Periodicity in a System of Differential Equations with Finite Delay

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ABSTRACT. The existence and uniqueness of a periodic solution of the system of differential equations

$$\frac{d}{dt}x(t) = A(t)x(t - \tau)$$

are proved. In particular the Krasnoselskii's fixed point theorem and the contraction mapping principle are used in the analysis. In addition, the notion of fundamental matrix solution coupled with Floquet theory is also employed.

Keywords: Fixed point; Fundamental matrix solution; Floquet theory; Periodic solution

1. INTRODUCTION

Periodic solutions of differential equations have recently been studied extensively. We refer to [5]-[14] and the references therein for a wealth of information on this subject.

In this paper, we study the existence and uniqueness of a periodic solution of the system of equations

$$\frac{d}{dt}x(t) = A(t)x(t - \tau), \tag{1.1}$$

where $A(t)$ is an $n \times n$ matrix with continuous real-valued functions as its elements and τ is a positive constant.

Floquet theory offers a lot of results on the periodicity of the system (1.1) when $\tau = 0$. In [16], the author extended Floquet theory to non-autonomous linear systems of the form $z' = A(x)z$, where $A : \mathbb{C} \to \mathbb{C}$ is an $\omega-$ periodic function in the complex variable x, whose solutions are meromorphic. There are however no corresponding results for system (1.1). The qualitative properties of the scalar version of (1.1) have been studied in [4]. Therefore, in this paper by using the notion of the fundamental solution coupled with Floquet theory we prove the existence and uniqueness of solutions of (1.1).

2. EXISTENCE OF PERIODIC SOLUTIONS

We begin this section by assuming that there exists a nonsingular $n \times n$ matrix $G(t)$ with continuous real-valued functions as its elements such that

$$\frac{d}{dt}x(t) = G(t)x(t) - \frac{d}{dt}\int_{t-\tau}^{t} G(s)x(s)ds + [A(t) - G(t - \tau)]x(t - \tau) \quad (2.1)$$

Lemma 2.1. *Equation* (1.1) *is equivalent to* (2.1).

Proof. By differentiating the integral term in (2.1) we obtain

$$\frac{d}{dt}\int_{t-\tau}^{t} G(s)x(s)ds = G(t)x(t) - G(t - \tau)x(t - \tau).$$

Substituting this into (2.1), we obtain

$$\begin{aligned}
\frac{d}{dt}x(t) &= G(t)x(t) - G(t)x(t) + G(t - \tau)x(t - \tau) \\
&\quad + [A(t) - G(t - \tau)]x(t - \tau) = A(t)x(t - \tau).
\end{aligned}$$

$\square$

For $T > 0$ let P_T be the set of all n-vector functions $x(t)$, periodic in t of period T. Then $(P_T, \|.\|)$ is a Banach space with the supremum norm

$$\|x(.)\| = \sup_{t \in \mathbb{R}} |x(t)| = \sup_{t \in [0,T]} |x(t)|,$$

where $|.|$ denotes the infinity norm for $x \in \mathbb{R}^n$. Also, if A is a $n \times n$ real matrix, then we define the norm of A by $|A| = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|$.

Definition 2.2. If the matrix $G(t)$ is periodic of period T, then the linear system

$$y' = G(t)y \quad (2.2)$$

is said to be noncritical with respect to T if it has no periodic solution of period T except for the trivial solution $y = 0$.

In this paper we assume that

$$A(t + T) = A(t), G(t + T) = G(t). \tag{2.3}$$

Throughout this paper it is assumed that the system (2.2) is noncritical. We next state some known results [7] about system (2.2) which will be useful in the rest of the paper.

Let $K(t)$ represent the fundamental matrix of the system (2.2) with $K(0) = I$, where I is the $n \times n$ identity matrix. Then:

(i) $\det K(t) \neq 0$.
(ii) There exists a constant matrix B such that $K(t+T) = K(t)e^{BT}$, by Floquet theory.
(iii) System (2.2) is noncritical if and only if $det(I - K(T)) \neq 0$.

Lemma 2.3. *Suppose* (2.3) *hold. If* $x(t) \in P_T$, *then* $x(t)$ *is a solution of* (2.1) *if and only if*

$$
\begin{aligned}
x(t) &= -\int_{t-\tau}^{t} G(s)x(s)ds + K(t)(K^{-1}(T) - I)^{-1}\bigg\{ \\
&\quad \int_{t}^{t+T} K^{-1}(u)[A(u)x(u-\tau) - G(u-\tau)x(u-\tau) \\
&\quad - G(u)\int_{u-\tau}^{u} G(s)x(s)ds]du\bigg\}.
\end{aligned} \tag{2.4}
$$

