UNIVERSITY OF CAPE COAST

HOMOGENIZATION OF ELLIPTIC EQUATIONS IN PERIODIC DOMAINS: THE CASE OF ELLIPTIC EQUATIONS OF THE CURL TYPE

ALBERT LANOR SACKITEY

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BY

ALBERT LANOR SACKITEY

Thesis submitted to the Department of Mathematics of the School of Physical Sciences, College of Agriculture and Natural Sciences, University of Cape Coast, in partial fulfilment of the requirements for the award of Doctor of Philosophy degree in Mathematics

FEBRUARY, 2020

DECLARATION

Candidate's Declaration

I hereby declare that this thesis is the result of my own original research and that no part of it has been presented for another degree in this university or elsewhere.

Candidate's Signature .... Date ....

Name: ALBERT LANOR SACKITEY

Supervisors' Declaration

We hereby declare that the preparation and presentation of the thesis were supervised in accordance with the guidelines on supervision of thesis laid down by the University of Cape Coast.

Principal Supervisor's Signature .... Date ....

Name: PROF. EMMANUEL K. ESSEL

Co-Supervisor's Signature .... Date ....

Name: PROF. ERNEST YANKSON

ABSTRACT

In this thesis, we homogenize elliptic equations in the periodically perforated domain. The two scale convergence method is used in this work for the homogenization. In particular, we homogenize the quasilinear elliptic equation with the dirichlet boundary condition, the time independent incompressible reynolds equation as well as the elliptic equation of the curl type of which the Maxwell type equations is a typical example. We obtain the cell problems and the homogenized equations for the problems which could easily be solved using any numerical method such as matlab or comsol in place of the original problems which contain the fast oscillating parameter ε.

KEY WORDS

Elliptic Equations

Homogenization Theory

Maxwell Type Equations

Multiple Scale Expansion

Reynolds Equations

Two-Scale Convergence

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DEDICATION

To my family

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CHAPTER ONE

INTRODUCTION

This chapter is made up of the background of the study, the statement of the problem, the objectives of this study as well as the organization of the chapters in the thesis

Background to the Study

Composite materials are very widely used nowadays in civil engineering, structural engineering, electrotechnics, sporting equipments, defense and the aerospace industry for spacecrafts. These composites are materials that combine two or more materials (a selected filler or reinforcing elements and compatible matrix binder) that have quite different properties that when combined offer properties which are more desirable than the properties of the individual materials. The different materials work together to give the composite unique properties, but within the composite you can easily see the different materials, they do not dissolve or blend into each other (Olsson, 2008).

They are not a single material but a family of materials whose stiffness, strength, density, and thermal and electrical properties can be tailored. The matrix, the reinforcement material, the volume and shape of the reinforcement, the location of the reinforcement, the fabrication method and others can all be varied to achieve required properties.

Composite materials (fibred, stratified, porous, among others) play an important role in many branches of Mechanics, Physics, Chemistry and Engineering. They are characterised by the fact that they contain two or more finely mixed constituents. They also have in general a 'better behaviour' than the average behaviour of their individual constituents (Defranchi, 1993). They have been used extensively in engineering applications due to their high strength to weight ratios. Natural materials, including wood and human bone tissue, are also composite materials with complex microstructures optimized for withstanding functional loads. For example, the low mass composite structures of bone tissue enable an organism to move efficiently and withstand high structural loads while minimizing metabolic costs. The prevalence of composite structures in nature along with their increasing engineering applications suggest that these materials will become the rule rather than the exception in structural design and analysis. In the era where there is an increase in the application of composite materials, there is the need for accurate yet feasible methods for analyzing composite material mechanics (Hollister & Kikuehi, 1992).

Partial differential equations (PDEs) which govern physical systems such as electrical, engineering, electromagnetism, geometric theory of diffraction etc. are mostly solved using numerical methods. Some of these methods are; the Finite Volume Method, the Finite Element Method, Method of Moments, the Finite Difference Time Domain Method, among many others. These methods are able to solve PDEs of big volume at high frequency and also problems with complicated boundary conditions. However, when the differential operators oscillate rapidly due to the heterogeneity of the material, it requires a long computational time and it is also not able to capture the behaviours within the microscopic structure. The mesh to capture the microscopic behaviour have to be very fine and this makes the solution very costly with respect to the computational power. These rapid oscillations tend to make direct numerical solutions of the PDEs very difficult, sometimes impossible to solve (Alapäa, 2004). In view of these development and to be able to solve these problems, one has to undertake asymptotic analysis or averaging methods which lead to the concept of homogenization. This theory facilitates the analysis of PDEs with rapidly oscillating coefficients (Bensoussan *et al*, 1978). See also Cioranescu and Donato (1999), Essel (2009) and Tartar (2009).

Homogenization is a branch within mathematics that involves the study of partial differential equations (PDEs) with rapidly oscillating coefficients. The homogenization theory is specifically designed to analyze the physics of microstructured materials (Bensoussan et al., 1978; Sanchez-Palencia, 1980).

Much understanding of the macroscopic behaviours of composite materials was introduced by Einstein, Maxwell, and Rayleigh who are themselves physicists (Callister, 2007; Maxwell 1865, 1954). The main problem is to determine effective properties (e.g. heat transfer, elasticity, electric conductivity, magnetic permeability, flow among others) of strongly heterogeneous multiphase materials.

In the seventies, physical problems of material structures both macroscopic and microscopic were formulated. These fomulations were made from a purely mathematical view point which lead to the theory of homogenization. The first results which De Giorgi and Spagnolo obtained around 1970 are very significant in this new dicsipline of mathematics. See De Giorgi and Spagnolo (1973) and Spagnolo (1968, 1976). Since then the theory of homogenization has developed very rapidly with many researchers researching into this distinct discipline in mathematics. See the works of Allaire, (1992); Bensoussan *et al*, (1978); Persson *et al* (1993) and Nandakumaran (2007b).

The motivation of this field was from the study of composite materials, more generally any medium or domain which involves microstructures. The wording is more or less self-explaining: the limit model has no microstructure any more since it was eliminated by letting its size ε tend to zero. Thus, it describes a simpler, homogeneous physical system.

Homogenization in a medium can either be stochastic or deterministic microstructures and therefore is also between stochastic homogenization or deterministic homogenization. An example for stochastic microstructures is foams. A special class of Fibre-reinforced composites of where short fibres are randomly embedded into a matrix material also have stochastic microstructes. Carbon fibre-reinforced ceramic brakes have this kind of microstructure. Important references for stochastic homogenization include Bourgeat et al. (1994), Bensoussan et al. (1978) and Zhikov et al. (1994).

Deterministic homogenization is mostly concerned with periodic homogenization. The homogenization of physical systems where there are periodic microstructure is often referred to as periodic homogenization. Some research contributions of Nguetseng (2003, 2004a, 2004b) and Braides et al. (2009) have however studied certain examples of non-periodic deterministic microstructures.

In simple terms, homogenization is a mathematical procedure to understand heterogeneous materials (or media) with highly oscillating heterogeneties (at the microscopic level) via a homogeneous material. Mathematically, it is a limiting analysis (Nandakumaran, 2007a). The physical problems described on such materials leads to the study of mathematical equations like: differential or integral equations, optimization problems, spectral problems, and so on, will exhibit high oscillations in the coefficients present in the equation or in the domain. This high frequency, oscillations, in turn, will reflect in the solutions. Thus, even if the well posedness of the problems were guaranteed, a numerical computation (to predict the behaviour of such heterogeneous media) of such solutions will be highly non-trivial; in fact, it is almost impossible. The homogenization deals with the study of asymptotic analysis of such solutions and obtain the equation satisfied by the limit. This limit equation will characterize the bulk/ overall behaviour of the material, which does not consist of microscopic heterogenities and can be solved or computed. This solved and computed solution will then be a good approximation, in a suitable sense, to the original solution.

It is the purpose of homogenization theory to describe these limit processes, when ε tends to zero. More precisely, homogenization deals with the asymptotic analysis of Partial Differential Equations of Physics in heterogeneous materials with a periodic structure, when the characteristic length ε of the period tends to zero. Thus, the heterogeneous material appears homogeneous as ε tends to zero as in Figure 1.

Typically, in such materials, the physical parameters (such as conductivity, elasticity coefficients, ...) are discontinuous and oscillate between the different values characterizing each of the components. When these components are intimately mixed, these parameters oscillate very rapidly and the microscopic structure becomes complicated. Analyzing large structures on a microstructural level, however, is clearly an intractable problem.

On the other hand we may think of getting a good approximation of the

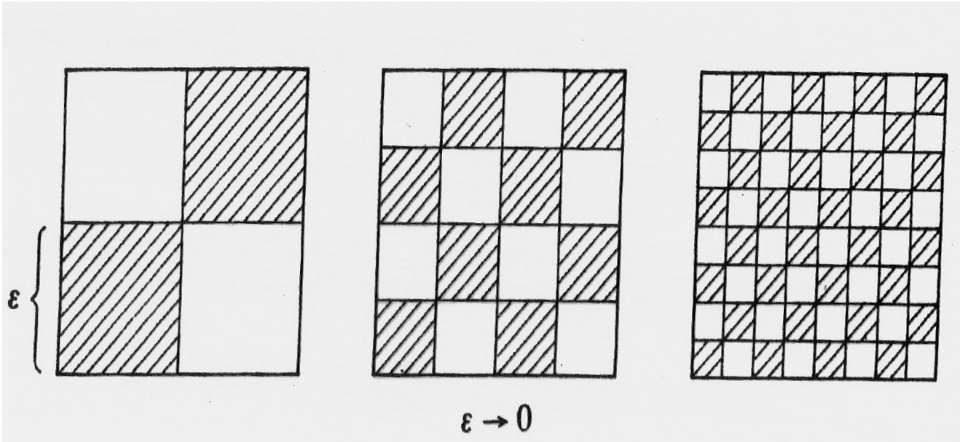

Figure 1: Homogenization as $\varepsilon \to 0$

macroscopic behaviour of such a heterogeneous material by letting the parameter ε (which describes the fineness of the microscopic structure) tend to zero in the equations describing the phenomena such as heat conduction and elasticity.

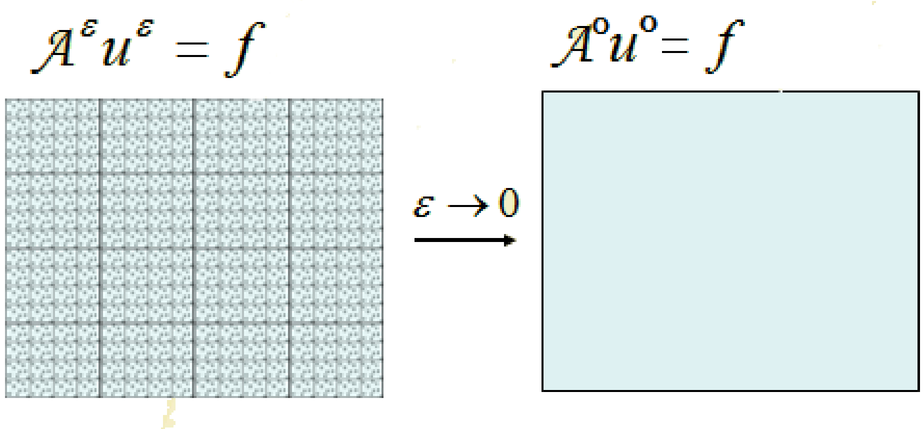

Figure 2: Homogenization of Composite Materials

According to Fabricius and Wall (2008), homogenization is a mathematical theory for studying; differential operators with rapidly oscillating coefficients, equations in perforated domains and also for boundary value problems with rapidly changing boundary conditions.
The classical methods in homogenization are:

- the multiple scale method which consists of looking (formally) for solutions in the form of asymptotic expansions with respect to the parameter ε [See books of Benssousan, Lions and Papanicolaou (1978) and Sanchez-Palencia (1980)],

- the oscillating test functions (also known as energy method) due to Tartar, a mathematical method based on the construction, for each problem to be studied, of appropriate oscillating test functions. The procedure includes a priori estimates and convergence results,

- the two-scale convergence method (Nguetseng and Allaire), based on a new notion of convergence that is tested on special test functions.

These methods are all based on the existence of the two scales; where x gives the position of a point in Ω and $y = x/\varepsilon$ describing what happens in the "magnified" cell Y.

Homogenization techniques include: multiscale convergence, asymptotic expansion method, Energy method via test functions, G-convergence, Fourier (Bloch wave) method, Γ-convergence, periodic unfolding method, p-connectedness, H-convergence, Two Scale (Multi-scale) Convergence, Young measures, compensated compactness and stochastic homogenization (Wall, 2007).

Applications of Homogenization

In this section, we shall name some applications of the homogenization theory to some science and engineering fields. The increased interest in homogenization theory is due the to possible application to other areas in mathematics and other science and engineering fields. Some areas in mathematics that it has been applied to are in perturbation theory, optimal control problems, numerical analysis, phase transitions among others (Persson et al., 1993). For other mathematical analysis of different physical and mechanical phenomenon in composites, perforated media, porous media and similar situations. The results of the homogenization of several types of partial differential equations with rapidly oscillating coefficients have been used in physical and engineering sciences such as heat conduction, elastic deformations, porous media and acoustics.

It has applications in different fields, e.g. composite engineering, material science, geophysics, fluid mechanics, elasticity, viscoelastic multilayered composites, in the analysis of vibrations of thin structures Anzellotti, Baldo and Percivale (1994); Bouchitté and Fragalá (2001); Amaziane, Goncharenko, Pankratov (2005), Friesecke and James (2002); Environmental Science, Mirrahimi and Souganidis (2013). In the study of properties of composites; some of these properties are structural, electro-magnetic, thermal properties etc. Also to the study of flow in porous media these include: flow of resins, water through subsurface, flow of oil, pollution of ground water etc. (Allaire, 1993; Choi et al., 1990).

Furthermore, in oscillating boundary (electromagnetic waves in a domain with a rough interface, flows over rough walls, etc.) (Dasht, 2005), Anisotropic Conduction in Electrocardiology, Kälz (2012).

It has been used extensively to analyze composite materials (Suquet, 1985; Guedes, 1990; Bakhvalov & Panasenko, 1989), and also in predicting optimal topology of microstructured materials (Bendsoe and Kikuchi, 1988; Lurie et al., 1982). In biomechanics, Crolet et al. (1988, 1990) applied the homogenization theory to model cortical bone mechanics. Hollister et al. (1989) applied homogenization analysis to trabecular bone mechanics. In orthopaedics, Hamed, Lee, and Jasiuk (2010) Barkaoui, Chamekh, and Merzouki, 2013; Hage, Shehadeh, Hamade (2014). In Biomedical Sciences Donovan, Chehreghanianzabi, Rathinam and Zustiak (2016) focused on a model where the solute is subjected to obstructed diffusion via stationary spherical obstacles and found that homogenization theory results agree well with computationally more expensive Monte Carlo simulations.

Another field in which homogenization have been applied successfully is in optimal bounds. The theory is often used to design materials with its desired properties. Finding the upper and lower bounds is well studied in homogenization theory by Essel, (2008); Dasht, (2005); Lukkassen, Meidell, and Wall (2007) and Almqvist, Essel, Persson and Wall (2007), the authors concluded that bounds are a very cost-effective method of estimating the effects of surface roughness in stationary hydrodynamic lubrication.

And has in recent times been applied to tribology, Tzandana (2014, 2016); Almqvist, Essel, Larsson and Wall (2007); Almqvist, Essel, Fabricius and Wall (2008a, 2008b, 2011), Almqvist, Essel, Persson and Wall (2007) and Canhanga and Tzandana (2009).

Statement of the Problem

Bensoussan et al., (1978) initiated the theory of homogenization to analyze the physics of microstructured materials and many other researchers have since then delved into this area obtaining many results by many methods. Elliptic equations in different domains and boundary conditions have been studied. For a fixed domain, in a general framework results have been proved for the G-convergence in Colombini and Spagnolo (1977) and for the H-convergence in Boccardo and Murat (1982). It has been extended to the case of perforated domains for the H-convergence of the linearized elasticity system in Donato and Haddadou (2006) who gave a simple proof for the case of a periodic matrix field and a periodically perforated domain using important theorems from Cioranescu and Donato (1999), concerning a domain without holes. Cabarrubias and Donato (2011) gave the existence and uniqueness for a quasilinear elliptic problem with the Robin condition in the periodically perforated domain.

Artola and Duvau (1982) proved that the (quasilinear) homogenized matrix field satisfies the same kind of assumptions as the original problem. For the homogenization of a periodic quasilinear elliptic problem with Lipschitz continuous coefficients in a fixed domain (Cabarrubias & Donato, 2012) was homogenized in the framework of the H-convergence.

For periodically perforated domains, linear equation with linear Robin condition has been studied in Cioranescu and P. Donato (1988) and with nonlinear Robin conditions in Cioranescu, Donato & Zaki (2007). The case of quasilinear elliptic equation with Lipschitz continuous coefficients and linear Robin conditions has been studied in Bendib (2004) and Bendib and Tcheugoué Tébou (1999). Cabarrubias and Donato (2012) prove the existence and the uniqueness of a solution of the problem. Suitable growth conditions are assumed on the nonlinear boundary term, as done in Cioranescu, Donato & Zaki (2007). On the quasilinear term, some assumptions on the modulus of continuity were introduced in Chipot, (2009).

The two-scale convergence method gives a better approximation for boundary value problems with rapidly oscillating coefficients Nguetseng (1989); Cioranescu and Donato (1999). In this study, we use the two-scale convergence to homogenize elliptic equations which earlier have been homogenized using the H-convergence. Also the incompressible reynolds equation time independent when a fluid is flowing through two surfaces where one is stationary and the other smooth and moving which was considered using the multiple scale method in Canhanga and Tzandana (2009).

Many physics and engineering problems are modelled using partial differential equations (PDE's). Their solutions are complicated when solving with numerical methods such as the Finite element method etc. This makes it difficult to solve numerically using computers.

A common feature which renders solving such problems numerically difficult is the occurrence of different length scales associated with such problems. This is where homogenization comes in. These problems could be solved using the theory of homogenization. This theory takes into account the different length scales and using averaging techniques to obtain a homogenized equation that is relatively easy to solve.

Elliptic equations of the divergence form (The Reynolds and Stokes equations) have been homogenized using different methods however not much work have been done on the elliptic equations which are of the curl type.

Objectives of the Study

The objectives of the study are to homogenize elliptic equations using the two-scale convergence method of homogenization. In particular, we will homogenize:

• the time independent reynolds equation

$$\nabla \cdot (h_\varepsilon^3 \nabla p_\varepsilon) = \wedge \frac{\partial h_\varepsilon}{\partial x_1} \quad \text{on} \quad \Omega,$$

• the quasilinear elliptic equation with the dirichlet boundary condition

$$\begin{aligned} -\nabla \cdot (\mathcal{A}^\varepsilon(x, u_\varepsilon)\nabla u_\varepsilon) &= f \;\; \text{for} \;\; x \in \Omega, \\ u_\varepsilon(x) &= 0, \;\; \text{for} \;\; x \in \partial\Omega \end{aligned} \tag{1.1}$$

• the elliptic equation of the curl type

$$\begin{aligned} \nabla_x \times [a^\varepsilon(x)(\nabla_x \times u_\varepsilon(x))] + b_0^\varepsilon u_\varepsilon(x) &= f \;\; \text{in} \;\; \Omega \\ u_\varepsilon(x) &= 0 \;\; \text{on} \;\; \partial\Omega. \end{aligned}$$

Organisation of the Thesis

This section outlines the contents within the chapters of this thesis and give a brief description of what is expected in each chapter.

In Chapter One, the background to the study , the problem statement and the objectives of the study are given. Chapter Two is on the review of literature. Some definitions, lemmas and theorems etc. are given. These include the boundary value problems, periodic functions and the Sobolev space.

In Chapter Three, the methods of homogenization which include: the Multiple Scale expansion, the oscillation test function method, G and H convergences and the two-scale convergence of homogenization are detailed. We also look at the properties of the two-scale convergence, weak convergence and its similarities to other convergences.

In Chapter Four, we give without proof the forms of the Reynold's equation and use the two scale convergence to homogenize the time independent incompressible reynolds equation as well as the quasilinear elliptic equation. The homogenization of the elliptic equation of the curl type is in Chapter Five.

Finally, in Chapter Six, we give a summary of the work and recommendations which will help in future works are drawn in the final chapter. Some of the observations and results were discussed and appropriate conclusions drawn from the observations. We finally give some recommendations which would be necessary for further work.

Chapter Summary

In this chapter, the background to the study was given. Homogenization theory which is the mathematical procedure to understand heterogeneous materials with highly oscillating heterogeneities detailed. A brief statement of the problem as well as the significance, objectives of the study and the organisation of the rest of the chapters in the work were stated.

CHAPTER TWO

LITERATURE REVIEW

Introduction

In this chapter, we give the mathematical definitions, theorems and spaces that will be used in the rest of the work. Also, the two scale convergence of homogenization will be reviewed, its properties, its similarities with the standard convergences i.e. weak and strong convergences and some examples to demonstrate this method.

In physics (as is the case in chemistry, materials science, mechanical, electrical, civil and other engineering disciplines) once a certain effect is observed for the first time, scientists develop theories to explain the effect and employ mathematical models to describe and predict it quantitatively. It is the way of things that the pursuing research on the effect requires the refinement of such theories or even calls for completely new explanations as scientists gain more and more insight into the mechanisms behind the effect. Often, an evolution of theories is triggered by the discovery of smaller, formerly undetected *length scales* (or *time scales*) of a physical system. As a consequence, the corresponding mathematical models have to be adapted to capture the newly discovered scales. While being closer to the actual physical nature, a mathematical model for a physical system that resolves smaller scales is usually more complicated and sometimes even virtually impossible to solve.

Homogenization is restricted to some selected approaches and methods for periodic homogenization that are either very inspiring or microstructured physical systems where both the macroscale and the microscale are continuous.

Homogenization exclusively applies to microstructured physical systems with all the scales; macroscale, microscale and other scale which are all continuous (Stelzig, 2012). Yeung et al. (2009) made an interesting progress concerning discrete-to-continuous limits which shows that one can very well pass from discrete crystalline systems to continuous solids without a priori assuming the crystals atoms to be arranged in some kind of periodic microstructure.

Lymphatic capillary drainage of interstitial fluid under both steady-state and inflammatory conditions is important for tissue fluid balance, cancer metastasis, and immunity. Lymphatic drainage function is critically coupled to the fluid mechanical properties of the interstitium, yet this coupling is poorly understood. Roose and Swartz (2011) effectively modelled the lymphatic-interstitial fluid coupling.Using homogenization method, which allows tissue-scale lymph flow to be integrated with the microstructural details of the lymphatic capillaries, thus gaining insight into the functionality of lymphatic anatomy to first describe flow in lymphatic capillaries using the Navier-Stokes equations and flow through the interstitium using Darcy's law. Then used multiscale homogenization to derive macroscale equations describing lymphatic drainage, with the mouse tail skin as a basis and found that the limiting resistance for fluid drainage is that from the interstitium into the capillaries rather than within the capillaries. Also they observed that between hexagonal, square, and parallel tube configurations of lymphatic capillary networks, the hexagonal structure is the most efficient architecture for coupled interstitial and capillary fluid transport; that is, it clears the most interstitial fluid for a given network density and baseline interstitial fluid pressure. And concluded that using homogenization theory, one can assess how vessel microstructure influences the macroscale fluid drainage by the lymphatics. They demonstrated why the hexagonal network of dermal lymphatic capillaries is optimal for interstitial tissue fluid clearance.

In 2016, Marigo and Maurel used a homogenization method based on matched asymptotic expansion technique to derive effective transmission conditions of thin structured films. The method led unambiguously to effective parameters of the interface which define jump conditions or boundary conditions at an equivalent zero thickness interface. The homogenized interface model in the context of electromagnetic waves for metallic inclusions associated with Neumann or Dirichlet boundary conditions for transverse electric or transverse magnetic wave polarization. By comparison with full-wave simulations, the model is shown to be valid for thin interfaces up to thicknesses close to the wavelength. And also compare the effective conditions with the two-sided impedance conditions obtained in transmission line theory and to the so-called generalized sheet transition conditions.

An alternative notion of two-scale convergence which gives a more natural modelling approach to the homogenization of partial differential equations with periodically oscillating coefficients: while removing the bother of the admissibility of test functions was introduced by Alouges and Di Fratta (2016). It nevertheless simplifies the proof of all the standard compactness results which made classical two-scale convergence very worthy of interest: bounded sequences in $L^2_{per}[Y, L^2(\Omega)]$ and $L^2_{per}[Y, H^1(\Omega)]$ are proven to be relatively compact with respect to this new type of convergence. The strengths of the notion are highlighted on the classical homogenization problem of linear second-order elliptic equations for which first order boundary corrector-type results are also established. Eventually, possible weaknesses of the method are pointed out on a nonlinear problem: the weak two-scale compactness result for $\mathbb{S}^2$-valued stationary harmonic maps.

Using the two-scale homogenization procedure to analyze three dimension composite structures by the finite element method, Otero, et al (2015) compared the results provided by three numerical models (Micro models, Mixing and Homogenization approaches), looking into the strengths and weaknesses of each one of them. It was observed by comparison that for linear analysis, homogenization is an excellent alternative to the other formulations considered. Based on the results obtained, they concluded that the homogenization method is an excellent alternative for the simulation of materials with complex micro structures.

In this section, we define some important functions and spaces which will be relevant for the rest of the thesis.

Dirichlet Problem

A Dirichlet boundary condition is a type of boundary condition when imposed on a partial or ordinary differential equation specifies the values a solution needs to take on the boundary of the domain. For example, $u(x) = 0, \quad x \in \partial\Omega$.

In solving a Dirichlet problem, one needs to find a function which solves a specified partial differential equation in the interior of a given region that takes prescribed values on the boundary of the region.

Smooth Function

A smooth function is a function that has continuous derivatives up to some desired order over some domain. A function can therefore be said to be smooth over a restricted interval such as (a, b) or $[a, b]$. The number of continuous derivatives necessary for a function to be considered smooth may vary from two to infinity. A function for which all orders of derivatives are continuous is called a C^∞ function.

Periodic Functions

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is Y-periodic if $f(x) = f(x + z_i)$ for every $x \in \mathbb{R}^n$ and for $i = 1, 2, ..., n$. In this case, we say that Y is a periodicity cell of the function f and that z_i is the period. If the periodicity cell is the unit cube with $e_1, ..., e_n$ denoting the canonical basis of $\mathbb{R}^n$ then $f : \mathbb{R}^n \to \mathbb{R}^m$ is p-periodic if $f(x) = f(x + e_i)$ for every $x \in \mathbb{R}^n$ and for $i = 1, 2, ..., n$. If $f(x)$ is a function with period p, then $f(ax)$ is periodic with period $\frac{p}{a}$, where a is a positive constant. In general, rapidly oscillating periodic functions converge weakly to their mean value $\mathcal{M}_Y(f)$, where $\mathcal{M}_Y(f)$ is defined by

$$\mathcal{M}_Y(f) = \frac{1}{|Y|} \int_Y f(y)\, dy. \tag{2.1}$$

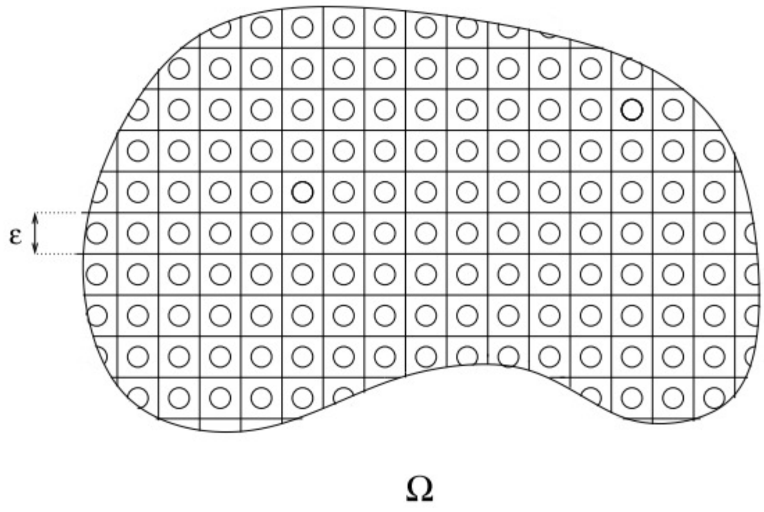

Figure 3: A Periodic Domain Ω

Sobolev Space

Let Ω be an open subset of $\mathbb{R}^n$ and $1 \leq p < \infty$. The Sobolev space $W^{1,p}(\Omega)$ is defined as the set of all functions $u \in L_p$ such that the weak partial derivative, $D^p u$ belongs to $L^p(\Omega)$. That is,

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) : Du \in L^p(\Omega; R^n)\},$$

where $Du = (D_1 u, D_2 u, ..., D_n u) = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_n}\right)$ denotes the first distributional derivative of the function u.

When $p = 2$ we get the $W^{1,2}(\Omega)$ space and is defined as the set of all $u \in L^2(\Omega)$ such that all the first partial derivatives $\frac{\partial u}{\partial x_i} \in L^2(\Omega)$. The exponent $1, 2$ in $W^{1,2}(\Omega)$ means the function u and its first partial derivatives of order 1 are square integrable. Functions belonging to $W^{1,2}(\Omega)$ do not have to be differentiable at every point. For example, it is enough if they are continuous with piecewise continuous partial derivatives in the domain of definition and satisfy the above conditions.

Definition 2.1

Let $1 \leq p < \infty$. $W_0^{1,p}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$. $W^{-1,q}(\Omega)$ with $\frac{1}{p} + \frac{1}{q} = 1$ indicates the dual space of $W_0^{1,p}(\Omega)$.

$W_0^{1,2}(\Omega) = \{u \in W^{1,2}(\Omega) : u \text{ is } 0 \text{ on the boundary } \partial\Omega\}$.

It is the set of all functions u which are elements of the space $W^{1,2}(\Omega)$ where u is zero on the boundary $\partial\Omega$.

In particular, the following hold,

$$W_0^{1,2}(Y) \subset W_{\text{per}}^{1,2}(Y) \subset W^{1,2}(Y) \subset L_2(Y)$$

and

$$W_0^{1,2}(\Omega) \subset W^{1,2}(\Omega) \subset L_2(\Omega) \subset L_1(\Omega).$$

The "$W-$spaces" are often called Sobolev spaces.

In this space the following definitions hold.

1. The inner product in $W^{1,2}(\Omega)$

$$(u,v)_{W^{1,2}(\Omega)} = \int_\Omega (uv + \nabla u \cdot \nabla v)\, d^n x \tag{2.2}$$

where $\nabla u \cdot \nabla v = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}$. If we denote the L_2-inner product by

$$(u,v)_{L_2(\Omega)} = \int_\Omega (uv)\, d^n x$$

then (2.2) reads

$$(u,v)_{W^{1,2}(\Omega)} = (u,v)_{L_2(\Omega)} + (\nabla u, \nabla v)_{L_2(\Omega)} \tag{2.3}$$

2. The norm of the sobolev space is given by

$$\begin{aligned} \|u\|^2_{W^{1,2}(\Omega)} &= (u,u)_{W^{1,2}(\Omega)} = (u,u)_{L_2(\Omega)} + (\nabla u, \nabla u)_{L_2(\Omega)} \\ &= \|u\|^2_{L_2(\Omega)} + \|\nabla u\|^2_{L_2(\Omega)} \\ &= \int_\Omega \left(|u|^2 + |\nabla u|^2\right) d^n x, \end{aligned} \tag{2.4}$$

where $\|\nabla u\|_{L_2(\Omega)} = \left(\sum_{i=1}^{n}\left|\frac{\partial u}{\partial x_i}\right|^2\right)$

3. The distance (or metric)

$$d(u,v)_{W^{1,2}(\Omega)} = \|u-v\|_{W^{1,2}(\Omega)}.$$

Remarks

1. (a) For the general Sobolev space $W^{(k),2}(\Omega)$ the inner product is given by

$$(u,v)_{W^{k,2}(\Omega)} = (u,v)_{L_2(\Omega)} + (u',v')_{L_2(\Omega)} + ... + \left(u^k,v^k\right)_{L_2(\Omega)}$$

   where $u' = \nabla u \quad v' = \nabla v$.

   (b) The norm is given by

$$\|u\|^2_{W^{(k),2}(\Omega)} = (u,u)_{W^{(k),2}(\Omega)}$$

   and the metric is given by

$$d(u,v)_{W^{(k),2}(\Omega)} = \|u-v\|_{W^{(k),2}(\Omega)}.$$

2. The space $W_0^{(k),2}(\Omega)$ is the subspace of $W^{(k),2}(\Omega)$ for which the following hold.

$$\begin{aligned} u(a) &= u'(a) = ... = u^{(k-1)}(a) = 0 \\ u(b) &= u'(b) = ... = u^{(k-1)}(b) = 0. \end{aligned}$$

   That is, u and its $(k-1)$ derivatives are all zero on the boundary $\partial\Omega$.

