An application of optimal control to the effective utilization of a renewable resource

Mahmud Ibrahim* and **Francis Benyah**

Department of Mathematics and Statistics, University of Cape Coast, Cape Coast, Ghana

Abstract. The study explores the optimal harvesting of renewable resources like fisheries. The fish biomass dynamics is described by a nonlinear growth model that maximizes the total net revenue whilst taking into consideration the sustainable and effective utilization of the resource. In addition, stability dynamics of the model is assessed through bifurcation analysis. Pontryagin's maximum principle is used to derive the optimality system and characterize the optimal control. A numeric iterative method employing the fourth order Runge-Kutta scheme facilitates the solution of the optimality system. The simulation results obtained are then discussed. The results show that the sum of the maximum allowable harvest and the final biomass level must not exceed the maximum sustainable yield (MSY).

Résumé. L'étude explore la récolte optimale des ressources renouvelables comme la pêche. La dynamique de la biomasse des poissons est décrite par un modèle de croissance non linéaire qui maximise le revenu net total tout en tenant compte de l'utilisation effective et durable de la ressource. En outre, la dynamique de la stabilité du modèle est évaluée grâce une analyse de bifurcation. Le principe maximal de Pontryagin est utilisé pour dériver le système d'optimalité et caractériser le contrôle optimal. Une méthode itérative numérique employant le schéma Runge-Kutta de quatrième ordre facilite la solution du système d'optimalité. Les résultats de simulation obtenus sont ensuite discutés. Ils montrent

*Corresponding author Mahmud Ibrahim: mibrahim@ucc.edu.gh
Francis Benyah : fbenyah@ucc.edu.gh

que la somme de la récolte maximale admissible et du niveau de biomasse finale ne doit pas dépasser le rendement maximal durable (MSY).

1. Introduction

In recent times, there has been a surge in the population of the world. This, invariably, has led to an increase in the exploitation of renewable resources like fisheries. The Food and Agriculture Organization (FAO) of the United Nations (UN) assessment of fisheries reveals that the share of fish biomass within biologically sustainable levels has exhibited a downward trend, declining from 90% in 1974 to 68.6% in 2013; see FAO (2016).

Renewable resources, by nature, possess self-regeneration capabilities and can provide mankind with an essentially endless supply of goods and services. However, mankind, in turn, possesses capacities for both conservation and destruction of the resource base, see Clark (1973). As Kompas (2005) rightly observed, inefficient fisheries are plagued by low profits and excessive fishing capacity, giving rise to the all too familiar outcome of "too many boats chasing too few fish". Therefore, there is the need for efficient and effective utilization of the resource to ensure sustainability. This calls for the use of appropriate and effective technology in the exploitation of the resource.

The basic Gordon-Schaefer model of renewable resource exploitation makes use of the following growth function with harvesting

$$\frac{dx(t)}{dt} = g(x) - h(t) , \qquad x(0) = x_0 \tag{1}$$

where the sate variable $x(t)$ denotes the biomass of fish population at time t, x_0 is the initial fish biomass, $g(x)$ is the natural growth rate of fish biomass, and $h(t)$ is the rate of harvesting of the fish biomass at time t; see Gordon (1954); Schaefer (1954).

The assumption is that the growth function g is differentiable and concave, and it satisfies, for $0 < x < K$,

$$g(0) = 0, \ g(K) = 0, \ g(x) > 0, \ \text{and} \ g''(x) < 0 \tag{2}$$

where K denotes the *carrying capacity* of the ecosystem; that is, the maximum sustainable fish biomass. Thus

$$\lim_{t \to \infty} x(t) = K \tag{3}$$

The most common growth function favoured by most researchers is the Pearl-Verhulst logistic function; see, for example, Clark (2010); Clark and Munro (1975); Craven (1995); Dubey and Patra (2013); Dubey *et al.* (2003); Tar and Chakraborty (2009). In a pioneering work, Clark and Munro (1975) proposed a nonlinear optimal control model that permitted the rate of fishing effort to be nonlinear, thereby introducing nonlinear costs into the

model. The model was also extended to make it non-autonomous in both the price and cost parameters. Craven (1995) in an exposition on the fishery resource, used a growth rate function that satisfies (2) and (3). Additionally, in determining the net revenue for the harvested fish, the model not only considered the unit price and unit cost of the fish but also the diminishing returns when there is a large amount of fish to sell. Dubey and Patra (2013) proposed a model that involves the interaction of the human population and the resource population. They considered the human population to be partially dependent on the resource, which is then harvested for the benefit of society. Also, Dubey *et al.* (2003) modeled the effect that a reserve zone has on the exploitation of a fishery resource. It is shown that even if the fishery is exploited continuously in the unreserved zone, the fish population can be maintained at an appropriate steady-state level in the habitat. Tar and Chakraborty (2009) developed a logistic growth model to assess the effect of harvesting on the Bangladesh shrimp fishery. Their work showed that it was feasible to assess the biological, social and economic impacts of the existing shrimp resource of the country, as well as providing appropriate measures to maintain long run sustainability of the resource.

In a recent paper, Wu *et al.* (2015) proposed an extension to the basic Gordon-Schaefer model in the determination of the net revenue. They emphasized on the need to include the effective utilization rate $s(t)$ in the objective functional to ensure that the fishing resources are prudently utilized. The effective utilization of the resource is a function of the technology used in harvesting the resource. The technological change envisaged may include a better engine, more efficient fishing gear, and navigation aids; see Kompas (2005). This change in technology would impact greatly on the fishing activity and hence the expected revenue. There is therefore the justification to modify the objective functional to accommodate the technology used in harvesting. For other related works with modified objective functional, see Clark and Munro (1975); Craven (1995); Hanson and Ryan (1998).

It is our intention to use the Gordon-Schaefer model with a modified objective functional as proposed by Wu *et al.* (2015). However, the model dynamics is explored and equilibrium points determined. The optimal control problem is then analyzed using the *current-value* Hamiltonian and the optimal harvesting policy and level of fish biomass are determined using numerical techniques. Also, the behavior of shadow price and net revenue of the resource through the various harvesting strategies is explored.

