Caspian Journal of Mathematical Sciences (CJMS)
University of Mazandaran, Iran
http://cjms.journals.umz.ac.ir
ISSN: 1735-0611

CJMS. **5**(1)(2016), 37-46

Existence and uniqueness of solutions for neutral periodic integro-differential equations with infinite delay

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Abstract. We prove the existence of solutions for the neutral periodic integro-differential equation with infinite delay

$$x'(t) = G(t, x(t), x(t-\tau(t))) + \frac{d}{dt}Q(t, x(t-\tau(t))) + \int_{-\infty}^{t} \Big(\sum_{j=1}^{n} g_j(t,s)\Big) f(x(s))ds,$$
$$x(t+T) = x(t).$$

A Krasnoselskii and Banach's fixed point theorems are employed in establishing our results.

Keywords: Krasnoselskii's Fixed point theorem, integro-differential neutral equation, periodic solution.

2000 Mathematics subject classification: 34A37, 34A12, 39A05.

1. Introduction

In this paper, we consider the neutral integro-differential equation

$$\begin{aligned} x'(t) &= G(t, x(t), x(t-\tau(t))) + \frac{d}{dt}Q(t, x(t-\tau(t))) + \int_{-\infty}^{t} \Big(\sum_{j=1}^{n} g_j(t,s)\Big) f(x(s))ds, \\ &\quad x(t+T) = x(t), \end{aligned} \tag{1.1}$$

[1] Corresponding author: ernestoyank@gmail.com
Received: 07 January 2015
Revised: 09 March 2016
Accepted: 09 April 2016

where $Q : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $G : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ and $g_j : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ for $j = 1, ..., n$ are continuous in their respective arguments.

This work is mainly motivated by the work of Althubiti, Makhzoum and Raffoul [1], in which they obtained sufficient conditions for the existence of periodic solutions for the equation

$$x'(t) \;=\; -a(t)x(t) + \frac{d}{dt}Q(t, x(t - \tau(t))) + \int_{-\infty}^{t} D(t, s)f(x(s))ds. \tag{1.2}$$

We refer to [1]-[17], and [19] for some qualitative results on neutral differential equations, integral equations and integro-differential equations.
The rest of the paper is organized as follows. In section 2, we provide some preliminary material needed for our work and in section 3 we state and prove our main results.

2. Preliminaries

Let $T > 0$ and define the set $P_T = \{\phi \in C(\mathbb{R}, \mathbb{R}) : \phi(t + T) = \phi(t)\}$, where C is the space of continuous real valued functions. Then $(P_T, ||.||)$ is a Banach space when it is endowed with the supremum norm $\|x\| = \sup_{t\in[0,T]} |x(t)|$.

In this paper we make the following assumptions.

$$\begin{aligned}&g_j(t + T, s + T) = g_j(t, s), \text{ for } j = 1, 2, ..., n,\\ &Q(t + T, x) = Q(t, x), \quad G(t + T, x, y) = G(t, x, y),\\ &\tau(t + T) = \tau(t).\end{aligned} \tag{2.1}$$

Also, there exist a continuous function $h : \mathbb{R} \to \mathbb{R}$ such that

$$h(t + T) = h(t), \qquad \int_0^T h(s)ds > 0. \tag{2.2}$$

We further assume that the functions $Q(t, x)$, $G(t, x, y)$ and $f(x)$ are globally Lipschitz. That is, there exist positive constants K_1, K_2, K_3, K_4 such that

$$| Q(t, x) - Q(t, y) | \leq K_1 \parallel x - y \parallel, \tag{2.3}$$

$$| G(t, x, y) - G(t, w, z) | \leq K_2 \parallel x - w \parallel + K_3 \parallel y - z \parallel . \tag{2.4}$$

and

$$| f(x) - f(y) | \leq K_4 \parallel x - y \parallel . \tag{2.5}$$

Also, there exist a constant K_5 such that

$$\int_{-\infty}^{t} \Big|\sum_{j=1}^{n} g_j(t,u)\Big| du < K_5 < \infty. \tag{2.6}$$