Dual space $H^{-1}(\Omega)$

$H^{-1}(\Omega)$ defines the dual space of $H_0^1(\Omega)$, i.e. the space of bounded, linear functionals on $H_0^1(\Omega)$. $H^{-1}(\Omega)$ is a Banach space with the norm

$$\|f\|_{H^{-1}(\Omega)} = \sup\{|\langle f, v\rangle| \mid v \in H_0^1(\Omega), \quad \|v\|_{H_0^1(\Omega)} \leq 1\} \tag{2.5}$$

where $\langle \cdot, \cdot \rangle$ denote the pairing between $H_0^1(\Omega)$ and $H_0^1(\Omega)$. Additionally, the following holds

$$|\langle f, v\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}| \leq \|f\|_{H^{-1}}\|v\|_{H_0^1} \quad \forall f \in H^{-1}(\Omega), \ \forall v \in H_0^1(\Omega). \tag{2.6}$$

Sobolev Space $H^1_{per}(Y)$. We denote by $C^\infty_{per}(Y)$ the space of infinitely differentiable functions in $\mathbb{R}^n$ that are 1-periodic. Then the space $H^1_{per}(Y)$ is defined to be the closure of $C^\infty_{per}(Y)$ with respect to the H^1-norm. The Poincaré inequality does not hold in the space $H^1_{per}(Y)$. This is due to the fact, that for constant functions the quantity in the Poincaré inequality will vanish, since the derivative of a constant function is zero. The inequality holds, however, if we add an additional condition that eliminates constant functions. The Poincaré inequality is important in the sense that it builds the framework, in which the Lax-Milgram theorem is applied in order to ensure the existence and uniqueness of solutions to boundary value problems. Hence, we define the following space

$$H = H^1_{per}(Y)/\mathbb{R} = \{u| \ u \in H^1_{per}(Y)| \int_Y u dy = 0\}. \tag{2.7}$$

By H we denote the subset of $H^1(\Omega)$ of all functions u in $H^1_{per}(Y)$ with mean value zero over the unit cell Y. As a consequence, the Poincaré inequality now holds for elements in H, i.e. there exists a constant $C_p > 0$ such that

$$\|u\|_{L^2(Y)} \leq C_p \|\nabla u\|_{L^2(Y)} \quad \forall \ u \in H. \tag{2.8}$$

This means that we can use

$$\|u\|_H = \|\nabla u\|_{L^2(Y)} \quad \forall \, u \in H,$$

as the norm in H.
The dual space H^* of H contains all elements of $(H^1_{per}(Y))^*$, which are orthogonal to constants:

$$H^* = \{u \in (H^1_{per}(Y))^* | \quad \langle u, 1\rangle = 0\} \tag{2.9}$$

where $\langle \cdot, \cdot\rangle$ denotes the pairing between $(H^1_{per}(Y))^*$ and $H^1_{per}(Y)$. We will see that the space $H^1(\Omega)$ and its subsets are appropriate spaces, in which we will look for weak solutions of boundary value problems for second order elliptic partial differential equations. The existence and uniqueness of such solutions is ensured by the Lax-Milgram lemma, which we state now.

Definition 2.2

Let $\mathcal{H}$ be a Hilbert space with norm $\|\cdot\|$ and inner product $(\cdot, \cdot)$.
A bilinear form $B : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is called continuous (or bounded) if there exists a constant $\beta \geq 0$ such that

$$|B(u, v)| \leq \beta\|u\|\|v\| \quad \forall\, u, v \in \mathcal{H}, \tag{2.10}$$

and coercive if there exists a constant $\alpha \geq 0$ such that

$$B(u, u) \geq \alpha\|u\|^2 \quad \forall u \in \mathcal{H}. \tag{2.11}$$

Some Important Definitions and Theorems

Strong Convergence

A sequence $\{u^h\}$ in a normed space X is said to converge strongly, or in norm, to u if $\|u^h - u\|_X \to 0$. This is denoted by $u^h \to u$ in X.

Weak Convergence

A sequence $\{u^h\}$ in a normed space X is said to converge weakly to u if

$$F(u^h) \to F(u) \text{ for all } F \in X', \text{ where } X' \text{ is the dual space of } X.$$

We denote this by $u^h \rightharpoonup u \quad \text{in} \quad X$.

Weak* Convergence

A sequence $\{F^h\}$ in the dual space X' of a normed space is said to converge weakly* to F if

$$F(u^h) \to F(u) \text{ for all } u \in X.$$

We use the notation

$$F(u^h) \overset{*}{\rightharpoonup} F(u) \text{ for all } X'.$$

Theorem 2.1

Let X be a reflexive Banach space. Then every bounded sequence $\{u^h\}$ in X has a weakly convergent subsequence.

Theorem 2.2

Let X be a seperable normed space. Then every bounded sequence $\{F^h\}$ in X' has a weakly* convergent subsequence.

Theorem 2.3

$L^p(\Omega)$ is a Banach space for $1 \leq p \leq \infty$. Furthermore, it is reflexive for $1 < p < \infty$ and seperable for $1 \leq p < \infty$.

Theorem 2.4

For $1 \leq p \leq \infty$, $W^{1,p}(\Omega)$ is a Banach space. It is reflexive for $1 < p < \infty$ and seperable for $1 \leq p < \infty$. Furthermore, the space $W_0^{1,p}(\Omega)$ is reflexive for $1 < p < \infty$, and for $1 \leq p \leq \infty$ it is seperable.

Theorem 2.5

For, $1 \leq p < \infty$, the dual of $L^p(\Omega)$ can be identified with the $L^q(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$. Moreover, $L^1(\Omega)'$ can be identified with $L^\infty(\Omega)$.

Rellich embedding theorem

Let $1 \leq p < \infty$. If $u^h \rightharpoonup u$ in $W^{1,p}(\Omega)$, then $u^h \to u$ in $L^p(\Omega)$. See proof of the theorem on page 285 of Brezis (2011).

Cauchy-Schwarz inequality

For any u, v in a Hilbert space H it holds that

$$|(u, v)_H| \leq \|u\|_H \|v\|_H. \tag{2.12}$$

Poincaré Inequality

There exist a constant C_Ω such that

$$\|u\|_{L^2(\Omega)} \leq C_\Omega \|Du\|_{L^2(\Omega)} \, \forall u \in W_0^{1,2},$$

where C_Ω is a constant depending on the diameter of Ω.

Lemma 2.6

Let Ω be a bounded open set and let $1 \leq p < +\infty$. Then there exists a constant $C > 0$ such that

$$\|u\|_{W_0^{1,2}(\Omega)} \leq C \|Du\|_{L^2(\Omega;\, \mathbb{R}^n)} \qquad \text{for every} \quad u \in W_0^{1,p}(\Omega).$$

Proof

The Sobolev norm is defined by

$$\|u\|_{W_0^{1,p}(\Omega)} = \left(\|u\|_{L^p(\Omega)}^p + \|Du\|_{L^p(\Omega)}^p\right)^{\frac{1}{p}}.$$

Applying the Poincaré inequality to $\|u\|_{L^2(\Omega)}^2$ for $p = 2$ we get

$$\begin{aligned}
\|u\|_{W_0^{1,2}(\Omega)} &\leq \left(C_\Omega^2 \|Du\|_{L^2(\Omega)}^2 + \|Du\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}} \\
&= \left(\left(C_\Omega^2 + 1\right) \|Du\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} \\
&= \left(C_\Omega^2 + 1\right)^{\frac{1}{2}} \|Du\|_{L^2(\Omega)} \\
&= K \|Du\|_{L^2(\Omega;\mathbb{R}^n)}. \text{ Where } K = \left(C_\Omega^2 + 1\right)^{\frac{1}{2}}.
\end{aligned} \tag{2.13}$$

This actually proves that the norm $\|u\|_{W_0^{1,2}(\Omega)}$ and $\|Du\|_{L^2(\Omega;\mathbb{R}^n)}$ are equivalent in $W_0^{1,2}(\Omega)$. From the sobolev norm, we have that

$$\begin{aligned}
\|u\|_{W_0^{1,2}(\Omega)} &= \left(\|u\|_{L^2(\Omega)}^2 + \|Du\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}} \geq \left(\|Du\|_{L^2(\Omega)}^2\right)^{\frac{1}{2}} \\
\Rightarrow \quad \|u\|_{W_0^{1,2}(\Omega)} &\geq \|Du\|_{L^2(\Omega;\mathbb{R}^n)}
\end{aligned} \tag{2.14}$$

Thus from (2.13) and (2.14) we see that the Poincaré inequality implies that

$$\|u\|_{W_0^{1,2}(\Omega)} = \|Du\|_{L^2(\Omega)}. \tag{2.15}$$

This equivalence does not hold in $W^{1,2}(\Omega)$ since for constant functions, the above quantity vanishes. This equivalence also holds for the subspace of functions with mean zero value.

Poincaré-Wirtinger inequality

Let $1 \leq p < \infty$. Then there exists a positive constant C such that

$$\|u - \mathcal{M}_\Omega(u)\|_{L^p(\Omega)} \leq \|Du\|_{L^p(\Omega)^N}, \quad \forall\, u \in W^{1,p}(\Omega) \tag{2.16}$$

where $\mathcal{M}_\Omega(u)$ denotes the integral mean value of u over Ω.

For Ω to satisfy the above condition, it has to be connected. Conversely, there are extra conditions to make this property hold. The simplest of such conditions is that the boundary of Ω be Lipschitz (by the compactness of the Rellich theorem).

Hölder's inequality

Let $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, where $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and Ω is a non-empty, measurable set in $\mathbb{R}^N$. Then

$$\begin{aligned} \left| \int u(x)v(x)\, dx \right| &\leq \int_\Omega |u(x)v(x)| dx \\ &\leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}. \end{aligned} \tag{2.17}$$

Moreover, if $u \in L^1(\Omega)$ and $v \in L^\infty(\Omega)$, then (2.17) for $p = 1$ and $q = \infty$ gives;

$$\int_\Omega |u(x)v(x)|\, dx \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)},$$

where L^q is the dual space of L^p such that $\frac{1}{p} + \frac{1}{q} = 1$ for $1 < p < +\infty$.

For $p = 2$ it is called the Cauchy-Schwarz inequality. Using the Cauchy Schwarz inequality we have

$$\int_\Omega |u(x)v(x)|\, dx \leq \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}.$$

For proof see Section 4.6 of Brezis (2011).

Bilinear form

Let H be a real Hilbert space. A mapping

$$a(\cdot,\cdot) : H \times H \longrightarrow \mathbb{R}$$

is called a bilinear form on H if it is linear in both arguments.

Definition 2.3

Let H be a Hilbert space. A bilinear form a on H is called continuous (or bounded) if there exists a positive constant K such that

$$|a(u,v)| \leq K\|u\|\|v\| \quad \forall\, u, v \in H,$$

and coercive if there exists a positive constant α such that

$$a(u,u) \geq \alpha \|u\|^2 \quad \forall\, u \in H.$$

Note that a *coercive function* is a function that grows rapidly at the extremes of the space on which it is defined. More precisely, a function $f : \mathbb{R}^n \to \mathbb{R}^n$ is called coercive if

$$\frac{f(x)\cdot x}{\|x\|} \to +\infty \text{ as } \|x\| \to +\infty. \tag{2.18}$$

Lax-Milgram Theorem

Let a be a bounded, coercive bilinear form on a Hilbert space H. Then for every bounded linear functional f in H^* there exists a unique element $u \in H$ such that

$$a(u,v) = \langle f, v\rangle \quad \forall v \in H.$$

Lemma 2.7

Let $\xi \in \mathbb{R}^n$. Then

$$|a^{\varepsilon}(x)\xi| = \left|a\left(\frac{x}{\varepsilon}\right)\xi\right| = |a(y)\,\xi| \leq K\,|\xi|$$

for some positive constant K.

Proof

Let A =max$|a_{ij}|$. Then

$$\begin{aligned}
|a^{\varepsilon}(x)\xi|^{2} &= (a_{11}\xi_1 + ...a_{1n}\xi_n)^2 + ... + (a_{n1}\xi_1 + ...a_{nn}\xi_n)^2 \\
&\leq n\left(a_{11}^2\xi_1^2 + ...a_{1n}^2\xi_n^2\right) + ... + n\left(a_{n1}^2\xi_1^2 + ...a_{nn}^2\xi_n^2\right) \\
&\leq n\left(\left(A^2\xi_1^2 + ...A^2\xi_n^2\right) + ... + \left(A^2\xi_1^2 + ...A^2\xi_n^2\right)\right) \\
&= n^2A^2\left(\xi_1^2 + ...\xi_n^2\right) = n^2A^2\left|\xi\right|^2 = K^2\left|\xi\right|^2.
\end{aligned}$$

where $K = nA$. Thus

$$|a^{\varepsilon}(x)\xi| \leq K\left|\xi\right|.$$

This proves boundedness (or continuity) of a^{ε}.

See detailed proof in section 3.3 of Emereuwa (2015).

Riesz representation theorem

Let F be a bounded linear functional on the Hilbert space H (i.e. Let $F \in H'$). Then there is a unique element $u \in H$ such that $F(v) = (u, v)_H$ for every $v \in H$ with $\|F\|'_H = \|u\|_H$.

See proof on page 97 of Brezis (2011).

Variational lemma

Let Ω be a non-empty open set in $\mathbb{R}^N$, let $u \in L^2(\Omega)$ and assume that

$$\int_{\Omega} u(x)v(x)\,dx = 0 \text{ for every } v \in C_0^{\infty}(\Omega). \tag{2.19}$$

Then $u(x) = 0$ for almost every $x \in \Omega$.

Lebesgue's generalised majorized convergence theorem

Let $\Omega \in \mathbb{R}^N$ be a measurable set and $f^h : \Omega \to \mathbb{R}$ be measurable for all h and assume that $\{f^h\}$ converges to f almost everywhere in Ω. Assume also that there are integrable functions $g^h : \Omega \to \mathbb{R}$ such that $|f^h(x)| \leq g^h(x)$ for almost every $x \in \Omega$,

$\{g^h\}$ converges to g almost everywhere in Ω and

$\int_\Omega g^h(x)\, dx \to \int_\Omega g(x)\, dx$.

Then

$$\lim_{h\to\infty} \int_\Omega f^h(x)\, dx = \int_\Omega \lim_{h\to\infty} f^h(x)\, dx.$$

Banach-Alaouglu theorem

Every bounded sequence in X^* has a weak* convergent subsequence whenever X is a Banach space.

For more theorems and proofs we refer the reader to Chapters 4 to 8 of Brezis (2011).

To homogenize the elliptic equations of curl type, we introduce the following lemmas.

Lemma 2.8 (Bensoussan et al, 1978)

Let $u = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ be a vector function with u_1, u_2 and u_3 having continuous second partial derivatives. Then the divergence of the curl of u is zero. That is,

$$\nabla_x \cdot (\nabla_x \times u) = 0. \tag{2.20}$$

$$
\begin{aligned}
& \nabla_x \cdot (\nabla_x \times u) \\
= \; & \nabla_x \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ u_1 & u_2 & u_3 \end{vmatrix} \\
= \; & \nabla_x \cdot \left[\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \right) \mathbf{i} - \left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3} \right) \mathbf{j} + \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) \mathbf{k} \right] \\
= \; & \left(\frac{\partial}{\partial x_1} \mathbf{i} + \frac{\partial}{\partial x_2} \mathbf{j} + \frac{\partial}{\partial x_3} \mathbf{k} \right) \cdot \\
& \left[\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \right) \mathbf{i} - \left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3} \right) \mathbf{j} + \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) \mathbf{k} \right] \\
= \; & \left(\frac{\partial^2 u_3}{\partial x_1 \partial x_2} - \frac{\partial^2 u_2}{\partial x_1 \partial x_3} \right) - \left(\frac{\partial^2 u_3}{\partial x_2 \partial x_1} - \frac{\partial^2 u_1}{\partial x_2 \partial x_3} \right) \\
& + \left(\frac{\partial^2 u_2}{\partial x_3 \partial x_1} - \frac{\partial^2 u_1}{\partial x_3 \partial x_2} \right)
\end{aligned}
$$

Since u_1, u_2 and u_3 are twice differentiable continuous functions we obtain,

$$
\begin{aligned}
\nabla_x \cdot (\nabla_x \times u) &= \left(\frac{\partial^2 u_3}{\partial x_2 \partial x_1} - \frac{\partial^2 u_2}{\partial x_3 \partial x_1} \right) - \left(\frac{\partial^2 u_3}{\partial x_1 \partial x_2} - \frac{\partial^2 u_1}{\partial x_3 \partial x_2} \right) \\
& \quad + \left(\frac{\partial^2 u_2}{\partial x_1 \partial x_3} - \frac{\partial^2 u_1}{\partial x_2 \partial x_3} \right) \\
&= \frac{\partial^2 u_3}{\partial x_2 \partial x_1} - \frac{\partial^2 u_2}{\partial x_3 \partial x_1} - \frac{\partial^2 u_3}{\partial x_1 \partial x_2} + \frac{\partial^2 u_1}{\partial x_3 \partial x_2} + \\
& \quad \frac{\partial^2 u_2}{\partial x_1 \partial x_3} - \frac{\partial^2 u_1}{\partial x_2 \partial x_3} \\
&= \frac{\partial^2 u_1}{\partial x_3 \partial x_2} - \frac{\partial^2 u_1}{\partial x_2 \partial x_3} - \frac{\partial^2 u_2}{\partial x_3 \partial x_1} + \\
& \quad \frac{\partial^2 u_2}{\partial x_1 \partial x_3} + \frac{\partial^2 u_3}{\partial x_2 \partial x_1} - \frac{\partial^2 u_3}{\partial x_1 \partial x_2} \\
&= 0
\end{aligned}
$$

Thus $$\nabla_x \cdot (\nabla_x \times u) = 0. \tag{2.21}$$

Lemma 2.9 (Bensoussan et al, 1978)

Let $u = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ be a vector function with u_1, u_2 and u_3 having continuous second partial derivatives. Then

$$\nabla_y \cdot (\nabla_x \times u) = -\nabla_x \cdot (\nabla_y \times u). \tag{2.22}$$

Proof

$$\begin{aligned}
& \nabla_y \cdot (\nabla_x \times u) \\
= \quad & \nabla_y \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ u_1 & u_2 & u_3 \end{vmatrix} \\
= \quad & \nabla_y \cdot \left[\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right)\mathbf{i} - \left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\right)\mathbf{j} + \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\mathbf{k}\right] \\
= \quad & \left(\frac{\partial}{\partial y_1}\mathbf{i} + \frac{\partial}{\partial y_2}\mathbf{j} + \frac{\partial}{\partial y_3}\mathbf{k}\right) \cdot \left[\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right)\mathbf{i} - \left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\right)\mathbf{j} \right. \\
& \left. + \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\mathbf{k}\right] \\
= \quad & \left(\frac{\partial^2 u_3}{\partial y_1 \partial x_2} - \frac{\partial^2 u_2}{\partial y_1 \partial x_3}\right) - \left(\frac{\partial^2 u_3}{\partial y_2 \partial x_1} - \frac{\partial^2 u_1}{\partial y_2 \partial x_3}\right) + \\
& \left(\frac{\partial^2 u_2}{\partial y_3 \partial x_1} - \frac{\partial^2 u_1}{\partial y_3 \partial x_2}\right)
\end{aligned}$$

Since u_1, u_2 and u_3 are twice differentiable continuous functions we obtain,

$$
\begin{aligned}
&\nabla_y \cdot (\nabla_x \times u) \\
= &\left(\frac{\partial^2 u_3}{\partial x_2 \partial y_1} - \frac{\partial^2 u_2}{\partial x_3 \partial y_1}\right) - \left(\frac{\partial^2 u_3}{\partial x_1 \partial y_2} - \frac{\partial^2 u_1}{\partial x_3 \partial y_2}\right) \\
&+\left(\frac{\partial^2 u_2}{\partial x_1 \partial y_3} - \frac{\partial^2 u_1}{\partial x_2 \partial y_3}\right) \\
= &-\frac{\partial^2 u_3}{\partial x_1 \partial y_2} + \frac{\partial^2 u_2}{\partial x_1 \partial y_3} + \frac{\partial^2 u_3}{\partial x_2 \partial y_1} - \frac{\partial^2 u_1}{\partial x_2 \partial y_3} \\
&+\frac{\partial^2 u_2}{\partial x_3 \partial y_1} - \frac{\partial^2 u_1}{\partial x_3 \partial y_2} \\
= &\frac{\partial}{\partial x_1}\left(-\frac{\partial u_3}{\partial y_2} + \frac{\partial u_2}{\partial y_3}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial u_3}{\partial y_1} - \frac{\partial u_1}{\partial y_3}\right) \\
&+\frac{\partial}{\partial x_3}\left(-\frac{\partial u_2}{\partial y_1} + \frac{\partial u_1}{\partial y_2}\right) \\
= &-\frac{\partial}{\partial x_1}\left(\frac{\partial u_3}{\partial y_2} - \frac{\partial u_2}{\partial y_3}\right) - \frac{\partial}{\partial x_2}\left(-\frac{\partial u_3}{\partial y_1} + \frac{\partial u_1}{\partial y_3}\right) \\
&-\frac{\partial}{\partial x_3}\left(\frac{\partial u_2}{\partial y_1} - \frac{\partial u_1}{\partial y_2}\right) \\
= &-\frac{\partial}{\partial x_1}\left(\frac{\partial u_3}{\partial y_2} - \frac{\partial u_2}{\partial y_3}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial u_1}{\partial y_3} - \frac{\partial u_3}{\partial y_1}\right) \\
&-\frac{\partial}{\partial x_3}\left(\frac{\partial u_2}{\partial y_1} - \frac{\partial u_1}{\partial y_2}\right) \\
= &-\left(\frac{\partial}{\partial x_1}\mathbf{i} + \frac{\partial}{\partial x_2}\mathbf{j} + \frac{\partial}{\partial x_3}\mathbf{k}\right) \cdot \left[\left(\frac{\partial u_3}{\partial y_2} - \frac{\partial u_2}{\partial y_3}\right)\mathbf{i}\right. \\
&\qquad\qquad \left. + \left(\frac{\partial u_1}{\partial y_3} - \frac{\partial u_3}{\partial y_1}\right)\mathbf{j} + \left(\frac{\partial u_2}{\partial y_1} - \frac{\partial u_1}{\partial y_2}\right)\mathbf{k}\right] \\
= &-\left(\frac{\partial}{\partial x_1}\mathbf{i} + \frac{\partial}{\partial x_2}\mathbf{j} + \frac{\partial}{\partial x_3}\mathbf{k}\right) \cdot \left[\left(\frac{\partial u_3}{\partial y_2} - \frac{\partial u_2}{\partial y_3}\right)\mathbf{i}\right. \\
&\qquad\qquad \left. - \left(\frac{\partial u_3}{\partial y_1} - \frac{\partial u_1}{\partial y_3}\right)\mathbf{j} + \left(\frac{\partial u_2}{\partial y_1} - \frac{\partial u_1}{\partial y_2}\right)\mathbf{k}\right] \\
= &-\nabla_x \cdot \left[\left(\frac{\partial u_3}{\partial y_2} - \frac{\partial u_2}{\partial y_3}\right)\mathbf{i} - \left(\frac{\partial u_3}{\partial y_1} - \frac{\partial u_1}{\partial y_3}\right)\mathbf{j}\right. \\
&\qquad \left. + \left(\frac{\partial u_2}{\partial y_1} - \frac{\partial u_1}{\partial y_2}\right)\mathbf{k}\right] \\
= &-\nabla_x \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial y_1} & \frac{\partial}{\partial y_2} & \frac{\partial}{\partial y_3} \\ u_1 & u_2 & u_3 \end{vmatrix} \\
= &-\nabla_x \cdot (\nabla_y \times u).
\end{aligned}
$$

Thus $$\nabla_y \cdot (\nabla_x \times u) = -\nabla_x \cdot (\nabla_y \times u). \tag{2.23}$$

Lemma 2.10

If f is a function of three variables that has continuous second order partial derivatives, then $\nabla \times (\nabla f) = 0$. The curl of the gradient of f is zero.
A conservative vector field is one for which $F = \nabla f$. If F is conservative, then $\nabla \times F = 0$. This gives us a way of verifying that a vector field is not conservative.

Chapter Summary

In this chapter, all the necessary functions and spaces needed for the work were stated. Important definitions, inequalities, lemmas and theorems from functional analysis which will be used were all defined. These include; Sobolev spaces, Riesz representation Theorem and Rellich embedding theorem among others.

CHAPTER THREE

RESEARCH METHODOLOGY

Introduction

In this chapter we give the overview of some methods of homogenization. These include: the two-sacle convergence method, the oscillating test functions method, the multiple scale method and the G and H convergences.

Two-Scale Convergence of Homogenization

On the introduction of homogenization as a branch of mathematics, many methods of solving these problems have been used. In 1989, Nguetseng introduced the two-scale convergence method of homogenization. This method was further developed by Allaire in his works in 1992 and 1994. It is an alternative way of dealing with the classical task of pairing two weakly convergent sequences together in an integral expression under special assumptions on one of the sequences (Flodén, 2009).

When using traditional techniques of compensated compactness type, one has certain conditions on $\{u^\varepsilon\}$ and $\{v^\varepsilon\}$ to obtain that

$$u^\varepsilon(x)v^\varepsilon(x) \longrightarrow u(x)v(x) \quad \text{in } \ D'(\Omega), \tag{3.1}$$

where u and v are the respective weak $L^2(\Omega)^N$-limits.

Two-scale convergence deviates mainly in two ways from this approach. The limit contains an extra scale that reflects certain types of micro-oscillations in $\{u^\varepsilon\}$. These micro-oscillations, which are not captured in the weak limit, are detected by functions $\{v^\varepsilon\}$ designed for this purpose. Hence, only one of the sequences in question needs to obey conditions other than boundedness in $L^2(\Omega)$. This is the other major difference compared to compensated compactness, for which special conditions on the derivatives of $\{u^\varepsilon\}$ and $\{v^\varepsilon\}$ are required.

Two-scale convergence is an important tool particularly in periodic homogenization theory which helps to overcome the problem of passing to the limit on a product of two weakly convergent sequences (Francü, 2010).

Two-scale convergence deals with integrals of the form

$$\int v^{\varepsilon}(x)\psi(x, \frac{x}{\varepsilon})dx,$$

where the sequence $\{v^{\varepsilon}\}$ is bounded in $L^2(\Omega)$ and $\psi(x, \frac{x}{\varepsilon})$ is a smooth function periodic with respect to y.

It enables us to overcome the following problem:

Let u^{ε} and v^{ε} be two sequences weakly converging in $L^2(\Omega)$. What is the limit of their product $u^{\varepsilon}v^{\varepsilon}$ or what is the limit of $\int_{\Omega} u^{\varepsilon}v^{\varepsilon}dx$?

If not more than one sequence converges weakly, then $\lim u^{\varepsilon}v^{\varepsilon} = \lim u^{\varepsilon} \lim v^{\varepsilon}$.

If both sequences converges weakly, then we cannot pass to the limit since the corresponding weak limits do not conserve enough information on the local behaviour of the functions u^{ε} and v^{ε}.

For example, the sequences $u^{\varepsilon} = v^{\varepsilon} = \sin(\frac{x}{\varepsilon})$, $(\varepsilon = 1, \frac{1}{2}, \ldots)$ converges weakly to zero functions but the limits $\lim u^{\varepsilon}v^{\varepsilon} = \frac{1}{2}$ while $\lim u^{\varepsilon} \lim v^{\varepsilon} = 0{\cdot}0 = 0$.

When two-scale convergence was first introduced in Nguetseng (1989), it was a totally new approach in the homogenization of partial differential equations. He proved that in $L^2(\Omega)$, bounded sequences have in a certain weak sense, a limit in $L^2(\Omega \times Y)$, where the second variable defined on Y represents the micro-oscillations of u^{ε} which are averaged away in the weak limit.

Two-scale convergence involves test functions of the form $\psi(x, \frac{x}{\varepsilon})$, which are traces in $L^2(\Omega)$ of $\psi \in L^2(\Omega; C_{per}(Y))$. The properties of ψ are of decisive importance for the two-scale convergence to work. The choice of test functions in the definition of the two-scale limit means that ψ should be Y -periodic in its second variable $y = \frac{x}{\varepsilon}$ for $x \in \Omega$ fixed. However, the function must still be measurable after y is replaced with $\frac{x}{\varepsilon}$.

Definition 3.1 (Test function)

A test function ψ is a smooth function with compact support $\psi \in C_0^\infty(\Omega)$. Test functions are needed for theoretical purposes only.

Definition 3.2 (Admissible Test function)

A function $\psi(x,y) \in L^2(\Omega)$ is called an admissible test function if it satisfies

$$\lim_{\varepsilon \to 0} \int_\Omega [\psi(x, \frac{x}{\varepsilon})]^2 dx = \frac{1}{|Y|} \int_\Omega \int_Y [\psi(x,y)]^2 dy dx. \tag{3.2}$$

Lemma 3.1 (Cioranescu and Donato, 1999)

(i) Let $\varphi \in L^p[\Omega; C_{per}(Y)]$ with $1 \leq p < \infty$. Then $\varphi(\cdot, \frac{\cdot}{\varepsilon}) \in L^p(\Omega)$ with

$$\|\varphi(\cdot, \frac{\cdot}{\varepsilon})\|_{L^p(\Omega)} \leq \|\varphi(\cdot,\cdot)\|_{L^p[(\Omega);C_{per}(Y)]}$$

and

$$\varphi(\cdot, \frac{\cdot}{\varepsilon}) \rightharpoonup \frac{1}{|Y|} \int_Y \varphi(\cdot, y) dy \quad \text{weakly in} \quad L^p(\Omega) \tag{3.3}$$

In particular, if $\varphi \in L^2[\Omega; C_{per}(Y)]$, then

$$\lim_{\varepsilon \to 0} \int_\Omega [\varphi(x, \frac{x}{\varepsilon})]^2 dx = \frac{1}{|Y|} \int_\Omega \int_Y [\varphi(x,y)]^2 dy dx. \tag{3.4}$$

(ii) Suppose that $\varphi(x,y) = \varphi_1(x)\varphi_2(y)$, $\varphi_1 \in L^s(\Omega)$, $\varphi_2 \in L^r_{per}(Y)$ with $1 \leq r, s < \infty$ and such that

$$\frac{1}{r} + \frac{1}{s} = \frac{1}{p},$$

then $\varphi(\cdot, \frac{\cdot}{\varepsilon}) \in L^p(\Omega)$ and

$$\varphi(\cdot, \frac{\cdot}{\varepsilon}) \rightharpoonup \frac{\varphi_1(\cdot)}{|Y|} \int_Y \varphi_2(y) dy \quad \text{weakly in} \quad L^p(\Omega).$$

Theorem 3.2

Assume that $v(x,y) = v_1(x)v_2(y)$, where $v_1 \in L^s(\Omega)$ and $v_2 \in L^t_{per}(Y)$ with $1 \leq s,t \leq \infty$, such that $\frac{1}{s} + \frac{1}{t} = \frac{1}{2}$. Then $v(x, \frac{x}{\varepsilon}) \in L^2(\Omega)$ and

$$v(x, \frac{x}{\varepsilon}) \rightharpoonup v_1(x) \int_Y v_2(y)dy \;\text{ in }\; L^2(\Omega). \tag{3.5}$$

Definition 3.3 (Lukkassen, Nguetseng & Wall, 2002)

Let $\{v^\varepsilon\}$ be a sequence of functions in $L^2(\Omega)$. We say that $\{v^\varepsilon\}$ two-scale converges to $v_0 = v_0(x,y) \in L^2$ and we write $v^\varepsilon \overset{2}{\rightharpoonup} v_0$ if for every admissible test function $\psi = \psi(x, \frac{x}{\varepsilon}) \in C_0^\infty[\Omega; C^\infty_{per}(Y)]$, one has

$$\lim_{\varepsilon \to 0} \int v^\varepsilon(x)\psi(x, \frac{x}{\varepsilon})dx = \frac{1}{|Y|} \int_Y \int_\Omega v_0(x,y)\psi(x,y)dxdy. \tag{3.6}$$

Definition 3.4

Let u^0 be an element of $L^2(\Omega \times Y)$. We say that a sequence u^ε from $L^2(\Omega)$, two-scale converges strongly to u^0 if $u^\varepsilon \overset{2}{\rightharpoonup} u^0$ and in addition,

$$\lim_{\varepsilon \to 0} \int_\Omega |u^\varepsilon(x)|^2 dx = \int_\Omega \int_Y |u^0(x,y)|^2 dydx, \tag{3.7}$$

i.e. $u^\varepsilon \to u^0$.

Lemma 3.3

If $u^\varepsilon \to u^0$ and $v^\varepsilon \rightharpoonup v^0$ where $u^0, v^0 \in L^2(\Omega \times Y)$, then also

$$\lim_{\varepsilon \to 0} \int_\Omega u^\varepsilon(x)v^\varepsilon(x)dx = \int_\Omega \int_Y u^0(x,y)v^0(x,y)dydx. \tag{3.8}$$

Proposition 3.4

Let $\{u^\varepsilon\}$ be a bounded sequence in $L^2(\Omega)$ such that

$$\lim_{\varepsilon \to 0} \int_\Omega u^\varepsilon(x)v^\varepsilon(x)dx = \int_\Omega \int_Y u^0(x,y)v^0(x,y)dydx, \tag{3.9}$$

for every $v \in D(\Omega; C^{\infty}_{per}(Y))$. Then $\{u^{\varepsilon}\}$ two-scale converges to u_0.

A sequence of functions u^{ε} is said to strongly two-scale converge to a limit $u^0(x, y) \in L^p(\Omega \times Y)$ with respect to the scale $\{\varepsilon\}$ if it converges two-scale weakly and moreover,

$$\|u^{\varepsilon}\|_{L^p(\Omega)} \to \frac{1}{|Y|^{1/p}} \|u^0\|_{L^p(\Omega \times Y)}. \tag{3.10}$$

Basic properties of two-scale convergence

Let X be a finite dimensional normed space, u_{ε} a sequence in $L^p(\Omega; X)$, and $U \in L^p(\mathbb{R}^N \times Y; X)$ with $U = 0$ almost everywhere outside $\Omega \times Y$and $1 < p < \infty$.

1. If u_{ε} is two-scale weakly convergent in $L^p(\Omega \times Y; X)$ as $\varepsilon \to 0$ then it is bounded in $L^p(\Omega; X)$ as $\varepsilon \to 0$.

