		

Source: Lee and Wang, 2003

These statistics offer insights into the central tendency, dispersion, and shape of the distribution of remission times. Specifically, we examine key measures including the minimum, first quartile, median, mean, third quartile, and maximum values, as well as variance, standard deviation, skewness, and kurtosis. This analysis helps in understanding the overall characteristics and distribution of remission times, which is crucial for interpreting the data and drawing meaningful conclusions.

The descriptive statistics for the remission times of 128 bladder cancer patients provide valuable insights into the distribution and characteristics of the data. The minimum remission time is 0.08 months, while the maximum ex-

Table 12: Descriptive Statistics of Remission Times

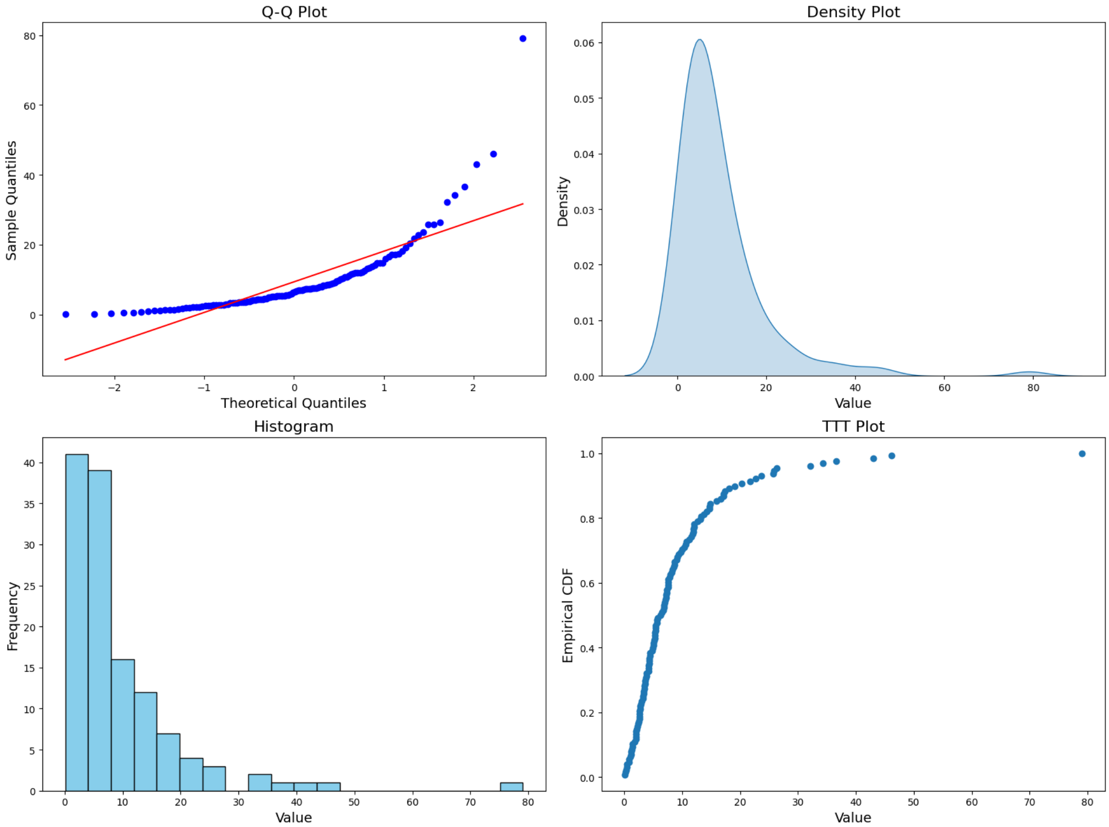
Min	1st Qu.	Median	Mean	3rd Qu.	Max	St. Dv.	Skew.	Kurt.
0.08	3.34	6.39	9.37	11.84	79.05	10.51	3.286	18.48

Source: Author, 2023

tends to 79.05 months, indicating a wide range in remission durations. The first quartile (3.348 months) and third quartile (11.838 months) suggest that 50% of patients have remission times between these values, with a median of 6.395 months indicating the midpoint of the distribution. The mean remission time is 9.366 months, which is higher than the median, reflecting a right-skewed distribution where a few extended remission times increase the average. The variance of 110.425 and standard deviation of 10.508 reveal considerable dispersion around the mean. The high skewness of 3.286 confirms a significant right skew, and the kurtosis of 18.483 suggests a distribution with heavy tails, implying that extreme remission times are more prevalent than in a normal distribution. Together, these statistics highlight the variability and asymmetry in the remission times of the patients, emphasizing the presence of a few outliers with very long remission periods.

To further explore the characteristics of the remission times, we present several plots that visually depict the distribution of the data. These plots include histograms, Q-Q plots, and probability density functions, which provide insights into the distribution, spread, and potential outliers in the remission times of the bladder cancer patients.

Figure 11: Some Statistical Plots for the Remission Data



Source: Author, 2023

The plots of the remission times offer a visual understanding of the dataset's distribution. The density curve and histogram illustrate the distribution of the data. The density curve reveals a right-skewed distribution, indicating a longer tail on the right side, which is corroborated by the histogram showing the majority of data clustered towards the lower end. The Q-Q plot compares the data's empirical quantiles to the theoretical quantiles of a normal distribution. The deviation of the points from the diagonal line suggests that the data does not conform to a normal distribution, aligning with the right skew observed in the density curve and histogram. The TTT plot assesses the underlying distribution by plotting the cumulative proportion of total time on test against the cumulative proportion of ordered values.

The goodness of fit of the QTLD is compared with the following distributions

- The Lindley distribution given in Equation (4.10)

- Cubic Transmuted Lindley Distribution (Sakthivel, Rajitha & Dhivakar, 2020).

$$g(x) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\Big[\lambda_1 + 2(\lambda_2-\lambda_1)\Big[(1-e^{-\theta x}-\frac{\theta x}{\theta+1}e^{-\theta x})\Big]^2 + (1-\lambda_2)\Big(1-e^{-\theta x}-\frac{\theta x}{\theta+1}e^{-\theta x}\Big)^3\Big]$$

where $x > 0, \theta > 0, \lambda_1 \in [0,3], \lambda_2 \in [-1,1]$

- Transmuted Lindley Distribution (TLD) (Merovci, 2013)

$$g(x) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\left[1-\lambda+2\lambda\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right]$$

where $x > 0, \lambda > 0,$ and $|\lambda| \leq 1$

Table 13 presents the MLE for the parameter values of the QTLD distribution, along with those of other distributions used for comparison. These estimates provide insights into the parameters that best fit each distribution to the data, enabling an assessment of how well the QTLD model performs relative to alternative distributions.

Table 13: MLEs of Selected Distributions

Distribution	Parameter	Estimate	SE
QTLD	$\theta, \lambda_1, \lambda_2, \lambda_3$	0.114, 0.818, 0.047, 0.038	0.041, 0.278, 0.600, 0.717
CTLD	$\theta, \lambda_1, \lambda_2$	0.122, 2.872, -0.382	0.020, 0.467, 0.450
TLD	θ, λ	0.156, 0.617	0.0447, 0.0707
Lindley	θ	0.196	0.0121

Source: Author, 2023

Table 14 displays the results for the LogLik, AIC, AICc, and BIC for the fitted distributions. These metrics are used to evaluate and compare the performance of different models, with lower values indicating a better fit of the model to the data while accounting for model complexity.

Table 14: Selection Criteria Values for Selected Models

Distributions	-LogLik	AIC	AICc	BIC
QTLD	405.063	818.126	818.451	829.534
CTLD	410.227	820.454	826.454	835.010
TD	415.150	834.310	834.406	840.010
Lindley	419.520	841.060	841.092	843.910

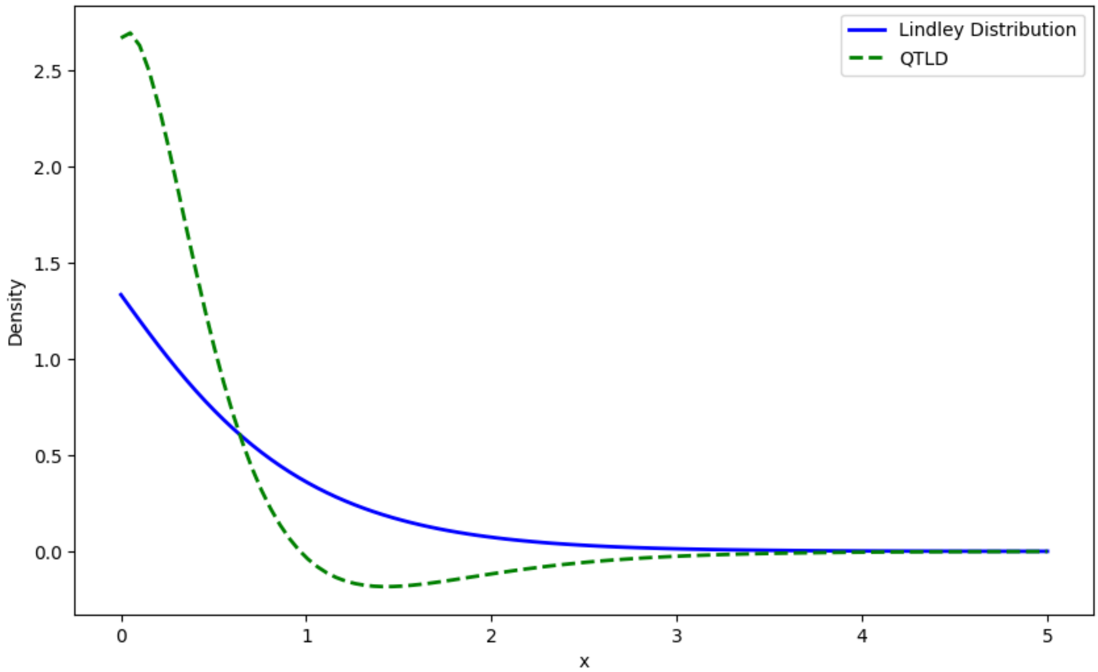
Source: Author, 2023

$$\text{Hessian Matrix} = \begin{bmatrix} 0.00413172 & -0.05723981 & 0.01241361 & 0.02979796 \\ -0.05723981 & 0.86175958 & -0.31993651 & -0.31473509 \\ 0.01241361 & -0.31993651 & 0.57546935 & -0.36918513 \\ 0.02979796 & -0.31473509 & -0.36918513 & 0.64513522 \end{bmatrix}$$

Distance Measure between QTLD and Lindley Distribution

Here, we analyze the relationship between the QTLD and the standard Lindley distribution by quantifying their differences using various statistical distance measures. Specifically, we compute the Kullback-Leibler (KL) Divergence, Hellinger Distance, and Total Variation Distance to assess the extent to which the QTLD deviates from the Lindley distribution in terms of their respective pdfs. These measures provide a rigorous comparison, highlighting both similarities and differences between the two distributions.

Figure 12: Comparison of QTLD and Lindley Distribution



Source: Author, 2023

The calculated distance measures between the $g(x)$ function and the Lindley distribution indicate significant differences in their respective pdfs. The KL Divergence of approximately 1.2335 quantifies the divergence between the two distributions, highlighting that the two distributions are relatively dissimilar, with the $g(x)$ function deviating considerably from the Lindley distribution in terms of their probability mass allocations. The Hellinger Distance of about

0.5431 further underscores the disparity, reflecting a moderate level of difference between the distributions in terms of their square root density functions. Lastly, the Total Variation Distance of approximately 4.2103 suggests a substantial deviation in the overall shape and distribution of probabilities between $g(x)$ and the Lindley distribution. These values are notably outside typical ranges, where a KL Divergence close to zero, Hellinger Distance below 0.5, and Total Variation Distance under 1 would generally indicate more similarity. The observed distances imply that the $g(x)$ function and the Lindley distribution have distinct characteristics.

Quartic Rank Transmuted Rayleigh Distribution

In this section, we introduce the quartic transmuted Rayleigh distribution (QTRD), which serves as a generalized version of the Rayleigh distribution. The QTRD is derived by employing the quartic rank transmutation map, offering a versatile extension of the conventional Rayleigh distribution. To establish context, we initially present a concise overview of the Rayleigh distribution, outlining its fundamental characteristics and properties. Subsequently, we proceed to introduce the pdf and cdf of the QTRD. These mathematical expressions shed light on the distinctions between the QTRD and the standard Rayleigh distribution, providing valuable insights into its behaviour. Additionally, we explore various statistical properties of the QTRD, such as moments, variance, and skewness. These statistical measures contribute to a comprehensive understanding of the QTRD's shape and central tendencies. Furthermore, we delve into the estimation procedures for the QTRD, with a particular focus on maximum likelihood estimation. This approach enables us to obtain optimal parameter estimates for the distribution, based on observed data. By applying this estimation technique, we can better model and analyse various phenomena using the QTRD. Throughout this section, our goal is to offer a thorough exploration of the QTRD, encompassing its theoretical foundations, statistical properties, estimation techniques, and practical applications. By examining real-world scenarios where the QTRD finds utility, we aim to demonstrate its relevance and potential advantages over the standard Rayleigh distribution in diverse fields. This comprehensive analysis aims to contribute to a broader understanding of the QTRD and its potential impact in various domains of research and practice.