In Section 2, the optimal control model is formulated consisting of the biomass dynamics and the complete bio-economic model. Bifurcation analysis of the model is performed in Section 3. Optimality of the model, which consists of the characterization of the optimal control as well as the singularity analysis of the model, is in Section 4. Numerical and graphical illustrations of the model are portrayed in Section 5 whilst the last chapter deals with the discussion and conclusion.

2. Model formulation

The formulation of the model takes into account the biological outcomes as well as the economic objectives of fisheries management.

2.1. The biomass dynamics

As mentioned previously, the growth function of choice in many fishery research is the logistic growth function

$$g(x) = rx(t) \left(1 - \frac{x(t)}{K} \right) \tag{4}$$

where $r > 0$ is the intrinsic growth rate or per capita growth rate of fish biomass. Clearly, (4) satisfies conditions in the model dynamics described by (2) and (3) regarding the growth function $g(x)$. Therefore, the fish biomass dynamics with harvesting, as proposed by Gordon (1954) and Schaefer (1954), is described by an initial value problem

$$\frac{dx(t)}{dt} = rx(t) \left(1 - \frac{x(t)}{K} \right) - h(t) \ , \qquad x(0) = x_0 \tag{5}$$

To obtain the equilibrium points, we set (5) to zero and solve the resulting quadratic equation to yield

$$x_1 = \frac{K}{2} \left(1 - \sqrt{1 - \frac{4h}{rK}} \right) \ , \qquad x_2 = \frac{K}{2} \left(1 + \sqrt{1 - \frac{4h}{rK}} \right) \tag{6}$$

provided $h < \frac{rK}{4}$. When $h \geq \frac{rK}{4}$, there is at most one equilibrium point. Therefore, $h = \frac{rK}{4}$ is a saddle-node bifurcation point for the model presented in (5). The equation for the sustainable yield is

$$h_S = rx \left(1 - \frac{x}{K} \right) \tag{7}$$

The MSY occurs when

$$\frac{dh_S}{dx} = r \left(1 - \frac{2x}{K} \right) = 0 \tag{8}$$

Thus the biomass level at MSY is

$$x_{MSY} = \frac{K}{2} \tag{9}$$

Additionally, substituting the value of the biomass level at MSY, (9), into (7) gives the value of MSY

$$h_{MSY} = \frac{rK}{4} \tag{10}$$

Therefore h_{MSY} is the maximum amount that can be harvested whilst keeping the population constant. That is, $\frac{dx}{dt} = 0$.

2.2. The bioeconomic model

Adding economic parameters into the afore-mentioned biological model, (5) gives the bioeconomic model. Let the discounted net revenue for the harvested or landed fish per unit time be expressed as

$$e^{-\delta t}(p - c)h(t) \tag{11}$$

where p is the price per unit biomass of landed fish, c is the cost per unit biomass of landed fish, $h(t)$ is the rate of harvesting, and δ is the social discount rate.

To ensure that the fishery is managed in a sustainable way so as to avoid wastage and pollution to the ecosystem, we incorporate the effective utilization rate, $s(t)$ into the objective functional. Then $s(t)$ should satisfy the following assumptions:

1. With the development of technology, the effective utilization rate $s(t)$ will gradually increase with respect to time t; that is, $s'(t) > 0$

2. The increase in $s(t)$ will become more difficult after it reaches a certain level in time (diminishing returns); that is, $s''(t) < 0$

3. The ideal is for the complete or total utilization of resources; that is, $\lim_{t \to \infty} s(t) = 1$

The function that satisfies the preceding assumptions and is therefore the effective utilization rate is

$$s(t) = 1 - ae^{-bt} \qquad a, b > 0 \tag{12}$$

In addition, let the initial utilization rate be $s(0) = s_0$ and the achievable effective utilization rate at the terminal time T be $s(T) = s_T$. Then, the parameters a and b in $s(t)$ can be obtained as follows:

$$
\begin{aligned}
a &= 1 - s_0 \\
b &= \frac{1}{T} \ln \frac{1 - s_0}{1 - s_T}
\end{aligned}
\tag{13}
$$

Thus the modified net revenue for the proposed model is

$$e^{-\delta t}(p - c)h(t)(1 - ae^{-bt})$$

and hence the optimal control problem can be cast as

$$\max_{h} J(h) = \int_0^T e^{-\delta t}(p - c)h(t)(1 - ae^{-bt})\,dt$$

$$\text{subject to} \quad \frac{dx(t)}{dt} = rx(t)\left(1 - \frac{x(t)}{K}\right) - h(t)$$

$$0 \leq h(t) \leq h_{max}$$

$$x(0) = x_0, \qquad x(T) = x_T \tag{14}$$

3. Bifurcation analysis

A bifurcation can be described as the change in the number of equilibrium points or periodic orbits, or in the stability properties of a dynamical system if a parameter is varied. The value of the parameter where the stability dynamics change is called a bifurcation point; see Daci and Spaho (2013).

As mentioned earlier, there are two equilibrium points associated with the state dynamics of the model described by (14) when the harvest is less than the bifurcation point. These are x_1 and x_2, and the bifurcation point is given by $h = \frac{rK}{4}$. The slope fields and solution curves of the state equation were plotted with the aid of the software dfield8 by John Polking. This is illustrated by using the following parameter values: $r = 4.4$ and $K = 100000$. Thus $h = 110000$ is the bifurcation point.

Figure 1 presents the solution curves for the case where the harvest $h = 80000$ is less than the bifurcation point. It is observed that there are two equilibrium points: $x_1 = 23888.35$ and $x_2 = 76111.65$. When the biomass level is greater than 76111.65, the biomass monotonically decreases towards this upper equilibrium point in the long run whilst for a biomass level less than the lower equilibrium point 23888.35, the biomass monotonically decreases towards zero. In other words, the biomass would be completely depleted in finite time. Also, when the biomass level is between the two equilibrium points, the biomass approaches the upper equilibrium point. Of course, biomass levels originating in the equilibrium points always maintain their course on these levels. Thus the point x_1 is unstable while x_2 is stable.