2. If u_{ε} is bounded in $L^p(\Omega; X)$ then u_{ε} is two-scale weakly convergent in $L^p(\Omega \times Y; X)$ along a suitable sequence $\varepsilon_n \to 0$.

3. $u_{\varepsilon} \xrightarrow{2} U$ two-scale strongly in $L^p(\Omega \times Y; X)$ as $\varepsilon \to 0$ if and only if $u_{\varepsilon} \xrightharpoonup{2} U$ two-scale weakly converges in $L^p(\Omega \times Y; X)$ and $\|u_{\varepsilon}\|_{L^p(\Omega; X)} \to \|u_{\varepsilon}\|_{L^p(\mathbb{R}^N \times Y; X)}$ as $\varepsilon \to 0$.

4. If $u_{\varepsilon} \xrightharpoonup{2} U$ two-scale weakly in $L^p(\Omega \times Y; X)$ as $\varepsilon \to 0$ then

$$\lim_{\varepsilon \to 0} \int_{\Omega} u_{\varepsilon}(x) \psi(x, \frac{x}{\varepsilon}) dx = \int_{\Omega} \int_{Y} U(x, y) \psi(x, y) dx dy,$$

for every $\psi \in L^q(\Omega; C^0_{per}(\overline{Y}; X))$.

5. If $u_{\varepsilon} \xrightharpoonup{2} U$ two-scale weakly converges in $L^p(\Omega \times Y)$ and $v_{\varepsilon} \xrightarrow{2} V$ two-scale strongly converges in $L^q(\Omega \times Y)$,

$$\lim_{\varepsilon \to 0} \int_{\Omega} u_{\varepsilon} v_{\varepsilon} dx = \int_{\Omega} \int_{Y} UV dx dy.$$

6. The weak or strong two-scale limit u^0 is unique as an element of $L^p(\Omega \times Y)$.

7. The weak or strong two-scale convergence of u_ε to $u^0(x,y)$ implies weak convergence in $L^p(\Omega)$ of u^ε to the limit $u^*(x) = \frac{1}{|Y|}\int_Y u^0(x,y)dy$.

Remark

(i) $C_0^\infty[\Omega; C_{per}^\infty(Y)]$ is dense in $L^q[\Omega; C_{per}^\infty(Y)]$ means that a sequence φ_n of functions in $C_0^\infty[\Omega; C_{per}^\infty(Y)]$ converges to $\varphi \in L^q[\Omega; C_{per}^\infty(Y)]$.

(ii) Due to density properties, it is easily seen that if $\{v^\varepsilon\}$ two-scale converges to v_0, (3.6) holds for any ψ of the form

$$\psi(x,y) = \psi_1(y)\psi_2(x,y)$$

with $\psi_1(y) \in L^\infty(Y)$ and $\psi_2(x,y) \in L^2_{per}[Y; C(\overline{\Omega})]$.

(iii) For the same reason, convergence in (3.6) is still true for any function ψ of the form $\psi(x,y) = \varphi_1(x)\varphi_2(y)$, where φ_1 and φ_2 are the same as in statement (ii) of Lemma 3.1.

Proposition 3.5

Two-scale convergence implies weak convergence in $L^2(\Omega)$.

Proof

If in Definition 3.3 we take ψ independent of y, that is, if $\psi(x,y) = \psi(x)$, then (3.6) reads as the following weak convergence

$$\begin{aligned}
\lim_{\varepsilon \to 0} \int v^\varepsilon(x)\psi(x)dx &= \frac{1}{|Y|}\int_\Omega \int_Y v_0(x,y)\psi(x)dxdy \\
&= \frac{1}{|Y|}\int_\Omega \left(\int_Y v_0(x,y)dy\right)\psi(x)dx \\
&= \frac{1}{|Y|}\int_\Omega v_0(x)\psi(x)dx
\end{aligned}$$

where $v_0(x) = \int_Y v_0(x,y)dy$ i.e.$(v^\varepsilon, \psi)_{L^p(\Omega),L^q(\Omega)} \rightharpoonup (v_0, \psi)_{L^p(\Omega),L^q(\Omega)}$. Thus,

$$v^\varepsilon \rightharpoonup v_0 = \frac{1}{|Y|}\int_Y v_0(x,y)dy \text{ weakly in } L^2(\Omega).$$

Definition 3.5

Let $\{v^\varepsilon\}$ be a sequence in $L^2(\Omega)$ that two-scale converges to $v_0 \in L^2(\Omega \times Y)$. Then,

$$v^\varepsilon \rightharpoonup \int_Y v_0(x,y)dy \text{ weakly in } L^2(\Omega). \tag{3.11}$$

Even though the two-scale limit belongs to another space from the sequence converging to it, all elements in $L^2(\Omega \times Y)$ are actually two-scale limits for some sequence in $L^2(\Omega)$.

Proposition 3.6

For every $v \in L^2(\Omega; C_{per}(Y))$, it holds that

$$v(x, \frac{x}{\varepsilon}) \rightharpoonup \int_Y v(x,y)\, dy \text{ in } L^2(\Omega). \tag{3.12}$$

Proof

If $g \in L^1(\Omega; C_{per}(Y))$, then

$$\lim_{\varepsilon\to 0}\int_\Omega g\left(x, \frac{x}{\varepsilon}\right)dx = \int_\Omega\int_Y g(x,y)dydx. \tag{3.13}$$

Since $vw \in L^1(\Omega; C_{per}(Y))$ when $v \in L^2(\Omega; C_{per}(Y))$ and $w \in L^2(\Omega; C_{per}(Y))$, it holds that

$$\lim_{\varepsilon\to 0}\int_\Omega v\left(x, \frac{x}{\varepsilon}\right)w(x)dx = \int_\Omega\int_Y v(x,y)w(x)dydx, \tag{3.14}$$

for all $w \in L^2(\Omega)$.

Clearly, the weak and the two-scale limits are equal if the two-scale limit does not depend on y or it is independent of y. This implies that if a sequence $\{v^\varepsilon\}$

two-scale converges, then it is bounded in $L^2(\Omega)$. This follows from the proposition which says that every weakly convergent sequence is bounded.

Theorem 3.7 (Flodén, 2009)

The two-scale limit is unique.

Proof

Assume that a sequence $u^\varepsilon \in L^2(\Omega)$ converges to two different functions η_0 and γ_0 in $L^2(\Omega \times Y)$, i.e. we have

$$\lim_{\varepsilon \to 0} \int_\Omega u^\varepsilon(x) v(x, \frac{x}{\varepsilon}) dx = \int_\Omega \int_Y \eta_0(x,y) v(x,y) dx dy \tag{3.15}$$

and

$$\lim_{\varepsilon \to 0} \int_\Omega u^\varepsilon(x) v(x, \frac{x}{\varepsilon}) dx = \int_\Omega \int_Y \gamma_0(x,y) v(x,y) dx dy, \tag{3.16}$$

for every $v \in L^2(\Omega; C_{per}(Y))$.

Taking the difference between (3.15) and (3.16)

$$\int_\Omega \int_Y \eta_0(x,y) v(x,y) dx dy - \int_\Omega \int_Y \gamma_0(x,y) v(x,y) dx dy = 0$$
$$\int_\Omega \int_Y \Big(\eta_0(x,y) - \gamma_0(x,y) \Big) v(x,y) dx dy = 0.$$

This implies from the variational lemma that $\eta_0(x,y) - \gamma_0(x,y) = 0$ and we get $\eta_0(x,y) = \gamma_0(x,y)$ almost everywhere in $L^2(\Omega \times Y)$.

Theorem 3.8 (Flodén et al, 2013)

Let $\{u_\varepsilon\}$ be a bounded sequence in $L^2(\Omega)$. Then it holds for some $u_0 \in L^2(\Omega \times Y)$ and up to a subsequence that

$$u^\varepsilon \overset{2}{\rightharpoonup} u_0(x,y).$$

See detailed proof in Flodén, et al (2013). We will give a simple proof here:

Let

$$F^{\varepsilon}(v) = \int_{\Omega} u^{\varepsilon}(x) v\left(x, \frac{x}{\varepsilon}\right) dx$$

where $v \in L^2(\Omega; C_{per}(Y))$. By the Hölder's inequality, we have that

$$\begin{aligned} |F^{\varepsilon}(v)| = \left| \int_{\Omega} u^{\varepsilon}(x) v\left(x, \frac{x}{\varepsilon}\right) dx \right| &\leq \|u^{\varepsilon}\|_{L^2(\Omega)} \left\| v\left(x, \frac{x}{\varepsilon}\right) \right\|_{L^2(\Omega)} \\ &\leq C \left\| v\left(x, \frac{x}{\varepsilon}\right) \right\|_{L^2(\Omega)}. \end{aligned}$$

By Property (iii) of Theorem **??**, $\{F^{\varepsilon}\}$ is bounded in X' and hence, up to a subsequence,

$$F^{\varepsilon} \xrightarrow{*} F \quad \text{in} \quad X'.$$

The property (iv) in the same theorem gives

$$|F(v)| \leq C \lim_{\varepsilon \to 0} \left\| v\left(x, \frac{x}{\varepsilon}\right) \right\|_{L^2(\Omega)} = C \|v(x,y)\|_{L^2(\Omega \times Y)}.$$

Hence, $F \in L^2(\Omega \times Y)'$ by the Riesz representation theorem

$$F(v) = \int_{\Omega} \int_{Y} u_0(x,y) v(x,y) dy dx \tag{3.17}$$

for every $v \in X$ and a unique $u_0 \in L^2(\Omega \times Y)$.

The proposition below yields that sequences created from admissible test functions two-scale converge.

Proposition 3.9

If $u \in \Psi(\Omega, Y)$, then

$$u\left(x, \frac{x}{\varepsilon}\right) \xrightarrow{2} u(x,y). \tag{3.18}$$

See Page 49 of Floden (2009) for proof.

Definition 3.6 (Ciouranescu and Donato, 1999)

Let $\{u_n\}$ be a sequence in $L^p(\Omega)$ with $1 < p < \infty$. The weak convergence $u_n \rightharpoonup u$ weakly in $L^p(\Omega)$ signifies that

$$\int_\Omega u_n \psi dx \rightharpoonup \int_\Omega u \psi dx \;\; \forall \psi \in L^q(\Omega),$$

with $\frac{1}{p} + \frac{1}{q} = 1$.

If $p = 1$, then $q = \infty$. If $p = \infty$, we take $q = 1$ and the convergence is called weak-* convergence in $L^\infty(\Omega)$.

Proposition 3.10

Let $1 < p < \infty$ and $\{u_n\}$ be a sequence in $L^p(\Omega)$. Then the following equivalence hold.

(a) $u_n \rightharpoonup u$ weakly $\in L^p(\Omega) \Leftrightarrow$

(b) **(i)** $\|u_n\|_{L^p(\Omega)} \leq c$ (independently of n)

(ii) $\int_I u_n dx \to \int_I u dx$ for any $I \subset \Omega$.

Proposition 3.11

Let $\{u_n\}$ be a sequence weakly convergent to x in E. Then

(i) $\{x_n\}$ is a bounded sequence in E, i.e. there exists a constant C independent of n such that $\forall\, n \in N$. That is, $\|x_n\|_E \leq C$,

(ii) the norm on E is lower semi-continuous with respect to the weak convergence. That is,

$$||x||_E \leq \liminf_{n\to\infty} \|x_n\|_E.$$

Lemma 3.12

Consider a function $v^\varepsilon \in L^2(\Omega)$ which admits the following two-scale expansion

$$v^\varepsilon(x) = v_0(x, \frac{x}{\varepsilon}) + \varepsilon v_1(x, \frac{x}{\varepsilon}) + ... \tag{3.19}$$

where $v_j(x,y) \in L^2(\Omega; C_{per}(Y)),\ j = 0, 1, ..., N$, Ω being a bounded domain in $\mathbb{R}^n$. Then $v^\varepsilon \xrightarrow{2} v_0$.

Proof

It is enough to consider the case where $N = 1$. Let $\psi(x,y) \in L^2[\Omega; C_{per}(Y)]$ and we define

$$\begin{aligned} f_j(x,y) &= v_j(x,y)\psi(x,y), \quad j = 0, 1. & (3.20)\\ f_j^\varepsilon(x) &= f_j(x, \frac{x}{\varepsilon}). & (3.21)\end{aligned}$$

Multiplying the two-scale expansion (3.19) by $\psi(x, \frac{x}{\varepsilon})$ and integrating over Ω, we obtain

$$\begin{aligned} \int_\Omega v^\varepsilon(x)\psi(x, \frac{x}{\varepsilon})dx &= \int_\Omega v_0(x, \frac{x}{\varepsilon})\psi(x, \frac{x}{\varepsilon})dx + \int_\Omega \varepsilon v_1(x, \frac{x}{\varepsilon})\psi(x, \frac{x}{\varepsilon})dx + \cdots \\ &= \int_\Omega f_0(x, \frac{x}{\varepsilon})dx + \int_\Omega \varepsilon f_1(x, \frac{x}{\varepsilon})dx + \cdots \\ &= \int_\Omega f_0^\varepsilon(x)dx + \int_\Omega \varepsilon f_1^\varepsilon(x)dx + \cdots \qquad (3.22)\end{aligned}$$

By Lemma 3.1, $f_0^\varepsilon(x)$ converges to its average over $Y = \int_Y f_0(x,y)dy$ weakly in $L^2(\Omega)$. We take notice that $1 \in L^2(\Omega)$ since Ω is a bounded subset of $\mathbb{R}^n$. Moreover, $\int_\Omega 1dx = |\Omega|$. Choosing $\psi = 1$ we obtain

$$\int_\Omega f_0^\varepsilon(x)dx \to \int_\Omega \int_Y f_0(x,y)dydx \tag{3.23}$$

$$= \int_\Omega \int_Y v_0(x,y)\psi(x,y)dydx.$$

We then consider the integral $\int_\Omega \varepsilon f_1^\varepsilon(x)dx$. Since the sequence f_1^ε is weakly convergent in $L^2(\Omega)$, it is bounded by Proposition (3.10).

Using the boundedness of Ω and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \int_\Omega \varepsilon f_1^\varepsilon(x) &\leq C\|f_1^\varepsilon\|_{L^2(\Omega)} \\ &\leq \varepsilon C \longrightarrow 0, \end{aligned} \tag{3.24}$$

as $\varepsilon \longrightarrow 0$.

Substituting (3.23) and (3.24) into (3.22) we obtain

$$\int_\Omega v^\varepsilon(x)\psi(x, \frac{x}{\varepsilon})dx \longrightarrow \int_\Omega \int_Y v_0(x,y)\psi(x,y)dydx.$$

Hence, v^ε two-scale converges to v_0.

Lemma 3.13 (Ciouranescu and Donato, 1999)

Compactness Lemma

Each bounded sequence v^ε in the dual space of a separable normed space (Banach space) contains a subsequence which converges weak*. The subsequence is still denoted by ε.

Theorem 3.14

Let $\{v^\varepsilon\}$ be a bounded sequence in $L^2(\Omega)$. Then there exists a subsequence, still denoted by $\{v^\varepsilon\}$ and a function $v_0(x,y) \in L^2(\Omega \times Y)$ such that $\{v^\varepsilon\}$ two-scale converges to $v_0(x,y)$.

Proof

Let X denote the space $L^2[\Omega; C_{per}(Y)]$. Also, let $\psi \in X$, then from Lemma 3.13, we have that

$$\|\psi(\cdot, \frac{\cdot}{\varepsilon})\|_{L^p(\Omega)} \leq \|\psi(\cdot,\cdot)\|_X.$$

1. From the Hölder inequality we have

$$\begin{aligned} \left| \int_\Omega v^\varepsilon \psi(x, \frac{x}{\varepsilon}) dx \right| &\leq \|v_\varepsilon\|_{L^2(\Omega)} \|\psi\|_{L^2(\Omega)} \\ &\leq C\|\psi(x,y)\|_X. \end{aligned} \tag{3.25}$$

2. From the above equation (3.25), we deduce that $\int_\Omega v^\varepsilon \psi(x, \frac{x}{\varepsilon}) dx$ defines a bounded linear functional over x, i.e. v^ε can be regarded as the element V^ε of the dual spaces of X (there exists $v^\varepsilon \in X^*$) such that

$$\langle V^\varepsilon, \psi \rangle_{X^*,X} = \int_\Omega v^\varepsilon \psi(x, \frac{x}{\varepsilon}) dx \ \ \forall \psi \ \in X.$$

Taking the supremum over all $x \in \Omega$ we obtain

$$\begin{aligned} \|V^\varepsilon\|_X^* &= \sup_{\|\psi\|=1} |\langle V^\varepsilon, \psi \rangle|_{X^*,X} \\ &\leq C \sup_{\|\psi\|=1} |\langle \psi \rangle|_X \\ &\leq C. \end{aligned}$$

Since X is a seperable Banach space, we can extract a weak* convergent subsequence still denoted by V^ε such that $V^\varepsilon \rightharpoonup V^0$ weakly* in X^* for some $V^0 \in X^*$. Consequently,

$$\lim_{\varepsilon \to 0} \int v^\varepsilon(x) \psi(x) dx = \lim_{\varepsilon \to 0} \langle V^\varepsilon, \psi \rangle_{X^*,X} \longrightarrow \langle V^0, \psi \rangle. \tag{3.26}$$

On the other hand, from the boundedness of $\{v^\varepsilon\}$, the Hölder inequality and convergence in Lemma 3.12 we find that

$$\lim_{\varepsilon \to 0} | \int v^\varepsilon(x) \psi(x) dx| \leq C \lim_{\varepsilon \to 0} \|\psi(x, \frac{x}{\varepsilon})\|_{L^2(\Omega)} = C\|\psi\|_{L^2(\Omega)} \tag{3.27}$$

From (3.26) and (3.27) we obtain

$$\langle V^0, \psi \rangle_{X^*,X} \leq C\|\psi\|_X \quad \forall \psi \in X. \tag{3.28}$$

Since X is dense in $L^2(\Omega \times Y)$ this implies that (3.28) actually holds for every $\psi \in L^2(\Omega \times Y)$. Hence $\langle V^0, \psi\rangle_{X^*,X}$ can be extended to become a bounded linear functional on $L^2(\Omega \times Y)$. Since this is a Hilbert space, the Riesz representation theorem holds. This enables us to identify the limiting bounded linear functional by a unique element $v_0(x, y)$ of $L^2(\Omega \times Y)$:

$$\begin{aligned} \langle V^0, \psi\rangle_{X^*,X} &= \int_\Omega \int_Y v_0(x, y)\psi(x, y)dydx \ \ \forall\psi \in X \\ &= \int_{\Omega\times Y} v_0(x, y)\psi(x, y)dydx \ \ \forall\psi \in X. \end{aligned} \tag{3.29}$$

From (3.26) and (3.29) we have that

$$\int_\Omega v^\varepsilon \psi(x, \frac{x}{\varepsilon})dx \longrightarrow \int_{\Omega\times Y} v_0(x, y)\psi(x, y)dydx \ \ \forall\psi \in X$$

which implies that v_0 is the two-scale convergence limit of the sequence v^ε.

See other proof in Theorem 9.7 of Ciouranescu and Donato (1999).

Proposition 3.15

The space $X = L^2(\Omega; C_{per}(Y))$ have the following properties.

i The space X is seperable.

ii The space X is dense in $L^2(\Omega \times Y)$.

iii If $v \in L^2(\Omega; C_{per}(Y))$. Then the function $x \mapsto v(x, \frac{x}{\varepsilon})$ defined on Ω is measurable and

$$\left\| v(x, \frac{x}{\varepsilon}) \right\|_{L^2(\Omega)} \leq \left(\int_\Omega \|v(x, \cdot)\|^2_{C_{per}(Y)} dx \right)^{\frac{1}{2}} = \|v\|_X, \tag{3.30}$$

iv For every $v \in X$, one has

$$\lim_{\varepsilon \to 0} \int_{\Omega} \left| v\left(x, \frac{x}{\varepsilon}\right) \right|^2 dx = \int_{\Omega} \int_{Y} |v(x,y)|^2 dydx. \tag{3.31}$$

Theorem 3.16

Let $B_p(\Omega, Y), 1 \leq p \leq \infty$ denote any of the spaces $L^p(\Omega; C_{per}(Y), L^p_{loc}(Y; C(\bar{\Omega}))$, $C(\Omega; C_{per}(Y))$. Then $B_p(\Omega, Y)$ has the following properties:

(i) $B_p(\Omega, Y)$ is a seperable Banach space.

(ii) $B_p(\Omega, Y)$ is dense in $L^p(\Omega \times Y)$

(iii) For every $f \in B_p(\Omega, Y)$, the function $x \mapsto f(x, \frac{x}{\varepsilon})$ defined on Ω is measurable and

$$\left\| f(x, \frac{x}{\varepsilon}) \right\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega; C_{per}(Y))} = \left(\int_{\Omega} \|f(x, \cdot)\|^p_{C_{per}(Y)} dx \right)^{\frac{1}{p}} \tag{3.32}$$

(iv.) For every $f \in B_p(\Omega, Y)$ we have,

$$\lim_{\varepsilon \to 0} \int_{\Omega} \left| f\left(x, \frac{x}{\varepsilon}\right) \right|^p dx = \int_{\Omega} \int_{Y} |f(x,y)|^p dydx. \tag{3.33}$$

For Ω bounded, $L^2(\Omega; C_{per}(Y)), L^2_{loc}(Y; C(\bar{\Omega})), C(\Omega; C_{per}(Y))$ are spaces of admissible test functions. We denote any of the spaces by $\Psi(\Omega, Y)$. The techniques used to prove the compactness results for two-scale convergence of sequences in $L^2(\Omega)$ given in this work apply for all these spaces. The functions in the spaces above generate weakly convergent sequences in $L^2(\Omega)$.

Remark

The functions in the space $L^2(\Omega; C_{per}(Y))$ can be used as test functions also for Ω unbounded, e.g. for $\Omega = R^n$; see Lemma 2.3 in Allaire (1993).

Proposition 3.6 holds true for all the spaces of test functions in Proposition 3.16. Just note that (iii) in Theorem 3.15 means that for v in any of those spaces,

$\{v(x, \frac{x}{\varepsilon})\}$ is bounded in $L^2(\Omega)$ and that for any $w \in D(\Omega)$, vw remains in the space of admissible test functions containing v.

Theorem 3.17

Let $\{v^\varepsilon\}$ be a sequence of functions in $L^2(\Omega)$ which two-scale converges to $v_0 \in L^2(\Omega \times Y)$. Suppose further that the admissible test function

$$\lim_{\varepsilon \to 0} \int_\Omega |v^\varepsilon(x)|^2 dx = \frac{1}{|Y|} \int_\Omega \int_Y |v_0(x, y)|^2 dy dx \tag{3.34}$$

holds, then for any sequence $\{w^\varepsilon\}$ that two-scale converges to a limit $w_0 \in L^2(\Omega \times Y)$, we have

$$v^\varepsilon w^\varepsilon \longrightarrow \frac{1}{|Y|} \int_Y v_0(\cdot, y) w_0(\cdot, y) dy. \tag{3.35}$$

Proof

Since the space $L^2(\Omega; C_{per}(Y))$ is dense in $L^2(\Omega \times Y)$ and that there exists a sequence $\{\varphi_n\} \subset L^2(\Omega; C_{per}(Y))$, such that as $n \to \infty$,

$$\varphi_n \to v_0 \quad \text{strongly in} \quad L^2(\Omega \times Y). \tag{3.36}$$

We then consider the integral

$$\begin{aligned} I_n^\varepsilon &= \int_\Omega [v^\varepsilon(x) - \varphi_n(x, \frac{x}{\varepsilon})]^2 dx \\ &= \int_\Omega [v^\varepsilon(x)]^2 dx - 2 \int_\Omega v^\varepsilon(x) \varphi_n(x, \frac{x}{\varepsilon}) dx + \int_\Omega [\varphi_n(x, \frac{x}{\varepsilon})]^2 dx, \end{aligned}$$

and we let $\varepsilon \longrightarrow 0$.

From (3.34) the first term will converge, while the second term will also converge by the two-scale definition. Also by Lemma 3.6 the third term will converge. We then obtain

$$\begin{aligned}\lim_{\varepsilon\longrightarrow 0} I_n^\varepsilon &= \frac{1}{|Y|}\int_\Omega\int_Y [v_0(x,y)]^2 dxdy - 2\frac{1}{|Y|}\int_\Omega\int_Y v_0(x,y)\varphi_n(x,y)dxdy \\ &\quad + \frac{1}{|Y|}\int_\Omega\int_Y [\varphi_n(x,y)]^2 dxdy \\ &= \frac{1}{|Y|}\int_\Omega\int_Y [v_0(x,y) - \varphi_n(x,y)]^2 dxdy.\end{aligned}$$

Due to (3.36) the last integral converges to 0 as $n \to \infty$. So that

$$\lim_{n\longrightarrow\infty} I_n^\varepsilon = \frac{1}{|Y|}\lim_{n\longrightarrow\infty}\int_\Omega\int_Y [v_0(x,y) - \varphi_n(x,y)]^2 = 0.$$

Also for any $\psi \in C^\infty(\Omega)$ one has that

$$\int_\Omega v^\varepsilon(x)w^\varepsilon(x)\psi(x)dx = \int_\Omega [v^\varepsilon(x) - \varphi_n(x,\frac{x}{\varepsilon})]w^\varepsilon(x)\psi(x)]dx$$

$$+\int_\Omega \varphi_n(x,\frac{x}{\varepsilon})w^\varepsilon(x)\psi(x)dx$$

we then make $\varepsilon \longrightarrow 0$ and then $n \longrightarrow \infty$. To do so, we observe that if a sequence $\{v^\varepsilon\}$ two-scale converges, and therefore converges weakly then it is bounded in $L^2(\Omega)$. By Proposition 3.11 and the Hölder's inequality, we obtain

$$\begin{aligned}&\lim_{n\longrightarrow\infty}\lim_{\varepsilon\longrightarrow 0}\left|\int_\Omega [v^\varepsilon(x) - \varphi_n(x,\frac{x}{\varepsilon})]w^\varepsilon(x)\psi(x)dx\right| \\ &\qquad \le C\lim_{n\longrightarrow\infty}\lim_{\varepsilon\longrightarrow 0}\left\{\int_\Omega [v^\varepsilon(x) - \varphi_n(x,\frac{x}{\varepsilon})]^2 dx\right\}^{\frac{1}{2}} = 0. \qquad (3.37)\end{aligned}$$

From the assumption that w^ε two-scale converges to w_0, we have that

$$\begin{aligned}&\lim_{n\longrightarrow\infty}\lim_{\varepsilon\longrightarrow 0}\int_\Omega [\varphi_n(x,\frac{x}{\varepsilon})]w^\varepsilon(x)\psi(x)dx \\ &= \lim_{n\longrightarrow\infty}\frac{1}{|Y|}\int_\Omega\int_Y w_0(x,y)\varphi_n(x,y)\psi(x)dxdy \\ &= \frac{1}{|Y|}\int_\Omega\int_Y w_0(x,y)v_0(x,y)\psi(x)dxdy.\end{aligned}$$

Therefore, from (3.37) we obtain

$$\int_{\Omega} v^{\varepsilon}(x) w^{\varepsilon}(x)\psi(x)dx = \frac{1}{|Y|}\int_{\Omega}\int_{Y} w_0(x,y)v_0(x,y)\psi(x)dxdy. \qquad (3.38)$$

So that with the necessary conditions,

$$v^{\varepsilon}w^{\varepsilon} \longrightarrow \frac{1}{|Y|}\int_{Y} v_0(x,y)w_0(x,y)dy. \qquad (3.39)$$

Theorem 3.18

Let $\{u^{\varepsilon}\}$ be a sequence in $L^2(\Omega)$ that two-scale converges to u_0. Then

$$\lim_{\varepsilon\to 0}\int_{\Omega} u^{\varepsilon}(x)v\left(x,\frac{x}{\varepsilon}\right)dx = \int_{\Omega}\int_{Y} u_0(x,y)v(x,y)dydx \qquad (3.40)$$

for all $v \in \Psi(\Omega, Y)$ and for all v of the form $v(x,y) = v_1(x)v_2(y)$, $v_1 \in L^s(\Omega)$, $v_2 \in L^t_{per}$ with $1 \leq s,t \leq \infty$ and such that $\frac{1}{s}+\frac{1}{t}=\frac{1}{2}$.

Relations between the weak, two-scale and strong convergences

There exists relations between the two-scale convergence and the other convergence. This convergences include the strong and weak convergence as well as other convergences. Strong convergences do not have oscillations which can affect the two-scale limit. The two scale limit of a sequence $\{u_{\varepsilon}\}$ converges to the strong limit u in $L^2(\Omega)$ if this limit exists. A sequence $\{u_{\varepsilon}\}$ that converges strongly will also converge two scale and the limits are the same. For a sequence $\{u_{\varepsilon}\}$ in the space $L^2(\Omega)$, there are weak, strong and two-scale convergences. The relation that exists between these convergences are as below:

Strong convergence $\Longrightarrow$ two-scale convergence $\Longrightarrow$weak convergence

The converse is however not true. i.e.

Weak Convergence $\nRightarrow$ Two-scale Convergence $\nRightarrow$ Strong convergence

The relation is between the convergences in $L^p(\Omega)$ can simply be written as:

strong $\Rightarrow$ strong two-scale $\Rightarrow$ weak two-scale $\Rightarrow$ weak.

See also Proposition 3.2 of Giacomini and Musesti (2011), Ganesh and Nandakumaran (2010) and Theorem 6.2 of Francü (2010) for other properties.

Theorem 3.19

If u_ε converges strongly to u_0 in $L^2(\Omega)$, then $u_\varepsilon(x) \stackrel{2}{\rightharpoonup} u_0(x)$.

Proof

We know from Analysis that if a sequence converges strongly, then it converges weakly and that both limits are the same. This theoerem is however its two-scale version.

Let $\psi \in L^2(\Omega; C_{per}(Y))$. Then

$$\left|\int_\Omega u_\varepsilon(x)\psi(x,\frac{x}{\varepsilon})dx - \int_\Omega\int_Y u_0(x)\psi(x,y)dydx\right| \leq \|u_\varepsilon - u_0\|_{L^2(\Omega)}\left\|\psi(x,\frac{x}{\varepsilon})\right\|_{L^2(\Omega)}$$
$$+\left|\int_\Omega u_0(x)\psi(x,\frac{x}{\varepsilon})dx - \int_\Omega\int_Y u_0(x)\psi(x,y)dydx\right|$$
$$\leq C\|u_\varepsilon - u_0\|_{L^2(\Omega)} + \left|\int_\Omega u_0(x)\Big(\psi(x,\frac{x}{\varepsilon})dx - \psi(x,y)\Big)dydx\right|.$$

The first term of the right hand side goes to zero since $\{u_\varepsilon\}$ converges strongly. Since $u_0(x)$ is in $L^1(\Omega; C_{per}(Y))$ and that the limit is unique, the second term also will go to zero. Therefore, the left hand side becomes zero and we conclude that the two-scale limit will also converge and converges to u_0. And we obtain that

$$\left|\int_\Omega u_\varepsilon(x)\psi(x,\frac{x}{\varepsilon})dx - \int_\Omega\int_Y u_0(x)\psi(x,y)dydx\right| \to 0$$

which gives the required result that if u_ε converges strongly to u_0 in $L^2(\Omega)$, then $u_\varepsilon(x) \stackrel{2}{\rightharpoonup} u_0(x)$.

Theorem 3.20

If $\{u_\varepsilon\}$ is a sequence in $L^2(\Omega)$ and $u_\varepsilon \stackrel{2}{\rightharpoonup} u_0(x,y)$, then $u_\varepsilon \rightharpoonup u$ weakly in $L^2(\Omega)$ where, $u(x) = \int_Y u_0(x,y)dy$ and $\{u_\varepsilon\}$ is bounded.

Proof

By definition of the two-scale convergence, it follows that

$$\int_{\Omega} u_{\varepsilon}(x)\phi(x,\frac{x}{\varepsilon}) \longrightarrow \int_{\Omega}\int_{Y} u(x,y)\phi(x,y)dydx, \tag{3.41}$$

for every $\phi \in L^q(\Omega; C_{per}(Y))$. For ϕ independent of y we obtain that

$$\int_{\Omega} u_{\varepsilon}(x)\phi(x) \longrightarrow \int_{\Omega}\int_{Y} u(x,y)\phi(x)dydx. \tag{3.42}$$

Since every function in $L^q(\Omega)$ can be identified with $\phi \in L^q(\Omega; C_{per}(Y))$ independent of y the result follows. That $\{u_{\varepsilon}\}$ is bounded in $L^p(\Omega)$ follows from the well-known fact that every weakly convergent sequence is bounded.

Remark

Usually the choice of the test function $L^q(\Omega; C_{per}(Y))$ is very essential for Theorem 3.20 to hold. The definition of two-scale convergence was given by taking the test function space $D(\Omega; C^{\infty}_{per}(Y))$ instead of $L^2(\Omega; C_{per}(Y))$. Such a definition will not be defined for Theorem 3.20.

Example 3.1

Let $\Omega = (0, 1)$, $u(x, y) = 0$ and define

$$u_{\varepsilon}(x) = \begin{cases} \frac{1}{\varepsilon} & \text{if} \quad 0 < x < \varepsilon, \\ 0 & \text{if} \quad \varepsilon < x < 1. \end{cases} \tag{3.43}$$

Then,

$$\int_{\Omega} u_{\varepsilon}(x)\phi(x,\frac{x}{\varepsilon}) \longrightarrow \int_{\Omega}\int_{Y} u(x,y)\phi(x,y)dydx = 0, \tag{3.44}$$

for all ϕ in $D(\Omega; C^{\infty}_{per}(Y))$. But $\{u_{\varepsilon}\}$ is neither bounded nor does it converge to 0 weakly in $L^p(\Omega)$. Choosing the function g in the dual as $g \equiv 1$,

$$\lim_{\varepsilon \to 0} \int_0^1 u_{\varepsilon} g dx = 1. \tag{3.45}$$

It is even not possible to also use a bigger space $C(\Omega; C^{\infty}_{per}(Y))$ in place of $D(\Omega; C^{\infty}_{per}(Y))$.

