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EXISTENCE AND UNIQUENESS OF PERIODIC SOLUTIONS FOR A SYSTEM OF DIFFERENCE EQUATIONS WITH FINITE DELAY

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Abstract. We consider a special class of system of delay difference equations. The fundamental matrix solution together with Floquet theory is used to convert the system of equations into an equivalent summation equation. Fixed point theorems due to Krasnoselski and Banach are then used to show the existence of a unique periodic solution of the system of difference equations.

1. Introduction

The study of the existence of periodic solutions for difference equations have gained the attention of many researchers in recent times, see for example [1] [2],[3],[4], [5],[7], [8], [9], [10], [11], [12], [13] and [15].

In this paper we consider the system of difference equations

$$\Delta x(n) = A(n)x(n-\tau), \tag{1}$$

where $A(n) \in \mathbb{R}^{s\times s}$ is a nonsigular matrix and τ is a positive constant. We are mainly motivated by the work of Raffoul in [13] where he obtained sufficient conditions for the existence and uniqueness of a periodic solution of a scalar version of (1). Thus, in this paper we establish sufficient conditions for (1) to have a unique periodic solution.

Floquet theory offers a lot of results on the periodicity of system (1) when $\tau = 0$. Throughout this paper Δ denotes the forward difference operator $\Delta x(n) = x(n+1) - x(n)$ for any sequence $\{x(n),\ n = 0, 1, 2, ...\}$. Also, we define the operator E by $Ex(n) = x(n + 1)$. For more on difference calculus we refer the reader to [6].

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2. Existence and uniqueness

We assume in this section that there exist a nonsingular $s \times s$ matrix $G(n)$ such that

$$\Delta x(n) = G(n)x(n) - \Delta_n \sum_{k=n-\tau}^{n-1} G(k)x(k) + \Big[A(n) - G(n-\tau)\Big]x(n-\tau). \tag{2}$$

Lemma 1 Equation (1) is equivalent to (2).

Proof. By taking the difference with respect to n of the summation term in (2) we obtain

$$\Delta_n \sum_{k=n-\tau}^{n-1} G(k)x(k) = G(n)x(n) - G(n-\tau)x(n-\tau). \tag{3}$$

Substituting (3) into (2) gives

$$\begin{aligned} \Delta x(n) &= G(n)x(n) - G(n)x(n) + G(n-\tau)x(n-\tau) + \Big[A(n) - G(n-\tau)\Big]x(n-\tau) \\ &= A(n)x(n-\tau). \end{aligned}$$

This completes the proof.

Let T be an integer such that $T \geq 1$. Let P_T be the set of all real-valued s-vector sequences $\{x(n),\ n = 0, 1, 2, ...\}$, periodic in n of period T. Then $(P_T, ||.||)$ is a Banach space with the maximum norm

$$||x(.)|| = \max_{n\in[0,T-1]} |x(n)|,$$

where $|.|$ denotes the infinity norm for $x \in \mathbb{R}^s$. Also, if A is an $s \times s$ real matrix, then we define the norm of A by $|A| = \max_{1\leq i\leq s} \sum_{j=1}^{s} |a_{ij}|$.

Definition 2 If the matrix $G(n)$ is periodic of period T, then the linear system

$$\Delta x(n) = G(n)x(n) \tag{4}$$

is said to be noncritical with respect to T if it has no periodic solution of period T except the trivial solution $x = 0$.

Throughout this paper it is assumed that system (4) is noncritical. The fundamental matrix of (4) has the following properties:
(i) det $\Phi(n) \neq 0$.
(ii) There exists a constant matrix B such that $\Phi(n - T + 1) = \Phi(n + 1)B^{-T}$, by Floquet theory.
(iii) System (4) is noncritical if and only if $\det(I - \Phi(n)) \neq 0$.

In this paper we assume that

$$A(n + T) = A(n), \qquad G(n + T) = G(n) \tag{5}$$

We now state and prove in the following lemma one of the fundamental properties of the difference operator.

