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Existence of bounded solutions for almost linear Volterra difference equations using fixed point theory and Lyapunov functionals

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Abstract. We obtain sufficient conditions for the boundedness of solutions of the almost linear Volterra difference equation

$$\Delta x(n) = a(n)h(x(n)) + \sum_{k=0}^{n-1} c(n,k)g(x(k))$$

using Krasnoselskii's fixed point theorem. Also, we will display a Lyapunov functional that yield boundedness of solution and compare both methods.

1 Introduction

In this paper we consider the scalar equation

$$\Delta x(n) = a(n)h(x(n)) + \sum_{k=0}^{n-1} c(n,k)g(x(k)),\ x(0) = x_0,\ n \geq 0. \tag{1.1}$$

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We assume that the functions h and g are continuous and that there exist positive constants H, H^*, G, G^* such that

$$| h(x) - Hx | \leq H^*, \tag{1.2}$$

and

$$| g(x) - Gx | \leq G^*. \tag{1.3}$$

Equation (1.1) will be called almost linear if (1.2) and (1.3) hold. In [3] Burton introduced this concept of almost linear equations for the continuous case and studied certain important properties of the resolvent kernel of a linear Volterra equation. Recently, Islam and Raffoul in the papers [7] and [8] also used the concept of almost linear equations for the continuous case to study boundedness of solutions of certain Volterra type equations. Our objective in this work is to extend the concept of almost linear equations to Volterra difference equations and prove that the solutions of these Volterra difference equations are also bounded if they satisfy (1.2) and (1.3). Due to (1.2) and (1.3) Contraction mapping principle can not be used since our mapping can not be made into a contraction. Therefore, we result to the use of Krasnoselskii's fixed point theorem. At the end of the paper we will construct a suitable Lyapunov functional and refer to [12] to deduce that all solutions of (1.1) are bounded. It turns out that either method has advantages and disadvantages.
This paper is organized as follow. In Section 2, we give a Lemma that is necessary for the construction of our mapping so that fixed point theory can be used. In Section 3, we state and prove our results by appealing to Krasnoselskii fixed point theorem. In Section 4,we provide an example as an application to our main results. Finally, in Section 5, we display a suitable Lyapunov functional that yield boundedness on all solutions.

2 Preliminaries

We begin this section by proving the following lemma, which is need for the construction of our mappings. Consider the general difference equation

$$\Delta x(n) - Ha(n)x(n) = f(n),\ x(0) = x_0,\ n \geq 0. \tag{2.1}$$

Lemma 2.1. Suppose $1 + Ha(n) \neq 0$ for all $n \in [0, \infty) \cap \mathbb{Z}$. Then $x(n)$ is a solution of equation (2.1) if and only if

$$x(n) = x(0) \prod_{s=0}^{n-1} (1 + Ha(s)) + \sum_{u=0}^{n-1} f(u) \prod_{s=u+1}^{n-1} (1 + Ha(s)) \tag{2.2}$$

Proof. First we note that (2.1) is equivalent to

$$\Delta \Big[\prod_{s=0}^{n-1} (1 + Ha(s))^{-1} x(n) \Big] = f(n) \prod_{s=0}^{n} (1 + Ha(s))^{-1} \tag{2.3}$$

Summing equation (2.3) from 0 to $n-1$ and dividing both sides by

$$\prod_{s=0}^{n-1} (1 + Ha(s))^{-1}$$

gives (2.2).

Lemma 2.2. Suppose $1+Ha(n) \neq 0$ for all $n \in [0,\infty) \cap \mathbb{Z}$. Then $x(n)$ is a solution of equation (1.1) if and only if

$$\begin{aligned} x(n) = x(0)\prod_{s=0}^{n-1}(1+Ha(s)) + \sum_{u=0}^{n-1}\Big[a(u)\Big(-Hx(u)+h(x(u))\Big)\Big]\prod_{s=u+1}^{n-1}(1+Ha(s)) \\ + \sum_{u=0}^{n-1}\sum_{k=0}^{u-1}c(u,k)\Big[g(x(k))-Gx(k)\Big]\prod_{s=u+1}^{n-1}(1+Ha(s)) \\ + \sum_{u=0}^{n-1}\sum_{k=0}^{u-1}c(u,k)Gx(k)\prod_{s=u+1}^{n-1}(1+Ha(s)). \end{aligned} \tag{2.4}$$