Rayleigh Distribution

The Rayleigh distribution holds significant importance in various physics-related fields, including the study of processes such as sound and light radiation, wave heights, and wind speed. Additionally, it finds practical applications in communication theory to characterize hourly median and instantaneous peak power of received radio signals. In the realm of renewable energy, the Rayleigh distribution has been utilized to model the frequency of different wind speeds at wind turbine sites over a year, as well as the daily average wind speed. Due to its versatility and applicability, the Rayleigh distribution is regarded as a valuable life distribution, finding widespread use in statistics and operations research. Across numerous disciplines, the Rayleigh distribution plays a pivotal role in diverse applications, spanning health, agriculture, biology, and other scientific domains. Researchers and practitioners frequently leverage this distribution to analyze and model various phenomena. It is noteworthy that the Rayleigh distribution constitutes a special case of the two-parameter Weibull distribution, where the shape parameter is fixed at 2. This relationship expands the distribution's versatility and its ability to adapt to different scenarios.

The origin of the Rayleigh distribution can be traced back to the pioneering work of Rayleigh in 1880 when he first introduced this probability distribution. Subsequently, researchers like Sinha and Howlader (1983) and Abd Elfattah et al. (2006) have delved into inference techniques for this distribution, contributing to a deeper understanding of its statistical properties and facilitating its application in various fields. The contributions of these researchers have further solidified the importance of the Rayleigh distribution in scientific investigations and practical problem-solving.

The pdf of the Rayleigh distribution is as given as

$$f(x;\theta) = \frac{x}{\theta^2} \exp\left(-\frac{x^2}{2\theta^2}\right), \quad \text{for } x \geq 0,\ \theta > 0 \tag{14}$$

The corresponding cdfF is given by

$$F(x;\theta) = 1 - \exp\left(-\frac{x^2}{2\theta^2}\right), \quad \text{for } x \geq 0,\ \theta > 0 \tag{15}$$

Derivation and Characteristics of the QTRD

Given the baseline distribution with cdf $G(x)$ in Equation (1), and using Equation (15), the cdf of the QTRD is given by

$$\begin{aligned} G(x) = &\left[1 - \exp\left(\frac{-x^2}{2\theta^2}\right)\right]\left[4\lambda_1 + 6(\lambda_2 - \lambda_1)\left(1 - \exp\left(\frac{-x^2}{2\theta^2}\right)\right)\right. \\ &+ 4(\lambda_1 - 2\lambda_2 + \lambda_3)\left(1 - \exp\left(\frac{-x^2}{2\theta^2}\right)\right)^2 \\ &\left.+ (1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)\left(1 - \exp\left(\frac{-x^2}{2\theta^2}\right)\right)^3\right] \end{aligned} \tag{16}$$

The corresponding pdf is given as

$$\begin{aligned} g(x) = &\left[\frac{x}{\theta^2} \exp\left(\frac{-x^2}{2\theta^2}\right)\right]\left[4\lambda_1 + 12(\lambda_2 - \lambda_1)\left(1 - \exp\left(\frac{-x^2}{2\theta^2}\right)\right)\right. \\ &+ 12(\lambda_1 - 2\lambda_2 + \lambda_3)\left(1 - \exp\left(\frac{-x^2}{2\theta^2}\right)\right)^2 \\ &\left.+ 4(1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)\left(1 - \exp\left(\frac{-x^2}{2\theta^2}\right)\right)^3\right] \end{aligned} \tag{17}$$

where $x > 0, \theta > 0, \lambda_i \in [0, 1]$

Proposition 3

Let X be a QTRD random variable. Then $g(x)$ is a valid pdf of X if and only if

1. $g(x) \geq 0, \quad \forall x$
2. $\int_0^\infty g(x)\,dx = 1$

Proof

1. It can be seen that $g(x) \geq 0$, for all values of x.

2. Also,

$$\int_0^\infty g(x)\,dx = \int_0^\infty \left[\frac{x}{\theta^2}\exp\left(-\frac{x^2}{2\theta^2}\right)\right]\left[4\lambda_1 + 12(\lambda_2 - \lambda_1)\left(1 - \exp\left(-\frac{x^2}{2\theta^2}\right)\right)\right.$$
$$+ 12(\lambda_1 - 2\lambda_2 + \lambda_3)\left(1 - \exp\left(-\frac{x^2}{2\theta^2}\right)\right)^2$$
$$\left.+ 4(1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)\left(1 - \exp\left(-\frac{x^2}{2\theta^2}\right)\right)^3\right] dx$$

Since each term under the integration sign is integrable with respect to x, we proceed to integrate each term individually. In other words, we will evaluate the integral term by term.

Integration of the First Term

$$\int_0^\infty \frac{x}{\theta^2}\exp\left(-\frac{x^2}{2\theta^2}\right)\,dx$$

Let $u = -\frac{x^2}{2\theta^2}$. Then:

$$du = -\frac{x}{\theta^2}\,dx \quad \text{or} \quad dx = -\frac{\theta^2}{x}\,du$$

Substitute du into the integral:

$$\int_0^\infty \frac{x}{\theta^2} \exp\left(-\frac{x^2}{2\theta^2}\right) dx = -\int_0^{-\infty} \exp(u)\, du$$

As x ranges from 0 to ∞, u ranges from 0 to $-\infty$. Hence, the integral becomes:

$$-\int_0^{-\infty} \exp(u)\, du = -\exp(u)\big|_0^{-\infty}$$

Evaluating this:

$$-\exp(u)\big|_0^{-\infty} = -\left(\exp(-\infty) - \exp(0)\right) = -(0 - 1) = 1$$

Thus:

$$\int_0^\infty \frac{x}{\theta^2} \exp\left(-\frac{x^2}{2\theta^2}\right) dx = 1$$

Integration of the Second Term

$$\int_0^\infty \frac{x}{\theta^2} \exp\left(-\frac{x^2}{2\theta^2}\right) \left[1 - \exp\left(-\frac{x^2}{2\theta^2}\right)\right] dx$$

Let $u = -\frac{x^2}{2\theta^2}$. Then:

$$du = -\frac{x}{\theta^2}\, dx \quad \text{or} \quad dx = -\frac{\theta^2}{x}\, du$$

Thus:

$$x\, dx = -\theta^2\, du$$

Substitute into the integral:

$$\int_0^\infty \frac{x}{\theta^2} \exp\left(-\frac{x^2}{2\theta^2}\right) \left[1 - \exp\left(-\frac{x^2}{2\theta^2}\right)\right] dx$$

becomes:

$$-\int_0^{-\infty} e^u \left(1 - e^u\right)\, du$$

Distributing e^u:

$$= -\left[\int_0^{-\infty} e^u\, du - \int_0^{-\infty} e^{2u}\, du\right]$$

Evaluating each integral, we have

$$\int_0^{-\infty} e^u\, du = -e^u\Big|_0^{-\infty} = -(0-1) = 1$$

$$\int_0^{-\infty} e^{2u}\, du = \frac{e^{2u}}{2}\Big|_0^{-\infty} = \frac{0-1}{2} = -\frac{1}{2}$$

Thus:

$$-\left[1 - \left(-\frac{1}{2}\right)\right] = -\left[1 + \frac{1}{2}\right] = -\frac{3}{2}$$

Therefore:

$$\int_0^{\infty} \frac{x}{\theta^2} \exp\left(-\frac{x^2}{2\theta^2}\right)\left[1 - \exp\left(-\frac{x^2}{2\theta^2}\right)\right] dx = \frac{1}{2}$$

Integration of the Third Term

$$\int_0^{\infty} \frac{x}{\theta^2} \exp\left(-\frac{x^2}{2\theta^2}\right)\left[1 - \exp\left(-\frac{x^2}{2\theta^2}\right)\right]^2 dx$$

Let $u = -\frac{x^2}{2\theta^2}$. Then:

$$du = -\frac{x}{\theta^2}\, dx \quad \text{or} \quad dx = -\frac{\theta^2}{x}\, du$$

Thus:

$$x\, dx = -\theta^2\, du$$

Substituting into the integral, we have

$$\int_0^\infty \frac{x}{\theta^2} \exp\left(-\frac{x^2}{2\theta^2}\right)\left[1-\exp\left(-\frac{x^2}{2\theta^2}\right)\right]^2 dx$$

becomes:

$$-\int_0^{-\infty} e^u \left(1-e^u\right)^2 du$$

Expand $(1-e^u)^2$:

$$(1-e^u)^2 = 1-2e^u+e^{2u}$$

Thus:

$$-\int_0^{-\infty} e^u \left(1-2e^u+e^{2u}\right) du$$

Separate the integral:

$$= -\left[\int_0^{-\infty} e^u\, du - 2\int_0^{-\infty} e^{2u}\, du + \int_0^{-\infty} e^{3u}\, du\right]$$

Evaluate each integral:

$$\int_0^{-\infty} e^u\, du = -e^u\big|_0^{-\infty} = -(0-1) = 1$$

$$\int_0^{-\infty} e^{2u}\, du = \frac{e^{2u}}{2}\bigg|_0^{-\infty} = \frac{0-1}{2} = -\frac{1}{2}$$

$$\int_0^{-\infty} e^{3u}\, du = \frac{e^{3u}}{3}\bigg|_0^{-\infty} = \frac{0-1}{3} = -\frac{1}{3}$$

Thus:

$$-\left[1-2\left(-\frac{1}{2}\right)-\frac{1}{3}\right] = -\left[2-\frac{1}{3}\right] = -\frac{6}{3}+\frac{1}{3} = -\frac{5}{3}$$

Therefore:

$$\int_0^{\infty} \frac{x}{\theta^2} \exp\left(-\frac{x^2}{2\theta^2}\right) \left[1 - \exp\left(-\frac{x^2}{2\theta^2}\right)\right]^2 dx = \frac{1}{3}$$

Integration of the Fourth Term

Consider the integral:

$$\int_0^{\infty} \frac{x}{\theta^2} \exp\left(-\frac{x^2}{2\theta^2}\right) \left[1 - \exp\left(-\frac{x^2}{2\theta^2}\right)\right]^3 dx$$

Let $u = -\frac{x^2}{2\theta^2}$. Then:

$$du = -\frac{x}{\theta^2}\, dx \quad \text{or} \quad dx = -\frac{\theta^2}{x}\, du$$

Thus:

$$x\, dx = -\theta^2\, du$$

Substitute into the integral:

$$\int_0^{\infty} \frac{x}{\theta^2} \exp\left(-\frac{x^2}{2\theta^2}\right) \left[1 - \exp\left(-\frac{x^2}{2\theta^2}\right)\right]^3 dx$$

becomes:

$$-\int_0^{-\infty} e^u \left(1 - e^u\right)^3\, du$$

Expand $(1 - e^u)^3$:

$$(1 - e^u)^3 = 1 - 3e^u + 3e^{2u} - e^{3u}$$

Thus:

$$-\int_0^{-\infty} e^u \left(1 - 3e^u + 3e^{2u} - e^{3u}\right) \, du$$

Separate the integral:

$$= -\left[\int_0^{-\infty} e^u \, du - 3\int_0^{-\infty} e^{2u} \, du + 3\int_0^{-\infty} e^{3u} \, du - \int_0^{-\infty} e^{4u} \, du\right]$$

Evaluate each integral:

$$\int_0^{-\infty} e^u \, du = -e^u\big|_0^{-\infty} = -(0-1) = 1$$

$$\int_0^{-\infty} e^{2u} \, du = \left.\frac{e^{2u}}{2}\right|_0^{-\infty} = \frac{0-1}{2} = -\frac{1}{2}$$

$$\int_0^{-\infty} e^{3u} \, du = \left.\frac{e^{3u}}{3}\right|_0^{-\infty} = \frac{0-1}{3} = -\frac{1}{3}$$

$$\int_0^{-\infty} e^{4u} \, du = \left.\frac{e^{4u}}{4}\right|_0^{-\infty} = \frac{0-1}{4} = -\frac{1}{4}$$

Thus:

$$-\left[1 - 3\left(-\frac{1}{2}\right) + 3\left(-\frac{1}{3}\right) - \left(-\frac{1}{4}\right)\right] = -\left[\frac{3}{2} - \frac{1}{4}\right]$$

$$= -\left[\frac{6}{4} - \frac{1}{4}\right] = -\frac{5}{4}$$

Therefore:

$$\int_0^{\infty} \frac{x}{\theta^2} \exp\left(-\frac{x^2}{2\theta^2}\right) \left[1 - \exp\left(-\frac{x^2}{2\theta^2}\right)\right]^3 dx = \frac{1}{4}$$