Lemma 3 For functions $y(n)$ and $z(n)$ of a real variable n,

$$\Delta(y(n)z(n)) = Ey(n)\Delta z(n) + [\Delta y(n)]z(n). \tag{6}$$

Proof.

$$\begin{aligned}
\Delta(y(n)z(n)) &= y(n+1)z(n+1) - y(n)z(n) \\
&= y(n+1)z(n+1) - y(n+1)z(n) + y(n+1)z(n) - y(n)z(n) \\
&= y(n+1)[z(n+1) - z(n)] + [y(n+1) - y(n)]z(n) \\
&= Ey(n)\Delta z(n) + [\Delta y(n)]z(n).
\end{aligned}$$

Lemma 4 Suppose (5) hold. Suppose further that $\Phi(0) = I$. If $x(n) \in P_T$ then $x(n)$ is a solution of (1) if and only if

$$\begin{aligned}
x(n) &= -\sum_{k=n-\tau}^{n-1} G(k)x(k) \\
&\quad + \Phi(n)(\Phi^{-1}(T) - I)^{-1} \sum_{u=n}^{n+T-1} \Phi^{-1}(u+1)\Big[A(u)x(u-\tau) - G(u)\sum_{k=u-\tau}^{u-1} G(k)x(k) \\
&\quad - G(u-\tau)x(u-\tau)\Big]. \qquad (7)
\end{aligned}$$

Proof. Let $x(n) \in P_T$ be a solution of (1) and $\Phi(n)$ is a fundamental system of solutions of (2). First we write (1) as

$$\begin{aligned}
\Delta[x(n) + \sum_{k=n-\tau}^{n-1} G(k)x(k)] &= G(n)[x(n) + \sum_{k=n-\tau}^{n-1} G(k)x(k)] \\
&\quad - G(n)\sum_{k=n-\tau}^{n-1} G(k)x(k) + A(n)x(n-\tau) \\
&\quad - G(n-\tau)x(n-\tau).
\end{aligned}$$

Since $\Phi(n)\Phi^{-1}(n) = I$, it follows from Lemma 3 that

$$\begin{aligned}
0 &= \Delta(\Phi(n)\Phi^{-1}(n)) = \Phi(n+1)\Delta\Phi^{-1}(n) + [\Delta\Phi(n)]\Phi^{-1}(n) \\
&= \Phi(n+1)\Delta\Phi^{-1}(n) + [G(n)\Phi(n)]\Phi^{-1}(n) \\
&= \Phi(n+1)\Delta\Phi^{-1}(n) + G(n).
\end{aligned}$$

This implies that

$$\Delta\Phi^{-1}(n) = -\Phi^{-1}(n+1)G(n).$$

If $x(n)$ is a solution of (1) with $x(0) = x_0$, then

$$\begin{aligned}
\Delta\Big[\Phi^{-1}(n)\Big(x(n) + \sum_{k=n-\tau}^{n-1} G(k)x(k)\Big)\Big] &= \Phi^{-1}(n+1)\Delta\Big(x(n) + \sum_{k=n-\tau}^{n-1} G(k)x(k)\Big) \\
&\quad + [\Delta\Phi^{-1}(n)][x(n) + \sum_{k=n-\tau}^{n-1} G(k)x(k)] \\
&= \Phi^{-1}(n+1)\Big[G(n)[x(n) + \sum_{k=n-\tau}^{n-1} G(k)x(k)] \\
&\quad - G(n)\sum_{k=n-\tau}^{n-1} G(k)x(k) + A(n)x(n-\tau) \\
&\quad - G(n-\tau)x(n-\tau)\Big] \\
&\quad - [\Phi^{-1}(n+1)G(n)][x(n) + \sum_{k=n-\tau}^{n-1} G(k)x(k)] \\
&= \Phi^{-1}(n+1)\Big[A(n)x(n-\tau) - G(n)\sum_{k=n-\tau}^{n-1} G(k)x(k) \\
&\quad - G(n-\tau)x(n-\tau)\Big].
\end{aligned}$$

Summing the above equation from 0 to $n-1$ gives,

$$\begin{aligned}
x(n) &= -\sum_{k=n-\tau}^{n-1} G(k)x(k) + \Phi(n)\Big(x_0 + \sum_{k=-\tau}^{-1} G(k)x(k)\Big) \\
&\quad + \Phi(n)\sum_{u=0}^{n-1}\Phi^{-1}(u+1)\Big[A(u)x(u-\tau) - G(u)\sum_{k=u-\tau}^{u-1} G(k)x(k) \\
&\quad - G(u-\tau)x(u-\tau)\Big].
\end{aligned} \tag{8}$$