Solution curves corresponding to the case where $h = h_{MSY} = 110000$, the bifurcation point, is presented in Figure 2. At the bifurcation point, there is only a single equilibrium point, $x_{MSY} = 50000$. That is, for any initial biomass level greater than 50000, the long-term population of fish stock approaches this equilibrium point. On the other hand, for a biomass level less than 50000, the biomass monotonically decreases towards zero and the resource would become extint in finite time. For an initial biomass level of 50000, the resource
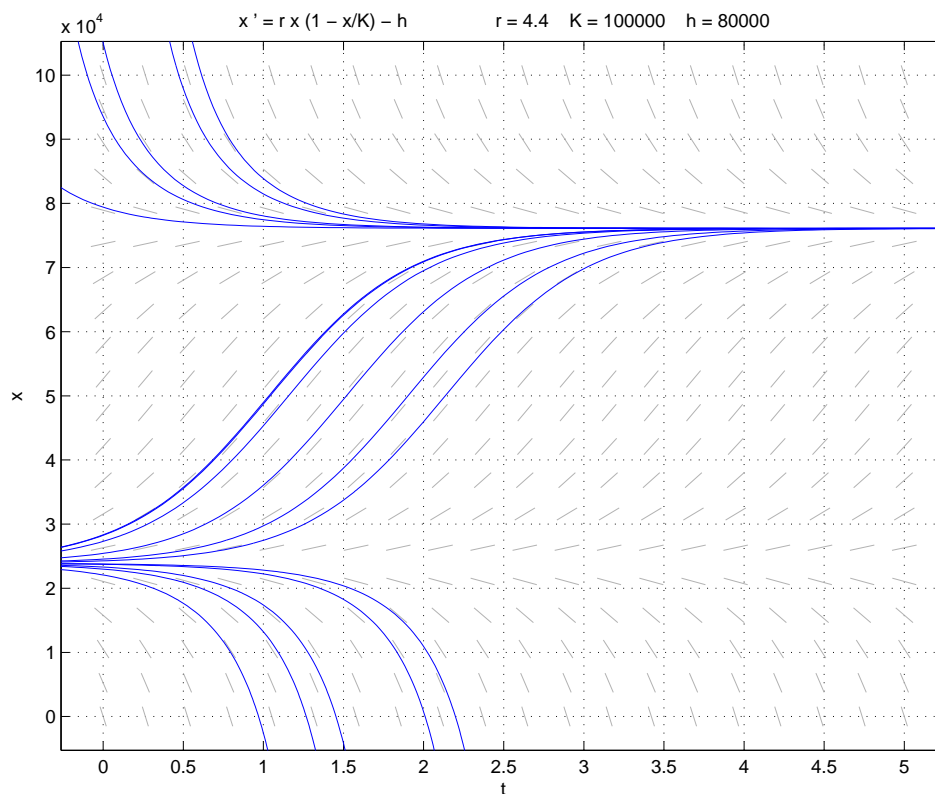
Fig. 1: Solution curves for $h = 800000$

stock remains on this level. Therefore, this point is semi-stable and known as a saddle-node bifurcation point.

The case where harvest level, $h = 150000$ is greater than the bifurcation point is shown in Figure 3. Corresponding to this harvest level, there exists no equilibrium point. In this situation, whatever the initial fish population, the fish will die out as a result of overfishing or excessive harvesting in finite time; see Edwards and Penney (2004); Suri (2008).

4. Optimality of the model

In this section, we first find a characterization for the optimal control h^* and then determine the presence of singular path or lack thereof in the model described by (14).
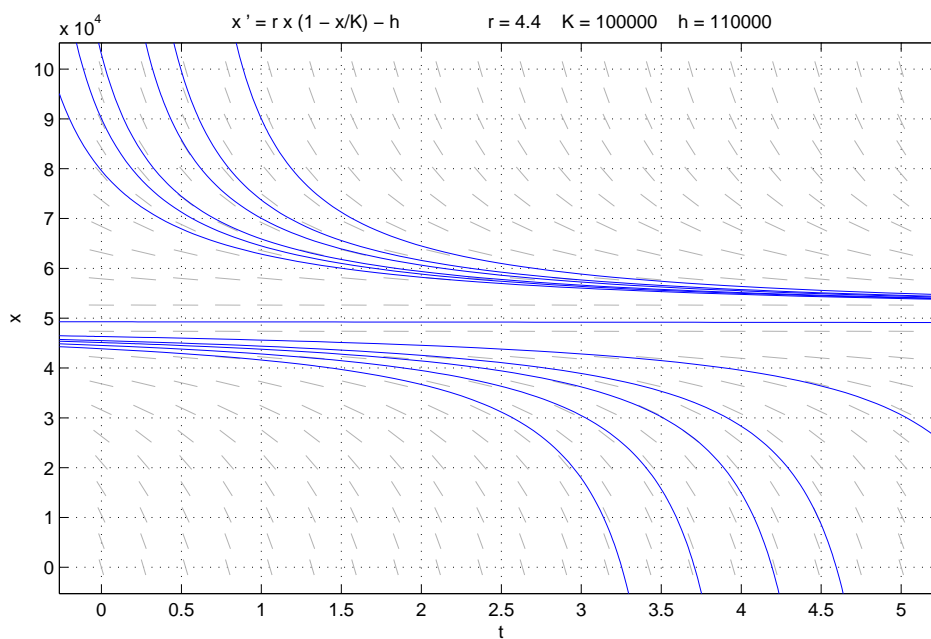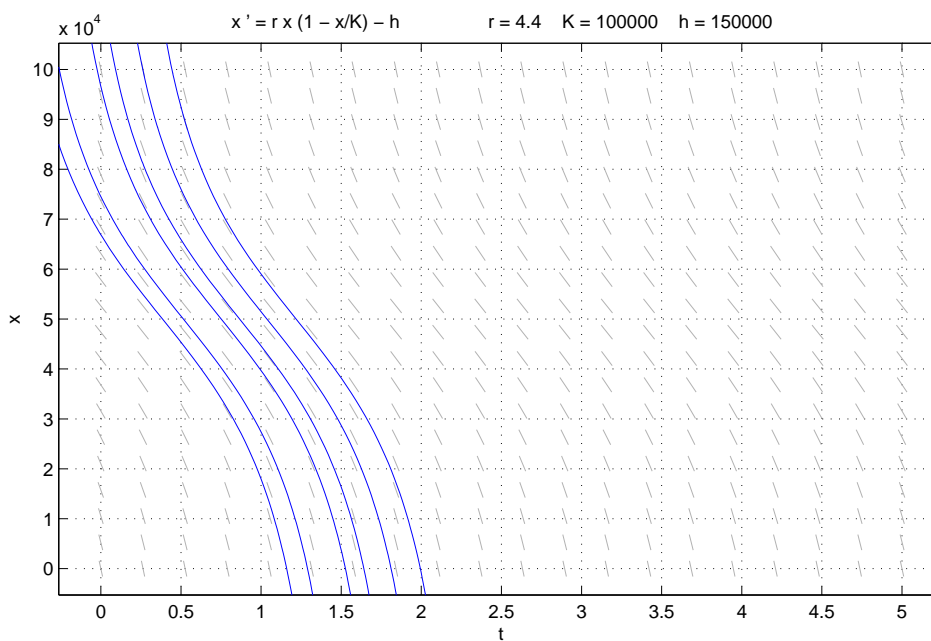
Fig. 2: Solution curves for $h = 110000$