Now putting all the constants and the integrated values back in the

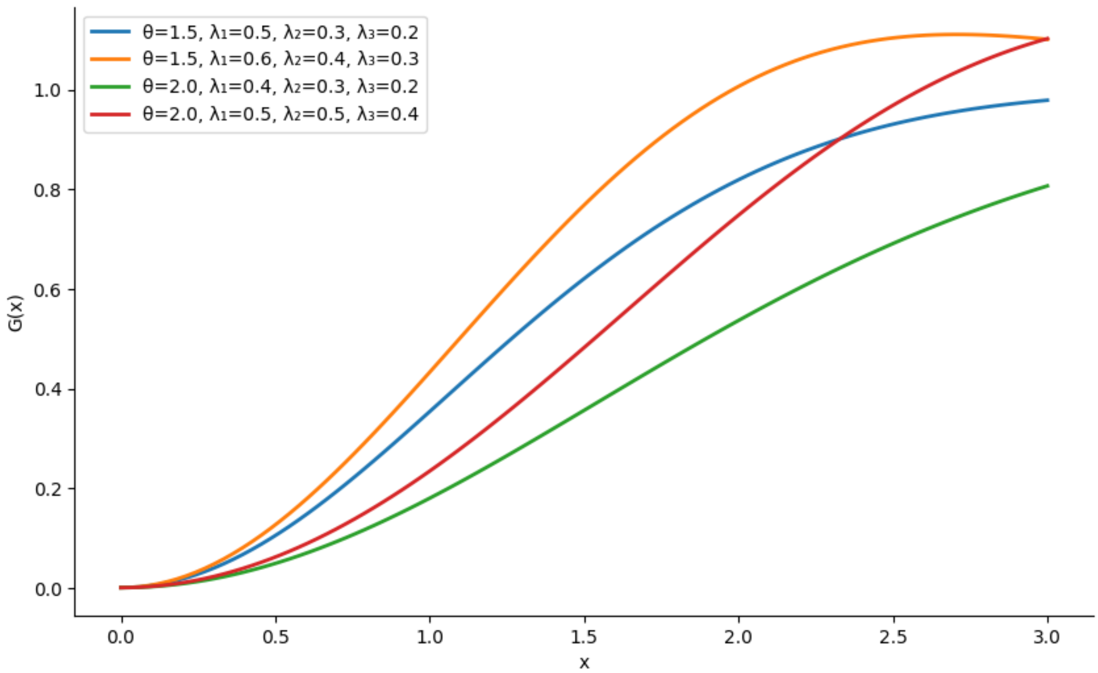
$$\int_o^\infty g(x)$$

we have

$$\begin{aligned}\int_0^\infty g(x)\,dx &= 4\lambda_1 \cdot 1 + 12(\lambda_2 - \lambda_1) \cdot \frac{1}{2} + 12(\lambda_1 - 2\lambda_2 + \lambda_3) \cdot \frac{1}{3} + 4(1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3) \cdot \frac{1}{4} \\ &= 4\lambda_1 + 6(\lambda_2 - \lambda_1) + 4(\lambda_1 - 2\lambda_2 + \lambda_3) + 1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3 \\ &= 4\lambda_1 - 6\lambda_1 + 4\lambda_1 - 2\lambda_1 + 6\lambda_2 - 4\lambda_2 + 2\lambda_2 + 4\lambda_3 - 4\lambda_3 \\ &= \underline{1}\end{aligned}$$

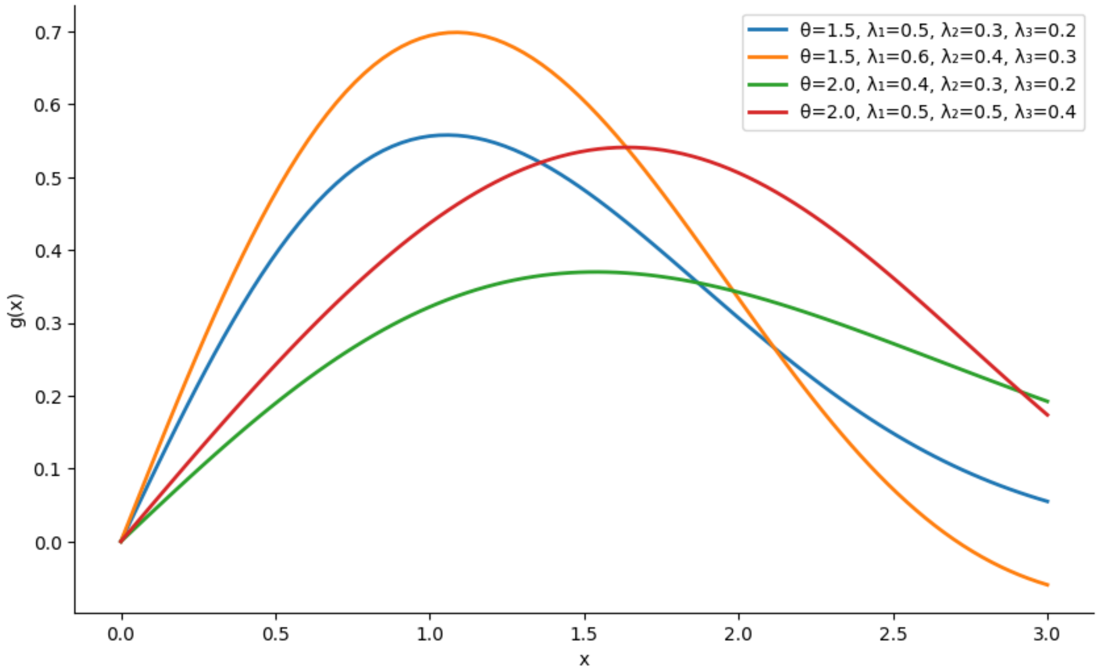
We present the graphical representations of the QTRD through its pdf and cdf. These plots provide a visual understanding of the distribution's behaviour, illustrating how the probability mass is distributed and accumulated across different values of the random variable. The graphs offer insights into the shape, spread, and key characteristics of the QTRD, serving as a foundation for further statistical analysis.

Figure 13: CDF Plot of QTRD



Source: Author, 2023

Figure 14: PDF Plot of QTRD



Source: Author, 2023

Figure 13 shows the cdf plot and the pdf plot of the QTRD for various parameter values. The rising nature of the cdf plot suggests an increasing distribution, underlining the progressive accumulation of probability as the random variable X advances. This rising trend in the cdf further buttresses the understanding that the QTRD distribution exhibits a positively skewed nature, emphasizing its ability to model data with increasing probability as values of X rise. The pdf plot displays QTRD across different parameter values. The observed shape of the plot indicates a positively skewed distribution. The variation in parameters contributes to the distinctive forms of the pdf, illustrating the flexibility of the QTRD in capturing different skewness patterns. This graphical representation offers valuable insights into the characteristics of the distribution under diverse parameter settings, aiding in the interpretation and understanding of its behaviour.

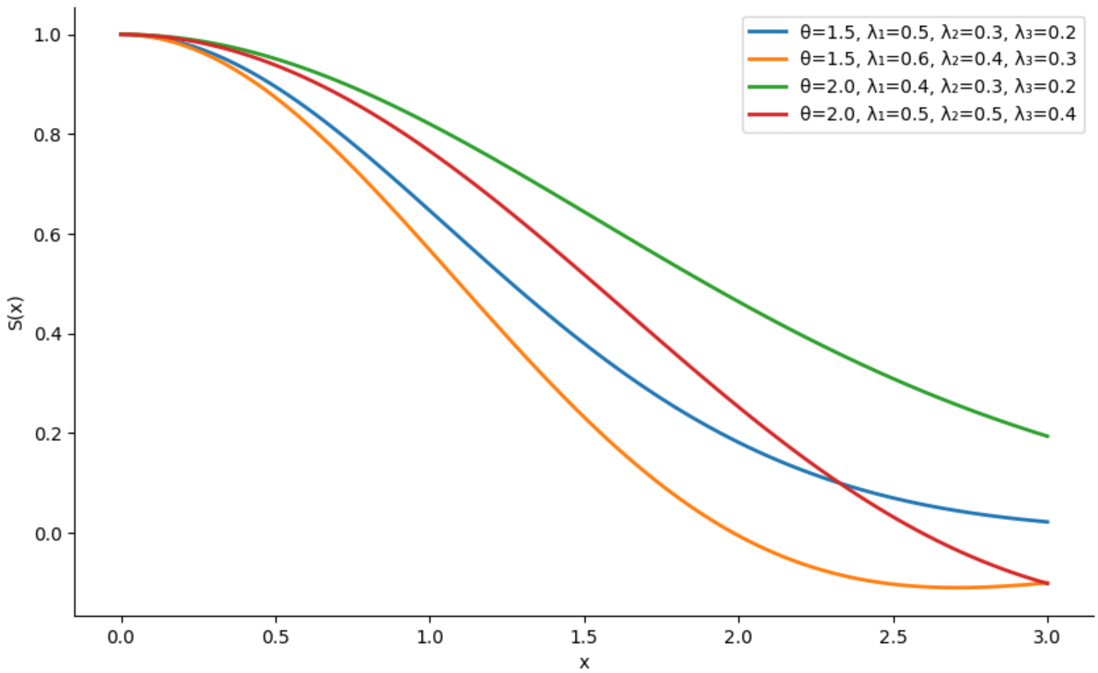
Survival Quantities of the QTRD

In this section, we present the reliability, hazard rate, reversed hazard, cumulative hazard, and odds functions for the quartic transmuted Rayleigh distribution.

Reliability Analysis

$$\begin{aligned}R(x) &= 1 - G(x)\\ &= 1 - \left\{\left[1-\exp\left(\frac{-x^2}{2\theta^2}\right)\right]\left[4\lambda_1 + 6(\lambda_2-\lambda_1)\left(1-\exp\left(\frac{-x^2}{2\theta^2}\right)\right)\right.\right.\\ &\quad +4(\lambda_1-2\lambda_2+\lambda_3)\left(1-\exp\left(\frac{-x^2}{2\theta^2}\right)\right)^2\\ &\quad \left.\left.+(1-2\lambda_1+2\lambda_2-4\lambda_3)\left(1-\exp\left(\frac{-x^2}{2\theta^2}\right)\right)^3\right]\right\}\end{aligned}$$

Figure 15: Survival Plot of QTRD



Source: Author, 2023

The survival plots in Figure 15 illustrate the survival function for the QTRD plotted under different parameter values. These plots showcase the probability of survival beyond various time points, providing insights into the distribution's tail behaviour. The declining nature of the survival curves indicates a decrease in survival probability over time, offering valuable information about the distribution's characteristics in terms of reliability and failure patterns. The variations in the plots demonstrate the impact of different parameter values on the survival function, emphasizing the flexibility of the QTRD in capturing diverse survival behaviours in real-world applications.