Since $x(T) = x_0 = x(0)$, using (8) we obtain

$$\begin{aligned}
x_0 + \sum_{k=-\tau}^{-1} G(k)x(k) &= (I - \Phi(T))^{-1}\sum_{u=0}^{T-1}\Phi(T)\Phi^{-1}(u+1)[A(u)x(u-\tau) - G(u)\sum_{k=u-\tau}^{u-1} G(k)x(k) \\
&\quad - G(u-\tau)x(u-\tau)].
\end{aligned} \tag{9}$$

Substituting (9) into (8) gives

$$\begin{aligned}
x(n) &= -\sum_{k=n-\tau}^{n-1} G(k)x(k) \\
&\quad + \Phi(n)(I - \Phi(T))^{-1}\sum_{u=0}^{T-1}\Phi(T)\Phi^{-1}(u+1)[A(u)x(u-\tau) - G(u)\sum_{k=u-\tau}^{u-1} G(k)x(k)
\end{aligned}$$

$$
\begin{aligned}
&- G(u-\tau)x(u-\tau)] \\
&+ \Phi(n)\sum_{u=0}^{n-1}\Phi^{-1}(u+1)\Big[A(u)x(u-\tau) - G(u)\sum_{k=u-\tau}^{u-1}G(k)x(k) \\
&- G(u-\tau)x(u-\tau)\Big].
\end{aligned} \tag{10}
$$

We will now show that (10) is equivalent to (7). Since

$$
(I-\Phi(T))^{-1} = (\Phi(T)(\Phi^{-1}(T)-I))^{-1} = (\Phi^{-1}(T)-I)^{-1}\Phi^{-1}(T),
$$

equation (10) becomes

$$
\begin{aligned}
x(n) \quad = \quad & -\sum_{k=n-\tau}^{n-1}G(k)x(k) \\
& + \Phi(n)(\Phi^{-1}(T)-I)^{-1}\sum_{u=0}^{T-1}\Phi^{-1}(u+1)[A(u)x(u-\tau) - G(u)\sum_{k=u-\tau}^{u-1}G(k)x(k) \\
& - G(u-\tau)x(u-\tau)] \\
& + \Phi(n)\sum_{u=0}^{n-1}\Phi^{-1}(u+1)\Big[A(u)x(u-\tau) - G(u)\sum_{k=u-\tau}^{u-1}G(k)x(k) \\
& - G(u-\tau)x(u-\tau)\Big] \\
= \quad & -\sum_{k=n-\tau}^{n-1}G(k)x(k) \\
& + \Phi(n)(\Phi^{-1}(T)-I)^{-1}\Big\{\sum_{u=0}^{T-1}\Phi^{-1}(u+1)[A(u)x(u-\tau) - G(u)\sum_{k=u-\tau}^{u-1}G(k)x(k) \\
& - G(u-\tau)x(u-\tau)] \\
& + (\Phi^{-1}(T)-I)\sum_{u=0}^{n-1}\Phi^{-1}(u+1)\Big[A(u)x(u-\tau) - G(u)\sum_{k=u-\tau}^{u-1}G(k)x(k) \\
& - G(u-\tau)x(u-\tau)\Big]\Big\} \\
= \quad & -\sum_{k=n-\tau}^{n-1}G(k)x(k) \\
& + \Phi(n)(\Phi^{-1}(T)-I)^{-1}\Big\{\sum_{u=n}^{T-1}\Phi^{-1}(u+1)[A(u)x(u-\tau) - G(u)\sum_{k=u-\tau}^{u-1}G(k)x(k) \\
& - G(u-\tau)x(u-\tau)] \\
& + \sum_{u=0}^{n-1}\Phi^{-1}(T)\Phi^{-1}(u+1)\Big[A(u)x(u-\tau) - G(u)\sum_{k=u-\tau}^{u-1}G(k)x(k) \\
& - G(u-\tau)x(u-\tau)\Big]\Big\}.
\end{aligned}
$$