Example 3.2

Let $\Omega = (0,1)$ and u_ε be defined as

$$u_\varepsilon(x) = \begin{cases} \tilde{u}_\varepsilon(\frac{x}{\varepsilon}) & \text{if } \frac{1}{4} < x \leq \frac{3}{4}, \\ 0 & \text{otherwise} \end{cases} \tag{3.46}$$

where $\tilde{u}_\varepsilon$ is the (0,1)-periodic extension to $\mathbb{R}$ of the function defined in Example 3.1 and

$$u(x) = \begin{cases} 1 & \text{if } \frac{1}{4} < x \leq \frac{3}{4} \\ 0 & \text{otherwise.} \end{cases} \tag{3.47}$$

Then

$$\int_\Omega u_\varepsilon(x)\psi(\frac{x}{\varepsilon})dx \to \int_\Omega \int_Y u(x,y)\psi(x,y)dydx, \tag{3.48}$$

is satisfied for all $\psi(x,y)$ in $C(\Omega; C^{\infty}_{per}(Y))$, but u_ε does not converge to $v(x) = \int_Y u(x,y)dy$ weakly in $L^p(\Omega)$ and is certainly not bounded. Note that $\{u_\varepsilon\}$ is not bounded in $L^2(\Omega)$.

The following result shows that it is possible to replace $L^q(\Omega; C_{per}(Y))$ by $D(\Omega; C^{\infty}_{per}(Y))$ in the definition of two-scale convergence provided we add the assumption that $\{u_\varepsilon\}$ is bounded in $L^p(\Omega)$.

Proposition 3.21

Let $\{u_\varepsilon\}$ be a bounded sequence in $L^p(\Omega)$ such that

$$\int_\Omega u_\varepsilon(x)\psi(\frac{x}{\varepsilon})dx \to \int_\Omega \int_Y u(x,y)\psi(x,y)dydx, \tag{3.49}$$

for every $\psi \in D(\Omega; C^{\infty}_{per}(Y))$. Then $\{u_\varepsilon\}$ two-scale converges to $u(x)$.

Example 3.3

Considering the sequence $\{u_\varepsilon = \sin(\frac{x}{\varepsilon})\}$, it has a two-scale limit, that is $u_0 = \sin(y)$. The sequence however does not converge strongly in $L^2(\Omega)$.

In order to see the global and the microscopic cell oscillative behaviour of a sequence $\{u_\varepsilon\}$ are treated in the limit in the cases of weak convergence and two-scale convergence, consider

$$u_\varepsilon(x) = 2x + x\sin(2\pi\varepsilon x), \quad x \in \Omega,$$

where $\Omega = (0, 1)$.

Concerning the weak convergence of $\{u_\varepsilon\}$ to some u in $L^2(\Omega)$, this is equivalent to saying that $\{u_\varepsilon\}$ is bounded in $L^2(\Omega)$ and $\int_I u$.

$$\int_I u_\varepsilon(x)dx \to \int_I u(x)dx$$

for any $I \subset \Omega$.

We will prove that $u(x) = 2x, \quad x \in \Omega$, is the weak limit. First we have

$$\begin{aligned}
\|u_\varepsilon\|^2_{L^2(\Omega)} &= \int_\Omega |u_\varepsilon(x)|^2 dx \\
&= \int_0^1 (2x + x\sin(2\pi\varepsilon x))^2 dx \\
&= \frac{3}{2} - \frac{2}{\pi\varepsilon} - \frac{1}{16\pi^2\varepsilon^2} \\
&\leq \frac{3}{2} \ \forall \ \varepsilon \in Z_+,
\end{aligned} \tag{3.50}$$

i.e. the boundedness property is satisfied.

Next, for the integral convergence property we have for any $I = (a, b) \subset \Omega$,

$$\begin{aligned}
\int_I u_\varepsilon(x)dx &= \int_a^b (2x + x\sin(2\pi\varepsilon x))dx \\
&= x^2 + \frac{x\cos(2\pi x\varepsilon)}{2\pi\varepsilon} + \frac{\sin(2\pi x\varepsilon)}{(2\pi\varepsilon)^2}\Bigg|_a^b \\
&= (b^2 - a^2) - \frac{b\cos(2\pi b\varepsilon) - a\cos(2\pi a\varepsilon)}{2\pi\varepsilon} \\
&\qquad\qquad + \frac{\sin(2\pi\varepsilon b) - \sin(2\pi\varepsilon a)}{4\pi^2\varepsilon^2} \\
&\longrightarrow b^2 - a^2 = \int_a^b 2x dx = \int_I u(x)dx \;\; as \quad \varepsilon \to \infty.
\end{aligned} \tag{3.51}$$

The weak limit of $u_\varepsilon(x) = 2x + x\sin(2\pi\varepsilon x)$ is $u(x) = 2x$.

Hence we have verified that $u_\varepsilon \rightharpoonup u$ in $L^2(\Omega)$.

The weak convergence mode is incapable of capturing the rapid oscillations. Instead, the weak convergence seems to be ideal to describe the global behaviour.

When the sequences of functions which are bounded in L^2 have rapid oscillations in only one microscale, then the two-scale is enough for the study. In the case where there are two microscales then the three-scale convergence will do. However, if the functions under study contains more than one microscale, then the two-scale convergence is will not be enough. A wider concept of multiscale convergence would then be needed. With the introduction of more than one scale, the multiscale convergence now covers all the microcscales. The idea is the same as that of the two-scale convergence, the only difference however, is that instead of the one local variable y, there are now many other variables $y_1, y_2, \ldots, y_n$, introduced in the problem under study. These variables take care of the oscillations in all for each microscales.

In 1996, Allaire and Briane made a generalization of two-scale convergence, to cover all the microscales. Which is known as the multiscale convergence. For more details on the concept of the multiscale convergence, we refer the reader to Section 4.2 of Flodén (2009).

Example 3.4

For example, both sequences $u^{\varepsilon}(x) = x\sin(2\pi x/\varepsilon)$,

$v^{\varepsilon}(x) = v(x, \frac{x}{\varepsilon}) = \frac{1}{x}\sin(2\pi x/\varepsilon)$ will converge weakly to $u = v = 0$ as $\varepsilon \to 0$

as seen in the graph below, but $u^{\varepsilon}(x)v^{\varepsilon}(x) = \sin^2(2\pi x/\varepsilon)$ and that for periodic

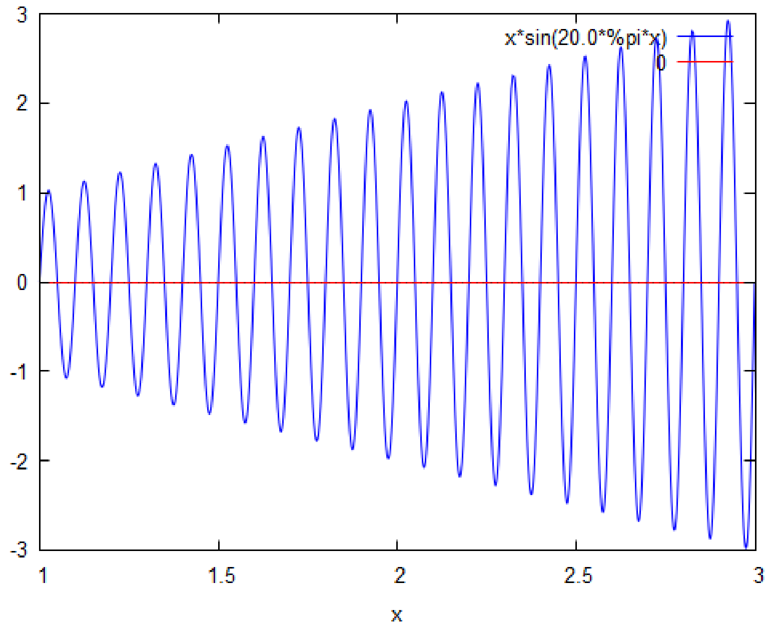

Figure 4: u^{ε} For $\varepsilon = 0.1$, With Its Weak Limit.

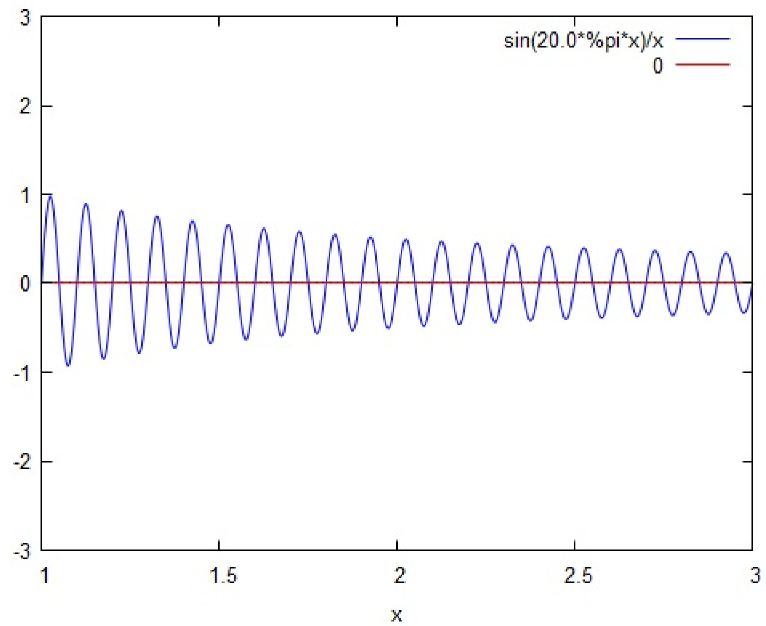

Figure 5: v^{ε} For $\varepsilon = 0.1$, With Its Weak Limit.

functions

$$u^\varepsilon(x)v^\varepsilon(x) = \sin^2(2\pi\frac{x}{\varepsilon}) \rightharpoonup \int_Y \sin^2(2\pi y)dy = 0.5 \quad \text{in} \quad L^2(\Omega), \tag{3.52}$$

and hence in $D'(\Omega)$.

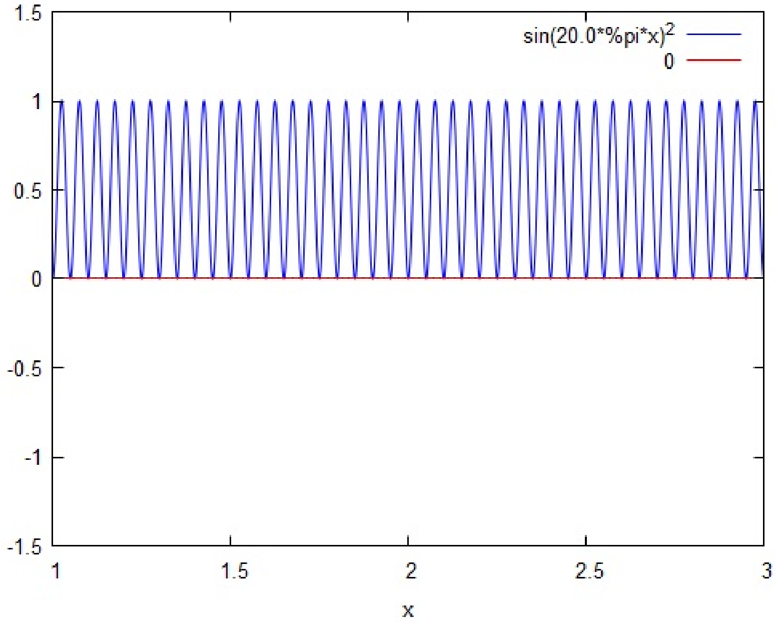

Figure 6: $u^\varepsilon v^\varepsilon$ for $\varepsilon = 0.1$, With Its Weak Limit uv.

So we see that

$$\int_\Omega u^\varepsilon(x)v^\varepsilon(x)dx \not\rightharpoonup \int_\Omega u(x)v(x)dx = 0. \tag{3.53}$$

There is however no information about the respective sequence contribution to the limit provided. The $D'(\Omega)$ limit also does not give any enlightenment about the oscillations of $u^\varepsilon v^\varepsilon$. The two-scale convergence solves this problem. We observe that

$$u^\varepsilon(x) \overset{2}{\rightharpoonup} u_0(x, y) = x\sin(2\pi y) \tag{3.54}$$

and that v^ε emanates from an admissible test function V. We now obtain using the two-scale convergence,

$$\begin{aligned}\int_\Omega u^\varepsilon(x)v(x,\frac{x}{\varepsilon})dx &\longrightarrow \int_\Omega\int_Y u_0(x,y)v(x,y)dydx\\ &= \int_\Omega\int_Y x\sin(2\pi y)\cdot\frac{1}{x}\sin(2\pi y)dydx\\ &= \int_\Omega\int_Y \sin^2(2\pi y)dydx,\end{aligned}$$

as $\varepsilon \to 0$. Hence the impact of the respective sequence is revealed. The product of two weakly convergent sequences will in general not converge to the product of their respective weak limit and hence

$$\int_\Omega u^\varepsilon(x)v^\varepsilon(x)dx \not\rightarrow \int_\Omega\Big(\int_Y u_0(x,y)dy\int_Y v(x,y)dy\Big)dx. \tag{3.55}$$

Whereas we have by the two-scale convergence that

$$\int_\Omega u^\varepsilon(x)v^\varepsilon(x)dx \longrightarrow \int_\Omega\int_Y u_0(x,y)v(x,y)dy. \tag{3.56}$$

The two-scale convergence handles the information about what the two sequences can do together.

Bounded Sequences in $H^1(\Omega)$ and their two-scale limits

Theorem 3.22 (Ganesh & Nandakumaran, 2010)

Let $\{u_\varepsilon\}$ be a sequence in $H^1(\Omega)$ such that $u_\varepsilon \rightharpoonup u$ weakly in $H^1(\Omega)$. Then

1. The sequence $\{u_\varepsilon\}$ two-scale converges to u.

2. There exists a subsequence of ε and $u_1 \in L^2(\Omega; H^1_{per}(Y)/\mathbb{R})$ such that

$$\nabla u_\varepsilon \stackrel{2}{\rightharpoonup} \nabla_x u(x) + \nabla_y u_1(x,y). \tag{3.57}$$

Proof

As the sequence $\{u_\varepsilon\}$ converges weakly in $H^1(\Omega)$, it is a bounded sequence. As a consequence $\{u_\varepsilon\}$ and $\{\nabla u_\varepsilon\}$ are bounded sequences in $L^2(\Omega)$ and $L^2(\Omega)^n$

respectively. By the compactness lemma 3.13, there exists a subsequence (still denoted by ε), $u_0 \in L^2(\Omega \times Y)$ and $\xi_0 \in L^2(\Omega \times Y)^n$ such that

$$u_\varepsilon \xrightharpoonup{2} u \quad \text{and} \quad \nabla u_\varepsilon \xrightharpoonup{2} \xi_0.$$

By Theorem 3.20 we know that $u(x) = \int_Y u_0(x,y)dy$.

If we prove that u_0 does not depend on $y \in Y$, we get that the whole sequence two-scale converges to u. This follows from the uniqueness of the weak limit. Let $\Psi \in [\mathcal{D}(\Omega; C^\infty_{per}(Y))]^n$. Then

$$\begin{aligned} \int_\Omega \nabla u_\varepsilon(x) \cdot \Psi(x, \frac{x}{\varepsilon})dx &= \int_\Omega u_\varepsilon(x)\Big[\nabla_x \cdot \Psi(x, \frac{x}{\varepsilon})\Big]dx + \\ & \frac{1}{\varepsilon}\int_\Omega u_\varepsilon(x)\nabla_y \cdot \Psi(x, \frac{x}{\varepsilon})dx \\ &= \int_\Omega u_\varepsilon(x)\big[\nabla_x \cdot \Psi(x, \frac{x}{\varepsilon})\big]dx. \end{aligned}$$

Multiplying by ε and then passing to the limit on both sides as $\varepsilon \to 0$ we obtain

$$0 = \int_\Omega \int_Y u_0(x,y)\nabla_y \cdot \Psi(x,y)dydx.$$

This shows that u_0 is independent of y.We further assume that Ψ also satisfies $\nabla_y \Psi(x,y) = 0$. Then we have

$$\int_\Omega \nabla u_\varepsilon(x) \cdot \Psi(x, \frac{x}{\varepsilon})dx = \int_\Omega u_\varepsilon(x)\big[\nabla_x \cdot \Psi(x, \frac{x}{\varepsilon})\big]dx. \tag{3.58}$$

Passing to the limit as $\varepsilon \to 0$ on both sides, we get

$$\int_\Omega \int_Y \xi_0(x,y)\Psi(x,y)dydx = \int_\Omega \int_Y u(x)\nabla_x \Psi(x,y)dydx.$$

Hence

$$\int_\Omega \int_Y [\xi_0(x,y) - \nabla_x u(x)]\Psi(x,y)dydx = 0,$$

for all divergence free Ψ.

It is well known that a vector field orthogonal to divergence free vector fields must be a gradient. Thus there exists $u_1(x, y)$ such that

$$\xi_0(x, y) - \nabla u(x) = \nabla_y u_1(x, y).$$

This complete the proof of Theorem 3.22.

Theorem 3.23

If $v^\varepsilon(x)$ is a sequence in $W_0^{1,2}(\Omega)$ such that $v^\varepsilon(x) \overset{2}{\rightharpoonup} v_0(x, y)$ and $\nabla v^\varepsilon(x) \overset{2}{\rightharpoonup} \nabla v_0(x) + \nabla v_1(x, y)$, then the weak two-scale limit v_0 is independent of y and belong to $W_0^{1,2}(\Omega)$, i.e. $v_0(x, y) = v_0(x) \in W_0^{1,2}(\Omega)$, and $v_1 \in L^2[\Omega; W_{per}^{1,2}(Y)]$.

Theorem 3.24

Assume that $\{v_\varepsilon\}$ is a bounded sequence in $H^1(\Omega)$. Then up to a subsequence we have that

$$v^\varepsilon \rightharpoonup v_0 \quad \text{in } \ H^1(\Omega), \tag{3.59}$$

$$\text{and} \qquad \nabla v^\varepsilon \overset{2}{\rightharpoonup} \nabla v_0 + \nabla_y v_1, \tag{3.60}$$

for some $v_o \in H^1(\Omega)$ and some $v_1 \in L^2(\Omega; H^1(Y)/\mathbb{R})$.

Theorem 3.25

Every bounded sequence in $L^2(\Omega)$ has a subsequence which is two-scale convergent. See proof in Persson, *et al* (1993).

The norm of the two-scale limit

The oscillations captured by the two-scale limit u_0 may cause the norm of u_0 to become larger than the norm of the weak $L^2(\Omega)$-limit u.

Theorem 3.26 (Lukkassen, Nguetseng & Wall, 2002)

Let $\{u^\varepsilon\}$ be a sequence in $L^2(\Omega)$ that two-scale converges to $u_0 \in L^2(\Omega \times Y)$. Then we have

$$\lim_{\varepsilon \to 0} \|u^\varepsilon\|_{L^2(\Omega)} \geq \|u_0\|_{L^2(\Omega \times Y)} \geq \|u\|_{L^2(\Omega)}, \tag{3.61}$$

where $u(x) = \int_Y u_0(x, y)dy$ and that $u^\varepsilon(x) \rightharpoonup u(x) \quad \text{in} \quad L^2(\Omega)$.

We will give a short proof for this theorem. We refer the reader to other proofs in Zhikov (2000), Theorem 17 of Lukkassen, Nguetseng & Wall (2002) and Theorem 1.24 of Ganesh & Nandakumaran (2010).

Proof

Note that $u^\varepsilon(x)$ can be written as

$$u^\varepsilon(x) = u(x) + \tilde{u}^\varepsilon(x) \tag{3.62}$$

where

$$\tilde{u}^\varepsilon(x) \rightharpoonup 0 \quad \text{in} \quad L^2(\Omega), \tag{3.63}$$

and that a similar decomposition

$$u_0(x, y) = u(x) + \tilde{u}(x, y) \tag{3.64}$$

where

$$\int_Y \tilde{u}(x, y)dy = 0, \tag{3.65}$$

is possible for the two-scale limit. Here u provides the global tendency while $\tilde{u}^\varepsilon$ and $\tilde{u}$ reflect the rapid oscillations.

It is a well-known result in $L^2(\Omega)$ weak convergence theory that if a sequence

$\{u_\varepsilon\}$ converges weakly to u, then

$$\|u\|_{L^2(\Omega)} \leq \liminf_{\varepsilon\to 0} \|u_\varepsilon\|_{L^2(\Omega)}. \tag{3.66}$$

Considering the classical inequality (3.66), it holds that

$$\liminf_{\varepsilon\to 0} \|u_\varepsilon\|^2_{L^2(\Omega)} = \liminf_{\varepsilon\to 0} \int_\Omega u^2(x) + 2u(x)\tilde{u}^\varepsilon(x) + (\tilde{u}^\varepsilon(x))^2 dx \tag{3.67}$$

where (3.63) makes the middle term vanish as $\varepsilon \to 0$, and hence

$$\liminf_{\varepsilon\to 0} \|u^\varepsilon\|^2_{L^2(\Omega)} = \int_\Omega u^2(x) + \liminf_{\varepsilon\to 0} \int_\Omega (\tilde{u}^\varepsilon(x))^2 dx \geq \|u\|^2_{L^2(\Omega)}. \tag{3.68}$$

When the inequality is strict, this must be due to the oscillations $\tilde{u}^\varepsilon(x)$. If

$$\lim_{\varepsilon\to 0} \int_\Omega (\tilde{u}^\varepsilon(x))^2 dx = 0, \tag{3.69}$$

we have from (3.68) that

$$\lim_{\varepsilon\to 0} \|u^\varepsilon\|_{L^2(\Omega)} = \|u\|_{L^2(\Omega)} \tag{3.70}$$

and thus a strong convergence in $L^2(\Omega)$.

In a similar way, with

$$\begin{aligned} \|u_0\|^2_{L^2(\Omega\times Y)} &= \int_\Omega \int_Y u_0^2(x,y) dy dx \\ &= \int_\Omega u^2(x) dx + \int_\Omega \int_Y 2u(x)\tilde{u}(x,y) dy dx + \int_\Omega \int_Y \tilde{u}^2 dy dx. \end{aligned}$$

The middle term will go to zero due to (3.65) and we get for the second inequality in (3.68).

$$\|u_0\|^2_{L^2(\Omega\times Y)} = \int_\Omega u^2(x) dx + \int_\Omega \int_Y \tilde{u}^2 dy dx \geq \|u\|^2_{L^2(\Omega)} \tag{3.71}$$

The appearance of a strict inequality depends on $\tilde{u}$, that is, on the oscillations captured by the two-scale limit.

Finally, we consider the left-hand inequality

$$\liminf_{\varepsilon\to 0} \|u^{\varepsilon}\|_{L^2(\Omega)} \geq \|u_0\|_{L^2(\Omega\times Y)} \tag{3.72}$$

which can, using (3.68) and (3.71), be expressed as

$$\begin{aligned}\int_{\Omega} u^2(x)dx + \liminf_{\varepsilon\to 0}\int_{\Omega}(\tilde{u}^{\varepsilon}(x))^2 dx &\geq \int_{\Omega} u^2(x)dx + \int_{\Omega}\int_{Y}\tilde{u}^2 dydx\\ \Rightarrow \qquad \liminf_{\varepsilon\to 0}\int_{\Omega}(\tilde{u}^{\varepsilon}(x))^2 dx &\geq \int_{\Omega}\int_{Y}\tilde{u}^2 dydx \end{aligned}\tag{3.73}$$

and thus

$$\liminf_{\varepsilon\to 0} \|\tilde{u}^{\varepsilon}\|_{L^2(\Omega)} \geq \|\tilde{u}\|_{L^2(\Omega\times Y)}. \tag{3.74}$$

This means that the limit of the $L^2(\Omega)$-norm of the oscillations $\tilde{u}^{\varepsilon}$ will always be greater than or equal to the $L^2(\Omega\times Y)$-norm for the oscillations $\tilde{u}$ of the two-scale limit.

For the special case when

$$u^{\varepsilon}(x) = \hat{u}\left(\frac{x}{\varepsilon}\right),$$

where $\hat{u} \in L^2(\Omega; C_{per}(Y))$, we have by Proposition **??**, that

$$u^{\varepsilon}(x) = \hat{u}\left(\frac{x}{\varepsilon}\right) \overset{2}{\rightharpoonup} \hat{u}(x,y) = u_0(x,y).$$

Rewriting (3.64) we get, for u the weak $L^2(\Omega)$-limit to u^{ε},

$$\tilde{u}(x,y) = \hat{u}(x,y) - u(x)$$

which means that $\tilde{u} \in L^2(\Omega; C_{per}(Y))$ and from (iv) in Proposition 3.15 we obtain,

$$\int_{\Omega}(\tilde{u}^{\varepsilon}(x))^2 dx = \tilde{u}^2\left(x, \frac{x}{\varepsilon}\right)dx \to \int_{\Omega}\int_{Y}\tilde{u}^2(x,y). \tag{3.75}$$

The oscillations contained in $\tilde{u}^\varepsilon$ and $\tilde{u}$, respectively, are of the same magnitude and we get

$$\liminf_{\varepsilon \to 0} \|u^\varepsilon\|_{L^2(\Omega)} = \|\hat{u}\|_{L^2(\Omega \times Y)} = \|u_0\|_{L^2(\Omega \times Y)}. \tag{3.76}$$

In this case there is a perfect match between the oscillations in u^ε and those of the test function $v(x, \frac{x}{\varepsilon})$ and the two-scale limit captures the oscillations in u^ε completely. The same holds true for any $\hat{u} \in \Psi(\Omega, Y)$.

Theorem 3.27

Let $\{u_\varepsilon\}$ be a sequence in $L^p(\Omega)$ which two-scale converges to $u \in L^p(\Omega \times Y)$. Then

$$\liminf_{\varepsilon \to 0} \|u^\varepsilon\|_{L^p(\Omega)} \geq \|u_0\|_{L^p(\Omega \times Y)} \geq \|v\|_{L^p(\Omega)}, \tag{3.77}$$

where $v(x) = \int_Y u(x, y)dy$. See also Theorem 1.24 of Ganesh & Nandakumaran (2010) and Flodén (2009) for more details.

The Oscillating Test Function Method

In this section, we give an overview of the oscillating test function method and use it to homogenize the general elliptic equation. The oscillating test function method (also known as the energy method) is a very elegant and efficient method for rigorously homogenizing partial differential equations. This method was introduced by Tartar (1977). It is very general and does not require any geometric assumptions on the behaviour of the partial differential equation coefficients neither periodicity nor statistical properties like ergodicity.

We give an overview of the method and refer the reader to Tartar (2009), Allaire (2012) and Emereuwa (2015) for a more detailed account of this method.

Given the elliptic equation

$$-\nabla_x \cdot \left(A\left(\frac{x}{\varepsilon}\right) \nabla u_\varepsilon \right) = f \text{ in } \Omega$$
$$u_\varepsilon = 0 \text{ on } \partial\Omega, \tag{3.78}$$

where $f(x) \in L^2(\Omega)$ is the source term.

Applying the Lax-Milgram theorem, (3.78) admits a unique solution u_ε in the space $H_0^1(\Omega)$ which satisfies the apriori estimate

$$\|u_\varepsilon\|_{H_0^1(\Omega)} \leq C\|f\|_{L^2(\Omega)}, \tag{3.79}$$

where C is a positive constant independent of ε.

This implies that the sequence u_ε, indexed by a sequence of periods $\varepsilon \to 0$ is bounded in the sobolev space $H_0^1(\Omega)$. Therefore up to a subsequence it converges weakly to a limit u in $H_0^1(\Omega)$.

To obtain (3.79), we multiply (3.78) by u_ε and integrate by parts then apply the Poincaré inequality. The variational formulation of (3.78) is given by

$$\int_\Omega A(\frac{x}{\varepsilon})\nabla u_\varepsilon(x) \cdot \nabla\varphi(x)dx = \int_\Omega f(x)\varphi(x)dx, \tag{3.80}$$

where $\varphi \in H_0^1(\Omega)$ is any test function. By the apriori in (3.79) a subsequence still denoted by ε can be extracted, such that u_ε converges weakly to a limit u in $H_0^1(\Omega)$. However, the left hand side of (3.80) involves the product of two weakly converging sequences in $L^2(\Omega)$, $A(\frac{x}{\varepsilon})$ and $\nabla u_\varepsilon(x)$, for which their limit may not converge to their product as its weak limit. We then need further arguments to pass to the limit in (3.80).

The idea of this method is to replace the fixed test function φ in (3.80) by a weakly converging sequence φ_ε (the so-called oscillating test function), chosen in a way such that the left hand side of (3.80) can pass to its limit. This phenomenon is an example of the compensated compactness theory which was developed by Murat (1978) and Tartar (1979), which allows one to pass to the limit in some products of weak convergences under additional conditions. The key idea of this method is the choice of the oscillating test function φ_ε.

Let $\varphi(x) \in D(\Omega)$ be a smooth function with compact support in Ω. Taking the first two terms of the asymptotic expansion of u_ε, the oscillating test function φ_ε is defined by:

$$\varphi_\varepsilon(x) = \varphi(x) + \varepsilon \sum_{i=1}^{N} \frac{\partial \varphi}{\partial x_i}(x) w_i^*(\frac{x}{\varepsilon}), \tag{3.81}$$

where $w_i^*(y)$ are solutions of the dual cell problems defined by

$$\begin{cases} -\nabla_y \cdot (A^t(y)(e_i + \nabla_y w_i^*)) = 0 \quad \text{in } Y \\ y \mapsto w_i^*(y) \quad Y - \text{periodic.} \end{cases} \tag{3.82}$$

Lemma 3.28

Let $w(x,y)$ be a continuous function in x, square integrable and Y -periodic in y, i.e. $w(x,y) \in L^2_\sharp(Y, C(\Omega))$. Then, the sequence $w(x, \frac{x}{\varepsilon})$ converges weakly in $L^2(\Omega)$ to $\int_\Omega w(x,y)dy$.

By periodicity in Y of w_i^*, it is easily seen that $\varepsilon w_i^*(\frac{x}{\varepsilon})$ is a bounded sequence in $H^1(\Omega)$ which converges weakly to zero.