Fig. 3: Solution curves for $h = 150000$

4.1. Characterization of the optimal control

The goal, as stated earlier, is to maximize the net revenue while taking into consideration the effective utilization of the resource. We therefore seek an optimal control h^* such that

$$J(h^*) = \max\{J(h) \,|\, h \in U\},$$

where the control set is Lebesgue measurable and defined by

$$U = \{h \,|\, 0 \le h \le h_{max}, \ t \in [0, T]\}.$$

In order to derive the necessary conditions for the optimal control, we employ Pontryagin's maximum principle; see Joshi *et al.* (2015); Pontryagin *et al.* (1962). The *current-value* Hamiltonian for the optimal control problem described by (14) is

$$H(x, \lambda, h, t) = (p - c)h(1 - ae^{-bt}) + \lambda \left[rx \left(1 - \frac{x}{K} \right) - h \right] \tag{15}$$

The adjoint variable λ is governed by

$$\begin{aligned} \lambda' &= \delta\lambda - \frac{\partial H}{\partial x} \\ &= \delta\lambda - \lambda \left(r - \frac{2rx}{K} \right) \end{aligned} \tag{16}$$

The switching function is defined as

$$\begin{aligned} \psi &= \frac{\partial H}{\partial h} \\ &= (p - c)(1 - ae^{-bt}) - \lambda \end{aligned} \tag{17}$$

Therefore the characterization of the optimal control is

$$h^* = \begin{cases} 0, & \text{if} \quad \psi < 0, \\ \in (0, h_{max}), & \text{if} \quad \psi = 0, \\ h_{max}, & \text{if} \quad \psi > 0. \end{cases} \tag{18}$$

4.2. Singularity analysis of the model

The singularity analysis would determine the choice of method for the solution of the model presented in (14). For a singular control, we assume that there is an interval I for all $t \in I \subset [0, T]$ such that

$$\psi(t) = 0 \tag{19}$$

Thus from (17) and (19),

$$(p - c)(1 - ae^{-bt}) - \lambda = 0 \tag{20}$$

So, solving for λ we find

$$\lambda = (p - c)(1 - ae^{-bt}) \tag{21}$$

Differentiating (21) with respect to t, it follows that

$$\lambda' = (p - c)abe^{-bt} \tag{22}$$

By plugging the λ expression in (21) into the adjoint, (16), we get

$$\lambda' = (p - c)(1 - ae^{-bt})\left(\delta - r + \frac{2rx}{K}\right) \tag{23}$$

Setting the (22) and (23) equal to each other and simplifying, we obtain

$$x = \frac{K}{2r}\left(r - \delta + \frac{abe^{-bt}}{1 - ae^{-bt}}\right) \tag{24}$$

Thus, on the singular interval I we can take the derivative of (24) with respect to t. This gives us

$$x' = \frac{K}{2r}\frac{-ab^2e^{-bt}}{(1 - ae^{-bt})^2} \tag{25}$$

comparing (25) to the state equation in the model described by (14), we see that they are not equal for all t. So h^* is nowhere singular and thus it is *bang-bang* with at most two switching times, occurring where the functions $\lambda(t)$ and $(p - c)(1 - ae^{-bt})$ intersect. Therefore, the inclusion of the utilization factor effectively rules out the singular path and only the *bang-bang* approach is feasible.

Hence the *bang-bang* control is

$$h^* = \begin{cases} 0, & \text{if} \quad \lambda > (p - c)(1 - ae^{-bt}), \\ h_{max}, & \text{if} \quad \lambda < (p - c)(1 - ae^{-bt}). \end{cases} \tag{26}$$

Therefore the optimal harvesting regime involves extreme policies only, either harvesting at full capacity or none at all. In other words, the resource should be harvested if and only if the net revenue per unit harvest, taking into consideration the effective utilization rate, exceeds the shadow price of the resource; see Tar and Chakraborty (2009). The computations were performed using using the symbolic algebra software `Maple`.

It is normal in optimal control problems to ensure the existence of the optimality system, which is simply the state equation and the adjoint equation together with the characterization of the optimal control and the boundary conditions. In this vein, it should be noted that the state equation, which is logistic with harvesting is *a priori* bounded. Also, the state equation and the objective functional are both linear in the control h. Therefore, by standard arguments, an optimal control as well as the optimal state exists; see Fleming and Rishel (1995); Joshi *et al.* (2015).

5. Numerical simulations

The optimality system is solved using an iterative method with a Runge-Kutta fourth order scheme. The state is solved forward in time and the adjoint equation is solved backward in time with an initial guess. At the end of each iteration, the control is updated using the derived formula for the optimal control. The iterations continue until convergence is achieved.

Using a modified `MATLAB` code, originally developed by Lenhart and Workman (2007), the numerical solutions to the optimality system are obtained with the following reasonably realistic parameter values in Table 1. The values illustrate the scenario of a reservoir inhabited by a fishery resource undergoing harvesting for a year, given the initial fish biomass and a pre-determined final biomass level.