By letting $u = s - T$, the above expression implies

$$
\begin{aligned}
x(n) \;=\;& -\sum_{k=n-\tau}^{n-1} G(k)x(k) \\
& + \Phi(n)(\Phi^{-1}(T) - I)^{-1}\Big\{\sum_{u=n}^{T-1}\Phi^{-1}(u+1)[A(u)x(u-\tau) - \; G(u)\sum_{k=u-\tau}^{u-1} G(k)x(k) \\
& - G(u-\tau)x(u-\tau)] \\
& + \sum_{s=T}^{n+T-1}\Phi^{-1}(T)\Phi^{-1}(s-T+1)\Big[A(s-T)x(s-T-\tau) \\
& - G(s-T)\sum_{k=s-T-\tau}^{s-T-1} G(k)x(k) - G(s-T-\tau)x(s-T-\tau)\Big]\Big\}. \qquad (11)
\end{aligned}
$$

By (ii) we have $\Phi(n-T+1) = \Phi(n+1)B^{-T}$ and $\Phi^{-1}(T) = B^{-T}$. Hence $\Phi^{-1}(T)\Phi^{-1}(s-T+1) = \Phi^{-1}(s+1)$. Consequently, equation (11) becomes

$$
\begin{aligned}
x(n) \;=\;& -\sum_{k=n-\tau}^{n-1} G(k)x(k) \\
& + \Phi(n)(\Phi^{-1}(T) - I)^{-1}\Big\{\sum_{u=n}^{T-1}\Phi^{-1}(u+1)[A(u)x(u-\tau) - \; G(u)\sum_{k=u-\tau}^{u-1} G(k)x(k) \\
& - G(u-\tau)x(u-\tau)] \\
& + \sum_{u=T}^{n+T-1}\Phi^{-1}(u+1)\Big[A(u)x(u-\tau) - \; G(u)\sum_{k=u-\tau}^{u-1} G(k)x(k) \\
& - G(u-\tau)x(u-\tau)\Big]\Big\} \\
=\;& -\sum_{k=n-\tau}^{n-1} G(k)x(k) \\
& + \Phi(n)(\Phi^{-1}(T) - I)^{-1}\sum_{u=n}^{n+T-1}\Phi^{-1}(u+1)[A(u)x(u-\tau) - \; G(u)\sum_{k=u-\tau}^{u-1} G(k)x(k) \\
& - G(u-\tau)x(u-\tau)].
\end{aligned}
$$

This completes the proof.

Define a mapping H by

$$
(H\varphi)(n) \;=\; -\sum_{k=n-\tau}^{n-1} G(k)\varphi(k)
$$

$$+ \Phi(n)(\Phi^{-1}(T) - I)^{-1} \sum_{u=n}^{n+T-1} \Phi^{-1}(u+1)[A(u)\varphi(u-\tau) - G(u) \sum_{k=u-\tau}^{u-1} G(k)\varphi(k) - G(u-\tau)\varphi(u-\tau)]. \tag{12}$$

It is clear that $H : P_T \to P_T$ by the way it was constructed in Lemma 4.

Next we state Krasnosel'skii's fixed point theorem which is the main mathematical tool that we will use to prove the existence of a periodic solution. We refer the reader to [14] for the proof of Krasnosel'skii's fixed point theorem.

Theorem 5 [Krasnosel'skii] Let $\mathbb{M}$ be a closed convex nonempty subset of a Banach space $(\mathbb{B}, ||.||)$. Suppose that C and B map $\mathbb{M}$ into $\mathbb{B}$ such that
(i) C is continuous and $C\mathbb{M}$ is contained in a compact set,
(ii) B is a contraction mapping.
(iii) $x, y \in \mathbb{M}$, implies $Cx + By \in \mathbb{M}$.
Then there exists $z \in \mathbb{M}$ with $z = Cz + Bz$.

Next we define $C, B : P_T \to P_T$ by

$$(B\varphi)(n) = - \sum_{k=n-\tau}^{n-1} G(k)\varphi(k), \tag{13}$$

and

$$(C\varphi)(n) = \Phi(n)(\Phi^{-1}(T) - I)^{-1} \sum_{u=n}^{n+T-1} \Phi^{-1}(u+1)[A(u)\varphi(u-\tau) - G(u) \sum_{k=u-\tau}^{u-1} G(k)\varphi(k) - G(u-\tau)\varphi(u-\tau)]. \tag{14}$$

It follows from (13) and (14) that $(H\varphi)(n) = (B\varphi)(n) + (C\varphi)(n)$.

Lemma 6 Suppose the assumptions of Lemma 4 hold. If C is defined by (14), then C is continuous and the image of C is contained in a compact set.