We replace φ by φ_ε which is the oscillating test function in (3.80) to obtain

$$\int_\Omega A(\frac{x}{\varepsilon}) \nabla u_\varepsilon(x) \cdot \nabla \varphi_\varepsilon(x) dx = \int_\Omega f(x) \varphi_\varepsilon(x) dx. \tag{3.83}$$

To take advantage of equation (3.82), we develop and integrate (3.83) by parts. We remark that

$$\nabla \varphi_\varepsilon = \sum_{i=1}^{N} \frac{\partial \varphi(x)}{\partial x_i} \left(e_i + \nabla_y w_i^* (\frac{x}{\varepsilon}) \right) + \varepsilon \sum_{i=1}^{N} \frac{\partial \nabla \varphi}{\partial x_i} w_i^* (\frac{x}{\varepsilon}). \tag{3.84}$$

Substituting (3.84) into the left hand side of (3.83) gives

$$\begin{aligned}
&\int_{\Omega} A(\frac{x}{\varepsilon})\nabla u_{\varepsilon}(x)\cdot\nabla\varphi_{\varepsilon}(x)dx \\
&= \int_{\Omega}\Big(A(\frac{x}{\varepsilon})\nabla u_{\varepsilon}(x)\cdot\sum_{i=1}^{N}\frac{\partial\varphi(x)}{\partial x_i}(e_i+\nabla_y w_i^*(\frac{x}{\varepsilon})) \\
&+\varepsilon\sum_{i=1}^{N}\frac{\partial\nabla\varphi}{\partial x_i}w_i^*(\frac{x}{\varepsilon})\Big)dx \qquad (3.85)\\
&= \int_{\Omega} A(\frac{x}{\varepsilon})\nabla u_{\varepsilon}(x)\cdot\sum_{i=1}^{N}\frac{\partial\varphi(x)}{\partial x_i}(e_i+\nabla_y w_i^*(\frac{x}{\varepsilon}))dx \\
&+\varepsilon\int_{\Omega} A(\frac{x}{\varepsilon})\nabla u_{\varepsilon}(x)\cdot\sum_{i=1}^{N}\frac{\partial\nabla\varphi}{\partial x_i}w_i^*(\frac{x}{\varepsilon})dx.
\end{aligned}$$

Making the substitution $I = I_1 + I_2$ where

$$I_1 = \int_{\Omega} A(\frac{x}{\varepsilon})\nabla u_{\varepsilon}(x)\cdot\sum_{i=1}^{N}\frac{\partial\varphi(x)}{\partial x_i}(e_i+\nabla_y w_i^*(\frac{x}{\varepsilon}))dx \qquad (3.86)$$

and

$$I_2 = \varepsilon\int_{\Omega} A(\frac{x}{\varepsilon})\nabla u_{\varepsilon}(x)\cdot\sum_{i=1}^{N}\frac{\partial\nabla\varphi}{\partial x_i}w_i^*(\frac{x}{\varepsilon})dx. \qquad (3.87)$$

Solving the equations term by term, we see that I_2 is bounded by a constant and thus it will be approaching zero as $\varepsilon \to 0$. Whereas,

$$\begin{aligned}
I_1 &= \int_{\Omega} A(\frac{x}{\varepsilon})\nabla u_{\varepsilon}(x)\cdot\sum_{i=1}^{N}\frac{\partial\varphi(x)}{\partial x_i}(e_i+\nabla_y w_i^*(\frac{x}{\varepsilon}))dx \\
&= -\int_{\Omega} u_{\varepsilon}(x)\nabla\cdot\Big(A^t(\frac{x}{\varepsilon})\sum_{i=1}^{N}\frac{\partial\varphi(x)}{\partial x_i}(e_i+\nabla_y w_i^*(\frac{x}{\varepsilon}))\Big)dx
\end{aligned} \qquad (3.88)$$

Simplifying the divergence which is a function of x and $y = \frac{x}{\varepsilon}$,

$$\begin{aligned}
&\nabla\cdot\Big(A^t(\frac{x}{\varepsilon})\sum_{i=1}^{N}\frac{\partial\varphi(x)}{\partial x_i}(e_i+\nabla_y w_i^*(\frac{x}{\varepsilon}))\Big)dx \\
&= \sum_{i=1}^{N}\frac{\partial\nabla\varphi(x)}{\partial x_i}\cdot A^t(y)(e_i+\nabla_y w_i^*(y)) \qquad (3.89)\\
&+\frac{1}{\varepsilon}\sum_{i=1}^{N}\frac{\partial\varphi(x)}{\partial x_i}\nabla_y\cdot A^t(Y)(e_i+\nabla_y w_i^*(y))).
\end{aligned}$$

By (3.82), the second term of (3.89) is zero and we obtain,

$$\begin{aligned}\nabla \cdot \Big(A^t(\frac{x}{\varepsilon}) \sum_{i=1}^{N} \frac{\partial \varphi(x)}{\partial x_i}(e_i + \nabla_y w_i^*(\frac{x}{\varepsilon})) \Big) dx \\ = \sum_{i=1}^{N} \frac{\partial \nabla \varphi(x)}{\partial x_i} \cdot A^t(y)(e_i + \nabla_y w_i^*(y))\end{aligned} \tag{3.90}$$

Therefore the divergence is bounded in $L^2(\Omega)$ and since it is a periodically oscillating function, it converges weakly to its mean value. We then obtain

$$I_1 = -\int_\Omega u_\varepsilon(x) \sum_{i=1}^{N} \frac{\partial \nabla \varphi(x)}{\partial x_i} \cdot A^t(y)(e_i + \nabla_y w_i^*(y)) \tag{3.91}$$

The left hand of (3.83) reduces to

$$I = -\int_\Omega u_\varepsilon(x) \sum_{i=1}^{N} \frac{\partial \nabla \varphi(x)}{\partial x_i} \cdot A^t(y)(e_i + \nabla_y w_i^*(y)). \tag{3.92}$$

The main point of this simplification is that we are now able to pass to the limit in the right hand side of (3.88). Since u_ε is bounded in $H_0^1(\Omega)$, by application of Rellich theorem, there exists a subsequence (still indexed by ε for simplicity) and a limit $u \in H_0^1(\Omega)$ such that u_ε converges strongly to u in $L^2(\Omega)$. The right hand side of (3.88) is the product of a weak convergence and a strong one $\{u_\varepsilon\}$, and thus its limit is the product of the two limits. In other words,

$$\begin{aligned}&\lim_{\varepsilon\to 0} \int_\Omega A(\frac{x}{\varepsilon}) \nabla u_\varepsilon(x) \cdot \nabla \varphi_\varepsilon(x) \\ &= -\int_\Omega u(x) \nabla_x \cdot \Big(\int_Y A^t(y) \sum_{i=1}^{N} \frac{\partial \varphi}{\partial x_i}(x)(e_i + \nabla_y w_i^*(y)) dy \Big) dx \\ &= -\int_\Omega u(x) \nabla_x \cdot \Big(A^{*t} \nabla \varphi(x) \Big) dx,\end{aligned} \tag{3.93}$$

where

$$A^* = \int_Y A(y)(e_i + \nabla_y w_i^*(y)) dy.$$

Taking the limit after integrating by parts yields

$$\lim_{\varepsilon \to 0} \int_{\Omega} A(\frac{x}{\varepsilon}) \nabla u_{\varepsilon}(x) \cdot \nabla \varphi_{\varepsilon}(x) = \int_{\Omega} A^{*} \nabla u(x) \cdot \nabla \varphi(x) dx. \tag{3.94}$$

Hence we obtain the variational formulation of (3.83) as

$$\int_{\Omega} A^{*} \nabla u(x) \cdot \nabla \varphi(x) dx = \int_{\Omega} f(x) \varphi dx. \tag{3.95}$$

By density of smooth functions in $H_0^1(\Omega)$, (3.95) is valid for any test function $\varphi \in H_0^1(\Omega)$. Since A^* satisfies the same coercivity condition as A, Lax-Milgram lemma shows that (3.95) admits a unique solution in $H_0^1(\Omega)$. This last result proves that any subsequence of u_ε converges to the same limit u. Therefore, the entire sequence u_ε, and not only a subsequence, converges to the homogenized solution u.

G – Convergence

This is a method considered by Spagnolo's (1968) early work in this field for second order elliptic and parabolic operators. It was introduced in the late sixties and developed further by Murat (1978, 1997) Murat and Tartar (1997). See also Chiadó Piat, Dal Maso, Defranceschi (1990) and Pankov (1997).

The G means Green since this type of convergence corresponds roughly to the convergence of the associated Green functions.

It is a more general concept than the periodic homogenization developed to express the convergence of partial differential operators. A key difference is that the G- convergence does not include any technique for calculating the coefficient in the limit operator. The main result of the G-convergence is a compactness theorem in the homogenization theory which states that, for any bounded and uniformly coercive sequence of coefficients of a symmetric second order elliptic equation, there exist a subsequence and a G-limit (i.e. homogenized coefficients) such that, for any source term, the corresponding subsequence of solutions converges to the solution of the homogenized equation. In practical terms, it means that the mechanical properties of an heterogeneous medium (like its conductivity, or elastic moduli) can be well approximated by the properties of a homogeneous or homogenized medium if the size of the heterogeneities are small compared to the overall size of the medium. The G-convergence can be seen as a mathematically rigorous version of the so-called representative volume element method for computing effective or averaged parameters of heterogeneous media (Allaire, 2012). It is an operator convergence defined as follows. A sequence $\{A_\varepsilon\} \subset M(\alpha, \beta, \Omega)$ G-convergences to A_0 denoted by $A_\varepsilon \xrightarrow{G} A_0$ if and only if for any $g \in H^{-1}(\Omega)$ the u_ε of

$$\begin{aligned} -\nabla_x \cdot (A_\varepsilon \nabla u_\varepsilon) = g \quad \text{in } \Omega \\ u_\varepsilon = 0 \quad \text{on } \partial\Omega \end{aligned} \tag{3.96}$$

is such that $u_\varepsilon \rightharpoonup u_0$ in $H_0^1(\Omega)$, where u_0 is the unique solution of

$$\begin{aligned} -\nabla_x \cdot (A_0 \nabla u_0) = g \quad \text{in } \Omega \\ u_0 = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.97}$$

The matrix A_0 is called the G-limit of the sequence $\{A_\varepsilon\}$. The G-limit has properties that make (3.97) solvable. G- convergence handles problems with symmetric matrices only and periodicity is not a necessary condition.

H – Convergence

The H-convergence is a generalization of the G-convergence to the case of non-symmetric operators. The G-convergence is a notion of convergence associated to sequences of symmetric operators. The H stands for Homogenization since it is an important tool of that theory. For the sake of simplicity, we restrict ourselves to the case of symmetric operators (i.e. diffusion equations with symmetric coefficients). In such a case, G- and H-convergence coincide. Therefore in the sequel, we use only the notation G-convergence. A sequence of matrices H-converges to $\{A_0\}$ if it G-converges and in addition, for any $g \in H^{-1}(\Omega)$ we have

$$A_\varepsilon \nabla u_\varepsilon \rightharpoonup A_0 \nabla u_0 \quad \text{weakly in } (L^2(\Omega))^n. \tag{3.98}$$

where u_ε and u_0 are as in (3.97).

Let $A_\varepsilon \in M(a, b, \Omega)$, $B_\varepsilon \in M(c, d, \Omega)$ and χ_ε^i be a function with properties:

$$\chi_\varepsilon^i \rightharpoonup x_i \qquad \text{weakly in } \; H^1(\Omega)$$

$$A_\varepsilon \nabla \chi_\varepsilon^i \rightharpoonup A_0 e_i \qquad \text{weakly in } \; (L^2(\Omega))^n$$

$$\nabla \cdot A_\varepsilon(\nabla \chi_\varepsilon^i) \;\; \text{converges strongly in } \; H^{-1}(\Omega) \qquad \text{and}$$

$$D_\varepsilon e_i = \nabla \chi_i^\varepsilon.$$

Also let A_ε H-converge to A_0 then the following are true.

(i) There exists a $B^\sharp$ (depending only on $\{A_\varepsilon\}$ and $\{B_\varepsilon\}$) such that:

$$^t D_\varepsilon B_\varepsilon D_\varepsilon \rightharpoonup B^\sharp \quad \text{in } \; (\mathcal{D})^{n \times n}.$$

Definition 3.7

Let g_ε be a sequence in $H^{-1}(\Omega)$ which converges to g strongly in $H^{-1}(\Omega)$. If v_ε is the solution of

$$\begin{cases} -\nabla \cdot (A(x, \dfrac{x}{\varepsilon}) \nabla v_\varepsilon) = g_\varepsilon & \text{in} \quad \Omega, \\ v_0 = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.99}$$

then, there exists v_0 and a matrix A_0 such that

$$\begin{aligned} v\varepsilon &\rightharpoonup v_0 & &\text{weakly in} \quad H_1^0(\Omega) \\ A(x, \frac{x}{\varepsilon}) \nabla v_\varepsilon &\rightharpoonup A_0 \nabla v_0 & &\text{weakly in } (L^2(\Omega))^n. \end{aligned} \tag{3.100}$$

Here, $v_0 \in H_0^1(\Omega)$ is the unique solution of the homogenized problem

$$\begin{cases} -\nabla \cdot (A_0 \nabla v_0) = g & \text{in} \quad \Omega, \\ v_0 = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{3.101}$$

Further, the ijth entry of the matrix A_0 is given by

$$(A_0)_{ij} = \int_Y A(x,y)[\nabla_y \mu_i(x,y) + e_i] \cdot [\nabla_y \mu_j(x,y) + e_j] dy.$$

The function μ_i, for $1 \leq i \leq n$, is the solution of the cell problem.

$$\begin{cases} -\nabla_y (A(x,y)[\nabla_y u_i(x,y) + e_i]) = 0 & \text{in } Y, \\ y \mapsto \mu_i(x,y) & \text{is Y-periodic,} \end{cases} \tag{3.102}$$

where $\{e_1, e_2, \ldots, e_n\}$ is the standard basis $\mathbb{R}^n$.

Multiple Scale Expansion

We will use the multiple scales method to homogenize an elliptic partial differential equation with Dirichlet boundary conditions. Let

$$-\nabla_x \cdot (\mathbf{A}_\varepsilon \nabla_x u^\varepsilon) = f \;\text{ for }\; x \in \Omega \tag{3.103}$$
$$u^\varepsilon(x) = 0, \;\text{ for }\; x \in \partial\Omega,$$

where the matrix

$$\mathbf{A}_\varepsilon(\mathbf{x}) = \begin{pmatrix} a^\varepsilon_{11}(x) & a^\varepsilon_{12}(x) \\ a^\varepsilon_{21}(x) & a^\varepsilon_{22}(x) \end{pmatrix} = \mathbf{A}(\frac{x}{\varepsilon}).$$

We work out for the homogenized equation together with the cell problems. To do this we make the following assumptions.

We take $\Omega \in \mathbb{R}^2$, open, bounded with its boundary smooth. We will assume that the coefficients $\mathbf{A}(y) = \{a_{ij}(y)\}^2_{i,j=1}$ are smooth, 1-periodic and uniformly elliptic. Furthermore, let $f(x)$ be smooth and independent of ε. We collect all the assumptions made as below.

$$a_{ij}(y), f(x) \in C^\infty(\mathbb{R}^2), \;\; i,j = 1,2 \;\;, \tag{3.104}$$
$$a_{ij}(y + \hat{e}_k) = a_{ij}(y), \;\; i,j,k = 1,2 \;\;, \tag{3.105}$$
$$\sum_{i,j=1}^{2} a_{ij}(y)\xi_i\xi_j \geq \alpha|\xi|^2, \;\; \alpha > 0, \forall\, y \;\in Y \;\; \forall\, \xi \in \; \mathbb{R}^2. \tag{3.106}$$

In order to study the asymptotic behaviour of the solutions to the problem (3.103) an efficient technique consists in applying asymptotic expansions using multiple scales.

We thus assume a solution of the form:

$$u^\varepsilon = u_0(x, \frac{x}{\varepsilon}) + \varepsilon u_1(x, \frac{x}{\varepsilon}) + \varepsilon^2 u_2(x, \frac{x}{\varepsilon}) + \cdots \tag{3.107}$$

where $u_j(x,y), \;\; j = 0, 1, \ldots$ are periodic in y.

In order to solve equations of the form

$$\mathcal{A}_0 u = h, \text{where } \ u \ \text{ is } \ 1\text{-periodic} \tag{3.108}$$

and $\mathcal{A}_0 = -\nabla_y \cdot (\mathbf{A}\nabla_y)$, where $\mathbf{A}$ is a periodic matrix function, we need the following lemmas.

Lemma 3.29 (Pavliotis, 2007)

Let $F(y)$ be 1-periodic function and also a smooth. Then

$$\int_Y \nabla_y F(y) dy = 0. \tag{3.109}$$

This is a result of the fundamental theorem of calculus and the periodicity of Y.

Lemma 3.30

A necessary condition for the existence of a solution to equations of the form

$$\mathcal{A}_0 u = h, \tag{3.110}$$

u being 1-periodic is that

$$\int_Y h(y) dy = 0. \tag{3.111}$$

This is known as the solvability condition.

Proof

Let u be a solution to (3.110). Integrating the left hand side of (3.110) and using Lemma 3.29, we obtain

$$\begin{aligned} \int_Y \mathcal{A}_0 u dy &= -\int_Y \nabla_y (A \nabla_y u) dy, \\ &= 0 \ \text{ (by Lemma 3.29)}, \\ &= \int_Y h(y) dy. \end{aligned}$$

This makes sense only if (3.111) holds.

Corollary 3.31

Let $F(y)$ and $G(y)$ be smooth 1-periodic functions. Then using the integration by parts formula becomes:

$$\int_Y \Big(\nabla_y F(y)\Big) G(y) dy = -\int_Y F(y) \Big(\nabla_y G(y)\Big) dy. \tag{3.112}$$

Proposition 3.32

The only solutions of the homogeneous equation

$$\mathcal{A}_0 u = 0, \tag{3.113}$$

are constants in y.

Proof

Let u be a solution of (3.113). Multiplying through by u and integrating over Y using Corollary 3.31, we have

$$\begin{aligned} 0 &= \int_Y u\mathcal{A}_0 u dy \\ &= -\int_Y \nabla_y \cdot \Big(A\nabla_y u\Big) u \, dy. \end{aligned}$$

Applying Corollary 3.31, we obtain

$$\begin{aligned} 0 &= \int_Y \Big(A\nabla_y u\Big) \nabla_y u \, dy \\ &= \int_Y A|\nabla_y u|^2 dy \\ &\geq \alpha \int_Y |\nabla_y u|^2 dy \end{aligned}$$

$$\Rightarrow \int_Y |\nabla_y u|^2 dy \leq 0.$$

This is true only if u is a constant in y. Thus the only solutions of the homogeneous equation (3.113) are constants in y.

From the above proposition, we state and prove the following corollary

Corollary 3.33

All solutions of (3.110) differ by a constant in y.

Proof

Let u_1 and u_2 be two solutions of (3.110) and let $u = u_1 - u_2$. Then, u is also a solution to (3.110). Since $\mathcal{A}_0 u_1 = h;\ \mathcal{A}_0 u_2 = h$

$\mathcal{A}_0(u_1 - u_2) = h - h = 0$

$\mathcal{A}_0(u_1 - u_2) = \mathcal{A}_0 u = 0$

and so proposition 3.32 implies that $u = u_1 - u_2$ is a constant in y.

The assumption that all terms in the expansion above depend on both x and $\frac{x}{\varepsilon}$ explicitly is the underlying fact for one to use the multiple scale expansion.

Furthermore, since the coefficients of the PDE are periodic functions of $\frac{x}{\varepsilon}$, it is reasonable to expect that the solution is also a periodic function of its argument $\frac{x}{\varepsilon}$.

The variable x represents the "slow" (macroscopic) and $y = \frac{x}{\varepsilon}$ the "fast" (microscopic) scales of the problem. The scales are assumed to be separated. As $\varepsilon \to 0$ the variable y changes much more rapidly than x and we can think of x as being a constant, when looking at the problem on the microscopic scale.

The fact that $y = \frac{x}{\varepsilon}$ implies that the partial derivatives with respect to x_j become

$$\frac{\partial}{\partial x_j} \to \frac{\partial}{\partial x_j} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_j}, \quad j = 1, ..., d \tag{3.114}$$

In other words, the total derivative (abusing slightly notation) of a function $f^\varepsilon(x_j) := f(x_j, \frac{x_j}{\varepsilon})$ can be expressed as

$$\frac{df^\varepsilon(x_j)}{dx_j} = \frac{\partial f(x_j, y_j)}{\partial x_j}\bigg|_{y_j = \frac{x_j}{\varepsilon}} + \frac{1}{\varepsilon}\frac{\partial f(x_j, y_j)}{\partial y_j}\bigg|_{y_j = \frac{x_j}{\varepsilon}}, \tag{3.115}$$

where the notation $f(x, y)|_{y=z}$ is the value of $f(x, y)$ at $y = z$. In gradient form (3.115) becomes

$$\nabla_x f^\varepsilon = \nabla_x f + \frac{1}{\varepsilon}\nabla_y f. \tag{3.116}$$

Let $\mathcal{A}^\varepsilon = -\nabla_x \cdot \Big(\mathbf{A}_\varepsilon(\mathbf{x})\nabla_x\Big)$ in (3.103). Then plugging in (3.116) we get

$$\begin{aligned}\mathcal{A}^\varepsilon &= -\nabla_x \cdot (\mathbf{A}\nabla_x)\\ &= -(\nabla_x + \varepsilon^{-1}\nabla_y)\cdot \Big(\mathbf{A}(\nabla_x + \varepsilon^{-1}\nabla_y)\Big)\\ &= -\tfrac{1}{\varepsilon^2}[\nabla_y \cdot (\mathbf{A}\nabla_y)] - \tfrac{1}{\varepsilon}[\nabla_x \cdot (\mathbf{A}\nabla_y) + \nabla_y \cdot (\mathbf{A}\nabla_x)]\\ &\quad -[\nabla_x \cdot (\mathbf{A}\nabla_x)]\\ &= \tfrac{1}{\varepsilon^2}\mathcal{A}_0 + \tfrac{1}{\varepsilon}\mathcal{A}_1 + \mathcal{A}_2,\end{aligned} \tag{3.117}$$

where

$$\mathcal{A}_0 = -\nabla_y \cdot (\mathbf{A}\nabla_y) \tag{3.118}$$

$$\mathcal{A}_1 = -\nabla_y \cdot (\mathbf{A}\nabla_x) - \nabla_x \cdot (\mathbf{A}\nabla_y) \tag{3.119}$$

$$\mathcal{A}_2 = -\nabla_x \cdot (\mathbf{A}\nabla_x). \tag{3.120}$$

Substituting (3.117) into (3.103), we obtain

$$\begin{aligned}(\frac{1}{\varepsilon^2}\mathcal{A}_0 + \frac{1}{\varepsilon}\mathcal{A}_1 + \mathcal{A}_2)u^\varepsilon = f, &\quad \text{for } \; x \in \Omega\\ u^\varepsilon(x) = 0, &\quad \text{for } \; x \in \partial\Omega.\end{aligned} \tag{3.121}$$

By the power series expansion in (3.107) we have

$$\left(\frac{1}{\varepsilon^2}\mathcal{A}_0 + \frac{1}{\varepsilon}\mathcal{A}_1 + \mathcal{A}_2\right)\left(u_0(x,\frac{x}{\varepsilon}) + \varepsilon u_1(x,\frac{x}{\varepsilon}) + \varepsilon^2 u_2(x,\frac{x}{\varepsilon}) + \ldots\right) = f$$

$$\frac{1}{\varepsilon^2}\mathcal{A}_0 u_0 + \frac{1}{\varepsilon}(\mathcal{A}_0 u_1 + \mathcal{A}_1 u_0) + (\mathcal{A}_0 u_2 + \mathcal{A}_1 u_1 + \mathcal{A}_2 u_0) + \varepsilon(\mathcal{A}_1 u_2 + \mathcal{A}_2 u_1) + \varepsilon^2 \mathcal{A}_2 u_2 + \ldots = f$$

Equating the powers of ε of order -2, -1 and 0, the following sequence of problems is obtained:

$$\mathcal{A}_0 u_0 = 0 \tag{3.122}$$

$$\mathcal{A}_0 u_1 + \mathcal{A}_1 u_0 = 0 \tag{3.123}$$

$$\mathcal{A}_0 u_2 = -\mathcal{A}_1 u_1 - \mathcal{A}_2 u_0 + f \tag{3.124}$$

which we now solve to obtain the homogenized equation and the cell problem. From (3.122) and Proposition 3.32, we see that $\mathcal{A}_0 u_0 = 0$ implies that $u_0(x, y) = u_0(x)$. Thus, $u_0(x, y)$ is independent of y.

From (3.123), we have that

$$\begin{aligned}
\mathcal{A}_0 u_1 &= -\mathcal{A}_1 u_0 \\
&= -\mathcal{A}_1 u_0(x) \\
&= -[-\nabla_y \cdot \big(\mathbf{A}\nabla_x u_0(x)\big) - \nabla_x \cdot \big(\mathbf{A}\nabla_y u_0(x)\big)] \\
&= \nabla_y \cdot \big(\mathbf{A}\nabla_x u_0(x)\big) + \nabla_x \cdot \big(\mathbf{A}\nabla_y u_0(x)\big)
\end{aligned}$$

However, u_0 is a function of x and that $\nabla_x \cdot \Bigg(\mathbf{A}\nabla_y u_0(x)\Bigg) = 0$ which gives

$$\mathcal{A}_0 u_1 = \nabla_y \cdot \Bigg(\mathbf{A}\nabla_x u_0(x)\Bigg) \tag{3.125}$$

It is possible to solve for u_1 in (3.125) since it satisfies Lemma 3.30.

$$\begin{aligned}
\text{i.e.} \quad \int_Y \mathcal{A}_0 u_1 dy &= \int_Y \nabla_y \cdot \Bigg(\mathbf{A}(\mathbf{y})\nabla_x u_0(x)\Bigg) dy \\
&= \nabla_x u_0(x) \int_Y \nabla_y \cdot \Bigg(\mathbf{A}(\mathbf{y})\Bigg) dy = 0 \qquad (3.126)
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}_0 u_1 &= \frac{\partial u_0(x)}{\partial x_1}\frac{\partial \mathbf{A}(\mathbf{y})}{\partial y_1} + \frac{\partial u_0(x)}{\partial x_2}\frac{\partial \mathbf{A}(\mathbf{y})}{\partial y_2} \\
&= \begin{pmatrix} \frac{\partial u_0(x)}{\partial x_1} \\ \frac{\partial u_0(x)}{\partial x_2} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial \mathbf{A}(\mathbf{y})}{\partial y_1} \\ \frac{\partial \mathbf{A}(\mathbf{y})}{\partial y_2} \end{pmatrix} \\
&= \nabla_x u_0(x) \cdot \nabla_y \mathbf{A}(\mathbf{y}) \\
&= \nabla_x u_0(x) v(y).
\end{aligned}$$

From (3.125) the following deductions can be made

$$
\begin{aligned}
\mathcal{A}_0 u_1 &= \nabla_y \cdot \Big(\mathbf{A}\nabla_x u_0(x)\Big) \\
&= \begin{pmatrix} \frac{\partial}{\partial y_1} \\ \frac{\partial}{\partial y_2} \end{pmatrix} \cdot \left[\begin{pmatrix} a_{11}(y) & a_{12}(y) \\ a_{21}(y) & a_{22}(y) \end{pmatrix} \begin{pmatrix} \frac{\partial u_0}{\partial x_1} \\ \frac{\partial u_0}{\partial x_2} \end{pmatrix} \right] \\
&= \begin{pmatrix} \frac{\partial}{\partial y_1} \\ \frac{\partial}{\partial y_2} \end{pmatrix} \cdot \begin{pmatrix} a_{11}(y)\frac{\partial u_0}{\partial x_1} + a_{12}(y)\frac{\partial u_0}{\partial x_2} \\ a_{21}(y)\frac{\partial u_0}{\partial x_1} + a_{22}(y)\frac{\partial u_0}{\partial x_2} \end{pmatrix} \\
&= \frac{\partial}{\partial y_1}\left(a_{11}(y)\frac{\partial u_0}{\partial x_1} + a_{12}(y)\frac{\partial u_0}{\partial x_2}\right) + \frac{\partial}{\partial y_2}\left(a_{21}(y)\frac{\partial u_0}{\partial x_1} + a_{22}(y)\frac{\partial u_0}{\partial x_2}\right) \\
&= \frac{\partial u_0}{\partial x_1}\left(\frac{\partial a_{11}(y)}{\partial y_1} + \frac{\partial a_{21}(y)}{\partial y_2}\right) + \frac{\partial u_0}{\partial x_2}\left(\frac{\partial a_{12}(y)}{\partial y_1} + \frac{\partial a_{22}(y)}{\partial y_2}\right) \\
&= v_1(y)\frac{\partial u_0}{\partial x_1} + v_2(y)\frac{\partial u_0}{\partial x_2} = \sum_{j=1}^{2} v_j(y)\frac{\partial u_0}{\partial x_j}\,.
\end{aligned}
$$

Thus, by linearity and Proposition 3.32, $u_1(x, y)$ has a solution of the form,

$$
u_1(x, y) = v_1\frac{\partial u_0}{\partial x_1} + v_2\frac{\partial u_0}{\partial x_2} + \hat{u}_1(x). \tag{3.127}
$$

Substituting (3.127) into (3.125), we obtain

$$
\mathcal{A}_0\left(v_1\frac{\partial u_0}{\partial x_1} + v_2\frac{\partial u_0}{\partial x_2} + \hat{u}_1(x)\right) = \nabla_y \cdot \Big(\mathbf{A}\nabla_x u_0\Big).
$$

But $\mathcal{A}_0\hat{u}_1(x) = 0$ since $\mathcal{A}_0$ is a differential operator involving y only.

$$
\begin{aligned}
\therefore\ \mathcal{A}_0\left(v_1\frac{\partial u_0}{\partial x_1} + v_2\frac{\partial u_0}{\partial x_2}\right) &= \nabla_y \cdot \Big(\mathbf{A}\nabla_x u_0\Big) \\
\text{also, } \nabla_x u_0(x) = \frac{\partial u_0}{\partial x_1}e_1 + \frac{\partial u_0}{\partial x_2}e_2 & \\
\mathcal{A}_0\left(v_1\frac{\partial u_0}{\partial x_1} + v_2\frac{\partial u_0}{\partial x_2}\right) &= \nabla_y \cdot \left(\mathbf{A}(\frac{\partial u_0}{\partial x_1}e_1 + \frac{\partial u_0}{\partial x_2}e_2)\right) \\
\nabla_y \cdot \left(\mathbf{A}\nabla_y[v_1\frac{\partial u_0}{\partial x_1} + v_2\frac{\partial u_0}{\partial x_2}]\right) &= \nabla_y \cdot \left(\mathbf{A}(\frac{\partial u_0}{\partial x_1}e_1 + \frac{\partial u_0}{\partial x_2}e_2)\right).
\end{aligned}
$$

Expanding and comparing coefficients, we obtain the following equations which give the cell problems as;

$$\begin{aligned}\nabla_y\cdot\Big(\mathbf{A}\nabla_y v_1\Big) &= \nabla_y\cdot\mathbf{A}e_1,\\ \nabla_y\cdot\Big(\mathbf{A}\nabla_y v_2\Big) &= \nabla_y\cdot\mathbf{A}e_2,\end{aligned}\tag{3.128}$$

where $v_i = v_i(x,y)$ are their solutions.

We then solve (3.124). Averaging over the period Y, we have

$$\int_Y \mathcal{A}_0 u_2 dy = \int_Y(-\mathcal{A}_1 u_1 - \mathcal{A}_0 u_2 + f)dy.\tag{3.129}$$

By periodicity, $\int_Y \mathcal{A}_0 u_2 dy = 0$ and thus, we have

$$\int_Y(\mathcal{A}_2 u_0 + \mathcal{A}_1 u_1)dy = \int_Y f dy\tag{3.130}$$

$$\begin{aligned}f &= \int_Y(\mathcal{A}_2 u_0 + \mathcal{A}_1 u_1)dy\\ &= -\int_Y\Big[\nabla_y\cdot(\mathbf{A}\nabla_x u_1) + \nabla_x\cdot\big(\mathbf{A}\nabla_y u_1\big) + \nabla_x\cdot\big(\mathbf{A}\nabla_x u_0\big)\Big]dy\\ &= -\int_Y\Big[\nabla_x\cdot(\mathbf{A}\nabla_y u_1) + \nabla_x\cdot\big(\mathbf{A}\nabla_x u_0\big)\Big]dy.\end{aligned}$$

Since $\int_Y \nabla_y\cdot(\mathbf{A}\nabla_x u_1)dy = 0$.

$$\begin{aligned}f &= -\int_Y\Big[\nabla_x\cdot\big(\mathbf{A}\nabla_y(v_1\frac{\partial u_0}{\partial x_1} + v_2\frac{\partial u_0}{\partial x_2} + \hat{u}_1(x))\big) + \nabla_x\cdot\big(\mathbf{A}\nabla_x u_0\big)\Big]dy\\ &= -\int_Y\Big[\nabla_x\cdot\Big(\mathbf{A}\nabla_y(v_1\frac{\partial u_0}{\partial x_1} + v_2\frac{\partial u_0}{\partial x_2})\Big) + \nabla_x\cdot\big(\mathbf{A}(\frac{\partial u_0}{\partial x_1}e_1 + \frac{\partial u_0}{\partial x_2}e_2)\big)\Big]dy\\ &= -\nabla_x\cdot\int_Y\Big[\Big(\mathbf{A}\nabla_y(v_1\frac{\partial u_0}{\partial x_1} + v_2\frac{\partial u_0}{\partial x_2})\Big) + \Big(\mathbf{A}(\frac{\partial u_0}{\partial x_1}e_1 + \frac{\partial u_0}{\partial x_2}e_2)\Big)\Big]dy\\ &= -\nabla_x\cdot\Big(\int_Y\mathbf{A}(\mathbf{y})\big[\nabla_y v_1 + e_1\big]dy\frac{\partial u_0}{\partial x_1} + \int_Y\mathbf{A}(\mathbf{y})\big[\nabla_y v_2 + e_2\big]dy\frac{\partial u_0}{\partial x_2}\Big)\\ &= -\nabla_x\cdot\Big(\frac{\partial u_0}{\partial x_1}\int_Y\mathbf{A}(\mathbf{y})\big(\nabla_y v_1 + e_1\big)dy + \frac{\partial u_0}{\partial x_2}\int_Y\mathbf{A}(\mathbf{y})\big[\nabla_y v_2 + e_2\big]dy\Big).\end{aligned}$$

We let

$$\begin{pmatrix} b_{11}(x) \\ b_{12}(x) \end{pmatrix} = \int_Y \mathbf{A}(\mathbf{y})\big(\nabla_y v_1 + e_1\big)dy,$$
$$\begin{pmatrix} b_{21}(x) \\ b_{22}(x) \end{pmatrix} = \int_Y \mathbf{A}(\mathbf{y})\big(\nabla_y v_2 + e_2\big)dy, \tag{3.131}$$

and substitute back into the above equation to obtain

$$\begin{aligned} f &= -\nabla_x \cdot \left[\frac{\partial u_0}{\partial x_1}\begin{pmatrix} b_{11}(x) \\ b_{21}(x) \end{pmatrix} + \frac{\partial u_0}{\partial x_1}\begin{pmatrix} b_{12}(x) \\ b_{22}(x) \end{pmatrix}\right] \\ &= -\nabla_x \cdot \left[\begin{pmatrix} b_{11}(x) & b_{12}(x) \\ b_{21}(x) & b_{22}(x) \end{pmatrix}\begin{pmatrix} \frac{\partial u_0}{\partial x_1} \\ \frac{\partial u_0}{\partial x_1} \end{pmatrix}\right] \\ f &= -\nabla_x \cdot \{\mathbf{B}(\mathbf{x})\nabla_x u_0\}, \end{aligned} \tag{3.132}$$

which is the homogenized equation of (3.103), where

$$\mathbf{B}(\mathbf{x}) = \begin{pmatrix} b_{11}(x) & b_{12}(x) \\ b_{21}(x) & b_{22}(x) \end{pmatrix}. \tag{3.133}$$

To obtain the solution of the homogenized equation (3.132), the following steps will be taken. First, the cell problem (3.128) will be solved. Then, calculate the integrals in (3.131) and compute the effective coefficients $\mathbf{B}(\mathbf{x})$ by evaluating the integrals. Finally, we solve the homogenized equation (3.132) after substituting the results of the cell problem and the integrals into (3.132) to solve the homogenized equation. Depending of course on the domain Ω, it is not so difficult to solve the homogenized equation. The above shows that it is very advantageous to solve the homogenized equation (3.132) which does not contain the oscillating coefficients, as opposed to solving the original equation (3.103).

Chapter Summary

We defined in this chapter the two-scale convergence and gave some of its properties, stated other existing methods of homogenization and the multiple scale method as well, which is the method of homogenization to be used in the work.