Lemma 2.1. *Suppose* (2.1) *hold. Let* $h : \mathbb{R} \to \mathbb{R}$ *be an arbitrary continuous function such that* (2.2) *also hold. If* $x(t) \in P_T$, *then* $x(t)$ *is a solution of equation* (1.1) *if and only if*

$$\begin{aligned} x(t) \;=\;& Q(t, x(t-\tau(t)) + \big(1 - e^{-\int_{t-T}^{t} h(u)du}\big)^{-1} \\ &\times \int_{t-T}^{t} \Big[- h(s)Q(s, x(s-\tau(s))) \\ &+ h(s)x(s) + \int_{-\infty}^{s} \Big(\sum_{j=1}^{n} g_j(s,u)\Big) f(x(u))du \\ &+ G(s, x(s), x(s-\tau(s)))\Big] e^{-\int_s^t h(u)du} ds\,. \end{aligned} \tag{2.7}$$

Proof. Let $x(t) \in P_T$ be a solution of (1.1). Rewrite (1.1) as

$$\begin{aligned} \Big(x(t) - Q(t, x(t-\tau(t)))\Big)' \;=\;& -h(t)[x(t) - Q(t, x(t-\tau(t)))] - h(t)Q(t, x(t-\tau(t))) \\ &+ h(t)x(t) + \int_{-\infty}^{t} \Big(\sum_{j=1}^{n} g_j(t,s)\Big) f(x(s))ds \\ &+ G(t, x(t), x(t-\tau(t))). \end{aligned} \tag{2.8}$$

Multiply both sides of (2.8) by $e^{\int_0^t h(u)du}$ and then integrate from $t-T$ to t to obtain

$$\begin{aligned} &\int_{t-T}^{t} \Big[\Big(x(s) - Q(s, x(s-\tau(s)))\Big) e^{\int_0^s h(u)du}\Big]' ds \\ &= \int_{t-T}^{t} \Big[- h(s)Q(s, x(s-\tau(s))) \\ &\quad + h(s)x(s) + \int_{-\infty}^{s} \Big(\sum_{j=1}^{n} g_j(s,u)\Big) f(x(u))du + G(s, x(s), x(s-\tau(s)))\Big] e^{\int_0^s h(u)du} ds. \end{aligned}$$

Thus we obtain,

$$\begin{aligned}
&\Big[(x(t) - Q(t, x(t-\tau(t)))e^{\int_0^t h(u)du} \\
&\quad - (x(t-T) - Q(t-T, x(t-T-\tau(t-T)))e^{\int_0^{t-T} h(u)du} \\
&= \int_{t-T}^{t} \Big[- h(s)Q(s, x(s-\tau(s))) \\
&\quad + h(s)x(s) + \int_{-\infty}^{s} \Big(\sum_{j=1}^{n} g_j(s,u)\Big) f(x(u))du \\
&\quad + G(s, x(s), x(s-\tau(s)))\Big] e^{\int_0^s h(u)du} ds.
\end{aligned}$$

By dividing both sides of the above equation by $\exp(\int_0^t h(u)du)$ and using the fact that $x(t) = x(t-T)$ together with condition (2.1), we obtain the desired result.
Since each step in the above work is reversible, the proof is complete.

$\square$

We next state Krasnoselskii's Theorem which can be found in [18].

Theorem 2.2. *(Krasnoselskii's) Let* $\mathbb{M}$ *be a closed convex nonempty subset of a Banach space* $(\mathbb{S}, ||.||)$. *Suppose that* J *and* H *map* $\mathbb{M}$ *into* $\mathbb{S}$ *such that*
(i) $x, y \in \mathbb{M}$, *implies* $Jx + Hy \in \mathbb{M}$,
(ii) H *is continuous and* $H\mathbb{M}$ *is contained in a compact set,*
(iii) J *is a contraction mapping.*
Then there exists $z \in \mathbb{M}$ *with* $z = Jz + Hz$.

Define the mappings $J : P_T \to P_T$ and $H : P_T \to P_T$ by

$$(Jx)(t) \quad = \quad Q(t, x(t-\tau(t))), \tag{2.9}$$

and

$$\begin{aligned}
(Hx)(t) \quad = \quad & \big(1 - e^{-\int_{t-T}^{t} h(u)du}\big)^{-1} \int_{t-T}^{t} \Big[- h(s)Q(s, x(s-\tau(s))) \\
& + h(s)x(s) + \int_{-\infty}^{s} \Big(\sum_{j=1}^{n} g_j(s,u)\Big) f(x(u))du \\
& + G(s, x(s), x(s-\tau(s)))\Big] e^{-\int_s^t h(u)du} ds
\end{aligned} \tag{2.10}$$

respectively.