Proof. Rewrite equation (1.1) as

$$\begin{aligned} \Delta x(n) - Ha(n)x(n) = -Ha(n)x(n) + a(n)h(x(n)) \\ + \sum_{k=0}^{n-1}c(n,k)\Big[g(x(k))-Gx(k)\Big] + \sum_{k=0}^{n-1}c(n,k)Gx(k) \end{aligned}$$

If we let

$$\begin{aligned} f(n) = -Ha(n)x(n) + a(n)h(x(n)) + \sum_{k=0}^{n-1}c(n,k)\Big[g(x(k))-Gx(k)\Big] \\ + \sum_{k=0}^{n-1}c(n,k)Gx(k), \end{aligned}$$

then the results follow from Lemma 1.

We next state Krasnoselskii's fixed theorem which will be used to prove boundedness of solutions of (1.1).

Theorem 2.1 (Krasnosel'skii). [13] Let $\mathbb{M}$ be a closed convex nonempty subset of a Banach space $(\mathbb{B}, ||.||)$. Suppose that C and B map $\mathbb{M}$ into $\mathbb{B}$ such that

(i) C is continuous and $C\mathbb{M}$ is contained in a compact set,
(ii) B is a contraction mapping.
(iii) $x, y \in \mathbb{M}$, implies $Cx + By \in \mathbb{M}$.

Then there exists $z \in \mathbb{M}$ with $z = Cz + Bz$.

We rely on the following theorem for the relative compactness criterion since the Arzella-Ascolli Theorem can not be utilized here due to the unbounded domain.

Theorem 2.2. [1] Let M be the space of all bounded continuous (vector-valued) functions on $[0,\infty)$ and $S \subset M$. Then S is relatively compact in M if the following conditions hold:

(i) S is bounded in M;
(ii) the functions in S are equicontinuous on any compact interval of $[0,\infty)$;
(iii) the functions in S are equiconvergent, that is, given $\varepsilon > 0$, there exists a $T = T(\varepsilon) > 0$ such that $\| \phi(t) - \phi(\infty) \|_{\mathbb{R}^n} < \varepsilon$, for all $t > T$ and all $\phi \in S$.

In this paper we assume that

$$\lim_{n\to\infty} a(n) = 0, \tag{2.5}$$

and for some positive constant L,

$$0 \le \sum_{k=0}^{u-1} |c(u,k)| \le L|a(u)| \text{ for all } u \in [0,\infty)\cap\mathbb{Z}, \tag{2.6}$$

and

$$H|a(n)| \le 1 - \big|1 + Ha(n)\big| \text{ for all } n \in [0,\infty)\cap\mathbb{Z}, \tag{2.7}$$

Moreover, we assume

$$\sum_{u=0}^{n-1}\Big|\prod_{s=u+1}^{n-1}(1+Ha(s))\Big|\sum_{k=0}^{u-1} G|c(u,k)| \le \alpha < 1, \tag{2.8}$$

and

$$\sum_{u=0}^{n-1}\Big|\prod_{s=u+1}^{n-1}(1+Ha(s))\Big|\Big[|a(u)|H^* + \sum_{k=0}^{u-1} G^*|c(u,k)|\Big] \le \beta < \infty. \tag{2.9}$$

Finally, choose a constant $\rho > 0$ such that

$$|x_0|\Big|\prod_{s=0}^{n-1}(1+Ha(s))\Big| + \alpha\rho + \beta \le \rho \tag{2.10}$$

for all $n \ge 0$. Let S be the Banach space of bounded sequences with the maximum norm. Let

$$M = \{\psi \in S,\ \psi(0) = x_0\ : ||\psi|| \le \rho\}. \tag{2.11}$$

Then M is a closed convex subset of S.