CHAPTER FOUR

HOMOGENIZATION OF ELLIPTIC EQUATIONS IN THE DIVERGENCE FORM

Introduction

Elliptic equations of the divergence form are homogenized in this chapter. Specifically, we review the forms of the Reynolds equation and homogenize the time independent incompressible Reynolds equation using the multiple scale convergence method. Finally, the quasilinear elliptic equation with boundary conditions is also homogenized using the two-scale convergence method.

Homogenization of Reynolds Equations

In this section the concept of homogenization that enables efficient analysis of the effects of surface roughness representations obtained by measurements in applications modelled by the Reynolds equation is introduced.

Reynolds equation is the mathematical statement of the classical theory of lubrication. Physically, the Reynold's equation can be thought of as an expression of conservation principles for a system made up of lubricant flowing in between two surfaces which are parallel.

Examples of such applications are trust-bearings and journal- bearings. The numerical analysis of these types of applications requires an extremely dense computational mesh in order to resolve the surface roughness, suggesting some type of averaging.

The generalised Reynolds equation is given by

$$\nabla \cdot \left[\frac{\rho(p(x))h^3(x)}{12\eta}\nabla p(x)\right] = \frac{u_1+u_2}{2}\frac{\partial}{\partial x_1}\left[\rho(p(x))h(x)\right]. \tag{4.1}$$

The terms on the right hand side represent flows which are induced by motions of the bounding surfaces and shear induced flow by the sliding velocities u_1 and u_2 while the terms on the left-hand side of the equation are the flow due to pressure gradients across the entire domain. Reynolds equations are mostly applicable in the field of tribology.

Tribology is a multidisciplinary field, which deals with the science, practice and technology of lubrication, wear prevention and friction control in machines. This enable lubrication engineers to minimize cost of moving parts. In this way machinery can be made more efficient, more reliable and more cost effective. In the field of hydrodynamic lubrication, the flow of fluid through machine elements such as bearings, gearboxes and hydraulic systems may be governed by the Reynolds equation.

Reynolds equations are often used in analysing the influence of texture and surface roughness on the hydrodynamic performance of different machine elements when a lubricant is flowing through it. Figure 7 shows a bearing with two smooth surfaces s_1 and s_2 with a fluid flowing through the surfaces.

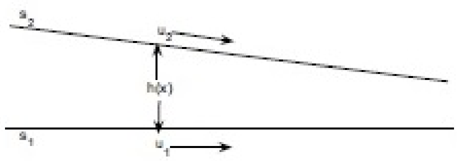

Figure 7: Bearing With Two Smooth Surfaces s_1 and s_2

The two surfaces through which a lubricant flows, may have any of the following characteristics:

(i) both surfaces are smooth and moving,

(ii) both surfaces are smooth and stationary,

(iii) both surfaces are rough and moving,

(iv) both surfaces are rough and stationary,

(v) one surface is rough and stationary while the other is smooth and moving,

(vi) one surface is rough and moving while the other is smooth and moving,

(vii) one surface is rough and stationary while the other is rough and moving,

(viii) one surface is smooth and moving while the other is smooth and stationary, among others

In Case (v), the governing Reynolds type equation will be time independent. This is due to the fact that the film thickness at any position x within the machine element remains the same at any time t. In Case (iii), due to the motion of the rough surfaces, the governing Reynolds equation will be time dependent. As a result of this motion, the film thickness h will be changing rapidly with respect to position x and time t, thus giving rise to a rapidly oscillating (changing) lubricant pressure within the machine element. In both cases, due to the surface roughness, the coefficient h in the Reynolds equation will be oscillating rapidly and therefore we may consider the possibility of solving the problem by using an averaging process, and here homogenization theory is a very useful method.

The generalized Reynolds equation is one of the models used for thin film lubricant flow when it is flowing between two parallel surfaces. The Reynolds equation is a simplified statement of several conservation principles. It is derived by solving the equation of continuity (expresses conservation of mass), $\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho v) = 0$, simultaneously with the simplified Navier-Stokes Equations (express conservation of linear momentum).

When a fluid flows between two surfaces s_1 and s_2, the governing equation is given by

$$\nabla \cdot \left[\frac{\rho(p(x))h^3(x)}{12\eta}\nabla p(x)\right] = \frac{u_1 + u_2}{2}\frac{\partial}{\partial x_1}\left[\rho(p(x))h(x)\right]. \tag{4.2}$$

Where,

u_1, u_2 as the velocities of the surfaces s_1 and s_2,

η is the viscosity of the lubricant,

ρ density of the lubricant,

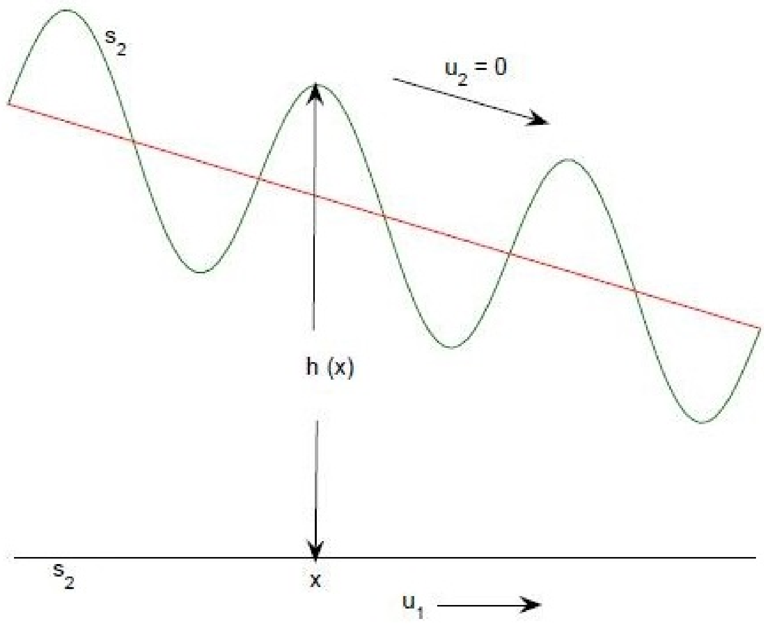

Figure 8: One Rough Stationary Surface and One Smooth Moving Surface

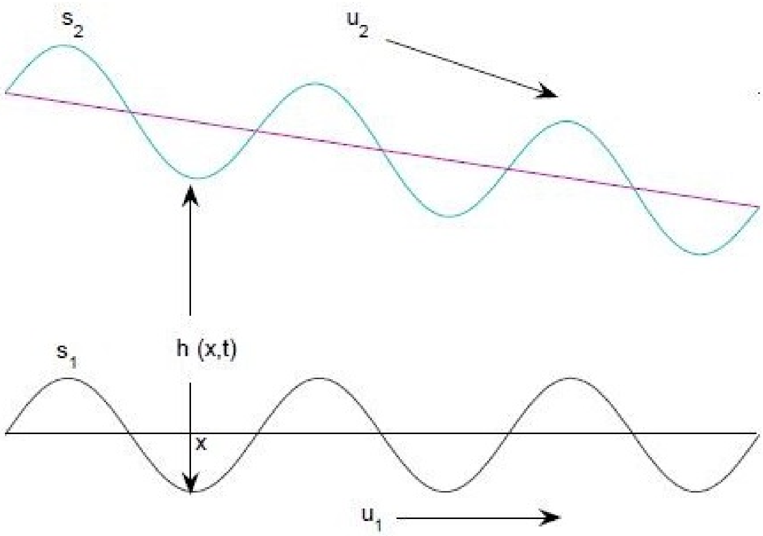

Figure 9: Both Surfaces Rough and Moving

$h(x)$ the film thickness between the two surfaces s_1 and s_2,

$p(x)$ the pressure built up between the surfaces when the lubricant passes through it, and

ε the roughness wavelength.

The bearing domain is Ω and the space variable $x \in \Omega \subset \mathbb{R}^2$. The density

of the lubricant ρ is a function of the pressure, i.e., $\rho = \rho(p(x))$. So that with a converging film thickness $h(x)$, the pressure also changes and as the pressure changes, the density also changes. Due to the periodic roughness of s_2, the film thickness will depend on the roughness wavelength ε. Where ε is a positive sequence which converges to zero as n increases. (For example, $\varepsilon = \frac{1}{2^n}$).
As a result, we replace $h(x)$ by $h_\varepsilon(x)$ in (4.2) to obtain

$$\nabla \cdot \left[\frac{\rho(p_\varepsilon(x))h_\varepsilon^3(x)}{12\eta}\nabla p_\varepsilon(x)\right] = \frac{u_1+u_2}{2}\frac{\partial}{\partial x_1}\left[\rho(p_\varepsilon(x))h_\varepsilon(x)\right]. \tag{4.3}$$

For, $h_\varepsilon(x) = h(x, \frac{x}{\varepsilon}) = h(x, y)$, $p_\varepsilon(x) = p(x, \frac{x}{\varepsilon}) = p(x, y)$ and ε describes how fast (rapid) the oscillations are. We then homogenize the system as $\varepsilon \to 0^+$.

Equation (4.3) is then the Reynolds equation, which takes into account the roughness contribution to the pressure build up in the bearing. Assuming that the rough surface is stationary, while the moving surface is smooth, then the film thickness $h_\varepsilon(x)$ at any position x within the bearing will remain the same at any time t and, hence, $h_\varepsilon(x)$ will be independent of time t. This explains why the Reynolds equation (4.3) does not involve time. Figure 9 is a pictorial description of case (iii) above. Here we consider the case where both surfaces are rough and moving. As a consequence of this motion, the film thickness will be changing rapidly, depending on the relative positions of the corresponding rough surfaces.

In Figure 10, we see that the film thickness $h_\varepsilon(x)$ at the position x is different for the two time steps t_1 and t_2. This is due to the relative positions of the corresponding rough surfaces. This shows clearly that the film thickness $h_\varepsilon(x)$ which is dependent on ε, is a function of both x and t in case (iii), i.e., $h_\varepsilon(x,t) = h(x, t, x/\varepsilon, t/\varepsilon) = h(x, t, y, \tau)$,

$p_\varepsilon(x,t) = p(x, t, x/\varepsilon, t/\varepsilon) = p(x, t, y, \tau)$, where $y = x/\varepsilon$ and $\tau = t/\varepsilon$.

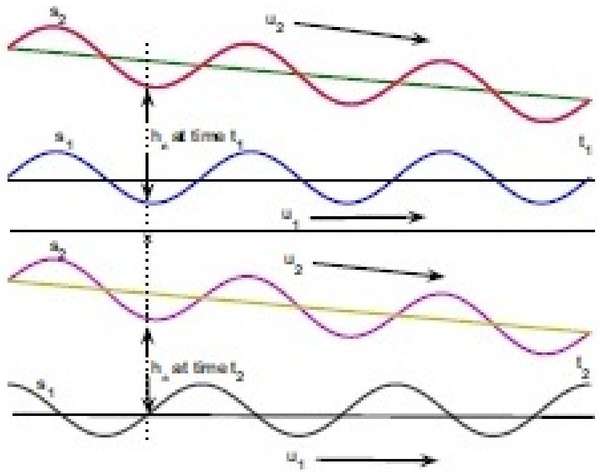

Figure 10: Time Dependent Surfaces in Motion

The Reynolds equation describing such a time dependent situation is given by

$$\frac{\partial}{\partial t}[\rho(p_\varepsilon(x,t))h_\varepsilon(x,t)] = \nabla \cdot \left[\frac{(\rho(p_\varepsilon(x,t)))h_\varepsilon^3(x,t)}{12\eta}\nabla p_\varepsilon(x,t)\right] - \left(\frac{u_1+u_2}{2}\right)\frac{\partial}{\partial x_1}[\rho(p_\varepsilon(x,t))h_\varepsilon(x,t)]. \quad (4.4)$$

In both the time independent and time dependent cases described above, we can deduce that the pressure varies rapidly due to the rapidly changing nature of the film thickness. As the roughness wavelength ε tends to zero, we expect to have a rapidly oscillating pressure. This means that we will need such a fine mesh that it is impossible to solve it directly with any numerical method.

Forms of the Reynolds Equation

1. Time independent (stationary) compressible Reynolds equation

$$\nabla \cdot \left(h_\varepsilon^3 \nabla w_\varepsilon(x)\right) = \lambda \frac{\partial}{\partial x_1}\left(w_\varepsilon(x) h_\varepsilon(x)\right) \text{ in } \Omega, \quad (4.5)$$

where $\lambda = 6\eta\mu\beta^{-1}$ and $w_\varepsilon(x)$ is a dimensionless density function as $w_\varepsilon(x) = \frac{\rho(p_\varepsilon(x))}{\rho_a}$.

2. Time independent (stationary) incompressible Reynolds equation

The compressibility of the fluid depends on $\rho\Big(p_\varepsilon(x)\Big)$. If $\rho(p_\varepsilon)$ is a constant then from equation (4.3) we have that

$$\begin{aligned}
\nabla\cdot\left[\frac{h_\varepsilon^3(x)}{12\eta}\nabla p_\varepsilon(x)\right] &= \frac{u_1+u_2}{2}\frac{\partial}{\partial x_1}\Big[h_\varepsilon(x)\Big] \\
\nabla\cdot h^3\nabla p &= 6\eta(u_1+u_2)\frac{\partial}{\partial x_1}\Big[h_\varepsilon(x)\Big] \\
\nabla\cdot h^3\nabla p &= \Lambda\frac{\partial}{\partial x_1}h_\varepsilon(x) \\
\nabla\cdot\Big(h_\varepsilon^3\nabla p_\varepsilon(x)\Big) &= \Lambda\frac{\partial}{\partial x_1}\Big(h_\varepsilon(x)\Big) \quad \text{on} \quad \Omega\subset\mathbb{R}^2, \qquad (4.6)
\end{aligned}$$

where $\Lambda = 6\eta u$ and $u = u_1+u_2$.

3. Time dependent (unstationary) compressible Reynolds equation

$$\gamma\frac{\partial}{\partial t}\Big(w_\varepsilon(x,t)h_\varepsilon(x,t)\Big) = \nabla\cdot\Big(h_\varepsilon^3(x,t)\nabla w_\varepsilon(x,t)\Big) - \lambda\frac{\partial}{\partial x_1}\Big(w_\varepsilon(x,t)h_\varepsilon(x,t)\Big) \qquad (4.7)$$

where $\gamma = 12\eta\beta^{-1}$ and $\lambda = 6\eta u\beta^{-1}$.

4. Time dependent (unstationary) incompressible Reynolds equation

If $\rho(p)$ is a constant, then we have that

$$\Gamma\frac{\partial}{\partial t}\Big(\rho h_\varepsilon(x,t)\Big) = \nabla\cdot\Big(h_\varepsilon^3(x,t)\nabla p_\varepsilon(x,t)\Big) - \Lambda\frac{\partial}{\partial x_1}h_\varepsilon(x,t). \qquad (4.8)$$

Linearization of Equation (4.3)

We note that the incompressible equations (4.3) and (4.4) are nonlinear. This makes them more difficult to solve. They could be linearized under the assumption that the dependence of ρ on pressure obeys the relation

$$\rho(p_\varepsilon(x)) = \rho_a e^{\frac{p_\varepsilon - p_a}{\beta}} \qquad (4.9)$$

Here, ρ is the atmospheric density of fluid at atmospheric pressure p_a

β the bulk modulus of the liquid, ($\beta > 0$).

This assertion is valid for reasonably low pressures.

We define a dimensionless density function $w_\varepsilon(x)$ as:

$$w_\varepsilon(x) = \frac{\rho(p_\varepsilon(x))}{\rho_a}. \tag{4.10}$$

Substituting (4.9) into (4.10), we obtain

$$w_\varepsilon(x) = \frac{\rho_a e^{\frac{p_\varepsilon - p_a}{\beta}}}{\rho_a} = e^{\frac{p_\varepsilon - p_a}{\beta}} \tag{4.11}$$

so that

$$\begin{aligned}
\nabla w_\varepsilon(x) &= e^{\frac{p_\varepsilon - p_a}{\beta}} \frac{\nabla p_\varepsilon}{\beta} \\
&= \frac{1}{\beta} e^{\frac{p_\varepsilon - p_a}{\beta}} \nabla p_\varepsilon \\
&= \frac{1}{\beta \rho_a} \rho_a e^{\frac{p_\varepsilon - p_a}{\beta}} \nabla p_\varepsilon \\
&= \frac{1}{\beta \rho_a} \rho(p_\varepsilon(x)) \nabla p_\varepsilon(x) \\
&= \beta^{-1} \rho_a^{-1} \rho(p_\varepsilon(x)) \nabla p_\varepsilon(x) \\
\Rightarrow \quad \beta \rho_a \nabla w_\varepsilon(x) &= \rho(p_\varepsilon(x)) \nabla p_\varepsilon(x). \qquad (4.12)
\end{aligned}$$

From (4.10), we have that

$$\rho\Big(p_\varepsilon(x)\Big) = \rho_a w_\varepsilon(x). \tag{4.13}$$

Substituting (4.12) and (4.13) into (4.3)

$$
\begin{aligned}
\nabla \cdot \left[\frac{\rho(p_\varepsilon(x))h_\varepsilon^3(x)}{12\eta}\nabla p_\varepsilon(x)\right] &= \frac{u_1+u_2}{2}\frac{\partial}{\partial x_1}\Big(\rho(p_\varepsilon(x))h_\varepsilon(x)\Big)\\
\nabla \cdot \left[\frac{\rho_a w_\varepsilon(x)h_\varepsilon^3(x)}{12\eta}\frac{\beta\rho_a \nabla w_\varepsilon(x)}{\rho_a(w_\varepsilon(x))}\right] &= \frac{u_1+u_2}{2}\frac{\partial}{\partial x_1}\Big(\rho_a w_\varepsilon(x)h_\varepsilon(x)\Big)\\
\nabla \cdot \Big(h_\varepsilon^3(x)\nabla w_\varepsilon(x)\Big)\frac{\beta\rho_a}{12\eta} &= \rho_a\frac{u_1+u_2}{2}\frac{\partial}{\partial x_1}\Big(w_\varepsilon(x)h_\varepsilon(x)\Big)\\
\nabla \cdot \Big(h_\varepsilon^3(x)\nabla w_\varepsilon(x)\Big)\frac{\beta}{12\eta} &= \frac{u_1+u_2}{2}\frac{\partial}{\partial x_1}\Big(w_\varepsilon(x)h_\varepsilon(x)\Big)\\
\nabla \cdot \Big(h_\varepsilon^3(x)\nabla w_\varepsilon(x)\Big) &= \frac{6\eta}{\beta}(u_1+u_2)\frac{\partial}{\partial x_1}\Big(w_\varepsilon(x)h_\varepsilon(x)\Big)\\
\nabla \cdot \Big(h_\varepsilon^3(x)\nabla w_\varepsilon(x)\Big) &= \lambda\frac{\partial}{\partial x_1}\Big(w_\varepsilon(x)h_\varepsilon(x)\Big) \qquad (4.14)
\end{aligned}
$$

where $\lambda = 6v\eta\beta^{-1}$ and $v = u_1 + u_2$.

Also, substituting (4.12) and (4.13) into (4.4)

$$
\begin{aligned}
\gamma\frac{\partial}{\partial t}\Big((w_\varepsilon(x,t)h_\varepsilon(x,t)\Big) = \nabla \cdot \Big(h_\varepsilon^3(x)\nabla w_\varepsilon(x)\Big)&\\
-\lambda\frac{\partial}{\partial x_1}\Big((w_\varepsilon(x,t)h_\varepsilon(x,t)\Big),& \qquad (4.15)
\end{aligned}
$$

where $\gamma = 12v\eta\beta^{-1}, \lambda = 6v\eta\beta^{-1}$ and $v = u_1 + u_2$.

Multiple Scale Expansion of Reynolds Equation

The Reynolds equation given by

$$
\nabla \cdot \Big(h_\varepsilon^3 \nabla w_\varepsilon(x)\Big) = \lambda\frac{\partial}{\partial x_1}\Big(w_\varepsilon(x)h_\varepsilon(x)\Big) \text{ in } \Omega \qquad (4.16)
$$

is used to describe the flow of thin fluids between two surfaces in relative motion. We will use the multiple scale expansion to derive the homogenized equation which does not contain fast oscillating roughness wavelength ε which can then be solved using any numerical method. We will however assume that the stationary surface is rough.

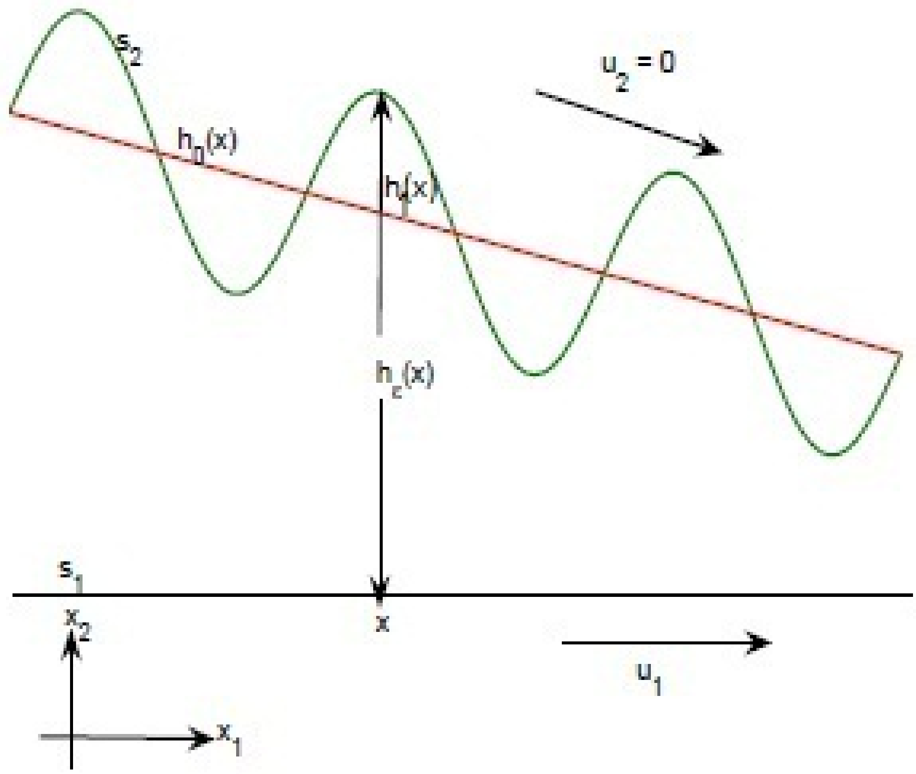

Figure 11: Bearing Geometry and Surface Roughness

It is possible to model the film thickness h_ε by

$$h_\varepsilon(x) = h(x, \frac{x}{\varepsilon}), \quad \varepsilon > 0.$$

To express the thickness of the film, we introduce the auxilliary function

$$h(x, y) = h_0(x) + h_1(y).$$

Here,

h_0 describes the global film thickness

h_1 represents the roughness contribution of the surface and is assumed to be periodic

ε describes the roughness wavelength.

We express the multiple scale expansion of the solution $w_\varepsilon(x)$ in the power series form as

$$w_\varepsilon(x) = w_0(x, \frac{x}{\varepsilon}) + \varepsilon w_1(x, \frac{x}{\varepsilon}) + \varepsilon^2 w_2(x, \frac{x}{\varepsilon}) + \cdots \tag{4.17}$$

where $w_i(x,y)$ is periodic, $\quad i = 1, 2, \ldots$

If $y_j = \frac{x_j}{\varepsilon_j}$ then applying the chain rule on the smooth function $\psi_\varepsilon(x) = \psi_\varepsilon(x, \frac{x}{\varepsilon})$

$$\frac{d\psi^\varepsilon(x_j)}{dx_j} = \frac{\partial \psi(x_j, y_j)}{\partial x_j}\bigg|_{y_j=\frac{x_j}{\varepsilon}} + \frac{1}{\varepsilon}\frac{\partial \psi(x_j, y_j)}{\partial y_j}\bigg|_{y_j=\frac{x_j}{\varepsilon}},$$

which we can re-write in the gradient form as

$$\nabla_x \psi = \nabla_x \psi + \frac{1}{\varepsilon}\nabla_y \psi. \tag{4.18}$$

Substituting equations (4.17) and (4.18) into equation (4.16) we obtain

$$\begin{aligned} & (\nabla_x + \frac{1}{\varepsilon}\nabla_y) \cdot [h^3(\nabla_x + \frac{1}{\varepsilon}\nabla_y)](w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \cdots) \\ = \; & \lambda\left(\frac{\partial}{\partial x_1} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_1}\right)(hw_0 + \varepsilon h w_1 + \varepsilon^2 h w_2 + \cdots) \end{aligned} \tag{4.19}$$

If we let

$$\mathcal{A}_0 = \nabla_y \cdot (h^3 \nabla_y) \tag{4.20}$$

$$\mathcal{A}_1 = \nabla_y \cdot (h^3 \nabla_x) + \nabla_x \cdot (h^3 \nabla_y) \tag{4.21}$$

$$\mathcal{A}_2 = \nabla_x \cdot (h^3 \nabla_x) \tag{4.22}$$

and expanding (4.19), we then obtain

$$\begin{aligned} & (\varepsilon^{-2}\mathcal{A}_0 + \varepsilon^{-1}\mathcal{A}_1 + \mathcal{A}_2)(w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \cdots) \\ & = \varepsilon^{-1}\lambda\frac{\partial}{\partial y_1}(hw_0) + \lambda\left(\frac{\partial}{\partial x_1}(hw_0) + \frac{\partial}{\partial y_1}(hw_1)\right) + \\ & \quad \varepsilon\lambda\left(\frac{\partial}{\partial y_1}(hw_2) + \frac{\partial}{\partial x_1}(hw_1)\right) + \varepsilon^2\lambda\frac{\partial}{\partial x_1}(hw_2) + \cdots \end{aligned} \tag{4.23}$$

Equating the powers of ε,

$$\begin{aligned} \mathcal{A}_0 w_0 &= 0, && (4.24)\\ \mathcal{A}_1 w_0 + \mathcal{A}_0 w_1 &= \lambda \frac{\partial}{\partial y_1}(h w_0), && (4.25)\\ \mathcal{A}_0 w_2 + \mathcal{A}_1 w_1 + \mathcal{A}_2 w_0 &= \lambda \left(\frac{\partial}{\partial x_1}(h w_0) + \frac{\partial}{\partial y_1}(h w_1) \right). && (4.26) \end{aligned}$$

We now solve (4.24)- (4.26).

From (4.20), the operator $\mathcal{A}_0$ involves derivatives with respect to y and therefore x is parameter in the solution of (4.20). So, we let $w_0(x, y) = w_0(x)$. (i.e., w_0 is a constant in y.)

From (4.25) we have that

$$\begin{aligned} \mathcal{A}_1 w_0 + \mathcal{A}_0 w_1 &= \lambda \frac{\partial}{\partial y_1}(h w_0) && (4.27)\\ \mathcal{A}_0 w_1 = \lambda \frac{\partial}{\partial y_1}(h w_0) &- \mathcal{A}_1 w_1. \end{aligned}$$

Substituting (4.20), (4.21) into (4.27) we have

$$\nabla_y \cdot (h^3 \nabla_y w_1) = \lambda \frac{\partial}{\partial y_1}(h w_0) - \nabla_y \cdot (h^3 \nabla_x w_0) - \nabla_x \cdot (h^3 \nabla_y w_0). \tag{4.28}$$

But w_0 is a function of x and therefore, $\nabla_x \cdot (h^3 \nabla_y w_0) = 0$. Thus we have that

$$\nabla_y \cdot (h^3 \nabla_y w_1) = \lambda \frac{\partial}{\partial y_1}(h w_0) - \nabla_y \cdot (h^3 \nabla_x w_0) \tag{4.29}$$

Since the right hand side of (4.29) consists of three terms, then by superposition, we expect that $w_1(x, y)$ should be a linear function of three terms. So we let

$$w_1(x, y) = \frac{\partial w_0}{\partial x_1} v_1(x, y) + \frac{\partial w_0}{\partial x_2} v_2(x, y) + w_0 v_3(x, y) \tag{4.30}$$

and write $v_i = v_i(x, y)$ for $i = 1, 2, 3$. Substituting (4.30) into (4.29) we have that

$$\nabla_y \cdot \left(h^3 \nabla_y \left(\tfrac{\partial w_0}{\partial x_1} v_1 + \tfrac{\partial w_0}{\partial x_2} v_2 + w_0 v_3 \right) \right) \\ = \lambda \tfrac{\partial}{\partial y_1}(h w_0) - \nabla_y \cdot (h^3 \nabla_x w_0). \tag{4.31}$$

But

$$\nabla_y \cdot (h^3 \nabla_x w_0) = \nabla_y \cdot \left(h^3 \frac{\partial w_0}{\partial x_1} e_1 + h^3 \frac{\partial w_0}{\partial x_2} e_2 \right) \tag{4.32}$$

where $\{e_1, e_2\}$ is the canonical basis in $\mathbb{R}^2$.

Substituting (4.30) and (4.32) into (4.31) we obtain

$$\nabla_y \cdot \left[h^3 \nabla_y \left(\frac{\partial w_0}{\partial x_1} v_1 + \frac{\partial w_0}{\partial x_2} v_2 + w_0 v_3 \right) \right] \\ = \lambda \frac{\partial}{\partial y_1}(h w_0) - \nabla_y \cdot \left(h^3 \frac{\partial w_0}{\partial x_1} e_1 + h^3 \frac{\partial w_0}{\partial x_2} e_2 \right) \tag{4.33}$$

Comparing the corresponding terms we obtain the following three local cell problems

$$\begin{aligned} \nabla_y \cdot \left(h^3 \nabla_y v_3 \right) &= \lambda \frac{\partial}{\partial y_1}(h) \\ \nabla_y \cdot \left(h^3 \nabla_y v_1 \right) &= -\nabla_y \cdot (h^3 e_1) \\ \nabla_y \cdot \left(h^3 \nabla_y v_2 \right) &= -\nabla_y \cdot (h^3 e_2). \end{aligned} \tag{4.34}$$

From (4.26) we have that

$$\mathcal{A}_0 w_2 + \mathcal{A}_1 w_1 + \mathcal{A}_2 w_0 = \lambda \frac{\partial}{\partial x_1}(h w_0) + \lambda \frac{\partial}{\partial y_1}(h w_1). \tag{4.35}$$

Averaging over the period Y we get

$$\int_Y \left(\mathcal{A}_0 w_2 + \mathcal{A}_1 w_1 + \mathcal{A}_2 w_0 - \lambda \frac{\partial}{\partial x_1}(h w_0) - \lambda \frac{\partial}{\partial y_1}(h w_1) \right) dy = 0. \tag{4.36}$$

Substituting (4.20)-(4.22) into (4.36) then

$$\int_Y \Big(\nabla_y \cdot (h^3 \nabla_y w_2) + \nabla_y \cdot (h^3 \nabla_x w_1) + \nabla_x \cdot (h^3 \nabla_y w_1) + \nabla_x \cdot (h^3 \nabla_x w_0) - \lambda \frac{\partial}{\partial x_1}(h w_0) - \lambda \frac{\partial}{\partial y_1}(h w_1) \Big) dy = 0 \quad (4.37)$$

which we rewrite as

$$\int_Y \Big(\nabla_y \cdot (h^3 \nabla_y w_2) + \nabla_x \cdot (h^3 \nabla_y w_1) + \nabla_y \cdot (h^3 \nabla_x w_1) + \nabla_x \cdot (h^3 \nabla_x w_0) \Big) dy = \int_Y \Big(\lambda \frac{\partial}{\partial x_1}(h w_0) + \lambda \frac{\partial}{\partial y_1}(h w_1) \Big) dy. \quad (4.38)$$

By periodicity, $\int_Y \nabla_y \cdot (h^3 \nabla_y w_2) dy = 0$ and we have that

$$\int_Y \Big(\nabla_x \cdot (h^3 \nabla_y w_1) + \nabla_y \cdot (h^3 \nabla_x w_1) + \nabla_x \cdot (h^3 \nabla_x w_0) \Big) dy = \int_Y \Big(\lambda \frac{\partial}{\partial x_1}(h w_0) + \lambda \frac{\partial}{\partial y_1}(h w_1) \Big) dy. \quad (4.39)$$

Now, $h^3 \nabla_x w_1$ and $h w_1$ are periodic in Y so that $\int_Y \nabla_y \cdot (h^3 \nabla_x w_1) dy = 0$ and that $\frac{\partial}{\partial y_1}(h w_1) dy = 0$.