3. Main result

In this section we state and prove our main results.

Lemma 3.1. *Assume that* (2.1), (2.3)-(2.6) *hold. Assume further that there exist a continuous function* $h : \mathbb{R} \to \mathbb{R}$ *such that* (2.2) *is satisfied. Then* $H : P_T \to P_T$*, as defined by* (2.10)*, is continuous and compact.*

Proof. We will first show that $H : P_T \to P_T$, where $(P_T, ||.||)$ is a Banach space. It must be noted that a subset of P_T which is closed and convex is defined in Theorem 3.3 and is denoted by $\mathbb{M}$. Evaluating (2.10) at $T + t$ we obtain,

$$\begin{aligned}(Hx)(t+T) &= \big(1 - e^{-\int_t^{t+T} h(u)du}\big)^{-1} \int_t^{t+T} \Big[- h(s)Q(s, x(s - \tau(s))) \\ &\quad + h(s)x(s) + \int_{-\infty}^{s} \Big(\sum_{j=1}^{n} g_j(s,u) \Big) f(x(u))du \\ &\quad + G(s, x(s), x(s-\tau(s))) \Big] e^{-\int_s^{t+T} h(u)du} ds.\end{aligned}$$

With $k = s - T$ and $v = u - T$ we obtain,

$$\begin{aligned}(Hx)(t+T) &= \big(1 - e^{-\int_t^{t+T} h(u)du}\big)^{-1} \int_t^{t+T} \Big[- h(s)Q(s, x(s - \tau(s))) \\ &\quad + h(s)x(s) + \int_{-\infty}^{s} \Big(\sum_{j=1}^{n} g_j(s,u) \Big) f(x(u))du \\ &\quad + G(s, x(s), x(s-\tau(s))) \Big] e^{-\int_s^{t+T} h(u)du} ds \\ &= \big(1 - e^{-\int_{t-T}^{t} h(v+T)dv}\big)^{-1} \int_{t-T}^{t} \Big[- h(k+T)Q(k+T, x(k+T-\tau(k+T))) \\ &\quad + h(k+T)x(k+T) + \int_{-\infty}^{k+T} \Big(\sum_{j=1}^{n} g_j(k+T, v+T) \Big) f(x(v+T))dv \\ &\quad + G(k+T, x(k+T), x(k+T-\tau(k+T))) \Big] e^{-\int_{k+T}^{t+T} h(u)du} dk \\ &= \big(1 - e^{-\int_{t-T}^{t} h(v)dv}\big)^{-1} \int_{t-T}^{t} \Big[- h(k)Q(k, x(k - \tau(k))) \\ &\quad + h(k)x(k) + \int_{-\infty}^{k} \Big(\sum_{j=1}^{n} g_j(k,v) \Big) f(x(v))dv \\ &\quad + G(k, x(k), x(k-\tau(k))) \Big] e^{-\int_k^{t} h(v)dv} dk \\ &= (Hx)(t).\end{aligned}$$

That is, $H : P_T \to P_T$.
We next show that H is continuous. To this end, we let

$$\eta = \sup_{t\in[0,T]} \mid (1 - e^{-\int_0^T h(v)dv})^{-1} \mid,\ \rho = \sup_{t\in[0,T]} \mid h(t) \mid,\ \gamma = \sup_{t\in[t-T,t]} e^{-\int_s^t h(v)dv}. \tag{3.1}$$

Let $\varphi, \psi \in P_T$, and $M = \eta\gamma T\Big[\rho K_1 + \rho + K_5K_4 + K_2 + K_3\Big]$. Given $\epsilon > 0$, choose $\delta = \frac{\epsilon}{M}$ such that $\| \varphi - \psi \| < \delta$. Thus,

$$\begin{aligned}
\| H\varphi - H\psi \| &\leq \eta\gamma \int_{t-T}^{t} \Big[\rho K_1 \| \varphi - \psi \| \\
&\quad + \rho \| \varphi - \psi \| + K_5K_4 \| \varphi - \psi \| \\
&\quad + K_2 \| \varphi - \psi \| + K_3 \| \varphi - \psi \| \Big] dk \\
&= \eta\gamma T\Big[\rho K_1 + \rho + K_5K_4 + K_2 + K_3\Big] \| \varphi - \psi \| \\
&\leq M \| \varphi - \psi \| < \epsilon.
\end{aligned}$$

Therefore, H is continuous.