Define mappings $\mathcal{A} : M \to S$ and $\mathcal{B} : M \to M$ as follows.

$$\begin{aligned}(\mathcal{A}\phi)(n) = &\sum_{u=0}^{n-1}\Big[a(u)\Big(-H\phi(u) + h(\phi(u))\Big)\Big]\prod_{s=u+1}^{n-1}(1+Ha(s)) \\ &+ \sum_{u=0}^{n-1}\sum_{k=0}^{u-1} c(u,k)\Big[g(\phi(k)) - G\phi(k)\Big]\prod_{s=u+1}^{n-1}(1+Ha(s)),\end{aligned} \tag{2.12}$$

$$\begin{aligned}(\mathcal{B}\phi)(n) = &\, x(0)\prod_{s=0}^{n-1}(1+Ha(s)) \\ &+ \sum_{u=0}^{n-1}\sum_{k=0}^{u-1} c(u,k)G\phi(k)\prod_{s=u+1}^{n-1}(1+Ha(s)).\end{aligned} \tag{2.13}$$

3 Main Results

In this section we state and prove our existence of bounded solutions result. We begin with the following lemma.

Lemma 3.1. Suppose (2.8) and (2.10) hold. The map $\mathcal{B}$ is a contraction from M into M.

Proof. Let $\phi \in M$. It follows from (2.8) and (2.10) that

$$|(\mathcal{B}\phi)(n)| \leq |x_0|\Big|\prod_{s=0}^{n-1}(1+Ha(s))\Big| + \alpha\rho \leq \rho. \tag{3.1}$$

Also, for $\phi, \psi \in M$, we obtain

$$\begin{aligned}|(\mathcal{B}\phi)(n)-(\mathcal{B}\psi)(n)| &\leq \sum_{u=0}^{n-1}\Big|\prod_{s=u+1}^{n-1}(1+Ha(s))\Big|\sum_{k=0}^{u-1}G|c(u,k)|||\phi-\psi||\\ &\leq \alpha||\phi-\psi||.\end{aligned}$$

Therefore proving that $\mathcal{B}$ is a contraction from M into M.

Lemma 3.2. The mapping $\mathcal{A}$ is a continuous mapping on M.

Proof. Let $\{\phi_n\}$ be any sequence of functions in M with $\parallel \phi_n - \phi \parallel \to 0$ as $n \to \infty$. Then one can easily verify that

$$\parallel \mathcal{A}\phi_n - \mathcal{A}\phi \parallel \to 0 \text{ as } n \to \infty.$$

Lemma 3.3. Suppose (1.2), (1.3), (2.5), (2.6), and (2.7) hold. Then $\mathcal{A}(M)$ is relatively compact.

Proof. We use Theorem 2.2 to prove the relative compactness of $\mathcal{A}(M)$ by showing that all three conditions of Theorem 2.2 hold. Thus to see that $\mathcal{A}(M)$ is uniformly bounded, we use conditions (1.2), (1.3), (2.6), (2.7) to obtain

$$\begin{aligned}|(\mathcal{A}\phi)(n)| &\leq \frac{H^*+LG^*}{H}\sum_{u=0}^{n-1}H|a(u)|\Big|\prod_{s=u+1}^{n-1}(1+Ha(s))\Big|\\ &\leq \frac{H^*+LG^*}{H}\sum_{u=0}^{n-1}\big(1-\big|1+Ha(u)\big|\big)\Big|\prod_{s=u+1}^{n-1}(1+Ha(s))\Big|\\ &= \frac{H^*+LG^*}{H}\sum_{u=0}^{n-1}\Delta_u\Big[\prod_{s=u}^{n-1}|(1+Ha(s))|\Big]\\ &\leq \frac{H^*+LG^*}{H}\Big[1-\prod_{s=0}^{n-1}|(1+Ha(s))|\Big] := \sigma \text{ for all } n \in [0,\infty)\cap\mathbb{Z}.\end{aligned}$$

Thus showing that $\mathcal{A}(M)$ is uniformly bounded.