Hazard Function

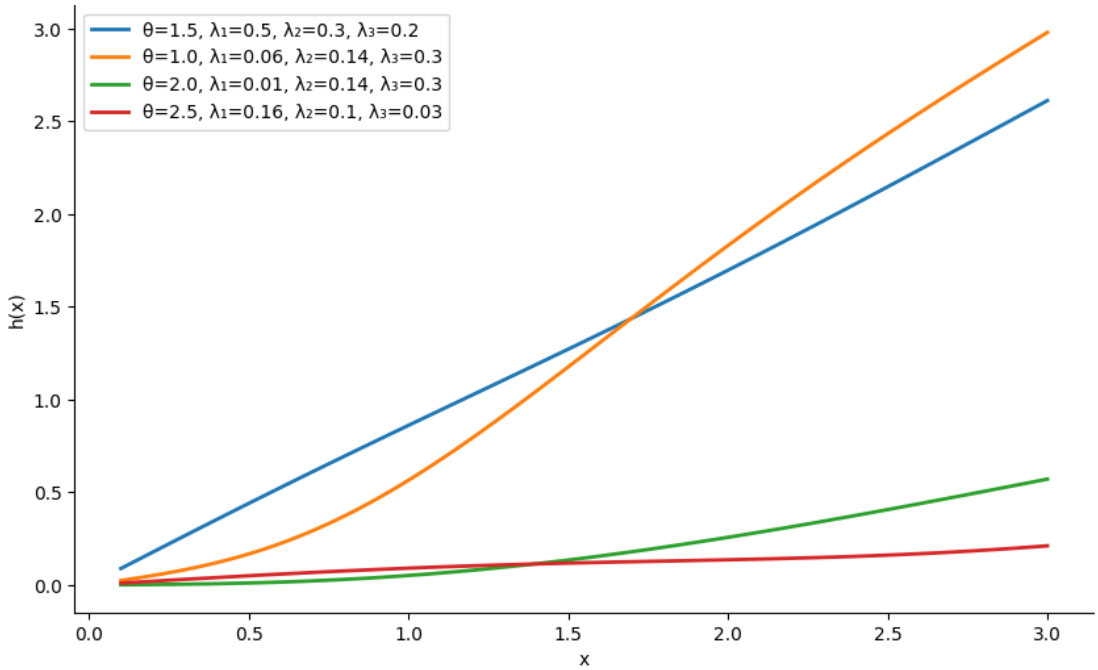
The hazard rate function is obtained mathematically as the ratio of the pdf, $g(x)$ to the reliability function, $G(x)$.

$$h(x) = \frac{\frac{x}{\theta^2} y \left[4\lambda_1 + 12(\lambda_2 - \lambda_1)(1-y) + 12(\lambda_1 - 2\lambda_2 + \lambda_3)(1-y)^2 + 4(1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)(1-y)^3\right]}{1 - (1-y)\left[4\lambda_1 + 6(\lambda_2 - \lambda_1)(1-y) + 4(\lambda_1 - 2\lambda_2 + \lambda_3)(1-y)^2 + (1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)(1-y)^3\right]}$$

where

$$y = \exp\left(\frac{-x^2}{2\theta^2}\right)$$

Figure 16: Hazard Plot of QTRD



Source: Author, 2023

Cumulative Hazard

The cumulative hazard function of the QTRD is defined as

$$\begin{aligned} H(x) &= -\ln\left[1 - G(x)\right] \\ &= -\ln\left[1 - \left\{\left[1 - \exp\left(-\frac{x^2}{2\theta^2}\right)\right]\left[4\lambda_1 + 6(\lambda_2 - \lambda_1)\left(1 - \exp\left(-\frac{x^2}{2\theta^2}\right)\right)\right.\right.\right. \\ &\quad +4(\lambda_1 - 2\lambda_2 + \lambda_3)\left(1 - \exp\left(-\frac{x^2}{2\theta^2}\right)\right)^2 \\ &\quad \left.\left.\left. + (1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)\left(1 - \exp\left(-\frac{x^2}{2\theta^2}\right)\right)^3\right]\right\}\right] \end{aligned}$$

The reversed hazard rate is defined as the ratio of the pdf to the cdf. Thus, the reversed hazard rate is given as:

Using the substitution $y = \exp\left(-\frac{x^2}{2\theta^2}\right)$, the hazard function $RH(x)$ is:

$$RH(x) = \frac{\frac{x}{\theta^2} y \left[4\lambda_1 + 12(\lambda_2 - \lambda_1)(1-y) + 12(\lambda_1 - 2\lambda_2 + \lambda_3)(1-y)^2 + 4(1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)(1-y)^3\right]}{(1-y)\left[4\lambda_1 + 6(\lambda_2 - \lambda_1)(1-y) + 4(\lambda_1 - 2\lambda_2 + \lambda_3)(1-y)^2 + (1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)(1-y)^3\right]}$$

The Odds Function

The odds function of the QTRD is given as

$$O(x) = \frac{\left[(1-y)\left[4\lambda_1 + 6(\lambda_2 - \lambda_1)(1-y) + 4(\lambda_1 - 2\lambda_2 + \lambda_3)(1-y)^2 + (1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)(1-y)^3\right]\right.}{1 - \left[(1-y)\left[4\lambda_1 + 6(\lambda_2 - \lambda_1)(1-y) + 4(\lambda_1 - 2\lambda_2 + \lambda_3)(1-y)^2 + (1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)(1-y)^3\right]\right]}$$

where $y = \exp\left(-\frac{x^2}{2\theta^2}\right)$.

Moment Based Measures of QTRD

This section presents some of the moment properties of the QTRD. These moment properties play a crucial role in characterizing the distribution and understanding its statistical behaviour. We will explore various statistics, such as the mean, variance, skewness, and kurtosis, which provide valuable insights into the shape and central tendencies of the QTRD. Additionally, we will examine other higher-order moments to gain a comprehensive understanding of the distribution's higher statistical moments and their implications in practical applications.

Moment Generating Function

Let X be a QTRD random variable with pdf $g(x)$ defined in equation (4.17). Thus, the moment generating function (MGF), by definition, is given by

$$
\begin{aligned}
M_X(t) = \mathbb{E}\left(e^{tX}\right) &= \int_0^\infty e^{tX} g(x)\,dx \\
&= \int_0^\infty e^{tX} \left[\frac{\theta^2}{2}\exp\left(-\frac{x^2}{2\theta^2}\right)\left\{4\lambda_1 + 12(\lambda_2-\lambda_1)\left[1-\exp\left(-\frac{x^2}{2\theta^2}\right)\right]\right.\right. \\
&\quad +12(\lambda_1 - 2\lambda_2 + \lambda_3)\left[1-\exp\left(-\frac{x^2}{2\theta^2}\right)\right]^2 \\
&\quad \left.\left. +4\left(1-2\lambda_1+2\lambda_2-4\lambda_3\right)\left[1-\exp\left(-\frac{x^2}{2\theta^2}\right)\right]^3\right\}\right] dx
\end{aligned}
$$

Now, since the expressions under the integral signs are integrable, we proceed to integrate it term by term. This approach involves separately integrating each term in the expression, which is possible due to the integrability of each individual term.

$$
\begin{aligned}
M_X(t) = 4\lambda_1 &\int_0^\infty e^{tX}\frac{\theta^2}{2}\exp\left(-\frac{x^2}{2\theta^2}\right)dx \\
&+ 12(\lambda_2-\lambda_1)\int_0^\infty e^{tX}\frac{\theta^2}{2}\exp\left(-\frac{x^2}{2\theta^2}\right)\left[1-\exp\left(-\frac{x^2}{2\theta^2}\right)\right]dx \\
&+ 12(\lambda_1-2\lambda_2+\lambda_3)\int_0^\infty e^{tX}\frac{\theta^2}{2}\exp\left(-\frac{x^2}{2\theta^2}\right)\left[1-\exp\left(-\frac{x^2}{2\theta^2}\right)\right]^2 dx \\
&+ 4\left(1-2\lambda_1+2\lambda_2-4\lambda_3\right)\int_0^\infty e^{tX}\frac{\theta^2}{2}\exp\left(-\frac{x^2}{2\theta^2}\right)\left[1-\exp\left(-\frac{x^2}{2\theta^2}\right)\right]^3 dx
\end{aligned}
$$

Hence,

$$
M_X(t) = \mathbb{E}\left(e^{tX}\right) = \alpha + \beta + \gamma + \zeta
$$

where

$$\alpha = \int_0^\infty 4\lambda_1 e^{tx} \left[\frac{x\theta^2}{2} \exp\left(-\frac{x^2}{2\theta^2}\right)\right] dx.$$

Rewrite x as $\theta^2 t - \theta^2 \left[t - \frac{x}{\theta^2}\right]$ and split the integral:

$$\alpha = \frac{4\lambda_1}{\theta^2} \int_0^\infty xe^{tx} \exp\left(-\frac{x^2}{2\theta^2}\right) dx$$

$$= \frac{4\lambda_1}{\theta^2} \left[\int_0^\infty \theta^2 t - \theta^2 \left[t - \frac{x}{\theta^2}\right] e^{tx} \exp\left(-\frac{x^2}{2\theta^2}\right) dx + \int_0^\infty e^{tx} \exp\left(-\frac{x^2}{2\theta^2}\right) dx\right].$$

Let $u = tx - \frac{x^2}{2\theta^2} \to du = \left(t - \frac{x}{\theta^2}\right) dx$.

Now substituting and integrating, we have:

$$\alpha = \frac{4\lambda_1}{\theta^2} \left[\frac{-\left[\left\{\left(\theta^3 \Gamma\left(\frac{1}{2}, \frac{\theta^2 t^2}{2}\right) - 2\theta^{\frac{7}{2}}\right) t - 2\theta^{\frac{7}{2}}\right\} - \sqrt{2}\theta^2 \Gamma\left(1, \frac{\theta^2 t^2}{2}\right)\right] e^{\frac{\theta^2 t^2}{2}}}{\theta^2}\right].$$

$$\beta = \int_0^\infty \frac{12(\lambda_2 - \lambda_1)}{\theta^2} e^{tx} \left[\frac{x\theta^2}{2} \exp\left(-\frac{x^2}{2\theta^2}\right)\left(1 - \exp\left(-\frac{x^2}{2\theta^2}\right)\right)\right] dx$$

$$= \frac{12(\lambda_2 - \lambda_1)}{\theta^2} \int_0^\infty e^{tx} x \exp\left(-\frac{x^2}{2\theta^2}\right)\left[1 - \exp\left(-\frac{x^2}{2\theta^2}\right)\right] dx.$$

Putting terms over a common denominator, applying linearity, and using integration by substitution, we have:

$$\beta = \frac{12(\lambda_2 - \lambda_1)}{\theta^2} \left[-\theta^3 \Gamma\left(\frac{1}{2}, \frac{\theta^2 t^2}{2}\right) te^{\frac{\theta^2 t^2}{2}} + 2\theta^{\frac{7}{2}} te^{\frac{\theta^2 t^2}{2}} + \theta^2 \Gamma\left(1, \frac{\theta^2 t^2}{2}\right) e^{\frac{\theta^2 t^2}{2}}\right.$$

$$\left. + \theta^3 \Gamma\left(\frac{1}{2}, \frac{\theta^2 t^2}{4}\right) te^{\frac{\theta^2 t^2}{4}} - \theta^{\frac{7}{2}} te^{\frac{\theta^2 t^2}{4}} - \theta^2 \Gamma\left(\frac{1}{2}, \frac{\theta^2 t^2}{4}\right) e^{\frac{\theta^2 t^2}{4}}\right].$$

$$\gamma = \int_0^\infty \frac{12(\lambda_1 - 2\lambda_2 + \lambda_3)}{\theta^2} e^{tx} \left[\frac{x\theta^2}{2} \exp\left(-\frac{x^2}{2\theta^2}\right)\left(1 - \exp\left(-\frac{x^2}{2\theta^2}\right)\right)^2\right] dx$$

$$= \frac{12(\lambda_1 - 2\lambda_2 + \lambda_3)}{\theta^2} \int_0^\infty e^{tx} x \exp\left(-\frac{x^2}{2\theta^2}\right) \left[1 - \exp\left(-\frac{x^2}{2\theta^2}\right)\right]^2 dx.$$

Putting terms over a common denominator, applying linearity, and using integration by substitution, we have:

$$\gamma = \frac{12(\lambda_1 - 2\lambda_2 + \lambda_3)}{\theta^2} \left[-\theta^3 \Gamma\left(\frac{1}{2}, \frac{\theta^2 t^2}{2}\right) t e^{\frac{\theta^2 t^2}{2}} + 2\theta^{\frac{7}{2}} t e^{\frac{\theta^2 t^2}{2}} + \theta^2 \Gamma\left(1, \frac{\theta^2 t^2}{2}\right) e^{\frac{\theta^2 t^2}{2}} \right.$$

$$+\theta^3 \Gamma\left(\frac{1}{2}, \frac{\theta^2 t^2}{4}\right) t e^{\frac{\theta^2 t^2}{4}} - \theta^{\frac{7}{2}} t e^{\frac{\theta^2 t^2}{4}} - \theta^2 \Gamma\left(\frac{1}{2}, \frac{\theta^2 t^2}{4}\right) e^{\frac{\theta^2 t^2}{4}}$$

$$\left. -\theta^3 \Gamma\left(\frac{1}{2}, \frac{\theta^2 t^2}{6}\right) t e^{\frac{\theta^2 t^2}{6}} + 2\theta^{\frac{7}{2}} t e^{\frac{\theta^2 t^2}{6}} + \theta^2 \Gamma\left(1, \frac{\theta^2 t^2}{6}\right) e^{\frac{\theta^2 t^2}{6}} \right].$$

$$\zeta = \int_0^\infty \frac{4(1 - 2\lambda_1 - 2\lambda_2 - 4\lambda_3)}{\theta^2} e^{tx} \left[\frac{x\theta^2}{2} \exp\left(-\frac{x^2}{2\theta^2}\right) \left(1 - \exp\left(-\frac{x^2}{2\theta^2}\right)\right)^3 \right] dx$$

$$= \frac{4(1 - 2\lambda_1 - 2\lambda_2 - 4\lambda_3)}{\theta^2} \int_0^\infty \left[x e^{tx} \exp\left(-\frac{x^2}{2\theta^2}\right) \left(1 - \exp\left(-\frac{x^2}{2\theta^2}\right)\right)^3 \right] dx.$$

Putting terms over a common denominator, applying linearity, and using integration by substitution, we have:

$$\zeta = \frac{4(1 - 2\lambda_1 - 2\lambda_2 - 4\lambda_3)}{\theta^2} \left[-\theta^3 \Gamma\left(\frac{1}{2}, \frac{\theta^2 t^2}{2}\right) t e^{\frac{\theta^2 t^2}{2}} + 2\theta^{\frac{7}{2}} t e^{\frac{\theta^2 t^2}{2}} + \theta^2 \Gamma\left(1, \frac{\theta^2 t^2}{2}\right) e^{\frac{\theta^2 t^2}{2}} \right.$$
$$+ 3\theta^3 \Gamma\left(\frac{1}{2}, \frac{\theta^2 t^2}{4}\right) t e^{\frac{\theta^2 t^2}{4}} - 3\theta^{\frac{7}{2}} t e^{\frac{\theta^2 t^2}{4}} - 3\theta^2 \Gamma\left(\frac{1}{2}, \frac{\theta^2 t^2}{4}\right) e^{\frac{\theta^2 t^2}{4}}$$
$$- \theta^3 \Gamma\left(\frac{1}{2}, \frac{\theta^2 t^2}{6}\right) t e^{\frac{\theta^2 t^2}{6}} + 2\theta^{\frac{7}{2}} t e^{\frac{\theta^2 t^2}{6}} + \theta^2 \Gamma\left(1, \frac{\theta^2 t^2}{6}\right) e^{\frac{\theta^2 t^2}{6}}$$
$$\left. + \theta^3 \Gamma\left(\frac{1}{2}, \frac{\theta^2 t^2}{8}\right) t e^{\frac{\theta^2 t^2}{8}} - \theta^{\frac{7}{2}} t e^{\frac{\theta^2 t^2}{8}} - \theta^2 \Gamma\left(1, \frac{\theta^2 t^2}{8}\right) e^{\frac{\theta^2 t^2}{8}} \right].$$

Now putting all the integrated terms together, we obtain the MGF of the QTRD.

