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Homogenization of the unstationary incompressible Reynolds equation

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Abstract

This paper is devoted to the effects of surface roughness during hydrodynamic lubrication. In the numerical analysis a very fine mesh is needed to resolve the surface roughness, suggesting some type of averaging. A rigorous way to do this is to use the general theory of homogenization. In most works about the influence of surface roughness, it is assumed that only the stationary surface is rough. This means that the governing Reynolds type equation does not involve time. However, recently, homogenization was successfully applied to analyze a situation where both surfaces are rough and the lubricant is assumed to have constant bulk modulus. In this paper we will consider a case where both surfaces are assumed to be rough, but the lubricant is incompressible. It is also clearly demonstrated, in this case that homogenization is an efficient approach. Moreover, several numerical results are presented and compared with those corresponding to where a constant bulk modulus is assumed to govern the lubricant compressibility. In particular, the result shows a significant difference in the asymptotic behavior between the incompressible case and that with constant bulk modulus.

Keywords: Reynolds equation; Surface roughness and homogenization

1. Introduction

To increase the hydrodynamic performance in different machine elements during lubrication, e.g. journal bearings and thrust bearings, it is important to understand the influence of surface roughness. To consider the surface effects in the numerical analysis, a very fine mesh is needed to resolve the surface roughness, suggesting some type of averaging. A rigorous way to do this is to use the general theory of homogenization. This theory facilitates the analysis of partial differential equations with rapidly oscillating coefficients, see e.g. [1]. Homogenization was recently applied to different problems connected to lubrication with much success, see e.g. [2–16].

In general, the density of a lubricant is a function of the pressure. In this paper we will consider two special cases, where the density is assumed to be constant, i.e. an incompressible lubricant, and where the compressibility of the lubricant is modelled, assuming that the lubricant has a constant bulk modulus, see e.g. [17].

If only one of the two surfaces is rough and the rough surface is stationary, then the governing Reynolds type equation is stationary. When at least one of the moving surfaces is rough, then the governing Reynolds type equations will then involve time. Most of the previous studies on the effects of surface roughness during lubrication are devoted to problems with no time dependency.

One technique within the homogenization theory is the formal method of multiple scale expansion, see e.g. [18,19]. Recently, the ideas in [2] were used to study the compressible unstationary Reynolds equation under the assumption of a constant bulk modulus. In this paper, the method of multiple scale expansion is applied to derive a homogenization result for the incompressible unstationary Reynolds equation, see also [6]. In particular, the result shows a significant difference in the asymptotic behaviors between the incompressible case and the case with constant bulk modulus. More precisely, the homogenized equation contains a fast parameter in the incompressible case. Hence

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the pressure distribution oscillates rapidly in time, while it is almost smooth with respect to the space variable. This is contrary to the case of constant bulk modulus where the homogenized pressure solution does not contain any fast parameters, i.e. the pressure solution is smooth in both space and time. Moreover, it is clearly demonstrated by numerical examples that the homogenization result permits the surface effects in lubrication problems to be efficiently analyzed.

We want to point out that in the more mathematical oriented works in [6,9], Reynolds type equations modelling roughness on both surfaces were analyzed by using the method known as two-scale convergence. Concerning the concept of two-scale convergence, the reader is also referred to e.g. [20–22]. However, in this work we use the more engineering oriented method of multiple scale expansions.

2. The governing Reynolds type equations

Let η be the viscosity of the lubricant and assume that the velocity of surface i is $V_i = (v_i, 0)$, where $i = 1, 2$ and v_i is constant. Moreover, the bearing domain is denoted by Ω, the space variable is represented by $x \in \Omega \subset \mathbb{R}^2$ and $t \in I \subset \mathbb{R}$ represents the time. To express the film thickness we introduce the following auxiliary function:

$$h(x,t,y,\tau) = h_0(x,t) + h_2(y - \tau V_2) - h_1(y - \tau V_1),$$

where h_1 and h_2 are assumed to be periodic. Without loss of generality it can also be assumed that for both h_1 and h_2 the cell of periodicity is $Y = (0,1) \times (0,1)$, i.e. the unit cube in $\mathbb{R}^2$. By using the auxiliary function h we can model the film thickness h_ε by

$$h_\varepsilon(x,t) = h(x,t,x/\varepsilon,t/\varepsilon), \quad \varepsilon > 0. \tag{1}$$

This means that h_0 describes the global film thickness, the periodic functions h_i, $i = 1, 2$, represent the roughness contribution of the two surfaces and ε is a parameter that describes the roughness wavelength, see Fig. 1.