Proof. Let $x(t) \in P_T$ be a solution of (2.1) and $K(t)$ is a fundamental system of solutions of (2.2). We first rewrite (2.1) as

$$
\begin{aligned}
\frac{d}{dt}\Big[x(t) + \int_{t-\tau}^{t} G(s)x(s)ds\Big] &= G(t)\Big[x(t) + \int_{t-\tau}^{t} G(s)x(s)ds\Big] \\
&\quad -G(t)\int_{t-\tau}^{t} G(s)x(s)ds \\
+ A(t)x(t-\tau) - G(t-\tau)x(t-\tau). &\tag{2.5}
\end{aligned}
$$

Since $K(t)K^{-1}(t) = I$, it follows that

$$
\begin{aligned}
0 &= \frac{d}{dt}(K(t)K^{-1}(t)) = \frac{d}{dt}(K(t))K^{-1}(t) + K(t)\frac{d}{dt}(K^{-1}(t)) \\
&= (G(t)K(t))K^{-1}(t) + K(t)\frac{d}{dt}(K^{-1}(t)) \\
&= G(t) + K(t)\frac{d}{dt}(K^{-1}(t)).
\end{aligned}
$$

This implies

$$\frac{d}{dt}(K^{-1}(t)) = -K^{-1}(t)G(t). \qquad (2.6)$$

If $x(t)$ is a solution of (2.1) with $x(0) = x_0$, then

$$\frac{d}{dt}\Big[K^{-1}(t)\Big(x(t) + \int_{t-\tau}^{t} G(s)x(s)ds\Big)\Big]$$

$$= \frac{d}{dt}K^{-1}(t)\Big[x(t) + \int_{t-\tau}^{t} G(s)x(s)ds\Big]$$

$$+ K^{-1}(t)\frac{d}{dt}\Big(x(t) + \int_{t-\tau}^{t} G(s)x(s)ds\Big).$$

Substituting (2.5) and (2.6) in the above equation, we obtain

$$\frac{d}{dt}\Big[K^{-1}(t)\Big(x(t) + \int_{t-\tau}^{t} G(s)x(s)ds\Big)\Big]$$

$$= -K^{-1}(t)G(t)\Big[x(t) + \int_{t-\tau}^{t} G(s)x(s)ds\Big] + K^{-1}(t)\Big\{G(t)\Big[x(t) +$$

$$\int_{t-\tau}^{t} G(s)x(s)ds\Big]; - G(t)\int_{t-\tau}^{t} G(s)x(s)ds + A(t)x(t-\tau)$$

$$- G(t-\tau)x(t-\tau)\Big\} = K^{-1}(t)A(t)x(t-\tau) - K^{-1}(t)G(t-\tau)x(t-\tau)$$

$$- K^{-1}(t)G(t)\int_{t-\tau}^{t} G(s)x(s)ds.$$

After integrating on $[0,t]$, we have

$$x(t) = -\int_{t-\tau}^{t} G(s)x(s)ds + K(t)\Big[x_0 + \int_{-\tau}^{0} G(s)x(s)ds\Big]$$

$$+ K(t)\int_{0}^{t} K^{-1}(u)\Big[A(u)x(u-\tau) - G(u-\tau)x(u-\tau)$$

$$- G(u)\int_{u-\tau}^{u} G(s)x(s)ds\Big]du. \qquad (2.7)$$

Since $x(T) = x_0 = x(0)$, we obtain from (2.7) that

$$x_0 + \int_{-\tau}^{0} G(s)x(s)ds = (I - K(T))^{-1}\int_{0}^{T} K(T)K^{-1}(u)\Big[A(u)x(u-\tau)$$

$$- G(u-\tau)x(u-\tau) - G(u)\int_{u-\tau}^{u} G(s)x(s)ds\Big]du.$$

$$(2.8)$$

Substituting (2.8) into (2.7), we obtain

$$x(t) = -\int_{t-\tau}^{t} G(s)x(s)ds + K(t)\Big((I - K(T))^{-1}\int_{0}^{T} K(T)K^{-1}(u)$$

$$[A(u)x(u-\tau) - G(u-\tau)x(u-\tau) - G(u)\int_{u-\tau}^{u} G(s)x(s)ds\Big]du\Big)$$