To show that H is compact, we consider the sequence of periodic functions $\varphi_n \in P_T$ and assume that the sequence is uniformly bounded. Let R be such that $\| \varphi_n \| \leq R$, for all $n \in \mathbb{N}$. Then

$$\begin{aligned}
\| (H\varphi_n) \| &= \| \big(1 - e^{-\int_{t-T}^t h(u)du}\big)^{-1} \int_{t-T}^{t} \Big[- h(s)Q(s, \varphi_n(s - \tau(s))) \\
&\quad + h(s)\varphi_n(s) + \int_{-\infty}^{s} \Big(\sum_{j=1}^{n} g_j(s,u)\Big) f(\varphi_n(u))du \\
&\quad + G(s, \varphi_n(s), \varphi_n(s - \tau(s)))\Big] e^{-\int_s^t h(u)du} ds \| \\
&\leq \eta\gamma \int_{t-T}^{t} \Big[\rho\Big(\mid Q(s, \varphi_n(s - \tau(s))) - Q(t, 0) \mid + \mid Q(t, 0) \mid \Big)
\end{aligned}$$

$$
\begin{aligned}
& + \rho \parallel \varphi_n \parallel + \int_{-\infty}^{s} \mid \Big(\sum_{j=1}^{n} g_j(s,u) \Big) \mid K_4 \parallel \varphi_n \parallel du \\
& + \mid G(s, \varphi_n(s), \varphi_n(s - \tau(s))) - G(s,0,0) \mid + \mid G(s,0,0) \mid \Big] ds \\
\leq \ & \eta\gamma \int_{t-T}^{t} \Big[\rho \Big(K_1 \parallel \varphi_n \parallel + \beta_1 \Big) \\
& + \rho \parallel \varphi_n \parallel + \int_{-\infty}^{s} \mid \Big(\sum_{j=1}^{n} g_j(s,u) \Big) \mid K_4 \parallel \varphi_n \parallel du \\
& + K_2 \parallel \varphi_n \parallel + K_3 \parallel \varphi_n \parallel + \beta_2 \Big] ds \\
\leq \ & \eta\gamma T \Big[\rho \Big(K_1 R + \beta_1 \Big) + \rho R + K_5 K_4 R + K_2 R + K_3 R + \beta_2 \Big] := D,
\end{aligned}
$$

where $\beta_1 = \sup_{t \in [0,T]} \mid Q(t,0) \mid$, and $\beta_2 = \sup_{t \in [0,T]} \mid G(t,0,0) \mid$. Thus, the sequence $H\varphi_n$ is uniformly bounded. Differentiating $H\varphi_n$ gives

$$
\begin{aligned}
(H\varphi_n)'(t) \ = \ & -h(t)(H\varphi_n)(t) - h(t) Q(t, \varphi_n(t - \tau(t))) \\
& + h(t)\varphi_n(t) + \int_{-\infty}^{t} \Big(\sum_{j=1}^{n} g_j(t,u) \Big) f(\varphi_n(u)) du \\
& + G(t, \varphi_n(t), \varphi_n(t - \tau(t))).
\end{aligned}
$$

Consequently,

$$
\mid (H\varphi_n)'(t) \mid \ \leq \ \rho D + \rho \Big(K_1 R + \beta_1 \Big) + \rho R + K_5 K_4 R + K_2 R + K_3 R + \beta_2 := F,
$$

for all n. Thus the sequence $\{H\varphi_n\}$ is uniformly bounded and equicontinuous. The Arzela-Ascoli Theorem implies that $\{H\varphi_{n_k}\}$ uniformly converges to a continuous T-periodic function φ^*. Hence H is compact. □

Lemma 3.2. *Let J be defined by* (2.9) *and*

$$
K_1 < 1. \tag{3.2}
$$

Then $J : P_T \to P_T$ is a contraction.