If the lubricant is compressible, i.e. the density ρ depends on the pressure, the pressure $p(x,t)$ satisfies then the unstationary compressible Reynolds equation

$$\frac{\partial}{\partial t}(\rho(p_\varepsilon)h_\varepsilon) = \nabla \cdot \left(\frac{h_\varepsilon^3}{12\eta}\rho(p_\varepsilon)\nabla p_\varepsilon\right) - \frac{v}{2}\frac{\partial}{\partial x_1}(\rho(p_\varepsilon)h_\varepsilon) \quad \text{on } \Omega \times I, \tag{2}$$

where $v = v_1 + v_2$. If the lubricant is incompressible, i.e. ρ is constant, Eq. (2) is then reduced to the unstationary incompressible Reynolds equation

$$\frac{\partial h_\varepsilon}{\partial t} = \nabla \cdot \left(\frac{h_\varepsilon^3}{12\eta}\nabla p_\varepsilon\right) - \frac{v}{2}\frac{\partial h_\varepsilon}{\partial x_1} \quad \text{on } \Omega \times I. \tag{3}$$

Note that Eq. (2) is non-linear and Eq. (3) is linear. This means that in general it is much more difficult to analyze the compressible case. The situation is rather simplified if the relation between density and pressure is assumed to be of the form

$$\rho(p_\varepsilon) = \rho_a \, e^{(p_\varepsilon - p_a)/\beta}, \tag{4}$$

where the constant ρ_a is the density at the atmospheric pressure p_a and β is a positive constant (bulk modulus). This relation is equivalent to the commonly used assumption that the lubricant has a constant bulk modulus β, see e.g. [17]. Note that this assumption is valid for reasonably low pressures. Due to the special form of relation (4) it is possible to transform the non-linear Eq. (2) into a linear equation. Indeed, if the function w_ε is defined as $w_\varepsilon(x,t) = \rho(p_\varepsilon(x,t))/\rho_a$, then

$$\nabla w_\varepsilon = \beta^{-1} e^{(p_\varepsilon - p_a)/\beta}\nabla p_\varepsilon = \beta^{-1}\rho_a^{-1}\rho(p_\varepsilon)\nabla p_\varepsilon$$

and equation (2) is converted into the linear equation

$$\gamma \frac{\partial}{\partial t}(w_\varepsilon h_\varepsilon) = \nabla \cdot \left(h_\varepsilon^3 \nabla w_\varepsilon\right) - \lambda \frac{\partial}{\partial x_1}(w_\varepsilon h_\varepsilon) \quad \text{on } \Omega \times I, \tag{5}$$

where $\gamma = 12\eta\beta^{-1}$ and $\lambda = 6\eta v\beta^{-1}$.

For small values of ε, the coefficients, including h_ε, are rapidly oscillating functions. This implies that a direct numerical analysis of the deterministic problems (2), (3) and (5) becomes difficult for small values of ε, because a very fine mesh is needed to resolve the surface roughness. This suggests some type of averaging. In this work, the multiple scale expansion method is used to homogenize the unstationary incompressible Reynolds equation (3), where h_ε is defined as in (1). These results will also be compared with known homogenization results for (5). A significant difference in the asymptotic behavior between the incompressible case and the case with constant bulk modulus will be seen.

Of note is that in the more mathematical oriented works [6,9] another method known as two-scale convergence was used to analyze Reynolds type equations modelling roughness on both surfaces. In particular, [9] considers air flow, where the air compressibility and slip-flow effects are considered. More precisely, the following non-linear

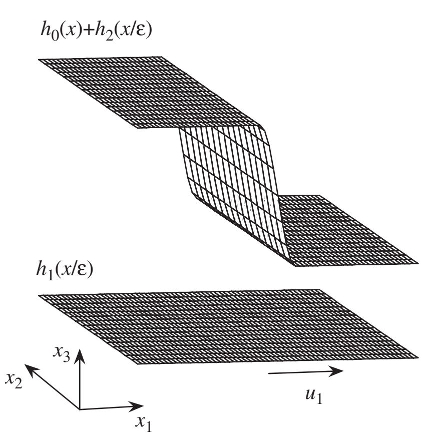

Fig. 1. Bearing geometry and surface roughness.

equation is homogenized:

$$a\frac{\partial}{\partial t}(p_\varepsilon h_\varepsilon) = \nabla\cdot((h_\varepsilon^3 p_\varepsilon + bh_\varepsilon^2)\nabla p_\varepsilon) - c\cdot\nabla(p_\varepsilon h_\varepsilon) \quad \text{on } \Omega\times I,$$

where a and b are positive constants and $c \in \mathbb{R}^2$.