Theorem 4

Let X be a QTRD random variable with pdf $g(x)$ as defined in Equation (4.17). Then the r^{th} moment of X, $\mathbb{E}(X^r)$, is given by:

$$\begin{aligned}\mathbb{E}(X^r) = &-\frac{\Gamma\left(\frac{r+2}{2}\right)\theta^r}{2^{\frac{r}{2}}3^{\frac{r}{2}}}\left[(4\cdot 3^{r/2}+12)\cdot 2^r + \left(-3\cdot 2^{r/2+2}-4\right)\cdot 3^{r/2}\right]\lambda_3\\ &+\left[3^{r/2}\cdot 2^{r+2}+\left(2-3\cdot 2^{r/2+1}\right)\cdot 3^{r/2}\right]\lambda_2\\ &+\left[\left(-3\cdot 2^{r/2+1}-2\right)\cdot 3^{r/2}\right]\lambda_1\\ &+\left[-4\cdot 3^{r/2}-4\right]\cdot 2^r+\left[3\cdot 2^{r/2+1}+1\right]\cdot 3^{r/2}.\end{aligned}$$

where $r+2>0$ and $\theta>0$.

Proof

$$\begin{aligned}\mathbb{E}(X^r) &= \int_0^\infty x^r g(x)\,dx\\ &= \int_0^\infty x^r\left[\frac{\theta^2}{2}\exp\left(-\frac{x^2}{2\theta^2}\right)\left\{4\lambda_1+12(\lambda_2-\lambda_1)\left(1-\exp\left(-\frac{x^2}{2\theta^2}\right)\right)\right.\right.\\ &\quad+12(\lambda_1-2\lambda_2+\lambda_3)\left(1-\exp\left(-\frac{x^2}{2\theta^2}\right)\right)^2\\ &\quad\left.+4\left(1-2\lambda_1+2\lambda_2-4\lambda_3\right)\left(1-\exp\left(-\frac{x^2}{2\theta^2}\right)\right)^3\right\}dx\\ &= \frac{4\lambda_1}{\theta^2}\int_0^\infty x^{r+1}\exp\left(-\frac{x^2}{2\theta^2}\right)dx\\ &\quad+\frac{12(\lambda_2-\lambda_1)}{\theta^2}\int_0^\infty x^{r+1}\exp\left(-\frac{x^2}{2\theta^2}\right)\left(1-\exp\left(-\frac{x^2}{2\theta^2}\right)\right)dx\\ &\quad+\frac{12(\lambda_1-2\lambda_2+\lambda_3)}{\theta^2}\int_0^\infty x^{r+1}\exp\left(-\frac{x^2}{2\theta^2}\right)\left(1-\exp\left(-\frac{x^2}{2\theta^2}\right)\right)^2dx\\ &\quad+\frac{4\left(1-2\lambda_1+2\lambda_2-4\lambda_3\right)}{\theta^2}\int_0^\infty x^{r+1}\exp\left(-\frac{x^2}{2\theta^2}\right)\left(1-\exp\left(-\frac{x^2}{2\theta^2}\right)\right)^3dx.\end{aligned}$$

Term by term integration yields

$$\begin{aligned}\mathbb{E}(X^r) = &-\frac{\Gamma\left(\frac{r+2}{2}\right)\theta^r}{2^{\frac{r}{2}}3^{\frac{r}{2}}}\left[(4\cdot 3^{r/2}+12)\cdot 2^r + \left(-3\cdot 2^{r/2+2}-4\right)\cdot 3^{r/2}\right]\lambda_3\\ &+\left[3^{r/2}\cdot 2^{r+2}+\left(2-3\cdot 2^{r/2+1}\right)\cdot 3^{r/2}\right]\lambda_2\\ &+\left[\left(-3\cdot 2^{r/2+1}-2\right)\cdot 3^{r/2}\right]\lambda_1\\ &+\left[-4\cdot 3^{r/2}-4\right]\cdot 2^r+\left[3\cdot 2^{r/2+1}+1\right]\cdot 3^{r/2}.\end{aligned}$$

Therefore, the first four moments of the QTRD are obtained by setting $r = 1, 2, 3, 4$ into $E(X^r)$.

$$E(X) = -\frac{\sqrt{\pi}\,\theta}{3 \cdot 2^{\frac{3}{2}}}\Bigg[\left(8 \cdot 3^{\frac{3}{2}} - 9 \cdot 2^{\frac{5}{2}} + 12\right)\lambda_3 + \left(30 - 9 \cdot 2^{\frac{3}{2}}\right)\lambda_2 + \left(8\sqrt{3} - 9 \cdot 2^{\frac{3}{2}} + 18\right)\lambda_1 - 8\sqrt{3} + 9 \cdot 2^{\frac{3}{2}} - 21\Bigg]$$

$$E(X^2) = -\frac{\theta^2\left(12\lambda_3 + 18\lambda_2 + 22\lambda_1 - 25\right)}{6}$$

$$E(X^3) = -\frac{\sqrt{\pi}\,\theta^3}{48}\Bigg[\left(2^{\frac{11}{2}} \cdot 3^{\frac{3}{2}} + 63 \cdot 2^{\frac{5}{2}} - 432\right)\lambda_3 + \left(153 \cdot 2^{\frac{3}{2}} - 216\right)\lambda_2 + \left(2^{\frac{11}{2}}\sqrt{3} + 135 \cdot 2^{\frac{3}{2}} - 216\right)\lambda_1 - 2^{\frac{11}{2}}\sqrt{3} - 279\sqrt{2} + 216\Bigg]$$

$$E(X^4) = -\frac{\theta^4\left(300\lambda_3 + 378\lambda_2 + 406\lambda_1 - 415\right)}{18}$$

Hence, the following statistical measures can be obtained.

$$Var(X) = E(X^2) - [E(X)]^2$$

$$\sigma_X = \sqrt{\text{Var}(X)}$$

$$\text{CV} = \frac{\sigma_X}{|\mathbb{E}(X)|}$$

$$\gamma_1 = \frac{\mathbb{E}(X^3) - 3\mathbb{E}(X)\mathbb{E}(X^2) + 2(\mathbb{E}(X))^3}{(\text{Var}(X))^{3/2}}$$

$$\gamma_2 = \frac{\mathbb{E}(X^4) - 4\mathbb{E}(X)\mathbb{E}(X^3) + 6(\mathbb{E}(X))^2\mathbb{E}(X^2) - 3(\mathbb{E}(X))^4}{(\text{Var}(X))^2} - 3$$

Table below presents the statistical measures calculated for various combinations of parameters θ, λ_1, λ_2, and λ_3. These measures include the mean, variance, standard deviation, coefficient of variation, skewness, and kurtosis, highlighting how different parameter settings influence the distribution characteristics.

Table 15: Calculated Statistics for Different Parameter Combinations

Parameters	Mean	Variance	Std. Dev.	CoV	Skewness	Kurt
1.0, 0.2, 0.0, 0.01	1.6907	0.5549	0.7449	0.4406	-0.0522	-0.2836
1.0, 0.2, 0.0, 0.03	1.6796	0.5523	0.7432	0.4425	-0.0325	-0.2875
1.0, 0.2, 0.05, 0.01	1.6432	0.5631	0.7504	0.4567	0.0292	-0.3752
1.0, 0.2, 0.05, 0.03	1.6321	0.5595	0.7480	0.4583	0.0488	-0.3711
1.0, 0.8, 0.0, 0.01	0.8885	0.4239	0.6511	0.7328	1.3892	1.7349
1.0, 0.8, 0.0, 0.03	0.8774	0.4035	0.6352	0.7240	1.4020	1.8739
1.0, 0.8, 0.05, 0.01	0.8410	0.3560	0.5966	0.7094	1.5134	2.5391
1.0, 0.8, 0.05, 0.03	0.8299	0.3345	0.5784	0.6969	1.5097	2.6682
2.5, 0.2, 0.0, 0.01	4.2267	3.4680	1.8623	0.4406	-0.0522	-0.2836
2.5, 0.2, 0.0, 0.03	4.1990	3.4519	1.8579	0.4425	-0.0325	-0.2875
2.5, 0.2, 0.05, 0.01	4.1081	3.5195	1.8760	0.4567	0.0292	-0.3752
2.5, 0.2, 0.05, 0.03	4.0803	3.4967	1.8700	0.4583	0.0488	-0.3711
2.5, 0.8, 0.0, 0.01	2.2213	2.6493	1.6277	0.7328	1.3892	1.7349
2.5, 0.8, 0.0, 0.03	2.1935	2.5219	1.5880	0.7240	1.4020	1.8739
2.5, 0.8, 0.05, 0.01	2.1026	2.2249	1.4916	0.7094	1.5134	2.5391
2.5, 0.8, 0.05, 0.03	2.0749	2.0908	1.4460	0.6969	1.5097	2.6682

Source: Author, 2023

The values provided represent various statistical measures calculated for different combinations of parameters θ, λ_1, λ_2, and λ_3. The mean increases with θ and decreases with higher values of λ_1 while keeping λ_2 constant. Higher values of λ_3 generally lead to slightly lower means. Variance and standard deviation show a similar pattern, generally decreasing as λ_1 increases and λ_2 and λ_3 are varied. Higher values of θ lead to larger variance and standard deviation, suggesting more spread in the data. The coefficient of variation (CV), which measures relative variability, remains relatively stable across different parameter values, hovering around $0.44 - 0.73$. This suggests that the variability of the dataset is consistent relative to the mean, regardless of the specific parameter configuration. Skewness values vary between negative and positive, with lower values of λ_1 and higher values of λ_3 leading to negative skewness, indicating a slight leftward tilt in the distribution. Conversely, higher λ_1 values lead to positive skewness, suggesting a rightward tilt, especially when combined with $\lambda_2 = 0.05$. The kurtosis values are negative for most combinations, indicating a platykurtic distribution (flatter than a normal distribution). This suggests that the distributions are less peaked and have lighter tails compared to a normal distribution. Higher values of λ_1 and λ_3 tend to produce higher kurtosis values, indicating heavier tails and increased outlier presence.