To show equicontinuity of $\mathcal{A}(M)$, without loss of generality, we let $n_1 > n_2$ for $n_1, n_2 \in [0,\infty)\cap\mathbb{Z}$ and use the notations

$$F(\phi(u)) = a(u)[H\phi(u) - h(\phi(u))],$$

and

$$J(\phi(u)) = \sum_{k=0}^{u-1} c(u,k)\Big[g(\phi(k)) - G\phi(k)\Big].$$

Then, we may write

$$(\mathcal{A})(n) = \sum_{u=0}^{n-1} \prod_{s=u+1}^{n-1} (1 + Ha(s))\Big[F(\phi(u)) + J(\phi(u))\Big]. \tag{3.2}$$

Hence we have

$$\begin{aligned}
|(A\phi)(n_1) - (A\phi)(n_2)| &= \Big|\sum_{u=0}^{n_1-1} \prod_{s=u+1}^{n_1-1} (1 + Ha(s))\Big[F(\phi(u)) + J(\phi(u))\Big] \\
&\quad - \sum_{u=0}^{n_2-1} \prod_{s=u+1}^{n_2-1} (1 + Ha(s))\Big[F(\phi(u)) + J(\phi(u))\Big]\Big| \\
&= \Big|\sum_{u=0}^{n_2-1} \Big[\prod_{s=u+1}^{n_1-1} (1 + Ha(s)) \\
&\quad - \prod_{s=u+1}^{n_2-1} (1 + Ha(s))\Big]\Big[F(\phi(u)) + J(\phi(u))\Big]\Big| \\
&\quad + \Big|\sum_{u=n_2}^{n_1-1} \prod_{s=u+1}^{n_1-1} (1 + Ha(s))\Big[F(\phi(u)) + J(\phi(u))\Big]\Big| \\
&= \sum_{u=0}^{n_2-1} \Big|\prod_{s=u+1}^{n_2-1} (1 + Ha(s)) \\
&\quad - \prod_{s=u+1}^{n_1-1} (1 + Ha(s))\Big|\Big|F(\phi(u)) + J(\phi(u))\Big| \\
&\quad + \sum_{u=n_2}^{n_1-1} \prod_{s=u+1}^{n_1-1} |(1 + Ha(s))|\Big|F(\phi(u)) + J(\phi(u))\Big| \\
&\leq \sigma \sum_{u=0}^{n_2-1} H|a(u)|\Big|\prod_{s=u+1}^{n_2-1} |(1 + Ha(s))| - \prod_{s=u+1}^{n_1-1} |(1 + Ha(s))|\Big| \\
&\quad + \sigma \sum_{u=n_2}^{n_1-1} H|a(u)| \prod_{s=u+1}^{n_1-1} |(1 + Ha(s))|
\end{aligned}$$

$$
\begin{aligned}
&\leq \sigma \sum_{u=0}^{n_2-1} [1-|1+Ha(u)|]\Big| \prod_{s=u+1}^{n_2-1} |(1+Ha(s))| - \prod_{s=u+1}^{n_1-1} |(1+Ha(s))|\Big| \\
&\quad + \sigma \sum_{u=n_2}^{n_1-1} [1-|1+Ha(u)|]\Big| \prod_{s=u+1}^{n_1-1} |(1+Ha(s))|\Big| \\
&\leq \sigma \sum_{u=0}^{n_2-1} \Big|\Delta_u\Big[\prod_{s=u}^{n_2-1} |(1+Ha(s))| - \prod_{s=u}^{n_1-1} |(1+Ha(s))|\Big]\Big| \\
&\quad + \sigma \sum_{u=n_2}^{n_1-1} \Delta_u\Big[\prod_{s=u}^{n_1-1} |(1+Ha(s))|\Big] \\
&\leq \sigma\Big[2-2\prod_{s=n_2}^{n_1-1} |(1+Ha(s))| - \prod_{s=0}^{n_2-1} |(1+Ha(s))| \\
&\quad + \prod_{s=0}^{n_1-1} |(1+Ha(s))|\Big] \to 0 \text{ as } n_2 \to n_1.
\end{aligned}
$$

This shows that $\mathcal{A}$ is equicontinuous.

To see that $\mathcal{A}$ is equiconvergent, we let

$$
\lim_{n\to\infty} \sum_{u=0}^{n-1} \prod_{s=u}^{n-1} (1+Ha(s))\Big[F(\phi(u))+J(\phi(u))\Big] =
\sum_{u=0}^{\infty} \prod_{s=u}^{\infty} (1+Ha(s))\Big[F(\phi(u))+J(\phi(u))\Big].
$$