3. Homogenization (constant bulk modulus)

The focus of this work is the homogenization of the incompressible unstationary Reynolds equation. However, the results will be compared with the corresponding homogenization results for the unstationary equation corresponding to the constant bulk modulus case recently obtained in [2], see also [23]. Therefore, for the readers convenience, we review the main conclusions in [2].

Let χ_i, $i = 1, 2, 3$ be the solutions of the local problems

$$\nabla_y\cdot(h^3\nabla_y\chi_1) = -\frac{\partial h^3}{\partial y_1} \quad \text{on } Y,$$

$$\nabla_y\cdot(h^3\nabla_y\chi_2) = -\frac{\partial h^3}{\partial y_2} \quad \text{on } Y,$$

$$\nabla_y\cdot(h^3\nabla_y\chi_3) = \gamma\frac{\partial h}{\partial \tau} + \lambda\frac{\partial h}{\partial y_1} \quad \text{on } Y.$$

Moreover, let $\overline{h}(x,t)$, the vector function $b(x,t)$ and the matrix function $A(x,t) = (a_{ij}(x,t))$ be defined as

$$\overline{h}(x,t) = \int_T\int_Y h(x,t,y,\tau)\,\mathrm{d}y\,\mathrm{d}\tau,$$

$$b(x,t) = \int_T\int_Y (\lambda h e_1 - h^3\nabla_y\chi_3)\,\mathrm{d}y\,\mathrm{d}\tau,$$

$$A(x,t) = \begin{pmatrix} \int_T\int_Y h^3\left(1+\frac{\partial\chi_1}{\partial y_1}\right)\mathrm{d}y\,\mathrm{d}\tau & \int_T\int_Y h^3\frac{\partial\chi_2}{\partial y_1}\mathrm{d}y\,\mathrm{d}\tau \\ \int_T\int_Y h^3\frac{\partial\chi_1}{\partial y_2}\mathrm{d}y\,\mathrm{d}\tau & \int_T\int_Y h^3\left(1+\frac{\partial\chi_2}{\partial y_2}\right)\mathrm{d}y\,\mathrm{d}\tau \end{pmatrix}.$$

The main result in [2] states that the deterministic solution w_ε of (5) can be approximated with high accuracy by $w_0(x,t)$, where w_0 is the solution of the homogenized (averaged) equation

$$\gamma\frac{\partial}{\partial t}(\overline{h}w_0) = -\nabla\cdot(bw_0) + \nabla\cdot(A\nabla w_0). \tag{6}$$

It was also clearly demonstrated that by using this homogenization result, an efficient method is obtained for analyzing the rough surface effects in problems where the lubricant has a constant bulk modulus and the governing equation is the time dependent compressible Reynolds equation (2).

Remark 1. If h is independent of t, i.e. $h = h(x,y,\tau)$, then the homogenized equation (6) has the form

$$0 = -\nabla\cdot(bw_0) + \nabla\cdot(A\nabla w_0). \tag{7}$$

4. Homogenization in the incompressible case

Consider the incompressible transient Reynolds equation

$$\Gamma\frac{\partial h_\varepsilon}{\partial t} + \Lambda\frac{\partial h_\varepsilon}{\partial x_1} - \nabla\cdot(h_\varepsilon^3\nabla p_\varepsilon) = 0, \tag{8}$$

where $\Gamma = 12\eta$ and $\Lambda = 6\eta v$. Assume the following multiple scale expansion of the solution p_ε:

$$p_\varepsilon = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \cdots, \tag{9}$$

where $p_i = p_i(x,y,t,\tau)$. The chain rule then implies that

$$\Gamma\left(\frac{\partial}{\partial t}+\frac{1}{\varepsilon}\frac{\partial}{\partial\tau}\right)h + \Lambda\left(\frac{\partial}{\partial x_1}+\frac{1}{\varepsilon}\frac{\partial}{\partial y_1}\right)h - \left(\frac{\partial}{\partial x_i}+\frac{1}{\varepsilon}\frac{\partial}{\partial y_i}\right) \times\left[h^3\left(\frac{\partial}{\partial x_i}+\frac{1}{\varepsilon}\frac{\partial}{\partial y_i}\right)(p_0+\varepsilon p_1+\varepsilon^2 p_2+\cdots)\right] = 0.$$

Let $\mathscr{A}_0$, $\mathscr{A}_1$ and $\mathscr{A}_2$ be defined as

$$\mathscr{A}_0 = \frac{\partial}{\partial y_i}\left(h^3\frac{\partial}{\partial y_i}\right) = \nabla_y\cdot(h^3\nabla_y),$$

$$\mathscr{A}_1 = \frac{\partial}{\partial x_i}\left(h^3\frac{\partial}{\partial y_i}\right) + \frac{\partial}{\partial y_i}\left(h^3\frac{\partial}{\partial x_i}\right) = \nabla_x\cdot(h^3\nabla_y) + \nabla_y\cdot(h^3\nabla_x),$$

$$\mathscr{A}_2 = \frac{\partial}{\partial x_i}\left(h^3\frac{\partial}{\partial x_i}\right) = \nabla_x\cdot(h^3\nabla_x).$$