Proof. Let $\varphi, \psi \in P_T$. Given $\epsilon > 0$, take $\delta = \epsilon/N$ with $N = rT(|A|+|G|^2\tau+|G|)$ where

$$r = \max_{n\in[0,T]} \Big(\max_{n\le u\le n+T-1} |[\Phi(u+1)(\Phi^{-1}(T) - I)\Phi^{-1}(n)]^{-1}| \Big). \tag{15}$$

Now for $\|\varphi - \psi\| < \delta$, we have that

$$\begin{aligned} \|C\varphi(.) - C\psi(.)\| &\le r \sum_{u=0}^{T-1} \Big[|A|\|\varphi - \psi\| + |G|^2\tau\|\varphi - \psi\| + |G|\|\varphi - \psi\| \Big] \\ &\le N\|\varphi - \psi\| < \epsilon. \end{aligned}$$

Thus, showing that C is continuous.
Next, we show that C maps bounded subsets into compact sets. Let J be given. Consider $S = \{\varphi \in P_T : ||\varphi|| \le J\}$ and $Q = \{(C\varphi)(n) : \varphi \in S\}$. Then S is a

subset of $\mathbb{R}^T$ which is closed and bounded thus compact. Since C is continuous in φ it maps compact sets into compact sets. Therefore $Q = C(S)$ is compact. This completes the proof.

Lemma 7 Suppose that

$$|G|\tau < 1,$$

then B is a contraction.

Proof. Let B be defined by (13). Then for $\varphi, \psi \in P_T$, we have

$$\begin{aligned} ||B\varphi(.) - B\psi(.)|| &= \max_{n\in[0,T-1]} |B\varphi(n) - B\psi(n)| \\ &\leq \tau|G|||\varphi - \psi||. \end{aligned}$$

Thus showing that B defines a contraction mapping.

Theorem 8 Suppose the hypothesis of Lemma 6 and Lemma 7 holds. Suppose further that (5) hold. Let r be given by (15). Moreover, let γ be a positive constant satisfying the inequality

$$rT\Big[|A| + |G|^2\tau + |G|\Big]\gamma + \tau|G|\gamma \leq \gamma. \tag{16}$$

Let $\mathbb{M} = \{\varphi \in P_T : \ ||\varphi|| \leq \gamma\}$. Then equation (1) has a T-periodic solution in $\mathbb{M}$.

Proof. Define $\mathbb{M} = \{\varphi \in P_T : \ ||\varphi|| \leq \gamma\}$. By lemma 6, C is continuous and $C\mathbb{M}$ is contained in a compact set. It follows also from Lemma 7 that the mapping B is a contraction and it is not difficult to see that $B : P_T \to P_T$. Finally we will show that if $\varphi, \psi \in \mathbb{M}$, we have $||C\varphi + B\psi|| \leq \gamma$. Let $\varphi, \psi \in \mathbb{M}$ with $||\varphi||, ||\psi|| \leq \gamma$. Then

$$\begin{aligned} ||C\varphi(.) + B\psi(.)|| &\leq r\sum_{u=0}^{T-1}\Big[|A|||\varphi|| + \tau|G|^2||\varphi|| + |G|||\varphi||\Big] + \sum_{k=n-\tau}^{n-1}|G|||\psi|| \\ &\leq rT\Big[|A| + \tau|G|^2 + |G|\Big]\gamma + \tau|G|\gamma \leq \gamma. \end{aligned}$$

Therefore, all the conditions of Krasnoselskii's theorem are satisfied. Thus, a fixed point z exist in $\mathbb{M}$ such that $z = Bz + Cz$. By Lemma 4, this fixed point is a solution of (1). Hence equation (1)has a T-periodic solution.

Theorem 9 Suppose that (5) hold. Suppose also that

$$rT\Big[|A| + |G|^2\tau + |G|\Big] + \tau|G| \leq< 1.$$

Then equation (1) has a unique T-periodic solution.

Proof. Let $\varphi, \psi \in P_T$. Using (12) we obtain

$$||H\varphi(.) - H\psi(.)|| \leq \Big(rT\Big[|A| + |G|^2\tau + |G|\Big] + \tau|G|\Big)||\varphi - \psi||.$$

Thus, by the contraction mapping principle, equation (1) has a unique T-periodic solution.

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