Table 1: Parameter values for model

Parameter	Description	Value	Unit
K	Carrying capacity of ecosystem	100000	tons
r	Per capita growth rate	4.4	year^{-1}
p	Price per unit harvest	38	$ ton^{-1}
c	Cost per unit harvest	20	$ ton^{-1}
δ	Discount rate	0.1	year^{-1}
h_{max}	Maximum rate of harvest	20000	tons year^{-1}
T	Terminal time	1	year
$x(0)$	Initial biomass level	5000	tons
$x(T)$	Terminal biomass level	70000	tons

The parameter values $a = 0.4$ and $b = 0.05$ are obtained from (13) using the values of $s_0 = 0.60$ and $s_T = 0.62$.

The adjoint variable, also known as the shadow price of the resource, can be interpreted as the additional revenue accruing to the fisher as a result of adding one more fish to the biomass at time t; see Lenhart and Workman (2007); Yusuf and Benyah (2012). We present the optimal harvesting strategies corresponding to three values of the maximum rate of harvest h_{max}. These are; 20000, 30000, and 40000 tons year^{-1} . Simulation results detailing the relationship between the shadow price and net revenue are illustrated.

Figure 4 depicts harvesting at the maximum rate of 20000 tons year^{-1} , where the shadow price is convex and the revenue, linear. At the start of the harvesting period, the shadow price of fish, \$34.25 is significantly higher than the net revenue, \$10.80. This signifies that the revenue due an additional fish being added to the biomass is greater than the expected revenue from harvesting the fish. So at this instance it is prudent not to harvest. As time progresses the shadow price experiences a sharp decline in value whilst the revenue slightly increases until the two intersect at the switching time $t^* = 0.34$. Thereafter, the shadow
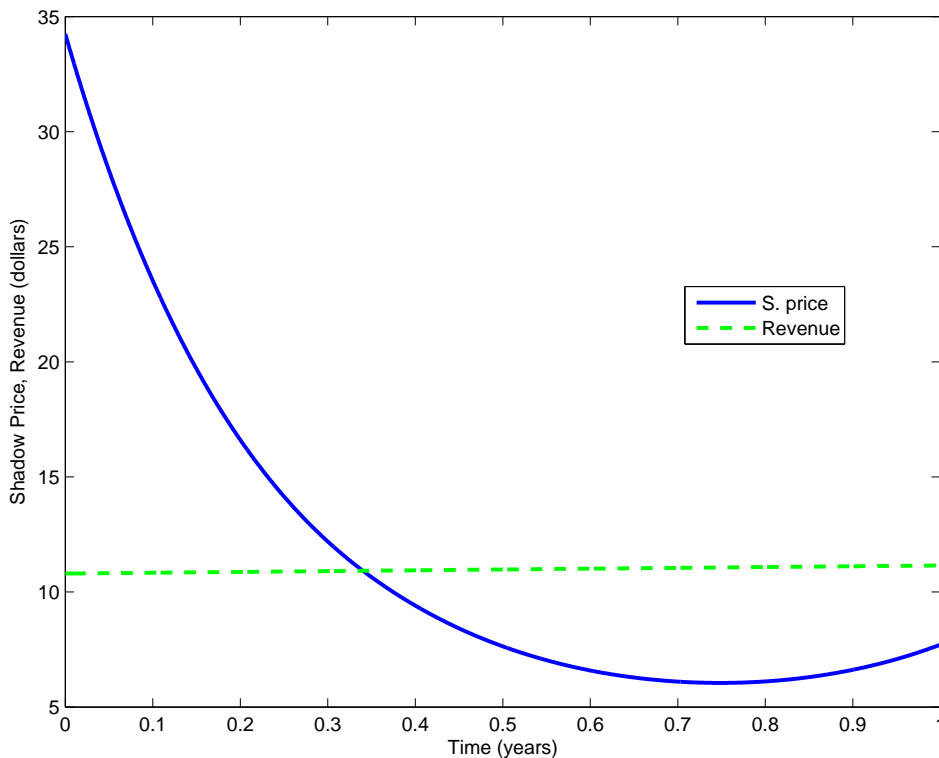
Fig. 4: Shadow price and revenue for $h_{max} = 20000$

price continues to fall to a low value of \$6.05 at $t^* = 0.75$ before rising slightly to end at a value of \$7.71. Meanwhile, the revenue has appreciated and ends with a terminal value of \$11.15. Thus, after 4 months (the switching time) it is now profitable to harvest the fish as the revenue would be greater than the shadow price.

In Figure 5 the revenue maintains it monotonicity, as well as the initial and terminal values. The shadow price also retains it parabolic shape; however, the initial and terminal values change, from \$47.22 to \$11.67. Again, at the initial period it is advisable not engage in any harvesting as the shadow price is higher than the revenue. With time, the revenue acts as secant line to the shadow price producing two switching times at $t^* = 0.48$ and $t^* = 0.99$. Therefore, harvesting is only recommended between the switching points where the revenue exceeds the shadow price. It is interesting to note that the lowest value of the shadow price is \$9.07 and occurs at $t^* = 0.73$.