Then (8) may be written as

$$\Gamma\left(\frac{\partial}{\partial t}+\frac{1}{\varepsilon^1}\frac{\partial}{\partial\tau}\right)h + \Lambda\left(\frac{\partial}{\partial x_1}+\frac{1}{\varepsilon}\frac{\partial}{\partial y_1}\right)h - (\varepsilon^{-2}\mathscr{A}_0 + \varepsilon^{-1}\mathscr{A}_1 + \mathscr{A}_2)(p_0+\varepsilon p_1+\varepsilon^2 p_2+\cdots) = 0.$$

The idea is now to collect terms of the same order of ε. For the homogenization it is sufficient to consider the orders -2, -1 and 0.

$$-\mathscr{A}_0 p_0 = 0, \tag{10}$$

$$\Gamma\frac{\partial h}{\partial\tau} + \Lambda\frac{\partial h}{\partial y_1} - \mathscr{A}_0 p_1 - \mathscr{A}_1 p_0 = 0, \tag{11}$$

$$\Gamma\frac{\partial h}{\partial t} + \Lambda\frac{\partial h}{\partial x_1} - \mathscr{A}_0 p_2 - \mathscr{A}_1 p_1 - \mathscr{A}_2 p_0 = 0. \tag{12}$$

It is well-known that equations of the form $\mathscr{A}_0 u = f$ have a unique solution up to an additive constant, if and only if the average over Y of the right-hand side is 0, see e.g. [24, p. 93]. Hence, it is clear from (10) that p_0 does not depend on y, i.e. $p_0 = p_0(x,t,\tau)$. Using this fact and averaging (11) with respect to y gives

$$\int_Y\left(\Gamma\frac{\partial h}{\partial\tau} + \Lambda\frac{\partial h}{\partial y_1} - \nabla_y\cdot\left(h^3\nabla_y p_1\right) - \nabla_y\cdot\left(h^3\nabla_x p_0\right)\right)\mathrm{d}y = 0.$$

By considering Y-periodicity, this is reduced to

$$\int_Y \frac{\partial h}{\partial \tau}\, \mathrm{d}y = 0. \tag{13}$$

Hence, the assumption that p_ε may be expanded as in (9) requires h to satisfy (13). We observe that h fulfills this condition in our case. Physically this means that the surface-to-surface volume does not depend on the relative position of the surface roughness. The fact that $p_0 = p_0(x, t, \tau)$ implies that Eq. (11) is

$$\nabla_y \cdot (h^3 \nabla_y p_1) = \Gamma \frac{\partial h}{\partial \tau} + \Lambda \frac{\partial h}{\partial y_1} - \nabla_y \cdot (h^3 \nabla_x p_0),$$

where x, t and τ are parameters. By linearity, p_1 is of the form

$$p_1(x,y,t,\tau) = v_1(x,y,t,\tau) + \frac{\partial p_0}{\partial x_1} v_2(x,y,t,\tau) + \frac{\partial p_0}{\partial x_2} v_3(x,y,t,\tau),$$

where v_i is the solutions of the following local problems:

$$\nabla_y \cdot (\Lambda h e_1 - h^3 \nabla_y v_1) = -\Gamma \frac{\partial h}{\partial \tau},$$

$$\nabla_y \cdot (h^3(e_1 + \nabla_y v_2)) = 0,$$

$$\nabla_y \cdot (h^3(e_2 + \nabla_y v_3)) = 0,$$

and $\{e_1, e_2\}$ is the canonical basis in $\mathbb{R}^2$.