Then we have

$$
\begin{aligned}
|(A\phi)(\infty)-(A\phi)(n)| = \Big| &\sum_{u=0}^{\infty} \prod_{s=u+1}^{\infty} (1+Ha(s))\Big[F(\phi(u))+J(\phi(u))\Big] \\
&- \sum_{u=0}^{n-1} \prod_{s=u+1}^{n-1} (1+Ha(s))\Big[F(\phi(u))+J(\phi(u))\Big]\Big|
\end{aligned}
$$

$$
\begin{aligned}
&= \Big| \sum_{u=0}^{n-1} \Big[\prod_{s=u+1}^{\infty} (1+Ha(s)) \\
&\quad - \prod_{s=u+1}^{n-1} (1+Ha(s)) \Big] \Big[F(\phi(u)) + J(\phi(u)) \Big] \Big| \\
&\quad + \Big| \sum_{u=n}^{\infty} \prod_{s=u+1}^{\infty} (1+Ha(s)) \Big[F(\phi(u)) + J(\phi(u)) \Big] \Big| \\
&= \sum_{u=0}^{n-1} \Big| \prod_{s=u+1}^{n-1} (1+Ha(s)) \\
&\quad - \prod_{s=u+1}^{\infty} (1+Ha(s)) \Big| \Big| F(\phi(u)) + J(\phi(u)) \Big| \\
&\quad + \sigma \sum_{u=n}^{\infty} \Delta_u \Big[\prod_{s=u}^{\infty} |(1+Ha(s))| \Big] \\
&\leq \sigma \sum_{u=0}^{n-1} \Big| \Delta_u \Big[\prod_{s=u}^{n-1} |(1+Ha(s))| - \prod_{s=u}^{\infty} |(1+Ha(s))| \Big] \Big| \\
&\quad + \sigma [1 - \prod_{s=n}^{\infty} |(1+Ha(s))|]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sigma \Big[2 - 2 \prod_{s=n}^{\infty} |(1+Ha(s))| - \prod_{s=0}^{n-1} |(1+Ha(s))| \\
&\quad + \prod_{s=0}^{\infty} |(1+Ha(s))| \Big] \to 0 \text{ as } n \to \infty,
\end{aligned}
$$

where we used (2.5) which yields $\lim_{n\to\infty} \prod_{s=n}^{\infty}(1+Ha(s)) = 1$.

Theorem 3.1. Assume (1.2), (1.3), (2.5)-(2.10) hold. Then (1.1) has a bounded solution.

Proof. For $\phi, \psi \in M$, we obtain

$$
|(\mathcal{A}\phi)(n) + (\mathcal{B}\psi)(n)| \leq |x_0| \Big| \prod_{s=0}^{n-1} (1+Ha(s)) \Big| + \alpha\rho + \beta \leq \rho.
$$

Thus, $\mathcal{A}\phi + \mathcal{B}\psi \in M$. Moreover, Lemma 3, Lemma 4 and Lemma 5 satisfy the requirements of Krasnoselskii's fixed point theorem and hence there exists a function $x(n) \in M$ such that

$$
x(n) = \mathcal{A}x(n) + \mathcal{B}x(n).
$$

This proves that (1.1) has a bounded solution $x(n)$.

4 An Example

Consider the Volterra difference equation

$$\Delta x(n) = -\frac{1}{2^n}h(x(n)) + \sum_{k=0}^{n-1}\frac{4^k}{4(2^n)n!}g(x(k)),\ x(0) = x_0,\ n \geq 0, \tag{4.1}$$

where the functions h and g satisfy conditions (1.2) and (1.3), respectively. Let H, G, H^*, and G^* be positive constants with $G < 1$ and $H = 1$. We choose $\rho > 0$ such that for any initial point x_0, the inequality

$$|x_0|\left|\prod_{s=0}^{n-1}(1-2^{-s})\right| + G\rho + (H^* + G^*) \leq \rho$$

holds. Then (4.1) has a bounded solution $x(n)$ satisfying $||x|| \leq \rho$.
We let $a(n) = -\frac{1}{2^n}$ and $c(n,k) = \frac{4^k}{4(2^n)n!}$.
Thus,

$$\begin{aligned}\sum_{u=0}^{n-1}|c(n,u)| &= \sum_{u=0}^{n-1}\frac{4^u}{4(2^n)n!}\\ &\leq \frac{1}{4(2^n)n!}\left(4^n - 1\right)\\ &\leq \frac{1}{2^n}.\end{aligned}$$