The shadow price starts from its highest value of \$50.92 to equally highest terminal value of \$12.62. The revenue remains unchanged and is the same as the previous two scenarios. This situation produces the greatest difference between the shadow price and revenue, with a value of \$40.12. Thus the preferable course of action is not to harvest the resource at the initial time. Similar to Figure 5, the revenue gives rise to two switching times, $t^* = 0.54$ and $t^* = 0.93$. However, the period between the switching times is shorter, indicating that the
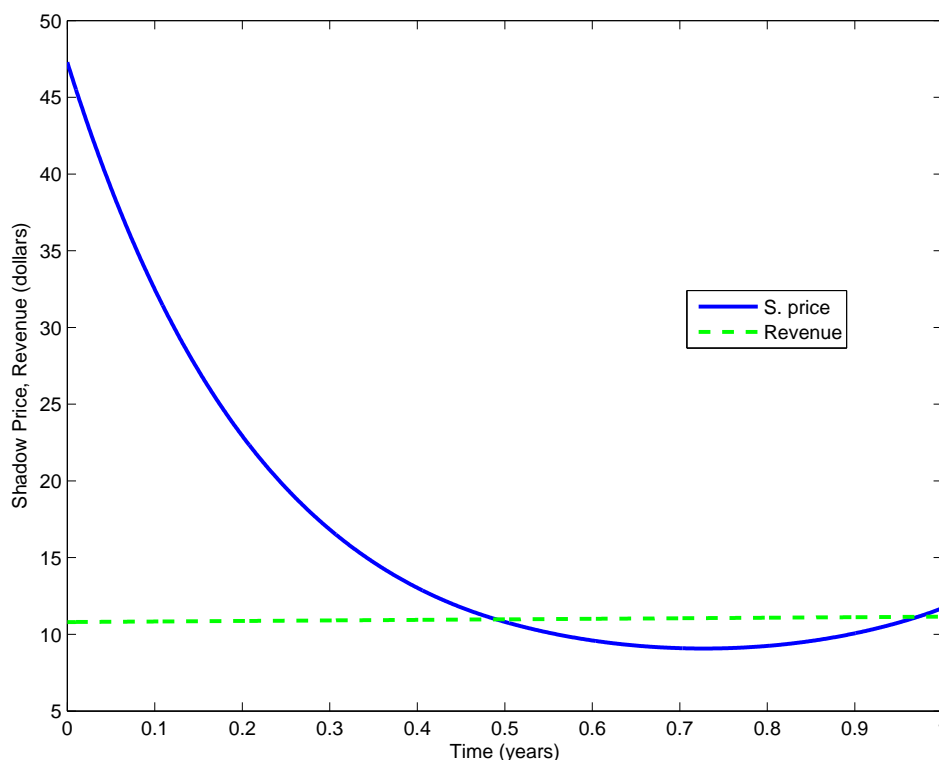
Fig. 5: Shadow price and revenue for $h_{max} = 30000$

harvest should start late and end early. Also, the shadow price attains its minimum value of $9.96 at $t^* = 0.73$. This is illustrated in Figure 6.

Simulation results for the harvesting strategy and biomass level relating to the case where $h_{max} = 20000$ tons year^{-1} are presented in Figures 7 and 8. In Figure 7, it is observed that the switching time occurs at $t^* = 0.34$ (see Figure 4) indicating that for the initial 4 months of the year, no harvesting should occur. Thereafter, the maximum rate of harvest should be applied. This harvesting policy allows the fishery resource to grow naturally for almost 4 months before full exploitation starts. A slight change in the growth of fish biomass can be observed once the maximum rate of harvest begin (see Figure 8).

The performance measure, which is the total net revenue per year, for this harvesting policy is computed and has a value of $136,104.20.

For the case where $h_{max} = 30000$ tons year^{-1} , the plots for the harvesting policy and biomass level are shown in Figures 9 and 10. Two switching times can be identified in the harvesting strategy at $t^* = 0.48$ and $t^* = 0.99$ (see Figure 5). The optimal harvesting policy, as shown in Figure 9, is therefore to wait for almost 6 months before starting any harvesting and to stop the harvest just before the close of the year. Figure 10 shows that
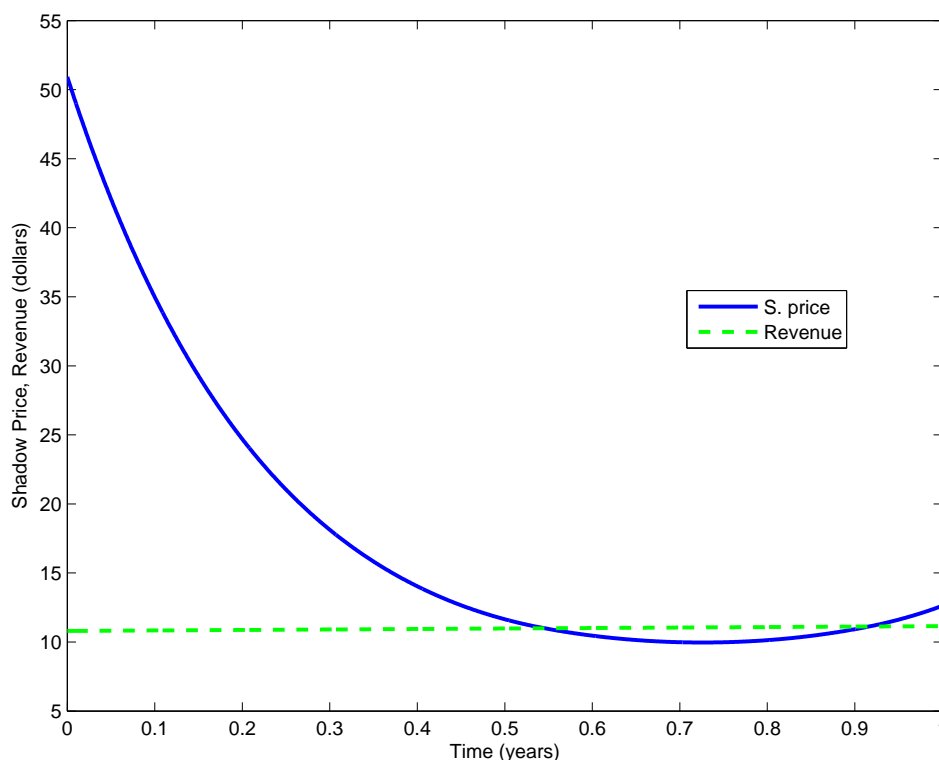
Fig. 6: Shadow price and revenue for $h_{max} = 40000$

the growth of fish biomass is affected at the switching points. The total net revenue per year for this harvesting regime is $148,710.80.

Figures 11 and 12 are the plots for the harvesting strategy and biomass level illustrating the case where $h_{max} = 40000$ tons year^{-1} . The optimal harvest policy also involves two switching times, at $t^* = 0.54$ and $t^* = 0.93$ (see Figure 6). The strategy in this case is to start the harvest a little over 6 months and to stop well before the year ends. This is illustrated in Figure 11. The fish biomass level is similarly affected once harvesting starts (see Figure 12). A value of $152.416.50 is obtained as the total net revenue per year.