Averaging Eq. (12) with respect to y gives the equation

$$\Gamma \frac{\partial}{\partial t} \int_Y h\, \mathrm{d}y + \nabla_x \cdot \int_Y \left(\Lambda h e_1 - h^3 \nabla_y v_1\right) \mathrm{d}y - \nabla_x \cdot \left(\frac{\partial p_0}{\partial x_1} \int_Y h^3 [e_1 + \nabla_y v_2]\, \mathrm{d}y + \frac{\partial p_0}{\partial x_2} \int_Y h^3 [e_2 + \nabla_y v_3]\, \mathrm{d}y \right) = 0. \tag{14}$$

If we introduce the notation $\overline{h}(x,t,\tau) = \int_Y h\, \mathrm{d}y$ and define the homogenized vector $b(x,t,\tau)$ and the homogenized matrix $A(x,t,\tau) = (a_{ij}(x,t,\tau))$ as

$$b = \int_Y (\Lambda h e_1 - h^3 \nabla_y v_1)\, \mathrm{d}y,$$

$$\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = \int_Y h^3 (e_1 + \nabla_y v_2)\, \mathrm{d}y \quad \text{and}$$

$$\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = \int_Y h^3 (e_2 + \nabla_y v_3)\, \mathrm{d}y,$$

then (14) takes the following form:

$$\Gamma \frac{\partial \overline{h}}{\partial t}(x,t,\tau) + \nabla_x \cdot b(x,t,\tau) - \nabla_x \cdot (A(x,t,\tau) \nabla p_0) = 0. \tag{15}$$

Note that t and τ are just parameters. The appearance of the fast parameter τ in the homogenized equation (15) means that for small wavelengths the pressure will oscillate rapidly in time. This should be compared with the case of liquid flow with a constant bulk modulus, see (6), where the pressure is almost smooth with respect to time, i.e. the amplitude of the oscillations in time, in the deterministic pressure solution p_ε, is very small for small wavelengths. In both cases, the pressure is almost smooth in the space variable.

It should be noted that if h is independent of t, i.e. $h = h(x, y, \tau)$, then the homogenized equation (15) has the form

$$\nabla_x \cdot (A(x,\tau) \nabla_x p_0(x,\tau)) = \nabla_x \cdot b(x,\tau). \tag{16}$$

It should also be noted that if only one of the surfaces is rough (either the moving or the stationary), i.e. h is of the form $h(x,y,t,\tau) = h_0(x,t) + h_i(y - \tau V_i)$, where $i = 1$ or 2, then $\overline{h}$, b and A are independent of τ. This means that the solution p_0 of the homogenized problem (15) is independent of τ and this simplifies problem (15).

5. Numerical results

In this section we present some numerical results based on the homogenized equations obtained in the previous sections. To perform the numerical analysis, the algorithms presented in [2,3] are used. In all examples the solution domain Ω is a subset of $\mathbb{R}^2$ such that $0 \leqslant x_1 \leqslant L$ and $-L/2 \leqslant x_2 \leqslant L/2$. For simplicity, the global film thickness h_0 is assumed to be time independent. More precisely,

$$h_0(x) = \begin{cases} h_{\min}(1+k), & x_1 < L/2, \\ h_{\min}, & x_1 > L/2 \end{cases}$$

and the roughness contribution is represented by

$$h_i(y - \tau v_i) = c_i h_{\min} \sin(2\pi(y - \tau v_i)).$$

This means that a step bearing with surface roughness is considered (in the numerical simulations the discontinuity has been smoothened). The specific parameters, common to all the numerical computations, may be found in Table 1.

5.1. Incompressible case

Fig. 2 depicts the deterministic solutions p_ε of (8) for a fixed ε and time t. In Fig. 3 the corresponding homogenized solution p_0 of (16) is plotted. It should be noted that the deterministic solution p_ε oscillates rapidly, while the homogenized solution is smooth (fixed time t and ε).

Table 1
Common problem specific parameters

Parameter	Value	Unit
$h_{\min}$	4×10^{-6}	m
k	1/4	
$c_1 = c_2$	1/8	
L	1×10^{-1}	m
v_1	1	$\mathrm{m\,s^{-1}}$
v_2	0	$\mathrm{m\,s^{-1}}$
η	0.14	Pa s
β	1×10^{11}	

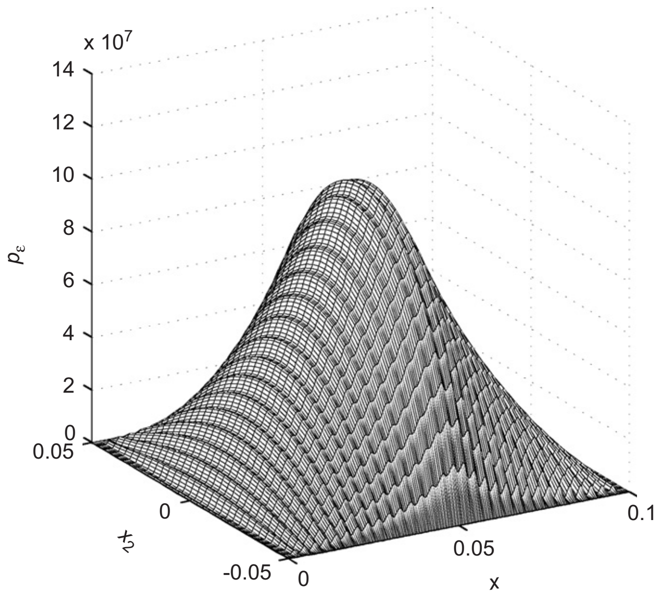

Fig. 2. Pressure distribution in the incompressible case for a fixed ε.