Proof. Trivially, $J : P_T \to P_T$. For $\varphi, \psi \in P_T$, we have

$$
\parallel J\varphi - J\psi \parallel \ \leq \ K_1 \parallel \varphi - \psi \parallel . \tag{3.3}
$$

Hence J defines a contraction mapping with contraction constant K_1. □

Theorem 3.3. *Let* $\beta_1 = \sup_{t \in [0,T]} \mid Q(t,0) \mid$, *and* $\beta_2 = \sup_{t \in [0,T]} \mid G(t,0,0) \mid$. *Let η, ρ, and γ be given by* (3.1). *Suppose* (2.2)-(2.6) *hold and there exist an arbitrary continuous function $h : \mathbb{R} \to \mathbb{R}$ such that*

(2.2) also hold. Let G be a positive constant such $\| x \| \leq G$ for $x \in P_T$ and that the inequality

$$\eta\gamma T\Big[\rho\Big(K_1G+\beta_1\Big)+\rho G+\ K_5K_4G+K_2G+K_3G+\beta_2\Big]+K_1G+\beta_1\leq G, \tag{3.4}$$

holds. Then (1.1) *has a T-periodic solution in $\mathbb{M} = \{\varphi \in P_T : \| \varphi \| \leq G\}$.*

Proof. In view of the fact that $\mathbb{M} = \{\varphi \in P_T : \| \varphi \| \leq G\}$, Lemma 3.1 implies that H is compact and continuous. Also, from Lemma 3.2, J is a contraction.

We next show that if $\varphi, \psi \in \mathbb{M}$ we have $\| H\varphi + J\psi \| \leq G$. Let $\varphi, \psi \in \mathbb{M}$, then we have that

$$\begin{aligned}\| H\varphi+J\psi \| \ &\leq\ \big(1-e^{-\int_{t-T}^{t}h(u)du}\big)^{-1}\int_{t-T}^{t}\Big[-h(s)Q(s,\varphi(s-\tau(s)))\\ &\quad+h(s)\varphi(s)+\int_{-\infty}^{s}\Big(\sum_{j=1}^{n}g_j(s,u)\Big)f(\varphi(u))du\\ &\quad+G(s,\varphi(s),\varphi(s-\tau(s)))\Big]e^{-\int_s^t h(u)du}ds+Q(t,\psi(t-\tau(t))\\ &\leq\ \eta\gamma T\Big[\rho\Big(K_1G+\beta_1\Big)+(\rho+\ K_5K_4+K_2+K_3)G+\beta_2\Big]\\ &\quad+K_1G+\beta_1\leq G.\end{aligned}$$

Thus, all the conditions of Krasnoselskii Theorem are satisfied. Thus, there exist a fixed point z in $\mathbb{M}$ such that $z = Hz + Jz$. By Lemma 2.1, this fixed point is a solution of (1.1) which is T-periodic. This completes the proof. □

Theorem 3.4. *Suppose* (2.2)-(2.6) *hold and there exist an arbitrary continuous function $h : \mathbb{R} \to \mathbb{R}$ such that* (2.2) *also hold. Let $\eta, \rho,$ and γ be given by* (3.1)*. If*

$$K_1+T\eta\gamma\Big[\rho K_1+\rho+K_4K_5+K_2+K_3\Big]\leq 1, \tag{3.5}$$

then (1.1) *has a unique $T-$periodic solution.*

Proof. Define the mapping $A : P_T \to P_T$ by

$$\begin{aligned}(A\varphi)(t) &= Q(t,\varphi(t-\tau(t))) + \big(1-e^{-\int_{t-T}^{t} h(u)du}\big)^{-1}\\ &\quad\times \int_{t-T}^{t}\Big[-h(s)Q(s,\varphi(s-\tau(s)))\\ &\quad+ h(s)\varphi(s)\\ &\quad+\int_{-\infty}^{s}\Big(\sum_{j=1}^{n} g_j(s,u)\Big)f(\varphi(u))du + G(s,\varphi(s),\varphi(s-\tau(s)))\Big]e^{-\int_s^t h(u)du}ds\,.\end{aligned}$$

Then, for $\varphi,\psi \in P_T$ we have,

$$\begin{aligned}\parallel (A\varphi)-(A\psi)\parallel &\le K_1 \parallel \varphi-\psi\parallel + \eta\gamma\int_{t-T}^{t}\Big[\rho K_1 \parallel \varphi-\psi\parallel\\ &\quad+\rho\parallel\varphi-\psi\parallel + K_4K_5\parallel\varphi-\psi\parallel + K_2\parallel\varphi-\psi\parallel\\ &\quad+K_3\parallel\varphi-\psi\parallel\Big]ds\\ &\le \Big(K_1 + T\eta\gamma\Big[\rho K_1 + \rho + K_4K_5 + K_2 + K_3\Big]\Big)\parallel\varphi-\psi\parallel\,.\end{aligned}$$

This completes the proof. □

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