Therefore (4.39) reduces to

$$\int_Y \Big(\nabla_x \cdot (h^3 \nabla_y w_1) + \nabla_x \cdot (h^3 \nabla_x w_0) - \lambda \frac{\partial}{\partial x_1}(h w_0) \Big) dy = 0 \quad (4.40)$$

Substituting (4.30) into (4.40) for w_1, we have

$$\int_Y \Big[\nabla_x \cdot \Big(h^3 \nabla_y \big(\frac{\partial w_0}{\partial x_1} v_1 + \frac{\partial w_0}{\partial x_2} v_2 + w_0 v_3 \big) \Big) + \nabla_x \cdot (h^3 \nabla_x w_0) - \lambda \frac{\partial}{\partial x_1}(h w_0) \Big] dy = 0 \quad (4.41)$$

$$\int_Y \Big[\nabla_x \cdot \Big(h^3 \nabla_y \big(\frac{\partial w_0}{\partial x_1} v_1 + \frac{\partial w_0}{\partial x_2} v_2 \big) \Big) + \nabla_x \cdot (h^3 \nabla_x w_0) + \nabla_x \cdot h^3 \nabla_y w_0 v_3 - \lambda \frac{\partial}{\partial x_1}(h w_0) \Big] dy = 0 \quad (4.42)$$

$$\Rightarrow \int_Y \left[\nabla_x \cdot \left(h^3 \nabla_y (\frac{\partial w_0}{\partial x_1} v_1 + \frac{\partial w_0}{\partial x_2} v_2)\right) + \nabla_x \cdot (h^3 \nabla_x w_0)\right] dy = \int_Y \left(\lambda \frac{\partial}{\partial x_1}(hw_0) - \nabla_x \cdot h^3 \nabla_y w_0 v_3\right) dy \quad (4.43)$$

and that

$$\int_Y \nabla_x \cdot \left[h^3 \nabla_y (\frac{\partial w_0}{\partial x_1} v_1 + \frac{\partial w_0}{\partial x_2} v_2)\right] dy + \int_Y \nabla_x \cdot (h^3 \nabla_x w_0) dy = \int_Y \left(\lambda \frac{\partial}{\partial x_1}(hw_0) - \nabla_x \cdot h^3 \nabla_y w_0 v_3\right) dy. \quad (4.44)$$

We note that

$$\begin{cases} \nabla_x w_0 = \frac{\partial w_0}{\partial x_1} e_1 + \frac{\partial w_0}{\partial x_2} e_2, \\ \lambda \frac{\partial}{\partial x_1}(hw_0) = \nabla_x \cdot \begin{pmatrix} \lambda h w_0 \\ 0 \end{pmatrix} \end{cases} \quad (4.45)$$

Substituting (4.45) into (4.44), we have

$$\int_Y \nabla_x \cdot \left[h^3 \nabla_y (\frac{\partial w_0}{\partial x_1} v_1 + \frac{\partial w_0}{\partial x_2} v_2)\right] dy + \int_Y \nabla_x \cdot \left[h^3 (\frac{\partial w_0}{\partial x_1} e_1 + \frac{\partial w_0}{\partial x_2} e_2)\right] dy = \int_Y \left(\lambda \frac{\partial}{\partial x_1}(hw_0) - \nabla_x \cdot h^3 \nabla_y w_0 v_3\right) dy. \quad (4.46)$$

Simplifying gives

$$\nabla_x \cdot \int_Y \left[h^3 \nabla_y (\frac{\partial w_0}{\partial x_1} v_1 + \frac{\partial w_0}{\partial x_2} v_2)\right] dy + \nabla_x \cdot \int_Y \left[h^3 (\frac{\partial w_0}{\partial x_1} e_1 + \frac{\partial w_0}{\partial x_2} e_2)\right] dy = \int_Y \left(\lambda \frac{\partial}{\partial x_1}(hw_0) - \nabla_x \cdot h^3 \nabla_y w_0 v_3\right) dy \quad (4.47)$$

$$\Rightarrow \quad \nabla_x \cdot \left[\frac{\partial w_0}{\partial x_1} \int_Y (h^3 e_1 + h^3 \nabla_y v_1) dy + \frac{\partial w_0}{\partial x_2} \int_Y (h^3 e_2 + h^3 \nabla_y v_2) dy\right] = \nabla_x \cdot \int_Y \left[\begin{pmatrix} \lambda h w_0 \\ 0 \end{pmatrix} - \begin{pmatrix} h^3 w_0 \frac{\partial v^3}{\partial y_1} \\ h^3 w_0 \frac{\partial v^3}{\partial y_2} \end{pmatrix}\right] dy, \quad (4.48)$$

We define a matrix function $\mathbf{B}(\mathbf{x}) = b_{ij}(x)$ in terms of v_1 and v_2 by

$$\begin{pmatrix} b_{11}(x) \\ b_{21}(x) \end{pmatrix} = \int_Y (h^3 e_1 + h^3 \nabla_y v_1) dy,$$
$$\begin{pmatrix} b_{12}(x) \\ b_{22}(x) \end{pmatrix} = \int_Y (h^3 e_2 + h^3 \nabla_y v_2) dy, \tag{4.49}$$

and $C(x) = (c_i(x))$ as a vector function defined in terms of v_3 by

$$\begin{pmatrix} c_1(x) \\ c_2(x) \end{pmatrix} = \begin{pmatrix} \int_Y \lambda h - h^3 \frac{\partial v^3}{\partial y_1} dy \\ \int_Y -h^3 \frac{\partial v^3}{\partial y_2} dy \end{pmatrix}. \tag{4.50}$$

Therefore (4.48) gives

$$\nabla_x \cdot \left[\frac{\partial w_0}{\partial x_1} \begin{pmatrix} b_{11}(x) \\ b_{21}(x) \end{pmatrix} + \frac{\partial w_0}{\partial x_2} \begin{pmatrix} b_{12}(x) \\ b_{22}(x) \end{pmatrix} \right] = \nabla_x \cdot w_0 \begin{pmatrix} \int_Y \lambda h - h^3 \frac{\partial v^3}{\partial y_1} dy \\ \int_Y -h^3 w_0 \frac{\partial v^3}{\partial y_2} dy \end{pmatrix},$$

which from (4.50) reduces to

$$\nabla_x \cdot \left[\frac{\partial w_0}{\partial x_1} \begin{pmatrix} b_{11}(x) \\ b_{21}(x) \end{pmatrix} + \frac{\partial w_0}{\partial x_2} \begin{pmatrix} b_{12}(x) \\ b_{22}(x) \end{pmatrix} \right] = \nabla_x \cdot w_0 \begin{pmatrix} c_1(x) \\ c_2(x) \end{pmatrix},$$

$$\nabla_x \cdot \left[\begin{pmatrix} b_{11}(x) & b_{12}(x) \\ b_{21}(x) & b_{22}(x) \end{pmatrix} \begin{pmatrix} \frac{\partial w_0}{\partial x_1} \\ \frac{\partial w_0}{\partial x_2} \end{pmatrix} \right] = \nabla_x \cdot w_0 \begin{pmatrix} c_1(x) \\ c_2(x) \end{pmatrix}. \tag{4.51}$$

The homogenized equation for (4.16) is then given by

$$\nabla_x \cdot [\mathbf{B}(\mathbf{x}) \nabla w_0] = \nabla_x \cdot [w_0 C(x)], \tag{4.52}$$

which describes the global behaviour of the solutions of (4.16) for a small value of ε. We then solve the homogenized equation which gives the approximate solution of the original equation given.

Homogenization of Reynold's Equation by Two-scale Convergence

Let Ω be an open bounded subset of $\mathbb{R}^2$ and Y the unit cube. Also let $h : \Omega \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be of the form $h(x,y) = h_0(x) + h_1(y)$, where $h_0 \in C(\overline{\Omega})$, $h_1 \in L^\infty(\mathbb{R}^N)$ and h_1 is a periodic function.

We also assume that there exists a constant $\alpha > 0$ such that $h(x,y) \geq \alpha$. Define

$$h_\varepsilon(x) = h(x, \frac{x}{\varepsilon}) = h_0(x) + h_1(\frac{x}{\varepsilon}). \tag{4.53}$$

Consider the Reynold's equation: Find $p_\varepsilon \in W_0^{1,2}$ such that

$$\text{div}(h_\varepsilon^3 \nabla p_\varepsilon) = \wedge \frac{\partial h_\varepsilon}{\partial x_1} \quad \text{on} \quad \Omega. \tag{4.54}$$

We choose $\phi = p_\varepsilon$ as test function in (4.54),

$$\int_\Omega h_\varepsilon^3 [\nabla p_\varepsilon]^2 dx = \wedge \int_\Omega h_\varepsilon \frac{\partial p_\varepsilon}{\partial x_1} dx \leq \wedge \int_\Omega h_\varepsilon \nabla p_\varepsilon dx.$$

By the assumption on h_0 and h_1, it follows that there exists a constant c such that

$$\begin{aligned}
\alpha \|\nabla p_\varepsilon\|^2_{L^2(\Omega:\mathbb{R}^2)} &\leq \int_\Omega h_\varepsilon^3 [\nabla p_\varepsilon]^2 dx = \wedge \int_\Omega h_\varepsilon \frac{\partial p_\varepsilon}{\partial x_1} dx \\
&\leq \wedge \int_\Omega h_\varepsilon \nabla p_\varepsilon dx \\
&\leq \wedge \|h_\varepsilon\|_{L^2(\Omega)} \|\nabla p_\varepsilon\|_{L^2(\Omega:\mathbb{R}^2)} \\
&\leq C \|\nabla p_\varepsilon\|_{L^2(\Omega:\mathbb{R}^2)}
\end{aligned}$$

$$\Rightarrow \|\nabla p_\varepsilon\|_{L^2(\Omega)} \leq c.$$

We take note that h_ε is bounded from the assumption that $h_0 \in C(\overline{\Omega}), h_1 \in L^\infty(\mathbb{R}^N)$.

By the Poincaré inequality, we have that

$$\|p_\varepsilon\|_{L^2(\Omega)} \leq c \|\nabla p_\varepsilon\|_{L^2(\Omega)^2} \leq c$$

which implies that the sequences $\{p_\varepsilon\}$ is bounded in $L^2(\Omega)$ and $\{\nabla p_\varepsilon\}$ is also bounded in $L^2(\Omega)^2$.

From this inequality, the sequences $\{p_\varepsilon\}$ and $\{\nabla p_\varepsilon\}$ is also bounded in the $L^2(\Omega)$ and $L^2(\Omega)^2$ respectively. By Theorem 3.23, there exists a subsequence such that $p_\varepsilon \overset{2}{\rightharpoonup} p_0(x,y)$ and $\nabla p_\varepsilon(x) \overset{2}{\rightharpoonup} \nabla p_0(x) + \nabla p_1(x,y)$, where $p_1 \in L^2[\Omega, W^{1,2}_{per}(Y)]$ and $p_0(x,y) = p_0(x) \in W^{1,2}_0(\Omega)$.
Moreover, by (3.25),

$$h^3(x, \frac{x}{\varepsilon})\nabla p_\varepsilon(x) \overset{2}{\rightharpoonup} h^3(x,y)[\nabla p_0(x) + \nabla_y p_1(x,y)]. \tag{4.55}$$

Since weak two-scale convergence implies weak convergence in $L^2(\Omega)$ we have that

$$h^3(x, \frac{x}{\varepsilon})\nabla p_\varepsilon(x) \rightarrow \int_Y h^3(x,y)[\nabla p_0(x) + \nabla_y p_1(x,y)]dy. \tag{4.56}$$

We can now pass to the limit (4.57).

By definition, p_ε is a solution of (4.54) if the following integral identity holds:

$$\int_\Omega (h^3_\varepsilon \nabla p_\varepsilon \cdot \nabla\phi)dx = \wedge \int_\Omega h_\varepsilon \frac{\partial \phi}{\partial x_1}dx, \quad \forall \phi \in W^{1,2}_0(\Omega) \tag{4.57}$$

and obtain

$$\int_\Omega \int_Y h^3(x,y)[\nabla p_0(x) + \nabla_y p_1(x,y)]dy \cdot \nabla\phi dx = \Lambda \int_\Omega \int_Y h(x,y)\frac{\partial \phi}{\partial x_1}dx, \tag{4.58}$$

for every ϕ in $C^\infty_0(\Omega)$.

We let $w_\varepsilon(x) = \varepsilon\psi(x)w(\frac{x}{\varepsilon})$ where $\psi \in C^\infty_0(\Omega)$ and $w \in C^\infty_0(Y)$. Then $w_\varepsilon(x) \in C^\infty_0(\Omega)$ and can thus be used as a test function in (4.57). Since w_ε is a product, we differentiate by the product rule to obtain

$$\begin{aligned} \nabla w_\varepsilon(x) &= \nabla[\varepsilon\phi(x)w(\frac{x}{\varepsilon})] \\ &= \varepsilon\phi(x)\nabla w(\frac{x}{\varepsilon})\cdot\frac{1}{\varepsilon}+\varepsilon\nabla\phi(x)w(\frac{x}{\varepsilon}) \\ &= \phi(x)\nabla w(\frac{x}{\varepsilon})+\varepsilon\nabla\phi(x)w(\frac{x}{\varepsilon}), \end{aligned} \tag{4.59}$$

which we substitute into (4.54) to obtain

$$\int_\Omega\Big(h^3(x,\frac{x}{\varepsilon})\nabla p_\varepsilon\cdot\big[\phi(x)\nabla w(y)+\varepsilon\nabla\phi(x)w(y)\big]\Big)dx=\wedge\int_\Omega h_\varepsilon\frac{\partial\phi}{\partial x_1}dx,\quad\forall\phi\in W_0^{1,2}(\Omega). \tag{4.60}$$

As $\varepsilon\to 0$, (4.60) then reduces to

$$\lim_{\varepsilon\to 0}\int_\Omega h^3(x,\frac{x}{\varepsilon})\nabla p_\varepsilon\cdot\phi(x)\nabla w(\frac{x}{\varepsilon})dx=\lim_{\varepsilon\to 0}\int_\Omega h^3(x,\frac{x}{\varepsilon})dx. \tag{4.61}$$

We note however that $w(x/\varepsilon)$ is a rapidly oscillating function, it converges to its mean value

$$\frac{\partial}{\partial x_1}w(x/\varepsilon)dx\to\frac{1}{|Y|}\int_Y\frac{\partial w}{\partial y_1}(y)dy \tag{4.62}$$

Since $\phi\in C_0^\infty(\Omega)$ is arbitrary, we have that $p_1(x,y)$ is a solution of the periodic problem, we then find $p_1\in L^2(\Omega;W^{1,2}_{per}(Y))$ such that

$$\begin{aligned} &\int_Y h^3(x,\frac{x}{\varepsilon})[\nabla p_0(x)+\nabla_y p_1(x,y)]\cdot\nabla w(y)dy \\ &\qquad=\wedge\int_\Omega\int_Y h(x,y)\frac{\partial w}{\partial y_1}(y)dxdy\ \text{ for any }\ w\in C^\infty_{per}(Y). \end{aligned} \tag{4.63}$$

We expand to obtain

$$\begin{aligned} &\int_Y h^3(x,\frac{x}{\varepsilon})\nabla p_0(x)\nabla w(y)dy+\int_Y h^3(x,\frac{x}{\varepsilon})\nabla_y p_1(x,y)\nabla w(y)dy \\ &\qquad=\wedge\int_\Omega\int_Y h(x,y)\frac{\partial w}{\partial y_1}(y)dxdy\ \text{ for any }\ w\in C^\infty_{per}(Y), \end{aligned} \tag{4.64}$$

which we re-write as

$$\int_Y h^3(x,\frac{x}{\varepsilon})\nabla_y p_1(x,y)\nabla w(y)dy = \wedge \int_\Omega \int_Y h(x,y)\frac{\partial w}{\partial y_1}(y)dxdy - \int_Y h^3(x,\frac{x}{\varepsilon})\nabla p_0(x)\nabla w(y)dy \text{ for any } w \in C^\infty_{per}(Y). \quad (4.65)$$

By linearity, p_1 is of the form

$$p_1(x,y) = v_1(x,y)\frac{\partial p_0}{\partial x_1} + v_2(x,y)\frac{\partial p_0}{\partial x_2} + v_3(x,y), \quad (4.66)$$

where $v_n \in L^2\Big(\Omega; W^{1,2}_{per}(Y)\Big)$, $n = 1,2,3$ solves a corresponding periodic problem

$$\int_Y h^3\nabla_y v_1(x,y)\nabla w(y)dy = -\int_Y h^3\frac{\partial w}{\partial y_1}(y)dy, \;\; \forall\, w \in C^\infty_{per}(Y) \quad (4.67)$$
$$\int_Y h^3\nabla_y v_2(x,y)\nabla w(y)dy = -\int_Y h^3\frac{\partial w}{\partial y_2}(y)dy, \;\; \forall\, w \in C^\infty_{per}(Y) \quad (4.68)$$
$$\int_Y h^3\nabla_y v_3(x,y)\nabla w(y)dy = -\int_Y h\frac{\partial w}{\partial y_1}(y)dy, \;\; \forall\, w \in C^\infty_{per}(Y) \quad (4.69)$$

We substitute (4.66) into (4.58) to obtain

$$\int_\Omega \int_Y h^3\Bigg[\nabla p_0(x)+\nabla_y\Bigg(v_1(x,y)\frac{\partial p_0}{\partial x_1}+v_2(x,y)\frac{\partial p_0}{\partial x_2}+v_3(x,y)\Bigg)\Bigg]dy\cdot\nabla\phi dx = \Lambda\int_\Omega\int_Y h(x,y)\frac{\partial \phi}{\partial x_1}dydx. \quad (4.70)$$

But

$$\nabla p_0(x) = \frac{\partial p_0}{\partial x_1}e_1 + \frac{\partial p_0}{\partial x_2}e_2$$

and (4.70) gives

$$\int_\Omega \int_Y h^3\Bigg[\frac{\partial p_0}{\partial x_1}e_1 + \frac{\partial p_0}{\partial x_2}e_2 + \nabla_y\Bigg(v_1(x,y)\frac{\partial p_0}{\partial x_1} + v_2(x,y)\frac{\partial p_0}{\partial x_2} + v_3(x,y)\Bigg)\Bigg]dy\cdot\nabla\phi dx = \Lambda\int_\Omega\int_Y h(x,y)\frac{\partial \phi}{\partial x_1}dydx \quad (4.71)$$

which by rearranging gives

$$\begin{aligned}
&\int_\Omega \left\{ \frac{\partial p_0}{\partial x_1}\left(\int_Y h^3(e_1+\nabla_y v_1)dy\right) + \frac{\partial p_0}{\partial x_2}\left(\int_Y h^3(e_2+\nabla_y v_2)dy\right)\right\}\cdot\nabla\phi dx \\
&= \int_\Omega\int_Y \Lambda h\frac{\partial \phi}{\partial x_1}dydx - \int_\Omega\int_Y h^3\nabla_y v_3 dy\cdot\nabla\phi dydx \\
&= \int_\Omega\int_Y \Lambda h e_1\cdot\nabla\phi dydx - \int_\Omega\int_Y h^3\nabla_y v_3 dy\cdot\nabla\phi dydx \qquad (4.72)\\
&= \int_\Omega\left(\int_Y(\Lambda h e_1 - h^3\nabla_y v_3)dy\right)\cdot\nabla\phi dx.
\end{aligned}$$

Let

$$\begin{pmatrix} b_{11}(x) \\ b_{21}(x) \end{pmatrix} = \int_Y (h^3 e_1 + h^3\nabla_y v_1)dy,$$

$$\begin{pmatrix} b_{12}(x) \\ b_{22}(x) \end{pmatrix} = \int_Y (h^3 e_2 + h^3\nabla_y v_2)dy, \qquad (4.73)$$

$$\begin{pmatrix} c_1(x) \\ c_2(x) \end{pmatrix} = \int_Y (\Lambda h - h^3\nabla_y v_3)dy. \qquad (4.74)$$

Substituting (4.73) and (4.74) into (4.72), we obtain

$$\int_\Omega\left\{\frac{\partial p_0}{\partial x_1}\begin{pmatrix} b_{11}(x) \\ b_{21}(x) \end{pmatrix} + \frac{\partial p_0}{\partial x_2}\begin{pmatrix} b_{12}(x) \\ b_{22}(x) \end{pmatrix}\right\}\cdot\nabla\phi dx = \int_\Omega\begin{pmatrix} c_1(x) \\ c_2(x) \end{pmatrix}\cdot\nabla\phi dx$$

$$\int_\Omega\begin{pmatrix} b_{11}(x) & b_{12}(x) \\ b_{21}(x) & b_{22}(x) \end{pmatrix}\begin{pmatrix} \frac{\partial p_0}{\partial x_1} \\ \frac{\partial p_0}{\partial x_2} \end{pmatrix}\cdot\nabla\phi dx = \int_\Omega\begin{pmatrix} c_1(x) \\ c_2(x) \end{pmatrix}\cdot\nabla\phi dx \qquad (4.75)$$

which we simplify to obtain

$$\int_\Omega \mathbf{B}(\mathbf{x})\nabla p_0(x)\cdot\nabla\phi dx = \int_\Omega \mathbf{c}(\mathbf{x})\nabla p_0(x)\cdot\nabla\phi dx \qquad (4.76)$$

where the matrix

$\mathbf{B}(\mathbf{x}) = \begin{pmatrix} b_{11}(x) & b_{12}(x) \\ b_{21}(x) & b_{22}(x) \end{pmatrix}$ and $\mathbf{c}(\mathbf{x}) = \begin{pmatrix} c_1(x) \\ c_2(x) \end{pmatrix}$

with $c_1(x) = \int_Y(\Lambda h - h^3\frac{\partial v_3}{\partial y_1})dy$ and $c_2(x) = \int_Y(-h^3\frac{\partial v_3}{\partial y_2})dy$.

The sequence of solutions p_ε of (4.54) converges weakly in $W_0^{1,2}(\Omega)$ of the homogenized equation

$$\int_\Omega \mathbf{B}(\mathbf{x})\nabla p_0 \cdot \nabla \phi dx = \int_\Omega \mathbf{c}(\mathbf{x}) \cdot \nabla \phi dx \quad \forall \phi \in C_{per}^\infty(\Omega), \tag{4.77}$$

where $\mathbf{B}(\mathbf{x})$ and $\mathbf{c}(\mathbf{x})$ are defined as in (4.73)-(4.74).

Moreover, $\nabla p^\varepsilon(x) \xrightarrow{2} \nabla p_0(x) + \nabla_y p_1(x, y)$, where $p_1 \in L^2[\Omega; W_{per}^{1,2}(Y)]$ may be expressed in the solutions of the periodic problems (4.67)-(4.69) having the form,

$$p_1(x, y) = v_1(x, y)\frac{\partial p_0}{\partial x_1} + v_2(x, y)\frac{\partial p_0}{\partial x_2} + v_3(x, y) \tag{4.78}$$

We note that $\mathbf{B}(\mathbf{x})$ is symmetric and that there is a constant $k > 0$ such that

$$k^{-1}|\xi|^2 \leq \mathbf{B}(\mathbf{x})\xi \cdot \xi \leq k|\xi|^2. \tag{4.79}$$

From this it follows by the Lax-Milgram theorem that the homogenized equation (4.77) has a unique solution and thus the theorem holds for the whole sequence.

Two Scale Convergence of Quasilinear Elliptic Equation

Given the following quasilinear elliptic problem:

$$\begin{aligned} -\nabla \cdot (\mathcal{A}^\varepsilon(x, u_\varepsilon)\nabla u_\varepsilon) = f \ \text{ for } \ x \in \Omega, \\ u_\varepsilon(x) = 0, \ \text{ for } \ x \in \partial\Omega. \end{aligned} \tag{4.80}$$

The space variable $x \in \Omega \subset \mathbb{R}^3$. The cell of periodicity is denoted by Y (i.e. the unit cube in $\mathbb{R}^3$) .

Let the matrix

$$\mathcal{A}_\varepsilon(x,u_\varepsilon) = \begin{pmatrix} a_{11}^\varepsilon(x) & a_{12}^\varepsilon(x) & a_{13}^\varepsilon(x) \\ a_{21}^\varepsilon(x) & a_{22}^\varepsilon(x) & a_{23}^\varepsilon(x) \\ a_{31}^\varepsilon(x) & a_{32}^\varepsilon(x) & a_{33}^\varepsilon(x) \end{pmatrix} = \mathbf{A}(x).$$

The unique solution can be obtained by solving the equation

$$-\nabla \cdot (\mathcal{A}^\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon) = f \;\text{ for }\; x \in \Omega_\varepsilon^*. \tag{4.81}$$

Let us choose an arbitrary test function $v \in V$,

$$-\nabla \cdot \mathcal{A}^\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon \nabla v = fv \tag{4.82}$$

$$-\int_\Omega \nabla \cdot \mathcal{A}^\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon \nabla v dx = \int_\Omega fv dx \tag{4.83}$$

Integrating over Ω, we obtain $\int_\Omega \mathbf{A}(x)\nabla u^\varepsilon(x)\cdot\nabla v dx = \int_\Omega f(x,y,u^\varepsilon)v(x)dx$ for all $v \in V = W_0^{1,2}(\Omega)$. We choose for the test function $v \in V$, the function $\varepsilon\psi(x)v(x/\varepsilon)$, where $\psi \in C_0^\infty(\Omega)$ and $v \in C_{per}^\infty(Y)$.

Then we get that

$$\int_\Omega \mathbf{A}(x)\nabla u^\varepsilon(x)\cdot\nabla[\varepsilon\psi(x)v(x/\varepsilon)]dx = \varepsilon\int_\Omega f(x,y,u^\varepsilon(x))\psi v(x/\varepsilon)dx. \tag{4.84}$$

Simplifying $\varepsilon\int_\Omega \mathbf{A}(x)\nabla u^\varepsilon(x)\cdot[\nabla\psi(x)v(x/\varepsilon)]dx = \varepsilon\int_\Omega f(x,y,u^\varepsilon)\psi(x)v(x/\varepsilon)dx$

As $\varepsilon \longrightarrow 0$, we find that

$$\int_\Omega\int_Y [\mathbf{A}(x)(\nabla u(x) + \nabla_Y\tilde{u}(x,y)\cdot[\psi(x)\nabla v(y)]dydx = 0. \tag{4.85}$$

Since $\psi \in C_0^\infty(\Omega)$ is arbitrary, we have that (for almost every x) $\tilde{u}(x,y)$ is the unique solution of the following periodic problem.

Find $\tilde{u}(x,y) \in [L^\infty(\Omega), W_{per}^{1,2}(Y)]$ such that

$$\int_Y [\mathbf{A}(x)(\nabla u(x) + \nabla_Y\tilde{u}(x,y))]\cdot\nabla v(y)dy = 0 \tag{4.86}$$

for almost all $x \in \mathbb{R}$.

Rearranging, we obtain

$$\begin{aligned}\int_Y \mathbf{A}(x)\nabla_Y \tilde{u}(x,y)\cdot\nabla v(y)dy &= -\int_Y \mathbf{A}(x)\nabla u(x)\cdot\nabla v(y)dy \qquad (4.87)\\ &= -\int_Y \mathbf{A}(x)\frac{\partial u}{\partial x_1}\frac{\partial v}{\partial y_1}dy - \int_Y \mathbf{A}(x)\frac{\partial u}{\partial x_2}\frac{\partial v}{\partial y_2}dy\\ &\quad -\int_Y \mathbf{A}(x)\frac{\partial u}{\partial x_3}\frac{\partial v}{\partial y_3}dy.\end{aligned}$$

By linearity,

$$\tilde{u}(x,y) = w_1(x,y)\frac{\partial u}{\partial x_1} + w_2(x,y)\frac{\partial u}{\partial x_2} + w_3(x,y)\frac{\partial u}{\partial x_3} + w_4(x) \quad (4.88)$$

where $w_i \in L^\infty(\Omega), \quad i = 1,2,3.$

Which we substitute into (4.87) thereby obtaining

$$\begin{aligned}\int_Y \mathbf{A}(x)\nabla_Y[w_1(x,y)\frac{\partial u}{\partial x_1} + w_2(x,y)\frac{\partial u}{\partial x_2} + w_3(x,y)\frac{\partial u}{\partial x_3} + w_4(x)]\cdot\nabla v(y)dy\\ = -\int_Y \mathbf{A}(x)\nabla u(x)\cdot\nabla v(y)dy\\ -\int_Y \mathbf{A}(x)\frac{\partial u}{\partial x_1}\frac{\partial v}{\partial y_1}dy - \int_Y \mathbf{A}(x)\frac{\partial u}{\partial x_2}\frac{\partial v}{\partial y_2}dy - \int_Y \mathbf{A}(x)\frac{\partial u}{\partial x_3}\frac{\partial v}{\partial y_3}dy\end{aligned}$$

$$\begin{aligned}\int_Y \mathbf{A}(x)\nabla_Y[w_1(x,y)\frac{\partial u}{\partial x_1} + w_2(x,y)\frac{\partial u}{\partial x_2} + w_3(x,y)\frac{\partial u}{\partial x_3} + w_4(x)]\cdot\nabla v(y)dy\\ = -\int_Y \mathbf{A}(x)\frac{\partial u}{\partial x_1}\frac{\partial v}{\partial y_1}dy - \int_Y \mathbf{A}(x)\frac{\partial u}{\partial x_2}\frac{\partial v}{\partial y_2}dy - \int_Y \mathbf{A}(x)\frac{\partial u}{\partial x_3}\frac{\partial v}{\partial y_3}dy\end{aligned}$$

Which reduces to

$$\begin{aligned}\int_Y \mathbf{A}(x)\nabla_Y[w_1(x,y)\frac{\partial u}{\partial x_1} + w_2(x,y)\frac{\partial u}{\partial x_2} + w_3(x,y)\frac{\partial u}{\partial x_3}]\cdot\nabla v(y)dy\\ = -\int_Y \mathbf{A}(x)\frac{\partial u}{\partial x_1}\frac{\partial v}{\partial y_1}dy - \int_Y \mathbf{A}(x)\frac{\partial u}{\partial x_2}\frac{\partial v}{\partial y_2}dy - \int_Y \mathbf{A}(x)\frac{\partial u}{\partial x_3}\frac{\partial v}{\partial y_3}dy\end{aligned}$$

Since $w_4(x)$ is independent of y.

Simplifying gives

$$\begin{aligned}\int_Y \mathbf{A}(x)\nabla_Y[w_1(x,y)\frac{\partial u}{\partial x_1} + w_2(x,y)\frac{\partial u}{\partial x_2} + w_3(x,y)\frac{\partial u}{\partial x_3}] \cdot \nabla v(y)dy \\ = -\int_Y \left(\mathbf{A}(x)[\frac{\partial u}{\partial x_1}e_1 + \frac{\partial u}{\partial x_2}e_2 + \frac{\partial u}{\partial x_3}e_3]\right) \cdot \nabla v(y)\,dy\end{aligned}$$

$$\begin{aligned}\int_Y \{\mathbf{A}(x)[(\nabla_Y w_1(x,y) + e_1)\frac{\partial u}{\partial x_1} + (\nabla_Y w_2(x,y) + e_2)\frac{\partial u}{\partial x_2} \\ + (\nabla_Y w_3(x,y) + e_3)\frac{\partial u}{\partial x_3}]\} \cdot \nabla v(y)dy = 0\end{aligned}$$

Which gives the following periodic problems

$$\int_Y \mathbf{A}(x)(\nabla_Y w_1(x,y) + e_1) \cdot \nabla v(y)dy = 0, \tag{4.89}$$

$$\int_Y \mathbf{A}(x)(\nabla_Y w_2(x,y) + e_2) \cdot \nabla v(y)dy = 0, \tag{4.90}$$

$$\int_Y \mathbf{A}(x)(\nabla_Y w_3(x,y) + e_3) \cdot \nabla v(y)dy = 0, \tag{4.91}$$

where $w_1, w_2, w_3 \in L^2(\Omega, W^{1,2}_{per}(Y))$ are the solutions of the periodic problems.

To homogenize the equation, we insert (4.88) into (4.86) to obtain

$$\int_\Omega \int_Y \mathbf{A}(x)\Big[\nabla u(x,y) + \nabla_Y \tilde{u}(x,y)\Big] \cdot \nabla v(x)dydx = \int_\Omega \int_Y f(x,y)v(x)dydx$$

$$\begin{aligned}\int_\Omega \int_Y \mathbf{A}(x)\Big[\nabla u(x,y) + \nabla_Y w_1\frac{\partial u}{\partial x_1} + \nabla_Y w_2\frac{\partial u}{\partial x_2} + \nabla_Y w_3\frac{\partial u}{\partial x_3}\Big] \cdot \nabla v(x)dydx \\ = \int_\Omega \int_Y f(x,y)v(x)dydx.\end{aligned} \tag{4.92}$$

But

$$\nabla u(x,y) = \frac{\partial u}{\partial x_1}e_1 + \frac{\partial u}{\partial x_2}e_2 + \frac{\partial u}{\partial x_3}e_3$$

where $\{e_1, e_2, e_3\}$ is the canonical basis in $\mathbb{R}^3$. Which we substitute into (4.92) to obtain

$$\int_\Omega \int_Y \mathbf{A}(x)\left[\frac{\partial u}{\partial x_1}e_1 + \frac{\partial u}{\partial x_2}e_2 + \frac{\partial u}{\partial x_3}e_3 + \nabla_Y w_1 \frac{\partial u}{\partial x_1} + \nabla_Y w_2 \frac{\partial u}{\partial x_2} + \nabla_Y w_3 \frac{\partial u}{\partial x_3}\right] \cdot \nabla v(x) dydx = \int_\Omega \int_Y f(x,y) v(x) dydx.$$

Simplifying, we obtain

$$\int_\Omega \int_Y \mathbf{A}(x)[(\nabla_Y w_1 + e_1)\frac{\partial u}{\partial x_1} + (\nabla_Y w_2 + e_2)\frac{\partial u}{\partial x_2} + (\nabla_Y w_3 + e_3)\frac{\partial u}{\partial x_3}] \cdot \nabla v(y) dydx = \int_\Omega \int_Y f(x,y) v(x) dydx.$$

$$\int_\Omega \left\{ \int_Y \mathbf{A}(x)(\nabla_Y w_1 + e_1)dy \frac{\partial u}{\partial x_1} + \int_Y \mathbf{A}(x)(\nabla_Y w_2 + e_2)dy \frac{\partial u}{\partial x_2} + \int_Y \mathbf{A}(x)(\nabla_Y w_3 + e_3)dy \frac{\partial u}{\partial x_3} \right\} \cdot \nabla v(x) dx = \int_\Omega \int_Y f(x,y) v(x) dydx.$$

$$\int_\Omega \left\{ \frac{\partial u}{\partial x_1}\Big(\int_Y \mathbf{A}(x)(\nabla_Y w_1 + e_1)dy\Big) + \frac{\partial u}{\partial x_2}\Big(\int_Y \mathbf{A}(x)(\nabla_Y w_2 + e_2)dy\Big) + \frac{\partial u}{\partial x_3}\Big(\int_Y \mathbf{A}(x)(\nabla_Y w_3 + e_3)dy\Big) \right\} \cdot \nabla v(x) dx = \int_\Omega \int_Y f(x,y) v(x) dydx.$$

$$\int_\Omega \left\{ \frac{\partial u}{\partial x_1}\begin{pmatrix} b_{11}(x) \\ b_{12}(x) \\ b_{13}(x) \end{pmatrix} + \frac{\partial u}{\partial x_2}\begin{pmatrix} b_{21}(x) \\ b_{22}(x) \\ b_{23}(x) \end{pmatrix} + \frac{\partial u}{\partial x_3}\begin{pmatrix} b_{31}(x) \\ b_{32}(x) \\ b_{33}(x) \end{pmatrix} \right\} \cdot \nabla v(x) dx = \int_\Omega \int_Y f(x,y) v(x) dydx.$$

$$\int_\Omega \left\{ \begin{pmatrix} b_{11}(x) & b_{21}(x) & b_{31}(x) \\ b_{12}(x) & b_{22}(x) & b_{32}(x) \\ b_{13}(x) & b_{23}(x) & b_{33}(x) \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x_1} \\ \frac{\partial u}{\partial x_2} \\ \frac{\partial u}{\partial x_3} \end{pmatrix} \right\} \cdot \nabla v(x) dx = \int_\Omega \int_Y f(x,y) v(x) dydx.$$

which reduces to

$$\int_\Omega \mathbf{B}(\mathbf{x})\nabla u \cdot \nabla v(x)dx = \int_\Omega \int_Y f(x,y)v(x)dydx, \tag{4.93}$$

where the matrix $\mathbf{B}(\mathbf{x}) = (b_{ij}(x))_{i,j=1,2,3}$ is defined by

$$\begin{pmatrix} b_{11}(x) \\ b_{12}(x) \\ b_{13}(x) \end{pmatrix} = \int_Y \mathbf{A}(x)(\nabla_Y w_1 + e_1)dy,$$

$$\begin{pmatrix} b_{21}(x) \\ b_{22}(x) \\ b_{23}(x) \end{pmatrix} = \int_Y \mathbf{A}(x)(\nabla_Y w_2 + e_2)dy$$

and

$$\begin{pmatrix} b_{31}(x) \\ b_{32}(x) \\ b_{33}(x) \end{pmatrix} = \int_Y \mathbf{A}(x)(\nabla_Y w_3 + e_3)dy.$$

If we let $\tilde{f}(x) = \int_Y f(x,y)dy$, then (4.93) gives

$$\int_\Omega \mathbf{B}(\mathbf{x})\nabla u \cdot \nabla v(x)dx = \int_\Omega \tilde{f}(x)v(x)dx, \tag{4.94}$$

which is the homogenized equation of (4.82).