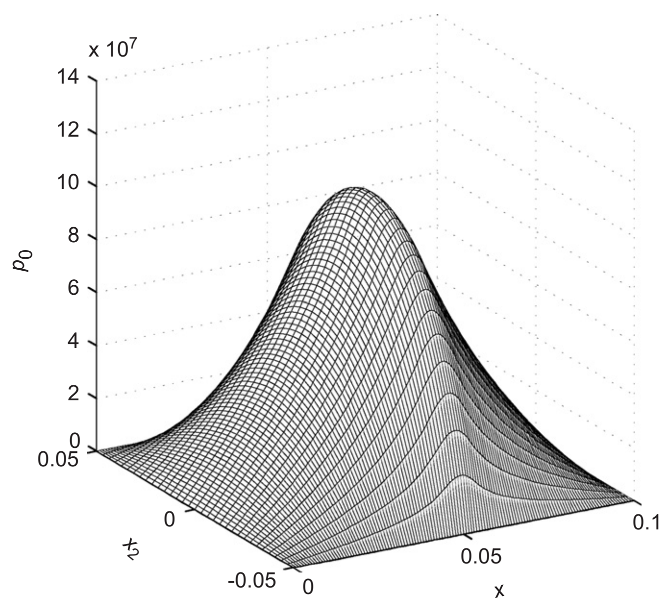

Fig. 3. Homogenized pressure distribution for the imcompressible case.

The convergence of the deterministic pressure towards the homogenized pressure p_0, as $\varepsilon \to 0$, was analyzed above by multiple scale expansions. This convergence will now be illustrated by means of numerical solutions. Indeed, Fig. 4 represent part of the pressure distribution between the two rough surfaces along the $x_2 = 0$ line at a particular point in time for different values of ε. As seen in the figure, the pressure distribution p_ε approaches that of the homogenized pressure as ε tends to zero. Fig. 5 represents an enlargement of a portion of Fig. 4, showing clearly the decrease in the amplitude of the pressure distribution towards the homogenized pressure solution as the roughness wavelength ε tends to zero.

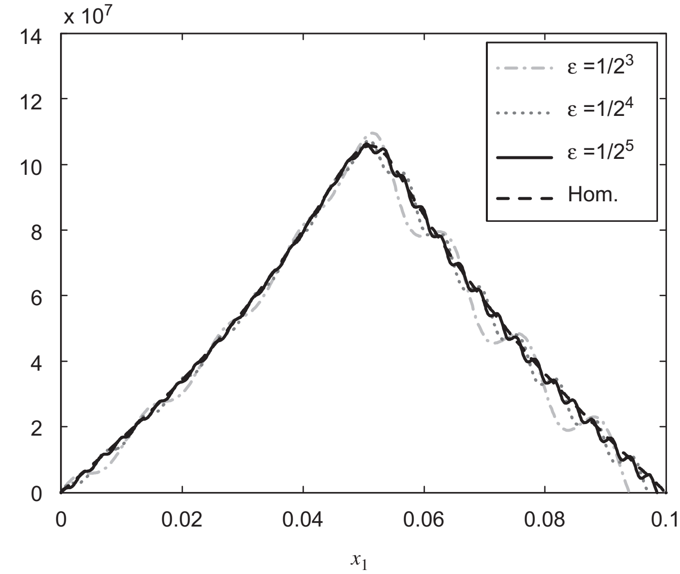

Fig. 4. Pressure solutions at $x_2 = 0$ for various ε as well as the corresponding homogenized solution at time $t = 0$ in the incompressible case.