6. Conclusion

The study considered an optimal control model that incorporated into the model the effective utilization of the fishery resource. The appropriate biological model was formulated and subsequently economic parameters were added to produce the bioeconomic model. Subjecting the model to bifurcation analysis showed that harvesting of the resource should only be contemplated when the rate of harvest is less than the MSY (bifurcation point).
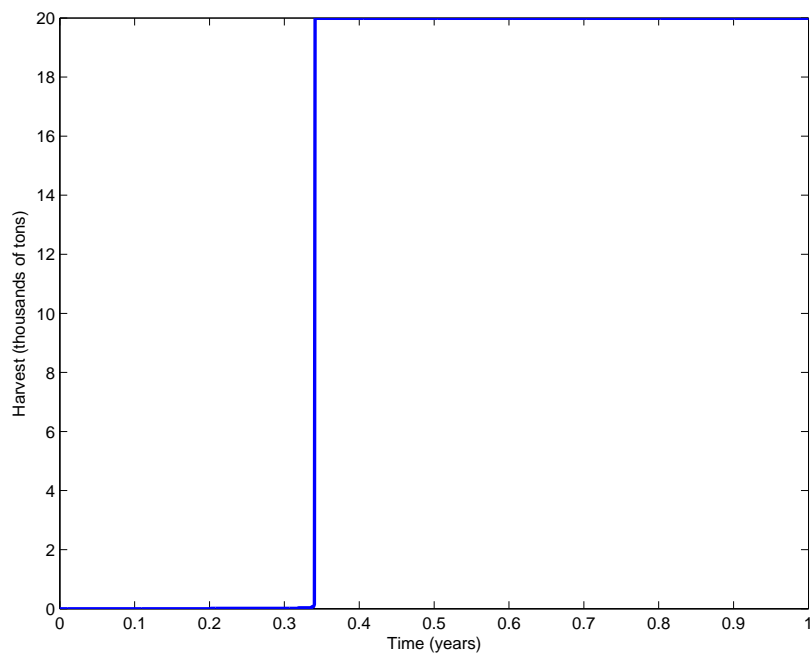
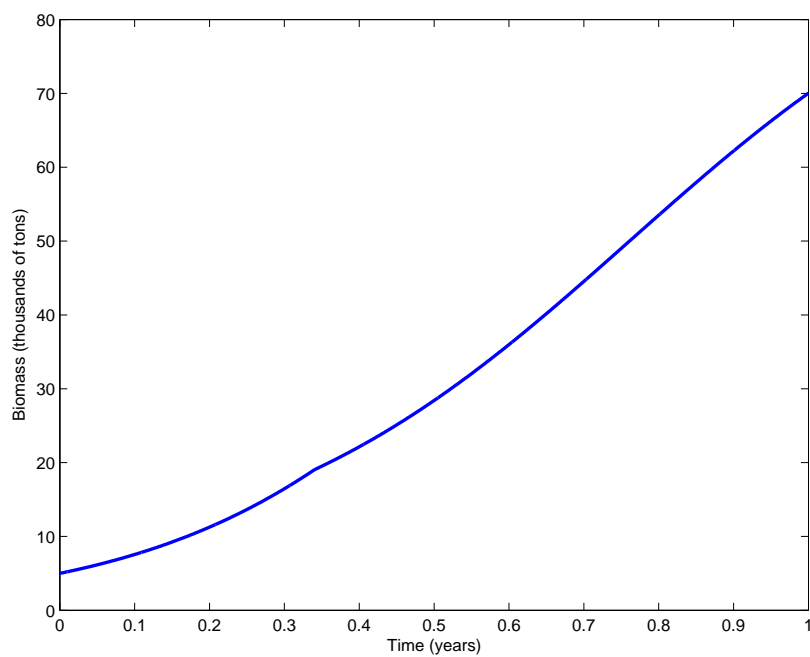Fig. 7: Harvesting strategy for $h_{max} = 20000$
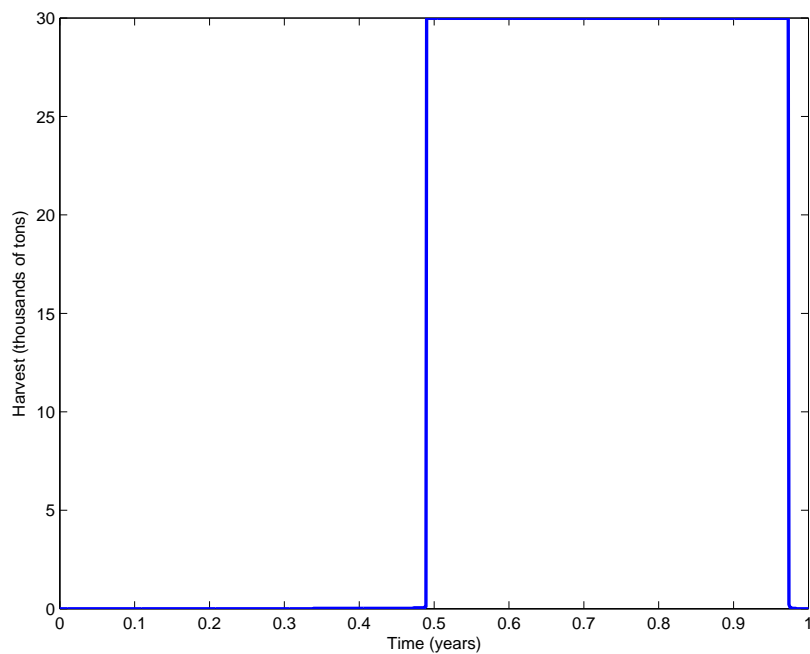