Order Statistics and Quantile Function

Order Statistic

Given the cdf $G(x)$ and pdf $g(x)$, the pdf of the k-th order statistic $X_{(k)}$ is given by:

$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!} [G(x)]^{k-1} [1 - G(x)]^{n-k} g(x)$$

Substituting the given cdf $G(x)$ and pdf $g(x)$:

$$
\begin{aligned}
f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!} &\left\{ \left[1 - \exp\left(\frac{-x^2}{2\theta^2}\right) \right] \right. \\
&\times \left[4\lambda_1 + 6(\lambda_2 - \lambda_1)\left(1 - \exp\left(\frac{-x^2}{2\theta^2}\right)\right) + 4(\lambda_1 - 2\lambda_2 + \lambda_3)\left(1 - \exp\left(\frac{-x^2}{2\theta^2}\right)\right)^2 \right. \\
&+ \left. \left. (1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)\left(1 - \exp\left(\frac{-x^2}{2\theta^2}\right)\right)^3 \right] \right\}^{k-1} \times \left\{ 1 - \left[1 - \exp\left(\frac{-x^2}{2\theta^2}\right) \right] \right. \\
&\times \left[4\lambda_1 + 6(\lambda_2 - \lambda_1)\left(1 - \exp\left(\frac{-x^2}{2\theta^2}\right)\right) + 4(\lambda_1 - 2\lambda_2 + \lambda_3)\left(1 - \exp\left(\frac{-x^2}{2\theta^2}\right)\right)^2 \right. \\
&+ \left. \left. (1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)\left(1 - \exp\left(\frac{-x^2}{2\theta^2}\right)\right)^3 \right] \right\}^{n-k} \\
&\times \left[\frac{x}{\theta^2} \exp\left(\frac{-x^2}{2\theta^2}\right) \right] \times \left[4\lambda_1 + 12(\lambda_2 - \lambda_1)\left(1 - \exp\left(\frac{-x^2}{2\theta^2}\right)\right) \right. \\
&+ 12(\lambda_1 - 2\lambda_2 + \lambda_3)\left(1 - \exp\left(\frac{-x^2}{2\theta^2}\right)\right)^2 \\
&+ \left. 4(1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)\left(1 - \exp\left(\frac{-x^2}{2\theta^2}\right)\right)^3 \right]
\end{aligned}
$$

For the minimum order statistic $X_{(1)}$:

$$
\begin{aligned}
f_{X_{(1)}}(x) &= n\left[1 - G(x)\right]^{n-1} g(x) \\
&= n \left\{ 1 - \left[1 - \exp\left(\frac{-x^2}{2\theta^2}\right) \right] \times \left[4\lambda_1 + 6(\lambda_2 - \lambda_1)\left(1 - \exp\left(\frac{-x^2}{2\theta^2}\right)\right) \right. \right. \\
&\quad + 4(\lambda_1 - 2\lambda_2 + \lambda_3)\left(1 - \exp\left(\frac{-x^2}{2\theta^2}\right)\right)^2 \\
&\quad + \left. \left. (1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)\left(1 - \exp\left(\frac{-x^2}{2\theta^2}\right)\right)^3 \right] \right\}^{n-1} \\
&\quad \times \left[\frac{x}{\theta^2} \exp\left(\frac{-x^2}{2\theta^2}\right) \right] \times \left[4\lambda_1 + 12(\lambda_2 - \lambda_1)\left(1 - \exp\left(\frac{-x^2}{2\theta^2}\right)\right) \right. \\
&\quad + 12(\lambda_1 - 2\lambda_2 + \lambda_3)\left(1 - \exp\left(\frac{-x^2}{2\theta^2}\right)\right)^2 \\
&\quad + \left. 4(1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)\left(1 - \exp\left(\frac{-x^2}{2\theta^2}\right)\right)^3 \right]
\end{aligned}
$$

For the maximum order statistic $X_{(n)}$:

$$
\begin{aligned}
f_{X_{(n)}}(x) &= n\left[G(x)\right]^{n-1} g(x) \\
&= n\left\{\left[1-\exp\left(\frac{-x^2}{2\theta^2}\right)\right] \times \left[4\lambda_1 + 6(\lambda_2-\lambda_1)\left(1-\exp\left(\frac{-x^2}{2\theta^2}\right)\right)\right.\right. \\
&\quad + 4(\lambda_1 - 2\lambda_2 + \lambda_3)\left(1-\exp\left(\frac{-x^2}{2\theta^2}\right)\right)^2 \\
&\quad \left.\left. + \ (1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)\left(1-\exp\left(\frac{-x^2}{2\theta^2}\right)\right)^3\right]\right\}^{n-1} \\
&\quad \times \left[\frac{x}{\theta^2}\exp\left(\frac{-x^2}{2\theta^2}\right)\right] \times \left[4\lambda_1 + 12(\lambda_2-\lambda_1)\left(1-\exp\left(\frac{-x^2}{2\theta^2}\right)\right)\right. \\
&\quad + 12(\lambda_1 - 2\lambda_2 + \lambda_3)\left(1-\exp\left(\frac{-x^2}{2\theta^2}\right)\right)^2 \\
&\quad \left. + 4(1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)\left(1-\exp\left(\frac{-x^2}{2\theta^2}\right)\right)^3\right]
\end{aligned}
$$

Quantile Function

To derive the quantile function from the given cdf $G(x)$ in Equation (4.16) we have;

The quantile function $Q(p)$ is defined such that:

$$G(Q(p)) = p$$

for $p \in [0, 1]$.

To find x in terms of p, set:

$$G(x) = p$$

This gives us:

$$\left[1-\exp\left(-\frac{x^2}{2\theta^2}\right)\right]\times\left[4\lambda_1+6(\lambda_2-\lambda_1)\left(1-\exp\left(-\frac{x^2}{2\theta^2}\right)\right)\right.$$
$$+4(\lambda_1-2\lambda_2+\lambda_3)\left(1-\exp\left(-\frac{x^2}{2\theta^2}\right)\right)^2$$
$$\left.+(1-2\lambda_1+2\lambda_2-4\lambda_3)\left(1-\exp\left(-\frac{x^2}{2\theta^2}\right)\right)^3\right]=p$$

Let $y=1-\exp\left(-\frac{x^2}{2\theta^2}\right)$. This substitution simplifies our equation. Hence, we have:

$$G(x)=y\left[4\lambda_1+6(\lambda_2-\lambda_1)y+4(\lambda_1-2\lambda_2+\lambda_3)y^2+(1-2\lambda_1+2\lambda_2-4\lambda_3)y^3\right]$$

To solve for y, we set

$$y\left[4\lambda_1+6(\lambda_2-\lambda_1)y+4(\lambda_1-2\lambda_2+\lambda_3)y^2+(1-2\lambda_1+2\lambda_2-4\lambda_3)y^3\right]=p$$

This is a polynomial equation in y:

$$(1-2\lambda_1+2\lambda_2-4\lambda_3)y^4+4(\lambda_1-2\lambda_2+\lambda_3)y^3+6(\lambda_2-\lambda_1)y^2+4\lambda_1y-p=0$$

Solving this quartic equation for y can be challenging and therefore numerical methods are be applied to find the roots y.

Once y is found, we can back substitute to find x:

$$y=1-\exp\left(-\frac{x^2}{2\theta^2}\right)$$

Rearranging gives:

$$\exp\left(-\frac{x^2}{2\theta^2}\right)=1-y$$

Taking the natural logarithm:

$$-\frac{x^2}{2\theta^2} = \ln(1-y)$$

Thus:

$$x^2 = -2\theta^2 \ln(1-y)$$

Finally, we have the quantile function:

$$Q(p) = \theta\sqrt{-2\ln(1-y)}$$

Where y is the solution to the quartic polynomial derived from setting $G(x) = p$.

The following table presents the quantiles for the distribution under consideration. These quantiles provide insights into the distribution of the data, illustrating the thresholds below which specific percentages of the data fall.

Table 16: Quantiles of the Distribution

Quantile	Value
10th quantile	0.3254
20th quantile	0.4748
30th quantile	0.6018
40th quantile	0.7220
50th quantile	0.8431
60th quantile	0.9717
70th quantile	1.1163
80th quantile	1.2929
90th quantile	1.5478

Source: Author, 2023

The quantiles presented in the table indicate the distribution of the underlying data. The 10th quantile (0.3254) suggests that 10% of the data falls below this value, indicating a relatively low threshold. As we progress to higher quantiles, such as the 50th (median) at 0.8431 and the 90th quantile at 1.5478, we observe an increasing trend, which suggests that the data is positively skewed. The

90th quantile being significantly higher than the median indicates that there are extreme values present in the upper range of the data. This can be useful for understanding the spread and potential outliers in the dataset.

Entropy Measures

We derive and present the Shannon, Tsallis, and Renyi entropy measures for the given probability density function $g(x)$. These entropy measures provide insights into the uncertainty and information content of the distribution described by $g(x)$.

Shannon Entropy

The Shannon entropy H is:

$$
\begin{aligned}
H &= -\int_0^\infty g(x)\log g(x)\,dx \\
&= -\int_0^\infty \Bigg\{\left[\frac{x}{\theta^2}\exp\left(-\frac{x^2}{2\theta^2}\right)\right]\Bigg[4\lambda_1 + 12(\lambda_2-\lambda_1)\left(1-\exp\left(-\frac{x^2}{2\theta^2}\right)\right) \\
&\quad +12(\lambda_1-2\lambda_2+\lambda_3)\left(1-\exp\left(-\frac{x^2}{2\theta^2}\right)\right)^2 \\
&\quad +4(1-2\lambda_1+2\lambda_2-4\lambda_3)\left(1-\exp\left(-\frac{x^2}{2\theta^2}\right)\right)^3\Bigg] \\
&\quad \times\Bigg[\log\left[\frac{x}{\theta^2}\exp\left(-\frac{x^2}{2\theta^2}\right)\right] \\
&\quad +\log\Bigg[4\lambda_1 + 12(\lambda_2-\lambda_1)\left(1-\exp\left(-\frac{x^2}{2\theta^2}\right)\right) \\
&\quad +12(\lambda_1-2\lambda_2+\lambda_3)\left(1-\exp\left(-\frac{x^2}{2\theta^2}\right)\right)^2 \\
&\quad +4(1-2\lambda_1+2\lambda_2-4\lambda_3)\left(1-\exp\left(-\frac{x^2}{2\theta^2}\right)\right)^3\Bigg]\,dx
\end{aligned}
$$

Tsallis Entropy

The Tsallis entropy H_q is:

$$
\begin{aligned}
H_q &= \frac{1}{q-1}\left(1-\int_0^\infty [g(x)]^q \, dx\right) \\
&= \frac{1}{q-1}\left(1-\int_0^\infty \left[\left[\frac{x}{\theta^2}\exp\left(-\frac{x^2}{2\theta^2}\right)\right]\left[4\lambda_1+12(\lambda_2-\lambda_1)\left(1-\exp\left(-\frac{x^2}{2\theta^2}\right)\right)\right.\right.\right. \\
&\quad +12(\lambda_1-2\lambda_2+\lambda_3)\left(1-\exp\left(-\frac{x^2}{2\theta^2}\right)\right)^2 \\
&\quad \left.\left.+\,4(1-2\lambda_1+2\lambda_2-4\lambda_3)\left(1-\exp\left(-\frac{x^2}{2\theta^2}\right)\right)^3\right]\right]^q dx
\end{aligned}
$$

Renyi Entropy

The Renyi entropy H_α is:

$$
\begin{aligned}
H_\alpha &= \frac{1}{1-\alpha}\log\left(\int_0^\infty [g(x)]^\alpha \, dx\right) \\
&= \frac{1}{1-\alpha}\log\left(\int_0^\infty \left[\left[\frac{x}{\theta^2}\exp\left(-\frac{x^2}{2\theta^2}\right)\right]\left[4\lambda_1+12(\lambda_2-\lambda_1)\left(1-\exp\left(-\frac{x^2}{2\theta^2}\right)\right)\right.\right.\right. \\
&\quad +12(\lambda_1-2\lambda_2+\lambda_3)\left(1-\exp\left(-\frac{x^2}{2\theta^2}\right)\right)^2 \\
&\quad \left.\left.+\,4(1-2\lambda_1+2\lambda_2-4\lambda_3)\left(1-\exp\left(-\frac{x^2}{2\theta^2}\right)\right)^3\right]\right)^\alpha dx
\end{aligned}
$$

Maximum Likelihood Estimation for QTRD

Given the pdf, $g(x)$, the likelihood function for a sample $x_1, x_2, \ldots, x_n$ is:

$$L(\theta, \lambda_1, \lambda_2, \lambda_3) = \prod_{i=1}^{n} g(x_i)$$

The log-likelihood function $\ell(\theta, \lambda_1, \lambda_2, \lambda_3)$ is given by:

$$\ell(\theta, \lambda_1, \lambda_2, \lambda_3) = \log L(\theta, \lambda_1, \lambda_2, \lambda_3) = \sum_{i=1}^{n} \log g(x_i)$$

where

$$\ell(\theta, \lambda_1, \lambda_2, \lambda_3) = \sum_{i=1}^{n} \log \left\{ \left[\frac{x_i}{\theta^2} \exp\left(-\frac{x_i^2}{2\theta^2} \right) \right] \left[4\lambda_1 + 12(\lambda_2 - \lambda_1) \left(1 - \exp\left(-\frac{x_i^2}{2\theta^2} \right) \right) \right. \right.$$
$$+12(\lambda_1 - 2\lambda_2 + \lambda_3) \left(1 - \exp\left(-\frac{x_i^2}{2\theta^2} \right) \right)^2$$
$$\left. \left. + 4(1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3) \left(1 - \exp\left(-\frac{x_i^2}{2\theta^2} \right) \right)^3 \right] \right\}$$

Partial Derivatives

To find the MLE, we compute the partial derivatives of the log-likelihood function with respect to each parameter.

$$\frac{\partial \ell(\theta, \lambda_1, \lambda_2, \lambda_3)}{\partial \theta} = \sum_{i=1}^{n} \frac{\partial \log g(x_i)}{\partial \theta}$$

$$\frac{\partial \ell(\theta, \lambda_1, \lambda_2, \lambda_3)}{\partial \lambda_1} = \sum_{i=1}^{n} \frac{\partial \log g(x_i)}{\partial \lambda_1}$$

$$\frac{\partial \ell(\theta, \lambda_1, \lambda_2, \lambda_3)}{\partial \lambda_2} = \sum_{i=1}^{n} \frac{\partial \log g(x_i)}{\partial \lambda_2}$$

$$\frac{\partial \ell(\theta, \lambda_1, \lambda_2, \lambda_3)}{\partial \lambda_3} = \sum_{i=1}^{n} \frac{\partial \log g(x_i)}{\partial \lambda_3}$$

Setting the partial derivatives to zero and solving, we obtain the MLEs of the parameters and . However, these results cannot be obtained analytically but it can be obtained numerically. Hence, python software is used in this study to estimate the parameters with respect to the available data sets.