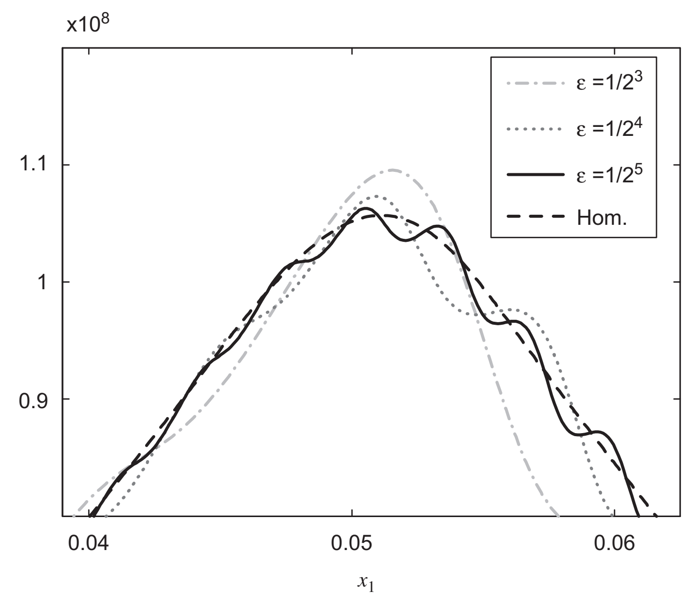

Fig. 5. Zoomed portion of Fig. 4.

As mentioned before in the analysis by multiple scale expansions, the appearance of the fast parameter τ in the homogenized equations (15) and (16) means that for small wavelengths the pressure will oscillate rapidly in time. This fact is illustrated in Fig. 6, which depicts the pressure distribution at some different times (within a period) for a fixed ε and the corresponding homogenized solutions.

In addition to the visual illustration of the convergence of p_ε to p_0, a more quantitative convergence analysis is considered here. For this purpose we consider what happens with the load carrying capacity as ε tends to 0. The load carrying capacity l_ε corresponding to p_ε and l_0

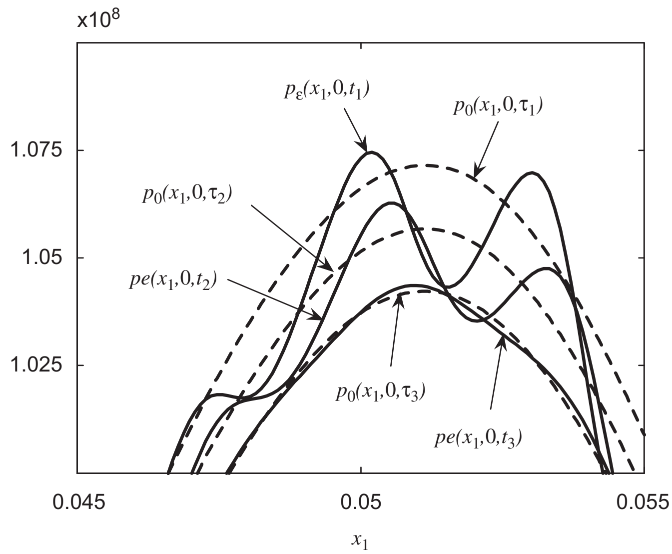

Fig. 6. Pressure solutions at $x_2 = 0$ for three different ε as well as the corresponding homogenized solution.

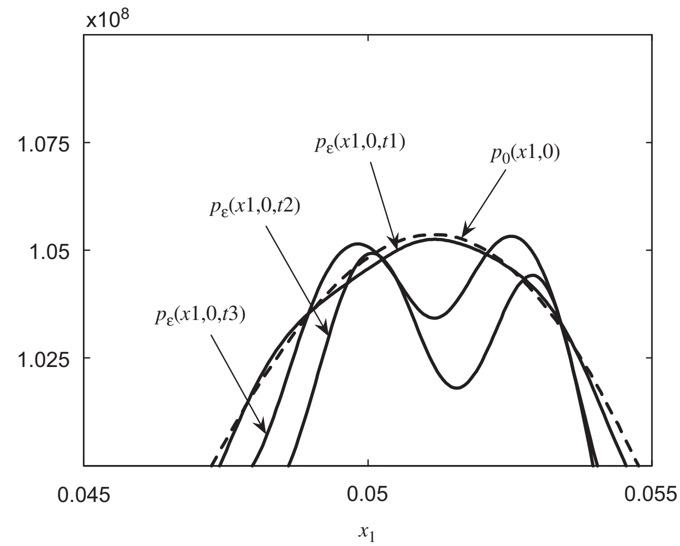

Fig. 8. The pressure solutions for a fixed ε at three different time steps and the homogenized solution in the compressible case.