Thus, showing that condition (2.6) is satisfied with $L = 1$. Condition (2.5) can easily be verified. Moreover,

$$H|a(n)| = 2^{-n} = 1 - (1 - 2^{-n}) \leq 1 - |1 + Ha(n)|,$$

thus, showing that condition (2.7) is satisfied. Next, we verify (2.8) as follows.

$$\begin{aligned}&\sum_{u=0}^{n-1}|\prod_{s=u+1}^{n-1}(1-2^{-s})|G\sum_{k=0}^{u-1}\frac{4^k}{4(2^n)n!}\\ &\leq G\sum_{u=0}^{n-1}\frac{1}{2^u} = G(1-\frac{1}{2^n})\\ &\leq G < 1\end{aligned}$$

Finally, we verify (2.9) as follows.

$$\begin{aligned}&\sum_{u=0}^{n-1}\left|\prod_{s=u+1}^{n-1}(1-2^{-s})\right|\left[2^{-u}H^* + \sum_{k=0}^{u-1}G^*\frac{4^k}{4(2^n)n!}\right]\\ &\leq \sum_{u=0}^{n-1}\left[2^{-u}H^* + G^*\frac{1}{2^u}\right]\\ &= (H^* + G^*)\sum_{u=0}^{n-1}\frac{1}{2^u}\\ &\leq (H^* + G^*)(1 - \frac{1}{2^n}) < (H^* + G^*).\end{aligned}$$

Thus, by Theorem 3.1, (4.1) has a bounded solution.

5 Boundedness Via Lyapunov Functional

In [12] the first author considered the functional difference equation

$$x(n+1) = G(n,x(s);\, 0 \le s \le n) \overset{def}{=} G(n,x(\cdot)) \tag{5.1}$$

where where $G : \mathbb{Z}^+ \times \mathbb{R}^k \to \mathbb{R}^k$ is continuous in x, and proved the following theorem

Theorem 5.1. [12] Let $\varphi(n,s)$ be a scalar sequence for $0 \le s \le n < \infty$ and suppose that $\varphi(n,s) \ge 0, \triangle_n \varphi(n,s) \le 0, \triangle_s \varphi(n,s) \ge 0$ and there are constants B and J such that $\sum_{s=0}^{n} \varphi(n,s) \le B$ and $\varphi(0,s) \le J$. Also, suppose that for each $n_0 \ge 0$ and each bounded initial function $\phi : [0,n_0] \to \mathbb{R}^k$, every solution $x(n) = x(n,n_0,\phi)$ of (5.1) satisfies

$$W_1(|x(n)|) \le V(n,x(\cdot)) \le W_2(|x(n)|) + \sum_{s=0}^{n-1} \varphi(n,s) W_3(|x(s)|)$$

and

$$\triangle V_{(5.1)}(n,x(\cdot)) \le -\rho W_3(|x(n)|) + K$$

for some constants ρ and $K \ge 0$ and $W_i, i = 1,2,3,4$ are wedges . Then solutions of (5.1) are uniformly bounded.

In this section, we construct a Lyapunov functional and then refer to the above theorem to deduce boundedness on all solutions of (5.2). First we rewrite (1.1) as

$$x(n+1) = b(n)h(x(n)) + \sum_{s=0}^{n-1} C(n,s)g(x(s)),\ x(0) = x_0,\ n \ge 0, \tag{5.2}$$

where $b(n) = 1 - a(n)$. Before we state the next theorem we note that as a consequence of (1.2) and (1.3) we have , respectively that

$$| h(x) | \le H | x | + H^*, \tag{5.3}$$

and

$$|g(x)| \le G | x | + G^*. \tag{5.4}$$

Theorem 5.2. Suppose (1.2) and (1.3) hold and for some $\alpha \in (0,1)$, we have that

$$H|b(n)| + G \sum_{j=n+1}^{\infty} |C(j,n)| - 1 \le -\alpha. \tag{5.5}$$

Also, assume that

$$\sum_{s=0}^{n} \sum_{j=n}^{\infty} |C(j,s)| < \infty, \tag{5.6}$$

and

$$\triangle_s |C(j,s)| \ge 0 \tag{5.7}$$

then solutions of (5.2) are bounded.