Hessian Matrix

The Hessian matrix H is composed of the second partial derivatives of the log-likelihood function.

$$H_{ij} = \frac{\partial^2 \ell(\theta, \lambda_1, \lambda_2, \lambda_3)}{\partial \theta_i \partial \theta_j}$$

The Hessian matrix is:

$$H = \begin{bmatrix} \frac{\partial^2 \ell}{\partial \theta^2} & \frac{\partial^2 \ell}{\partial \theta \partial \lambda_1} & \frac{\partial^2 \ell}{\partial \theta \partial \lambda_2} & \frac{\partial^2 \ell}{\partial \theta \partial \lambda_3} \\ \frac{\partial^2 \ell}{\partial \lambda_1 \partial \theta} & \frac{\partial^2 \ell}{\partial \lambda_1^2} & \frac{\partial^2 \ell}{\partial \lambda_1 \partial \lambda_2} & \frac{\partial^2 \ell}{\partial \lambda_1 \partial \lambda_3} \\ \frac{\partial^2 \ell}{\partial \lambda_2 \partial \theta} & \frac{\partial^2 \ell}{\partial \lambda_2 \partial \lambda_1} & \frac{\partial^2 \ell}{\partial \lambda_2^2} & \frac{\partial^2 \ell}{\partial \lambda_2 \partial \lambda_3} \\ \frac{\partial^2 \ell}{\partial \lambda_3 \partial \theta} & \frac{\partial^2 \ell}{\partial \lambda_3 \partial \lambda_1} & \frac{\partial^2 \ell}{\partial \lambda_3 \partial \lambda_2} & \frac{\partial^2 \ell}{\partial \lambda_3^2} \end{bmatrix}$$

Standard Errors

The standard errors of the estimated parameters are given by the square roots of the diagonal elements of the inverse of the Hessian matrix:

$$\text{SE}(\theta_i) = \sqrt{[\text{inv}(H)]_{ii}}$$

Simulation and Random Number Generation

This section details the methodology employed for simulating random samples from the specified pdf $g(x)$. The simulation process is crucial for evaluating the statistical properties of the distribution, testing hypotheses, and estimating parameters. By generating data that adheres to the distribution, we can assess the behaviour of various statistical methods under controlled conditions.

Probability Density Function

The probability density function used for simulation is given by:

$$\begin{aligned} g(x) = \left[\frac{x}{\theta^2} \exp\left(\frac{-x^2}{2\theta^2}\right)\right] \Bigg[& 4\lambda_1 + 12(\lambda_2 - \lambda_1)\left(1 - \exp\left(\frac{-x^2}{2\theta^2}\right)\right) \\ & + 12(\lambda_1 - 2\lambda_2 + \lambda_3)\left(1 - \exp\left(\frac{-x^2}{2\theta^2}\right)\right)^2 \\ & + 4(1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)\left(1 - \exp\left(\frac{-x^2}{2\theta^2}\right)\right)^3 \Bigg] \end{aligned}$$

This PDF is characterized by the parameters θ, λ_1, λ_2, and λ_3, each of which plays a significant role in shaping the distribution. The goal of the simulation is to generate random samples that follow this distribution for further analysis.

Random Number Generation Methodology

The process of random number generation involves the following steps:

Step 1: Generate Uniform Random Variables

To begin, we generate uniform random variables U_i on the interval $(0,1)$ for $i = 1, 2, \ldots, n$, where n is the desired sample size. These uniform random variables serve as the foundation for generating samples from the target distribution.

Step 2: Inverse Transform Sampling

The inverse transform sampling method is employed to generate random samples from the PDF $g(x)$. This method involves the following sub-steps:

- Compute the cumulative distribution function (CDF) $G(x)$ corresponding to the PDF $g(x)$. The CDF is expressed as:

$$
\begin{aligned}
G(x) = \left[1 - \exp\left(\frac{-x^2}{2\theta^2}\right)\right] \Bigg[& 4\lambda_1 + 6(\lambda_2 - \lambda_1)\left(1 - \exp\left(\frac{-x^2}{2\theta^2}\right)\right) \\
& + 4(\lambda_1 - 2\lambda_2 + \lambda_3)\left(1 - \exp\left(\frac{-x^2}{2\theta^2}\right)\right)^2 \\
& + (1 - 2\lambda_1 + 2\lambda_2 - 4\lambda_3)\left(1 - \exp\left(\frac{-x^2}{2\theta^2}\right)\right)^3 \Bigg]
\end{aligned}
$$

- Solve the equation $G(x) = U_i$ for x, which yields the random samples x_i. Depending on the complexity of $G(x)$, this may require numerical techniques.

Step 3: Data Simulation

With the generated random samples x_i, we simulate datasets of varying sizes. Each dataset represents observations drawn from the distribution described by $g(x)$. These simulated datasets are then used in subsequent analyses, including parameter estimation and model validation.

Implementation Example

The following Python code snippet illustrates the random number generation process:

```
import numpy as np
from scipy.stats import norm

def generate_samples(theta, lambda1, lambda2, lambda3, n):
# Generate uniform random variables
u = np.random.uniform(0, 1, n)

# Define the inverse CDF function
def inverse_cdf(u):
# Implement the inverse CDF transformation here
pass

# Generate samples
samples = inverse_cdf(u)
return samples

# Parameters
theta = 1.0
lambda1 = 1.0
```

```
lambda2 = 1.0
lambda3 = 1.0
n = 1000

# Generate samples
samples = generate_samples(theta, lambda1, lambda2, lambda3,
```

This code outlines the basic structure for generating random samples using the inverse transform sampling method. The `inverse_cdf(u)` function needs to be implemented based on the specific form of the CDF $G(x)$.

Simulation Results and Analysis

The simulated datasets are analyzed to estimate the parameters θ, λ_1, λ_2, and λ_3 of the distribution. The results are used to assess the performance of estimation techniques, validate theoretical findings, and explore the properties of the distribution under different scenarios. A simulation study was conducted by considering samples of size 50, 100, 150, 200, 300, 500, and 800 from the QTRD. A total of 1000 random samples were generated for each set up with the parameters fixed as $\theta = 2.0, \lambda_1 = 0.5, \lambda_2 = 0.5$ and $\lambda_3 = 0.1$. The results are provided in Table 17.

The table presents the estimates, bias, and standard errors (SE) for θ, λ_1, λ_2, and λ_3 for various sample sizes. As the sample size increases, the bias for θ decreases, indicating improved accuracy of the estimates, with bias reducing from 0.1655 at a sample size of 50 to 0.1399 at 800. The estimates for λ_1 exhibit relatively low bias across all sample sizes, with the highest bias being 0.0443 at sample size 200, suggesting good performance in estimating λ_1 even with smaller samples. In contrast, λ_2 shows more pronounced bias, especially for smaller sample sizes, reaching 0.3204 at a sample size of 200, though the bias

Table 17: Bias and SE for Different Sample Sizes

Sample Size	Estimate	Bias	SE
50	1.834466	0.165534	0.483899
	0.489160	0.010840	0.319076
	0.265132	0.234868	0.457043
	0.266455	0.166455	0.466531
100	1.882542	0.117458	0.478251
	0.503512	0.003512	0.303386
	0.217280	0.282720	0.495310
	0.260016	0.160016	0.489344
150	1.819921	0.180079	0.492086
	0.461481	0.038519	0.294978
	0.239273	0.260727	0.448983
	0.200632	0.100632	0.496096
200	1.806124	0.193876	0.491865
	0.455730	0.044270	0.299161
	0.179634	0.320366	0.446360
	0.217083	0.117083	0.475573
300	1.837274	0.162726	0.486249
	0.458164	0.041836	0.290240
	0.208873	0.291127	0.440451
	0.184807	0.084807	0.468604
500	1.792512	0.207488	0.507845
	0.480945	0.019055	0.302790
	0.139555	0.360445	0.443554
	0.213203	0.113203	0.455683
800	1.860068	0.139932	0.487477
	0.471239	0.028761	0.291074
	0.196388	0.303612	0.435872
	0.214715	0.114715	0.455672

Source: Author, 2023

decreases with larger samples. The parameter λ_3 demonstrates relatively low bias throughout, with slight improvements as the sample size increases, reflecting consistent accuracy in its estimates. Additionally, the SE values generally decrease as the sample size grows, indicating enhanced precision of the parameter estimates for larger samples.

Application

In this section, the QTRD is used to fit a real-life data set. This data set has previously been used by Choulakian and Stephens (2001); Merovci and Puka (2014), and Rahman (2019). The data set given in Table 18 contains the exceedances of flood peaks (in m3/s) of the Wheaton River near Carcross in Yukon Territory, Canada. The data consist of 72 exceedances for the years 1958–1984, rounded to one decimal place.

Table 18: Exceedances of Wheaton River Flood Data

1.7	2.2	14.4	1.1	0.4	20.6	5.3	0.7	13.0	12.0
9.3	1.4	18.7	8.5	25.5	11.6	14.1	22.1	1.1	2.5
14.4	1.7	37.6	0.6	2.2	39.0	0.3	15.0	11.0	7.3
22.9	1.7	0.1	1.1	0.6	9.0	7.0	20.1	0.4	14.1
9.9	10.4	10.7	30.0	3.6	5.6	30.8	13.3	4.2	25.5
3.4	11.9	21.5	27.6	36.4	2.7	64.0	1.5	2.5	27.4
1.0	27.1	20.2	16.8	5.3	9.7	27.5	2.5	27.0	1.9
2.8									

Source: Choulakian and Stephens (2001)

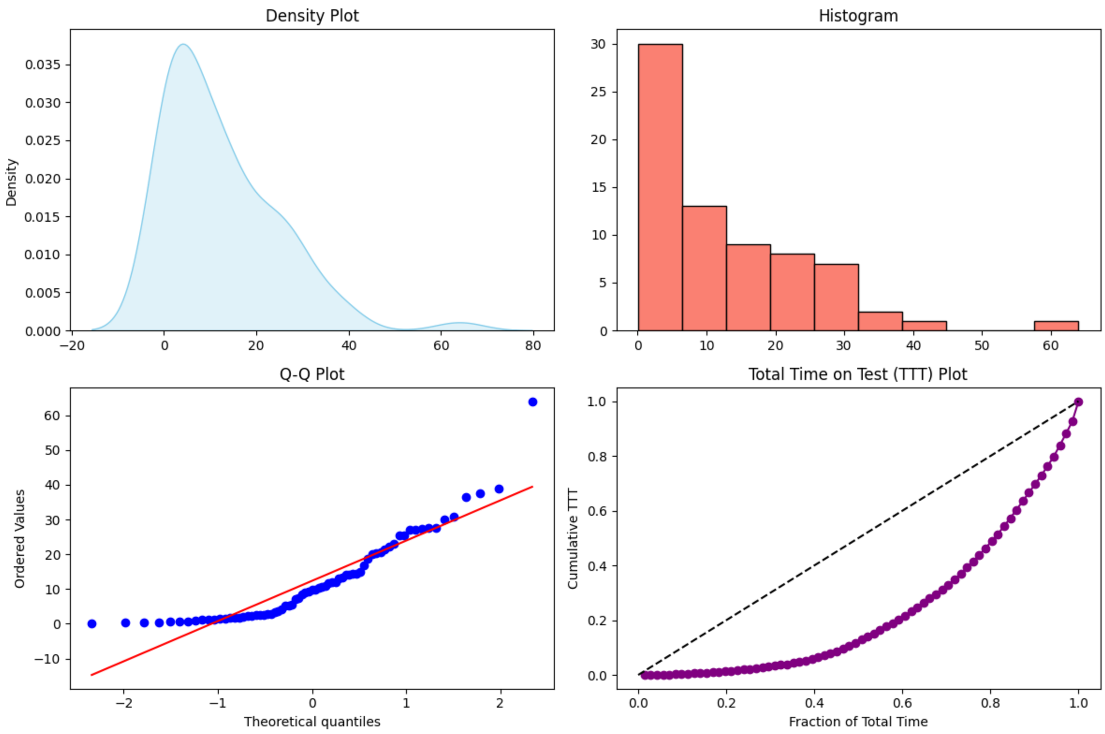
Table 19: Statistical Summary of Wheaton River Flood Data

Min	1st Qu.	Median	Mean	3rd Qu.	Max	St. Dv.	Skew.	Kurt.
0.1	2.125	9.5	12.17	20.125	64.00	12.87	1.44	2.73

Source: Author, 2023

Table 19 presents the descriptive statistics derived from the dataset. A kurtosis value of 2.73 indicates that the dataset's distribution has relatively heavier tails than a normal distribution (which has a kurtosis of 3). This means that the dataset has a higher frequency of extreme values (outliers) compared to a normal distribution. Furthermore, the obtained kurtosis value indicates that the distribution is platykurtic. We now plot the histogram, density and Q-Q plots for the data. These plots collectively offer insights into the characteristics of the exceedances in the Wheaton River flood data, helping to understand the distribution, identify patterns, and assess the fits to theoretical distributions.