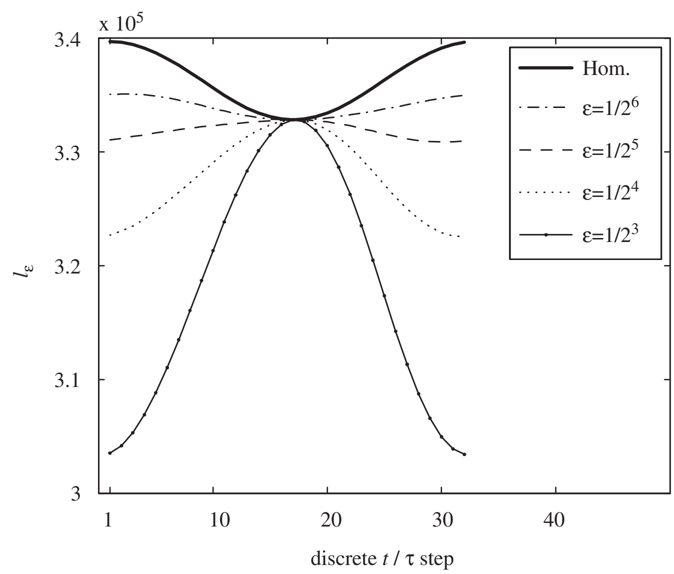

Fig. 7. Convergence of the load carrying capacity in the incompressible case.

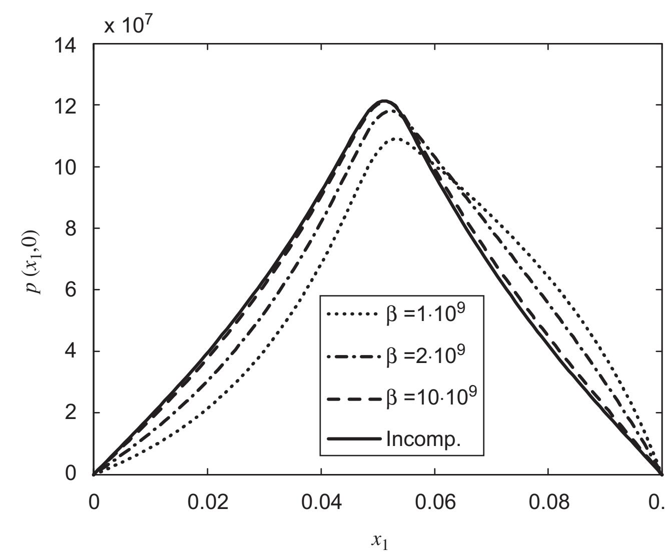

Fig. 9. Comparison of the pressure distribution between the incompressible and compressible case when the bearing surfaces are smooth for different values of β.

corresponding to p_0, are defined as

$$l_\varepsilon(t) = \int_\Omega p_\varepsilon(x,t)\,\mathrm{d}x \quad \text{and} \quad l_0(\tau) = \int_\Omega p_0(x,\tau)\,\mathrm{d}x. \tag{17}$$

In Fig. 7 we see that $l_\varepsilon \to l_0$ as ε approaches zero. The difference in load carrying capacity at $t = \tau = 0$, which is the worst case scenario, is approximately 1%. It is also noted that, in the case with perfectly sinusoidal surface roughness descriptions, for a specific value of ε between $\frac{1}{64}$ and $\frac{1}{32}$, a seemingly small variation of the load carrying capacity in time is obtained, i.e. it is possible to optimize the surfaces to reduce vibrations.

5.2. Constant bulk modulus case

In the analysis by multiple scale expansions we observed a significant difference in the asymptotic behavior between the incompressible case and the case with constant bulk modulus. No fast parameter τ is found in the homogenized equation of the constant bulk modulus case. This implies that we only have one homogenized solution in our example where $h_0 = h_0(x)$ contrary to the incompressible case where we have different homogenized solutions for different times t within a period. This fact is illustrated in Fig. 8, which corresponds to Fig. 6 in the incompressible situation.

In Fig. 9 we observe that, for perfectly smooth surfaces, as β is increased, the pressure distribution in the constant bulk modulus case, approaches that in the incompressible case. However, this does not seem to be the case for rough surfaces, due to the different asymptotic behavior between the constant bulk modulus case and the incompressible case.

6. Concluding remarks

We have clearly demonstrated that homogenization may be used to efficiently analyze the effects of surface roughness in incompressible thin film unstationary lubrication flow. This has been done using the method of asymptotic expansions and numerical examples where we visualize the convergence and give a quantitative convergence analysis of the load capacity. One important observation is that there is a difference in the asymptotic behavior between the incompressible case and the case with constant bulk modulus. When the lubricant is assumed to be incompressible, the homogenized (averaged) equation contains a fast parameter that is connected to the time. This means that for small wavelengths the pressure distribution oscillates rapidly in time, while it is almost smooth with respect to the space variable. For liquid flow of a lubricant with a constant bulk modulus, the pressure solution of the homogenized equation does not contain any of the fast parameters. Thus, for small wavelengths the pressure is almost smooth in both the space and time variables. There are many interesting directions to deepen our study of hydrodynamic lubrication, where both surfaces are assumed to be rough. For example, to include a model that regards cavitation, another would be to consider non-Newtonian lubricants.

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