Proof. Define

$$V(n,x(\cdot)) = |x(n)| + \sum_{s=0}^{n-1} \sum_{j=n}^{\infty} |C(j,s)||g(x(s))|. \tag{5.8}$$

Then along solutions of (1.1), we have

$$\Delta V(n,x(\cdot)) = |x(n+1)| - |x(n)| + \sum_{s=0}^{n} \sum_{j=n+1}^{\infty} |C(j,s)||g(x(s))|$$
$$- \sum_{s=0}^{n-1} \sum_{j=n}^{\infty} |C(j,s)||g(x(s))|$$

$$= \Big|b(n)h(x(n)) + \sum_{s=0}^{n-1} C(n,s)g(x(s))\Big|$$
$$-|x(n)| + \sum_{s=0}^{n} \sum_{j=n+1}^{\infty} |C(j,s)||g(x(s))|$$
$$- \sum_{s=0}^{n-1} \sum_{j=n}^{\infty} |C(j,s)||g(x(s))|$$
$$\leq \Big[H|b(n)| + G \sum_{j=n+1}^{\infty} |C(j,n)| - 1\Big] |x(n)| + M(n)$$
$$\leq -\alpha|x(n)| + M,$$

where $M = H^*|b(n)| + G^* \sum_{j=n+1}^{\infty} |C(j,n)|$.

Let $\varphi(n,s) = \sum_{j=n}^{\infty} |C(j,s)|$. Then, all the conditions of Theorem 5.2 are satisfied which implies that all solutions of (5.2) are bounded.

We note that Theorem 5.2 gives conditions under which all solutions of (5.2) are bounded. Unlike Theorem 3.1 from which one can only conclude the existence of a bounded solution.

Next, we use Example 1 and compare the conditions of Theorem 5.2 to those of Theorem 3.1. Let $a(n), G$, and H be given as in Example 1 and consider condition (5.5) for $n \geq 0$. Then,

$$H|b(n)| + G \sum_{j=n+1}^{\infty} |C(j,n)| - 1 = |1 - \frac{1}{2^n}| - 1 + \sum_{j=n+1}^{\infty} \frac{4^n}{4(2^j)j!}$$
$$= -\frac{1}{2^n} + 4^{n-1}\Big[\sum_{n=0}^{\infty} \frac{1}{(2^n)n!} - \sum_{j=0}^{n} \frac{1}{(2^j)j!}\Big]$$
$$= -\frac{1}{2^n} + 4^{n-1}\Big[\sqrt{e} - \sum_{j=0}^{n} \frac{1}{(2^j)j!}\Big]. \tag{5.9}$$

Next we perform the following calculations by using $n! > 2^n$ for $n \geq 4$.

$$
\begin{aligned}
-\sum_{j=0}^{n}\frac{1}{(2^j)j!} &= -\frac{3}{2}-\frac{1}{8}-\frac{1}{48}-\sum_{j=4}^{n}\frac{1}{(2^j)j!}\\
&\geq -\frac{3}{2}-\frac{1}{8}-\frac{1}{48}-\sum_{j=4}^{n}\frac{1}{(4^j)}\\
&= -\frac{3}{2}-\frac{1}{8}-\frac{1}{48}+\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}-\sum_{j=1}^{n}\frac{1}{(4^j)}\\
&= -\frac{78}{48}+\frac{21}{4^3}-\frac{1}{4}\Big(\frac{1-(1/4)^n}{1-1/4}\Big). \qquad (5.10)
\end{aligned}
$$

Thus, a substitution of (5.10) into (5.9) yields,

$$
\begin{aligned}
H|b(n)|+G\sum_{j=n+1}^{\infty}|C(j,n)|-1 &\geq -\frac{1}{2^n}+4^{n-1}\Big[\sqrt{e}-\frac{78}{48}+\frac{21}{4^3}-\frac{1}{4}\Big(\frac{1-(1/4)^n}{1-1/4}\Big)\Big]\\
&> 0, \text{ for } n=3. \qquad (5.11)
\end{aligned}
$$

This shows that condition (5.5) does not hold for $n \geq 0$. Hence, Theorem 5.2 gives no information regarding the solutions and yet Theorem 3.1 implies the existence of at least one bounded solution.

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