Fig. 8: Fish biomass for $h_{max} = 20000$

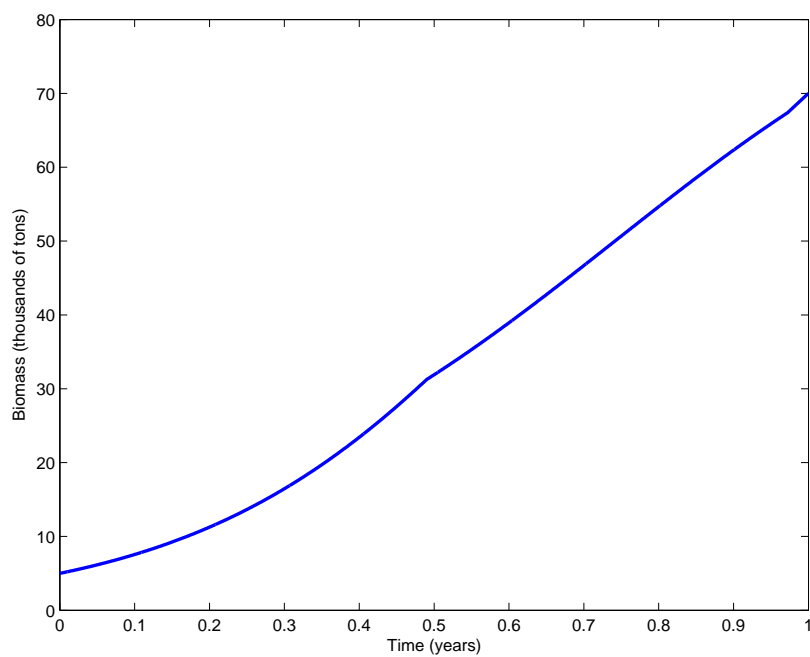Fig. 9: Harvesting strategy for $h_{max} = 30000$

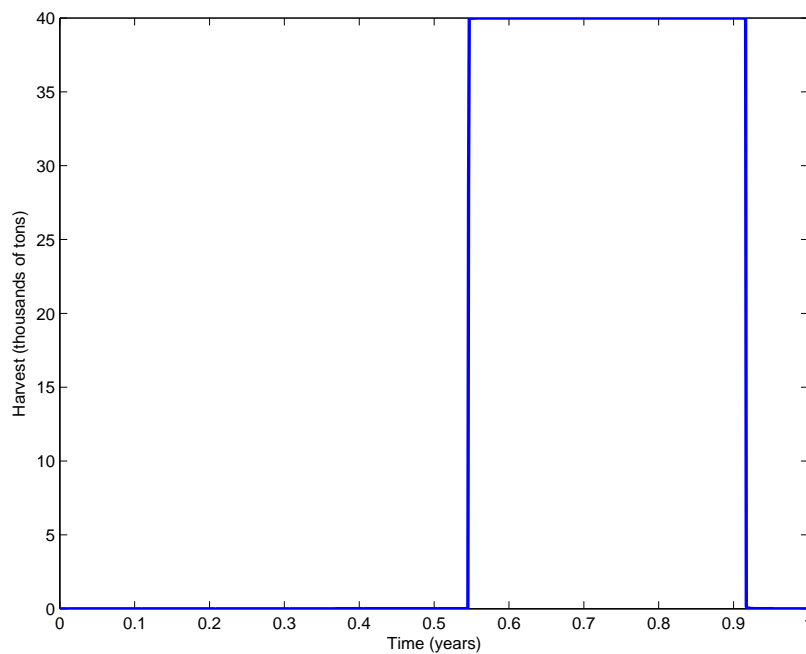Fig. 10: Fish biomass for $h_{max} = 30000$

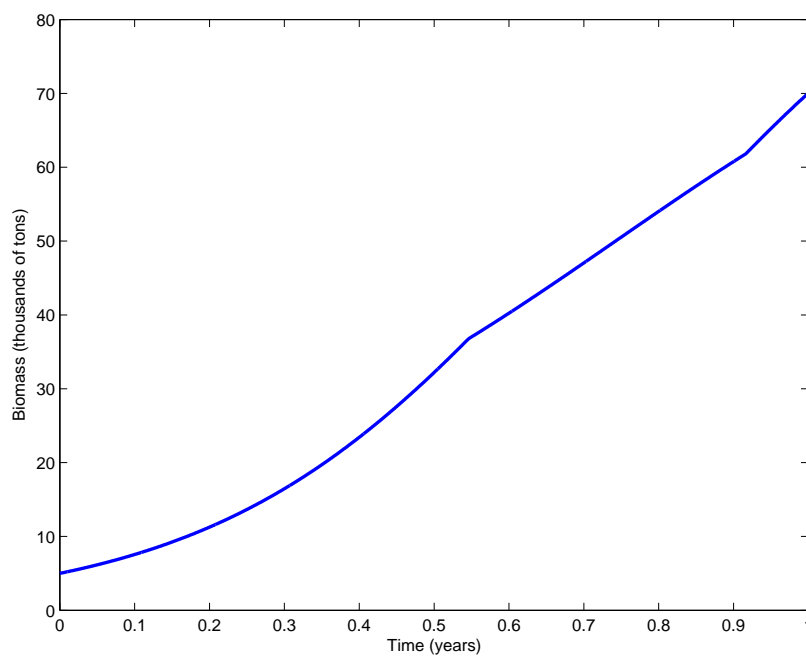Fig. 11: Harvesting strategy for $h_{max} = 40000$

Fig. 12: Fish biomass for $h_{max} = 40000$

In determining the optimality of the model, characterization of the optimal control, the harvesting rate, was performed. Also, the model was analyzed for singularity and the outcome showed that only the *bang-bang* path was applicable.

Afterwards, a simulation was carried out using realistic parameter values for the cases $h_{max} = 20000$ tons year^{-1} , $h_{max} = 30000$ tons year^{-1} and $h_{max} = 40000$ tons year^{-1} . The results indicated that the case where $h_{max} = 40000$ tons year^{-1} provided the greatest revenue. However, if early harvesting is desired because of market demands, then the case where $h_{max} = 20000$ tons year^{-1} should be given a consideration since it recommends the earliest commencement of harvesting among the three scenarios.

It is instructive to note that, for a terminal biomass level of 70000 tons, convergence of the iterates is not achieved for harvests greater than 40000 tons. This is not surprising, as the MSY for the model is 110000 tons. This value exactly equals the sum of the maximum allowable harvest per year, 40000 tons and the final biomass level, 70000 tons.

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