Figure 17: Some Statistical Plots of the Exceedance Data



Source: Author, 2023

From the density plot, the data appears to be right-skewed, with most values concentrated on the left. The density curve suggests a possible fit with an exponential or right-skewed distribution. Similar to the density plot, the histogram highlights the concentration of lower values and a long tail on the right, indicating right skewness. The Q-Q Plot compares the quantiles of the sample data against a theoretical normal distribution. The points deviate from the line, especially in the tails, indicating that the data does not follow a normal distribution. The departure in the upper tail further confirms right skewness. The TTT Plot is used to assess the shape of the distribution. The curve is convex, suggesting a distribution with a decreasing failure rate, typical of right-skewed distribution.

The goodness of fit of the QTRD is compared with the following distributions:

1. Rayleigh Distribution given in Equation (10)

2. Cubic Transmuted Rayleigh Distribution (Rahman, 2022):

$$f(x) = \frac{x}{\theta^2} e^{-\frac{3x^2}{2\theta^2}} \left[(1-\lambda) e^{-\frac{x^2}{\theta^2}} + 6\lambda e^{-\frac{x^2}{\theta^2}} - 6\lambda \right],$$

for $\quad x \in \mathbb{R}^+, \theta > 0, \lambda \in [-1, 1]$

3. Transmuted Rayleigh Distribution (Merovci, 2013):

$$f(x) = \frac{x}{\theta^2} \exp\left(-\frac{x^2}{2\theta^2}\right) \left[1 - \lambda + 2\lambda \exp\left(-\frac{x^2}{2\theta^2}\right)\right],$$

for $\quad x \in \mathbb{R}^+,\ \theta > 0,\ \lambda \in [-1, 1]$

In Table 20, the MLE of the parameters for the QTRD are presented alongside those for the comparing probability distributions. The table summarizes the estimated parameters and provides insights into the relative fit of each distribution to the dataset.

Table 20: MLEs of Selected Distributions

Distribution	Parameter	MLE Estimate
QTRD	$\theta, \lambda_1, \lambda_2, \lambda_3$	11.779,1.000, 0.242, 1.000
CTRD	θ, λ	11.951, -1
TRD	θ, λ	13.892, 0.634
Rayleigh	θ	12.909

Source: Author, 2023

In Table 21, we present the results of the Log-Likelihood, AIC, AICc, and BIC for the fitted distributions. The table provides a comprehensive comparison of the goodness-of-fit measures for each distribution, allowing for an assessment of their relative performance in modeling the data.

Standard Errors: $\theta = 0.7708, \lambda1 = 0.8843, \lambda2 = 1.6691, \lambda3 = 0.920$

Table 21: Selection Criteria Values for Selected Distributions

Distribution	Log-Lik	AIC	AICc	BIC
Quartic Transmuted Rayleigh	-214.1189	436.2379	436.8349	445.3445
Cubic Transmuted Rayleigh	-278.26781	574.8770	575.0510	579.4300
Transmuted Rayleigh	-296.56483	595.1300	597.3040	601.6830
Rayleigh	-302.83785	607.6760	607.7330	609.9520

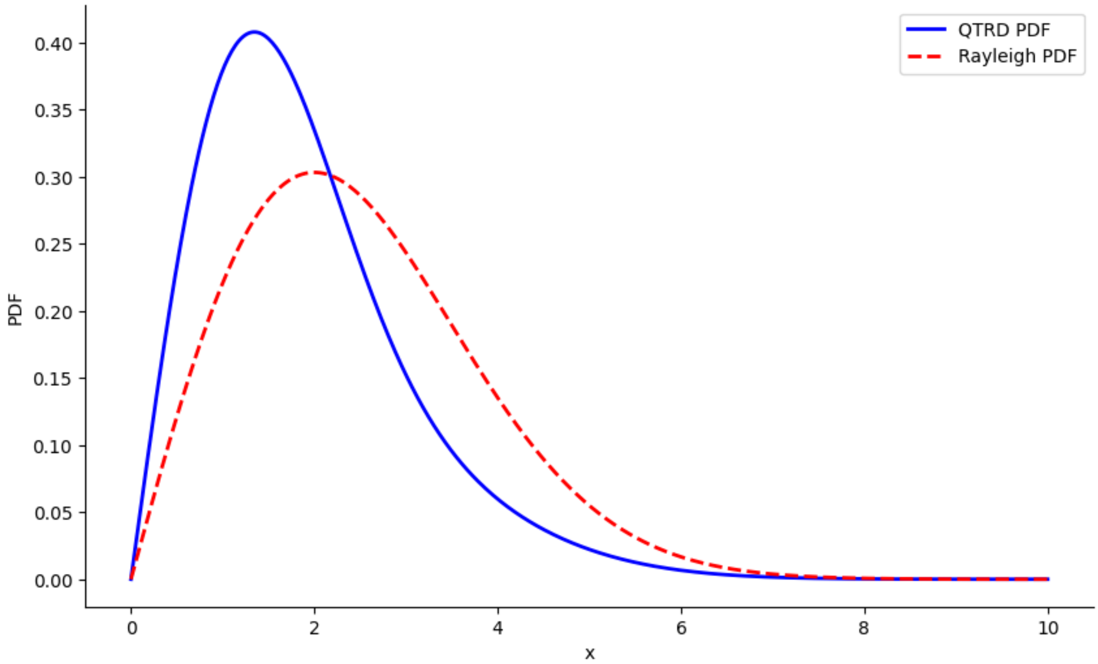
Source: Author, 2023

$$\text{Hessian Matrix}: \begin{bmatrix} 7.47248468 \times 10^{-6} & 1.60613458 \times 10^{-5} & -2.75647741 \times 10^{-3} \\ 1.60613458 \times 10^{-5} & 1.00000000 & -1.20825068 \\ -2.75647741 \times 10^{-3} & -1.20825068 & 2.68022265 \end{bmatrix}$$

Distance Measure between QTRD and Rayleigh Distribution

This section presents an analysis of the divergence between the QTRD and the Rayleigh distribution using several distance measures, including KL Divergence, Hellinger Distance, and Total Variation Distance.

Figure 18: Comparison of QTRD and Rayleigh



Source: Author, 2023

The calculated distance measures reveal the divergence between the QTRD and the Rayleigh distribution. The KL Divergence of 1.4062 indicates a moderate level of divergence between the two distributions, suggesting that QTRD and the Rayleigh distribution are not identical, but the difference is not extreme. The Hellinger Distance of 0.2987 shows a moderate degree of similarity; values closer to 0 would indicate higher similarity, while values around 0.3 suggest some substantial differences. Lastly, the Total Variation Distance of 0.5000, being at its maximum possible value of 1, indicates that there is a considerable difference in the probability mass between the two distributions, reflecting a significant divergence in their overall shapes. These metrics collectively illustrate that while there is some degree of overlap between the distributions, substantial differences remain.

Chapter Summary

In this chapter, we have introduced three novel probability distributions derived from the quartic transmuted distribution: the quartic transmuted exponential, quartic transmuted Rayleigh, and quartic transmuted inverse exponential distributions. We explored their structural properties, including moments and reliability measures, and employed maximum likelihood estimation for parameter estimation. To evaluate the performance of these new distributions, we compared them with existing ones using log-likelihood, AIC, AICc, and BIC metrics.

CHAPTER FIVE

SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS

Overview

This section of the thesis provides a summary, draws insightful conclusions, and offers valuable recommendations based on the findings and analyses presented in the preceding chapters

Summary

In the scope of this thesis, we have introduced three new statistical probability distributions: the quartic transmuted exponential distribution, the quartic transmuted Lindley distribution, and the quartic transmuted Rayleigh distribution. These new distributions were constructed by employing the quartic rank transmutation map. The foundational distributions used for comparison were the exponential distribution, the Lindley distribution, and the Rayleigh distribution. Each of the newly developed distributions is characterized by four distinct parameters. This study incorporates a thorough exploration of these distributions, encompassing various mathematical aspects such as probability density functions, survival and hazard functions, moments, means, variances, entropies, and order statistics. Additionally, visual representations such as cdfs, pdfs, and hazard rate functions were provided to aid in comprehending the distribution characteristics. For each of the developed distributions, an extensive simulation study was undertaken. The outcomes of this study underscored a consistent trend: as the sample size increases, the bias in maximum likelihood estimation diminishes, and the standard error becomes more refined. This simulation-based comparison served as an evaluation of estimator performance. The practical applicability of these newly proposed distributions were demonstrated using real-world datasets. The quartic transmuted exponential distribution was effectively employed to model the lifetime of 50 devices, referencing data from Aarset's study in 1987. Similarly, the quartic transmuted Lindley distribution was adeptly applied to remission times (measured in months) of 128 bladder cancer patients. Finally, the quartic transmuted Rayleigh distribution was successfully utilized to analyze a dataset comprising 72 instances of exceedance from the Wheaton River flood data near Carcoss in Yukon Territory, Canada. Based on the evaluation conducted with these datasets, it became apparent that the probability distributions proposed in this thesis consistently outperformed alternative distributions in terms of their ability fit to the observed data.

Conclusions

A profound understanding of selecting an appropriate statistical distribution for modelling lifetime data is vital across various academic disciplines. The foundation of many parametric inferences in these academic disciplines rests on specific distributional assumptions. However, datasets originating from these fields often deviate from the constraints of classical statistical distributions. Consequently, researchers in distribution theory are actively devising an array of methods to enhance and adapt classical statistical distributions, enabling them to better accommodate diverse datasets parametrically. Against this backdrop, a new statistical distribution generator, named the quartic rank transmuted distribution, has been developed and studied following the concept of the rank transmutation map. This generator was derived and studied with the objective of modifying existing well-established statistical distributions. It was employed to generalize three probability distributions: the Quartic Transmuted Exponential Distribution (QTED), the Quartic Transmuted Lindley Distribution (QTLD), and the Quartic Transmuted Rayleigh Distribution (QTRD). The QTED, for instance, was developed by adapting the exponential distribution as its foundation and then employing the quartic rank transmutation concept. This newly devised distribution was subjected to a comprehensive comparative analysis involving other related distributions, including the Cubic Transmuted Exponential Distribution (CTED), the Transmuted Exponential Distribution (TED), and the Exponential Distribution (ED). A similar approach was followed for the QTLD and QTRD, which were derived from the Lindley and Rayleigh distributions, respectively. The assessment of these distributions relied on various evaluation criteria, such as log-likelihood, Akaike Information Criterion (AIC), corrected AIC (AICc), and Bayesian Information Criterion (BIC). Through rigorous analysis, it was demonstrated that the proposed distributions within this thesis exhibit superior flexibility and performance. This research provides a substantial contribution to the field of distribution theory by introducing new methods for enhancing the adaptability of distributions in diverse applications.

Recommendations

In order to enhance adaptability in diverse applications, the proposed transmuted distributions are highly recommended. This thesis recommends the following:

- Emphasizing the remarkable adaptability and performance of the newly introduced distributions, researchers are encouraged to explore the robustness of these distributions in diverse scenarios. Further investigation into how these distributions respond to different data variations and noise levels could yield valuable insights.
- To further explore the potential of the quartic rank transmuted distributions, it is recommended to employ various estimation techniques. Comparing maximum likelihood estimation with alternative methods such as the method of moments and Bayesian estimation could provide a comprehensive view of the distributions' performance across different scenarios. This would contribute to a more holistic understanding of the distributions' behaviour under various estimation frameworks.

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