Also, along the boundary we have that $u_0(x) = 0$.

Chapter Summary

In this section, the elliptic equation of the divergence form were homogenized using the multiple-scale asymptotic expansion. The forms of the Reynold equation discussed, the time independent incompressible Reynolds equation and the quasi linear elliptic equation were homogenized to obtain the cell problems and the homogenized equation.

CHAPTER FIVE

RESULTS AND DISCUSSION

Introduction

In the previous chapters, we reviewed the homogenization of elliptic equations, in particular the Reynolds Equation which were in the divergence form. In this main work below, we will homogenize the elliptic type equation of the curl type to obtain the cell problem and the homogenized equation.

Homogenization of Elliptic Equations of the Curl Type

Theorem 5.1

The homogenized boundary value problem for the deterministic boundary value problem

$$\begin{aligned} \nabla_x \times [a^\varepsilon(x)(\nabla_x \times u_\varepsilon(x))] + b_0^\varepsilon u_\varepsilon(x) &= f \ \text{ in } \ \Omega \\ u_\varepsilon(x) &= 0 \ \text{ on } \ \partial\Omega \end{aligned} \tag{5.1}$$

is given by

$$\begin{aligned} \nabla_x \times [a^0(x)(\nabla_x \times u_0(x))] + \lambda u_0(x) &= f \ \text{ in } \ \Omega \\ u_0(x) &= 0 \ \text{ on } \ \partial\Omega. \end{aligned} \tag{5.2}$$

Proof

Let Ω be an open bounded subset of R^3, $Y = (0,1)^3$.
Introducing the auxiliary matrix $a = (a_{ij})$ where $a_{ij} = a_{ij}(x,y)$, and $i,j = 1,2,3$ are smooth functions which are Y-periodic in y. Assuming that there exists a constant $\alpha > 0$ such that

$$\sum_{i,j=1}^{3} a_{ij}(x,y)\xi_i\xi_j \geq \alpha|\xi|^2 \ \text{ for every } \xi \in R^3.$$

Let $\varepsilon > 0$, defining the matrix $a^\varepsilon(x)$ as

$$a^{\varepsilon}(x) = \begin{pmatrix} a_{11}^{\varepsilon}(x) & a_{12}^{\varepsilon}(x) & a_{13}^{\varepsilon}(x) \\ a_{21}^{\varepsilon}(x) & a_{22}^{\varepsilon}(x) & a_{23}^{\varepsilon}(x) \\ a_{31}^{\varepsilon}(x) & a_{32}^{\varepsilon}(x) & a_{33}^{\varepsilon}(x) \end{pmatrix} = a(x, \tfrac{x}{\varepsilon}) = a(x, y),$$

$b_0^{\varepsilon} = \lambda\mathbf{I}$, $\lambda > 0$, where $y = \frac{x}{\varepsilon}$.

In (5.1) let the operator $\mathcal{A}^{\varepsilon}$ be defined by

$$\mathcal{A}^{\varepsilon} = \nabla_x \times \Big[a^{\varepsilon}(\nabla_x \times \bullet) + \lambda\mathbf{I}\Big]. \tag{5.3}$$

Then (5.1) is of the form

$$\mathcal{A}^{\varepsilon} u_{\varepsilon} = f \quad \text{in } \Omega. \tag{5.4}$$

And this is an elliptic equation defined in terms of curl. Such equations are often described as Maxwell type equations. As the value of ε become smaller, the coefficients in (5.1) are rapidly oscillating. This suggest some type of averaging or asymptotic analysis. We shall prove that $u_{\varepsilon} \to u_0$ as $\varepsilon \to 0$ and that the solution u_0 can be found by solving a so-called homogenized equation (5.69) which does not contain any rapid oscillations. This means that u_0 may be used as an approximation of the solution u_{ε} for small values of ε. The multiple scale expansion method shall be used to derive a homogenized equation for (5.1).

We assume that the solution of u_{ε} in (5.4) has an expansion which is of the form

$$u_{\varepsilon}(x, y) = u_0(x, \frac{x}{\varepsilon}) + \varepsilon u_1(x, \frac{x}{\varepsilon}) + \varepsilon^2 u_2(x, \frac{x}{\varepsilon}) + \cdots = \sum_{i=0}^{\infty} \varepsilon^i u_i(x, \frac{x}{\varepsilon}) \tag{5.5}$$

with $u_i(x, y), \;\; y = \frac{x}{\varepsilon}$ for $i = 0, 1, 2, \ldots$ such that $u_i(x, y)$ is defined for $x \in \Omega$ and $y \in Y$ and that $u_i(\cdot, y)$ is Y-periodic.

Next we let $\phi = \phi(x, y)$ be a function depending on two variables of R^2. Denote also $\phi^{\varepsilon}(x)$ by the following: $\phi^{\varepsilon}(x) = \phi(x, \frac{x}{\varepsilon}) = \phi(x, y)$, where $y = \frac{x}{\varepsilon}$.

Then by the chain rule we have

$$\frac{\partial\phi(x)}{\partial x_i} = \frac{\partial}{\partial x_i}\phi(x,y) + \frac{\partial}{\partial y_i}\phi(x,y)\frac{dy}{dx} \tag{5.6}$$

$$= \frac{\partial}{\partial x_i}\phi(x,y) + \frac{1}{\varepsilon}\frac{\partial}{\partial y_i}\phi(x,y) \tag{5.7}$$

$$= \left(\frac{\partial}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_i}\right)\phi(x,y). \tag{5.8}$$

Which can be written in gradient notation as

$$\nabla_x\phi^\varepsilon(x) = (\nabla_x + \frac{1}{\varepsilon}\nabla_y)\phi(x,y). \tag{5.9}$$

Thus in general,

$$\nabla_x = (\nabla_x + \frac{1}{\varepsilon}\nabla_y). \tag{5.10}$$

Substituting (5.10) into (5.3), we obtain

$$\begin{aligned}
\mathcal{A}^\varepsilon &= (\nabla_x + \frac{1}{\varepsilon}\nabla_y) \times \left(a\left[\nabla_x + \frac{1}{\varepsilon}\nabla_y\right]\times\right) + \lambda\mathbf{I} \\
&= \nabla_x \times (a\nabla_x\times) + \frac{1}{\varepsilon}\nabla_x \times (a\nabla_y\times) + \frac{1}{\varepsilon}\nabla_y \times (a\nabla_x\times) \\
&\quad + \frac{1}{\varepsilon^2}\nabla_y \times (a\nabla_y\times) + \lambda\mathbf{I} \\
&= \frac{1}{\varepsilon^2}\nabla_y \times (a\nabla_y\times) + \frac{1}{\varepsilon}[\nabla_x \times (a\nabla_y\times) + \nabla_y \times (a\nabla_x\times)] \\
&\quad + \nabla_x \times (a\nabla_x\times) + \lambda\mathbf{I} \\
&= \frac{1}{\varepsilon^2}\mathcal{A}_0 + \frac{1}{\varepsilon}\mathcal{A}_1 + \mathcal{A}_2
\end{aligned} \tag{5.11}$$

where

$$\mathcal{A}_0 = \nabla_y \times (a\nabla_y\times) \tag{5.12}$$

$$\mathcal{A}_1 = \nabla_x \times (a\nabla_y\times) + \nabla_y \times (a\nabla_x\times) \tag{5.13}$$

$$\mathcal{A}_2 = \nabla_x \times (a\nabla_x\times) + \lambda\mathbf{I} \tag{5.14}$$

$$\mathcal{A}^\varepsilon = \frac{1}{\varepsilon^2}\mathcal{A}_0 + \frac{1}{\varepsilon}\mathcal{A}_1 + \mathcal{A}_2 \tag{5.15}$$

Substituting (5.15) into (5.4), we obtain

$$\left(\frac{1}{\varepsilon^2}\mathcal{A}_0 + \frac{1}{\varepsilon}\mathcal{A}_1 + \mathcal{A}_2\right)u^\varepsilon = f, \quad \text{for } x \in \Omega. \tag{5.16}$$

By substituting the power series expansion (5.5) in (5.15) we have

$$\left(\frac{1}{\varepsilon^2}\mathcal{A}_0 + \frac{1}{\varepsilon}\mathcal{A}_1 + \mathcal{A}_2\right)\left(u_0(x, \frac{x}{\varepsilon}) + \varepsilon u_1(x, \frac{x}{\varepsilon}) + \varepsilon^2 u_2(x, \frac{x}{\varepsilon}) + \cdots\right) = f$$

Rearranging we see that

$$\frac{1}{\varepsilon^2}\mathcal{A}_0 u_0 + \frac{1}{\varepsilon}(\mathcal{A}_0 u_1 + \mathcal{A}_1 u_0) + (\mathcal{A}_0 u_2 + \mathcal{A}_1 u_1 + \mathcal{A}_2 u_0)$$
$$+\varepsilon(\mathcal{A}_1 u_2 + \mathcal{A}_2 u_1) + \varepsilon^2 \mathcal{A}_2 u_2 + \ldots = f$$

Equating the powers of ε of order $-2, -1$ and 0, the following sequence of problems are obtained:

$$\mathcal{A}_0 u_0 = 0 \tag{5.17}$$

$$\mathcal{A}_0 u_1 + \mathcal{A}_1 u_0 = 0 \tag{5.18}$$

$$\mathcal{A}_0 u_2 + \mathcal{A}_1 u_1 + \mathcal{A}_2 u_0 = f \tag{5.19}$$

which we solve in place of the original equation given in (5.1) to obtain the cell problems and the homogenized equation.

We shall make use of the following well known result in solving (5.17)–(5.19) $\Gamma m = P$ has a solution if and only if

$$\int_Y P dy = 0. \tag{5.20}$$

A case for which m is unique up to an additive constant.

Here Γ represents one of the operators $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \ldots$. (Persson et al, 1993).

We solve (5.17) as follows,

$$\mathcal{A}_0 u_0 = \nabla_y \times a(\nabla_y \times u_0) = 0. \tag{5.21}$$

Taking the scalar product of $(\mathcal{A}_0 u_0, u_0)$ in $L^2(Y)^2$ we have,

$$
\begin{aligned}
(\mathcal{A}_0 u_0, u_0) &= \int_Y \Big(a(\nabla_y \times (\nabla_y \times u_0)) \cdot u_0 \Big) dy \\
&= \int_Y a(\nabla_y \times u_0) \cdot (\nabla_y \times u_0) dy = 0.
\end{aligned} \tag{5.22}
$$

Since a is a non zero matrix, we have that $(\mathcal{A}_0 u_0, u_0) = 0$ only if

$$
\nabla_y \times u_0 = 0. \tag{5.23}
$$

Next we consider equation (5.18),

$$
\begin{aligned}
&\mathcal{A}_0 u_1 + \mathcal{A}_1 u_0 = 0 \\
&\nabla_y \times a(\nabla_y \times u_1) + \nabla_x \times a(\nabla_y \times u_0) + \nabla_y \times a(\nabla_x \times u_0) = 0.
\end{aligned} \tag{5.24}
$$

By substituting (5.23) into (5.24) we obtain

$$
\nabla_y \times [a(\nabla_y \times u_1)] + \nabla_y \times [a(\nabla_x \times u_0)] = 0, \tag{5.25}
$$

which can further be written as

$$
\nabla_y \times \Big(a(\nabla_y \times u_1) + a(\nabla_x \times u_0) \Big) = 0. \tag{5.26}
$$

Finally, from (5.19) we have that

$$
\begin{aligned}
\nabla_y \times (a\nabla_y \times u_2) + \nabla_x \times (a\nabla_y \times u_1) + \nabla_y \times (a\nabla_x \times u_1) \\
+ \nabla_x \times (a\nabla_x \times u_0) + \lambda u_0 = f.
\end{aligned} \tag{5.27}
$$

Taking the divergence with respect to y on both sides of (5.27), we find that

$$
\begin{aligned}
\nabla_y \cdot (\nabla_y \times (a\nabla_y \times u_2)) + \nabla_y \cdot (\nabla_x \times (a\nabla_y \times u_1)) + \nabla_y \cdot (\nabla_y \times (a\nabla_x \times u_1)) \\
+ \nabla_y \cdot (\nabla_x \times (a\nabla_x \times u_0)) + \lambda \nabla_y \cdot u_0 = \nabla_y \cdot f
\end{aligned} \tag{5.28}
$$

$$\nabla_y \cdot (\nabla_y \times (a\nabla_y \times u_2)) + \nabla_y \cdot (\nabla_x \times (a\nabla_y \times u_1)) + \nabla_y \cdot (\nabla_y \times (a\nabla_x \times u_1)) + \nabla_y \cdot (\nabla_x \times (a\nabla_x \times u_0)) + \lambda\nabla_y \cdot u_0 = 0 \tag{5.29}$$

The right hand side of (5.28) equals zero since $\nabla_y \cdot f(x) = 0$.

Making use of Lemma 2.8 in (5.29) we obtain

$$\nabla_y \cdot (\nabla_x \times (a\nabla_y \times u_1)) + \nabla_y \cdot (\nabla_x \times (a\nabla_x \times u_0)) + \lambda\nabla_y \cdot u_0 = 0$$

$$\nabla_y \cdot \Big(\nabla_x \times (a\nabla_y \times u_1) + \nabla_x \times (a\nabla_x \times u_0)\Big) + \lambda\nabla_y \cdot u_0 = 0$$

$$\nabla_y \cdot \Big[\nabla_x \times \Big((a\nabla_y \times u_1) + (a\nabla_x \times u_0)\Big)\Big] + \lambda\nabla_y \cdot u_0 = 0 \tag{5.30}$$

Applying the result of Lemma 2.9 to (5.30) we see that

$$-\nabla_x \cdot \Big[\nabla_y \times \Big((a\nabla_y \times u_1) + (a\nabla_x \times u_0)\Big)\Big] + \lambda\nabla_y \cdot u_0 = 0. \tag{5.31}$$

Substituting (5.26) into (5.31), we find that

$$\lambda\nabla_y \cdot u_0 = 0 \ \text{ or } \ \nabla_y \cdot u_0(x, y) = 0 \tag{5.32}$$

which implies that

$$u_0(x, y) = u_0(x). \tag{5.33}$$

i.e. $u_0(x, y)$ is independent of y.

Substituting the result of (5.33) into (5.26) we can rewrite (5.26) as

$$\nabla_y \times \Big(a(\nabla_y \times u_1) + a(\nabla_x \times u_0(x))\Big) = 0. \tag{5.34}$$

In (5.34), we let

$$w = a\big(\nabla_y \times u_1 + \nabla_x \times u_0(x)\big), \tag{5.35}$$

so that we have

$$\nabla_y \times w = 0. \tag{5.36}$$

Multiplying both sides of (5.35) by a^{-1} we obtain

$$a^{-1}w = \big(\nabla_y \times u_1 + \nabla_x \times u_0(x)\big). \tag{5.37}$$

Taking the divergence of both sides, we get

$$\begin{aligned}\nabla_y \cdot (a^{-1}w) &= \nabla_y \cdot \big(\nabla_y \times u_1 + \nabla_x \times u_0(x)\big)\\ &= \nabla_y \cdot \big(\nabla_y \times u_1\big) + \nabla_y \cdot \big(\nabla_x \times u_0(x)\big).\end{aligned} \tag{5.38}$$

By Lemma 2.8 the first term on the right hand side is zero. Moreover, since $\nabla_x \times u_0(x)$ is a function of x only, its divergence with respect to y is zero hence the second term is also zero. Thus,

$$\nabla_y \cdot (a^{-1}w) = 0. \tag{5.39}$$

Integrating (5.35) over Y and making use of Y-periodicity yields

$$\overline{w}^y = \int_Y w dy = \int_Y a\big(\nabla_y \times u_1) dy + \int_Y a(\nabla_x \times u_0(x)\big) dy. \tag{5.40}$$

But $\int_Y a\big(\nabla_y \times u_1) dy = 0$ by Y-periodicity.

Hence

$$\overline{w}^y = \int_Y a(\nabla_x \times u_0(x)\big) dy. \tag{5.41}$$

Rearranging the terms we see that

$$\begin{aligned}\overline{w}^y =& \nabla_x \times u_0(x) \int_Y a dy\\ =& a\nabla_x \times u_0(x)\end{aligned}$$

where $\int_Y dy = 1$ for Y being 1-periodic and $a = a(x)$ after integrating over Y. Since $a(\nabla_x \times u_0(x))$ is a function of x only, let $a(\nabla_x \times u_0(x)) = \widetilde{w}(x) = \widetilde{w}$. Thus,

$$\overline{w}^y = \widetilde{w}(x) = \widetilde{w} \tag{5.42}$$

Since from (5.36) $\nabla_y \times w = 0$ and $\overline{w}^y = \widetilde{w}(x)$, it follows that $\nabla_y \times (w - \widetilde{w}) = 0$. And so by Lemma 2.10 there exists a Y-periodic function say $\psi(x, y)$ such that

$$w - \widetilde{w} = -\nabla_y \psi. \tag{5.43}$$

Making w the subject in (5.43) gives,

$$w = \widetilde{w} - \nabla_y \psi. \tag{5.44}$$

Substituting (5.44) into (5.39) and simplifying we obtain

$$\nabla_y \cdot (a^{-1}(\widetilde{w} - \nabla_y \psi)) = 0$$

$$\nabla_y \cdot (a^{-1}\widetilde{w} - a^{-1}\nabla_y \psi) = 0$$

$$\nabla_y \cdot (a^{-1}\widetilde{w}) - \nabla_y \cdot (a^{-1}\nabla_y \psi) = 0$$

$$-\nabla_y \cdot (a^{-1}\nabla_y \psi) = -\nabla_y \cdot (a^{-1}\widetilde{w}). \tag{5.45}$$

By linearity we let $\psi(x, y)$ be such that $\psi = v^1 w_1 + v^2 w_2 + v^3 w_3 = \sum_{i=1}^{3} v^i w_i$. Let $\widetilde{w}(x) = w_1 e_1 + w_2 e_2 + w_3 e_3$, $v = (v^1, v^2, v^3)$ and $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (1, 0, 0)$ are the canonical basis.

Plugging $\widetilde{w}(x)$ and v into (5.45), we find that

$$\nabla_y \cdot \Big(a^{-1} \nabla_y (v^1 w_1 + v^2 w_2 + v^3 w_3) \Big) = \nabla_y \cdot \Big(a^{-1} (w_1 e_1 + w_2 e_2 + w_3 e_3) \Big). \tag{5.46}$$

Comparing like terms we obtain the following cell problems

$$\nabla_y \cdot \Big(a^{-1}\nabla_y(v^1 w_1)\Big) = \nabla_y \cdot \Big(a^{-1}(w_1 e_1)\Big), \tag{5.47}$$

$$\nabla_y \cdot \Big(a^{-1}\nabla_y(v^2 w_2)\Big) = \nabla_y \cdot \Big(a^{-1}(w_2 e_2)\Big), \tag{5.48}$$

$$\nabla_y \cdot \Big(a^{-1}\nabla_y(v^3 w_3)\Big) = \nabla_y \cdot \Big(a^{-1}(w_3 e_3)\Big). \tag{5.49}$$

Taking (5.47), we have

$$\nabla_y \cdot \Big(a^{-1}\nabla_y(v^1 w_1)\Big) - \nabla_y \cdot \Big(a^{-1}(w_1 e_1)\Big) = 0$$

$$\nabla_y \cdot \Big(a^{-1}\nabla_y(v^1 w_1) - a^{-1}(w_1 e_1)\Big) = 0$$

$$\nabla_y \cdot \Big[a^{-1}\Big(\nabla_y(v^1 w_1) - (w_1 e_1)\Big)\Big] = 0$$

$$w_1\Big[\nabla_y \cdot \Big(a^{-1}\nabla_y v^1 - e_1\Big)\Big] = 0$$

which implies that

$$\nabla_y \cdot \Big[a^{-1}\Big(\nabla_y v^1 - e_1\Big)\Big] = 0, \tag{5.50}$$

with a weak formulation given by

$$\int_Y a^{-1}(\nabla_y v^1 - e_1) \cdot \nabla_y \mu dy = 0 \;\; \forall\, \mu \in C^\infty_{per}(Y).$$

Similarly, the weak formulation of (5.48) and (5.49) are

$$\int_Y a^{-1}(\nabla_y v^2 - e_2) \cdot \nabla_y \mu dy = 0 \;\; \forall\, \mu \in C^\infty_{per}(Y)$$

and

$$\int_Y a^{-1}(\nabla_y v^3 - e_3) \cdot \nabla_y \mu dy = 0 \;\; \forall\, \mu \in C^\infty_{per}(Y).$$

Generalizing we have the following cell problem whose solution $v^i, i = 1, 2, 3$ will be needed in obtaining the homogenized problem. That is,

$$\int_Y a^{-1}(\nabla_y v^i - e_i) \cdot \nabla_y \mu dy = 0 \ \forall \ \mu \in C^{\infty}_{per}(Y) \tag{5.51}$$

where $i = 1, 2, 3$.

$$\begin{aligned}
\text{With } \psi(x,y) &= v^1(y)w_1(x) + v^2(y)w_2(x) + v^3(y)w_3(x) \\
&= \begin{pmatrix} v^1(y) \\ v^2(y) \\ v^3(y) \end{pmatrix} \cdot \begin{pmatrix} w_1(x) \\ w_2(x) \\ w_3(x) \end{pmatrix} \\
&= v(y) \cdot \widetilde{w}(x) \\
&= v\widetilde{w}
\end{aligned}$$

$$\psi(x,y) = v\widetilde{w}. \tag{5.52}$$

Substituting (5.52) into (5.44), we have

$$\begin{aligned}
w =& \widetilde{w} - \nabla_y(v\widetilde{w}) \\
=& \widetilde{w} - v\nabla_y\widetilde{w} - \widetilde{w}\nabla_y v(y).
\end{aligned}$$

Since $\widetilde{w} = \widetilde{w}(x)$ is a function of x only $\nabla_y\widetilde{w} = 0$. So that

$$\begin{aligned}
w =& \widetilde{w} - \widetilde{w}\nabla_y v(y) \\
=& (\mathbf{I} - \nabla_y v)\widetilde{w}.
\end{aligned} \tag{5.53}$$

Multiplying (5.53) by a^{-1}, we see that

$$a^{-1}w = a^{-1}(\mathbf{I} - \nabla_y v)\widetilde{w}. \tag{5.54}$$

Integrating (5.54) with respect to y yields

$$\begin{aligned}\overline{(a^{-1}w)}^{y} &= \int_Y [a^{-1}(\mathbf{I} - \nabla_y v)\widetilde{w}]dy \\ &= \widetilde{w}\int_Y [a^{-1}(\mathbf{I} - \nabla_y v)]dy \\ &= A(x)\widetilde{w},\end{aligned} \tag{5.55}$$

where $A(x) = \int_Y [a^{-1}(\mathbf{I} - \nabla_y v(y))]dy$.

Also from (5.35)

$$w = a\big[(\nabla_y \times u_1) + (\nabla_x \times u_0(x))\big], \tag{5.56}$$

$$a^{-1}w = (\nabla_y \times u_1) + (\nabla_x \times u_0(x)). \tag{5.57}$$

Integrating (5.57) over Y gives

$$\begin{aligned}\overline{(a^{-1}w)}^{y} &= \int_Y (\nabla_x \times u_0(x))\, dy \quad + \int_Y (\nabla_y \times u_1)\, dy \\ &= \int_Y (\nabla_x \times u_0(x))\, dy,\end{aligned}$$

where $\int_Y (\nabla_y \times u_1)\, dy = 0$ by periodicity.

$$\begin{aligned}\overline{(a^{-1}w)}^{y} &= \int_Y (\nabla_x \times u_0(x))\, dy \\ &= (\nabla_x \times u_0(x))|Y| \\ &= \nabla_x \times u_0(x)\end{aligned}$$

since $|Y| = 1$. Thus we have the following relation:

$$\nabla_x \times u_0(x) = \overline{(a^{-1}w)}^{y} \tag{5.58}$$

Substituting (5.55) into (5.58) we obtain

$$\nabla_x \times u_0(x) = A(x)\widetilde{w}. \tag{5.59}$$

Making $\widetilde{w}$ the subject, we have

$$\begin{aligned}\widetilde{w} =& [A(x)]^{-1}(\nabla_x \times u_0(x)) \\ =& a^0(x)(\nabla_x \times u_0(x)).\end{aligned} \tag{5.60}$$

Where $a^0(x) = [A(x)]^{-1}$ is the homogenized matrix corresponding to the deterministic matrix $a^\varepsilon(x)$ in (5.1).

Finally to obtain the homogenized equation, we integrate each term of (5.19) with respect to y and simplify the terms before eventually plugging into (5.60).

$$\int_Y (\mathcal{A}_0 u_2 + \mathcal{A}_1 u_1 + \mathcal{A}_2 u_0) dy = \int_Y f dy$$

$$\int_Y \mathcal{A}_0 u_2 dy + \int_Y \mathcal{A}_1 u_1 dy + \int_Y \mathcal{A}_2 u_0 dy = \int_Y f dy \tag{5.61}$$

Integrating term by term and making use of Y periodicity gives,

$$\int_Y \mathcal{A}_0 u_2 dy = \int_Y \nabla_y \times a(\nabla_y \times u_2) dy = 0. \tag{5.62}$$

$$\begin{aligned}\int_Y \mathcal{A}_1 u_1 dy &= \int_Y \nabla_x \times a(\nabla_y \times u_1) dy + \int_Y \nabla_y \times a(\nabla_x \times u_1) dy \\ &= \int_Y \nabla_x \times a(\nabla_y \times u_1) dy\end{aligned} \tag{5.63}$$

since $\int_Y \nabla_y \times (a\nabla_x \times u_1) dy = 0$ by Y-periodicity.

Furthermore, the third term on the left hand side of (5.61) can be analyzed as follows,

$$\begin{aligned}\int_Y \mathcal{A}_2 u_0 dy &= \int_Y \Big(\nabla_x \times (a\nabla_x \times u_0(x)) + \lambda u_0(x)\Big) dy \\ &= \int_Y \Big(\nabla_x \times (a\nabla_x \times u_0(x)\Big) dy + \int_Y \lambda u_0(x) dy \\ &= \int_Y \Big(\nabla_x \times (a\nabla_x \times u_0(x)\Big) dy + \lambda u_0(x)|Y| \\ &= \int_Y \Big(\nabla_x \times (a\nabla_x \times u_0(x)\Big) dy + \lambda u_0(x)\end{aligned} \tag{5.64}$$

Substituting (5.62) – (5.64) into (5.61), we obtain

$$\int_Y \nabla_x \times a(\nabla_y \times u_1)dy + \int_Y \Big(\nabla_x \times a(\nabla_x \times u_0(x))\Big)dy + \lambda u_0(x) = f|Y| \quad (5.65)$$

$$\int_Y \nabla_x \times \Big((a\nabla_y \times u_1) + (a\nabla_x \times u_0(x))\Big)dy + \lambda u_0(x) = f|Y|$$

$$\int_Y \nabla_x \times a\Big(\nabla_y \times u_1 + \nabla_x \times u_0(x)\Big)dy + \lambda u_0(x) = f. \quad (5.66)$$

Substituting (5.35) into (5.66), we find that

$$\int_Y \Big(\nabla_x \times w\Big)dy + \lambda u_0(x) = f$$

$$\Big(\nabla_x \times \int_Y w\ dy\Big) + \lambda u_0(x) = f$$

$$\Big(\nabla_x \times \overline{w}^y\Big) + \lambda u_0(x) = f. \quad (5.67)$$

But from (5.42), $\overline{w}^y = \widetilde{w}(x) = \widetilde{w}$ and so (5.67) can be written as

$$\nabla_x \times \widetilde{w} + \lambda u_0(x) = f. \quad (5.68)$$

Inserting (5.60) into (5.68) we finally obtain ,

$$\nabla_x \times \Big[a^0(x)(\nabla_x \times u_0(x))\Big] + \lambda u_0(x) \quad = f \ \text{ in } \ \Omega, \quad (5.69)$$
$$u_0(x) = 0 \ \text{ on } \ \partial\Omega$$

as the homogenized equation corresponding to the deterministic boundary value problem given by (5.1) and has

$$\nabla_x \times \Big(a^0(x)(\nabla_x \times \bullet)\Big) + \lambda$$

as the homogenized operator.

Chapter Summary

In this chapter the elliptic equations of the curl type was homogenized, the cell problems and the homogenized equation obtained.

CHAPTER SIX

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

Overview

The thesis is summarized in this chapter, conclusions drawn from the work and some future works which could be explored listed.

Summary

In this thesis, we studied the homogenization of elliptic equations in periodic domains. We give the overview of the homogenization by the multiple scale and the two-scale convergences. The reynolds equation which is used in the flow of fluid through machine elements such as bearings, gearboxes and hydraulic systems was also homogenized.

In particular, we homogenize the quasi-linear elliptic equation using the two-scale convergence method to obtain the cell problems and the homogenized equation

$$\int_\Omega \mathbf{B}(\mathbf{x})\nabla u \cdot \nabla v(x)dx = \int_\Omega \tilde{f}(x)v(x)dx, \tag{6.1}$$

which is the homogenized equation of

$$-\nabla \cdot \mathcal{A}^\varepsilon(x, u_\varepsilon)\nabla u_\varepsilon \nabla v = fv \tag{6.2}$$

$$-\int_\Omega \nabla \cdot \mathcal{A}^\varepsilon(x, u_\varepsilon)\nabla u_\varepsilon \nabla v dx = \int_\Omega fv dx. \tag{6.3}$$

The elliptic equation of the curl form of which has the Maxwell type equation as an example was also homogenized to obtain the homogenized equation after obtaining the cell problems.

The homogenized boundary value problem for the deterministic boundary value problem

$$\begin{aligned} \nabla_x \times [a^\varepsilon(x)(\nabla_x \times u_\varepsilon(x))] + b_0^\varepsilon u_\varepsilon(x) = f \ \text{ in } \ \Omega \\ u_\varepsilon(x) = 0 \ \text{ on } \ \partial\Omega \end{aligned} \tag{6.4}$$

was also homogenized to obtain

$$\begin{aligned} \nabla_x \times [a^0(x)(\nabla_x \times u_0(x))] + \lambda u_0(x) = f \ \text{ in } \ \Omega \\ u_0(x) = 0 \ \text{ on } \ \partial\Omega, \end{aligned} \tag{6.5}$$

with the cell problem as

$$\int_Y a^{-1}(\nabla_y v^i - e_i) \cdot \nabla_y \mu dy = 0 \ \forall \ \mu \in C_{per}^\infty(Y). \tag{6.6}$$

where $i = 1, 2, 3.$ and $v^i, i = 1, 2, 3$ will be needed in solving the homogenized problem.

Conclusions and Recommendations

The quasilinear elliptic equation and the elliptic equation of the curl type were homogenized using the two scale convergence method of homogenization. The cell problems as well as the homogenized equations were obtained. In order to study the asymptotic expansion behaviour of u_ε as $\varepsilon \to 0$, we have considered the case where $a_0^\varepsilon = \lambda I, \quad \lambda > 0$. The more general case of the elliptic equations in the curl type be studied and other elliptic equations be homogenized using the two-scale method of homogenization. Also other researchers can consider the case where λ is complex.

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