$$+ K(t)\int_{0}^{t} K^{-1}(u)\Big[A(u)x(u-\tau) - G(u-\tau)x(u-\tau)$$

$$- G(u)\int_{u-\tau}^{u} G(s)x(s)ds\Big]du. \tag{2.9}$$

We will now show that (2.9) is equivalent to (2.4).
Since

$$(I - K(T))^{-1} = (K(T)(K^{-1}(T) - I))^{-1} = (K^{-1}(T) - I)^{-1}K^{-1}(T),$$

equation (2.9) turns to

$$\begin{aligned}
x(t) &= -\int_{t-\tau}^{t} G(s)x(s)ds + K(t)(K^{-1}(T) - I)^{-1}\int_{0}^{T} K^{-1}(u)\Big[\\
&\quad A(u)x(u-\tau) - G(u-\tau)x(u-\tau) - G(u)\int_{u-\tau}^{u} G(s)x(s)ds\Big]du \\
&\quad + K(t)\int_{0}^{t} K^{-1}(u)\Big[A(u)x(u-\tau) - G(u-\tau)x(u-\tau) \\
&\quad - G(u)\int_{u-\tau}^{u} G(s)x(s)ds\Big]du \\
&= -\int_{t-\tau}^{t} G(s)x(s)ds + K(t)(K^{-1}(T) - I)^{-1}\Big\{\int_{0}^{T} K^{-1}(u)\Big[\\
&\quad A(u)x(u-\tau) - G(u-\tau)x(u-\tau) - G(u)\int_{u-\tau}^{u} G(s)x(s)ds\Big]du \\
&\quad + \int_{0}^{t} K^{-1}(T)K^{-1}(u)\Big[A(u)x(u-\tau) - G(u-\tau)x(u-\tau) \\
&\quad - G(u)\int_{u-\tau}^{u} G(s)x(s)ds\Big]du \\
&\quad - \int_{0}^{t} K^{-1}(u)\Big[A(u)x(u-\tau) - G(u-\tau)x(u-\tau) \\
&\quad - G(u)\int_{u-\tau}^{u} G(s)x(s)ds\Big]du\Big\}
\end{aligned}$$

$$
\begin{aligned}
= \ & -\int_{t-\tau}^{t} G(s)x(s)ds + K(t)(K^{-1}(T) - I)^{-1}\Big\{\int_{t}^{T} K^{-1}(u)\Big[\\
& A(u)x(u-\tau) - \ G(u-\tau)x(u-\tau) - G(u)\int_{u-\tau}^{u} G(s)x(s)ds\Big]du \\
& + \int_{0}^{t} K^{-1}(T)K^{-1}(u)\Big[A(u)x(u-\tau) - G(u-\tau)x(u-\tau) \\
& - G(u)\int_{u-\tau}^{u} G(s)x(s)ds\Big]du\Big\}.
\end{aligned}
$$

By letting $u = i - T$, the above expression implies

$$
\begin{aligned}
x(t) \ = \ & -\int_{t-\tau}^{t} G(s)x(s)ds + K(t)(K^{-1}(T) - I)^{-1}\Big\{\int_{t}^{T} K^{-1}(u)\Big[\\
& A(u)x(u-\tau) - \ G(u-\tau)x(u-\tau) - G(u)\int_{u-\tau}^{u} G(s)x(s)ds\Big]du \\
& + \int_{T}^{t+T} K^{-1}(T)K^{-1}(i-T)\Big[A(i-T)x(i-T-\tau) \\
& -G(i-T-\tau)x(i-T-\tau) - \ G(i-T) \\
& \int_{i-T-\tau}^{i-T} G(s)x(s)ds\Big]di\Big\} \tag{2.10}
\end{aligned}
$$

Using condition (ii) we have $K(t-T) = K(t)e^{-BT}$ and $K(T) = e^{BT}$.
Hence, $K^{-1}(T)K^{-1}(i-T) = K^{-1}(i)$. Consequently, (2.10) becomes

$$
\begin{aligned}
x(t) \ = \ & -\int_{t-\tau}^{t} G(s)x(s)ds + K(t)(K^{-1}(T) - I)^{-1}\Big\{\int_{t}^{T} K^{-1}(u)\Big[\\
& A(u)x(u-\tau) - \ G(u-\tau)x(u-\tau) - G(u)\int_{u-\tau}^{u} G(s)x(s)ds\Big]du \\
& + \int_{T}^{t+T} K^{-1}(u)\Big[A(u)x(u-\tau) - G(u-\tau)x(u-\tau) \\
& - G(u)\int_{u-\tau}^{u} G(s)x(s)ds\Big]du\Big\}.
\end{aligned}
$$

Combining the two integrals in the above equation gives equation, we
obtain (2.4). This completes the proof of Lemma 2.3.

<div style="text-align:right">□</div>

Define a mapping H by

$$
\begin{aligned}
(H\varphi)(t) \;=\; & -\int_{t-\tau}^{t} G(s)\varphi(s)ds + K(t)(K^{-1}(T) - I)^{-1}\Big\{ \int_{t}^{t+T} K^{-1}(u)[\\
& A(u)\varphi(u - \tau) - \; G(u - \tau)\varphi(u - \tau) - G(u) \\
& \int_{u-\tau}^{u} G(s)\varphi(s)ds]du\Big\}.
\end{aligned}
\tag{2.11}
$$

It is clear from (2.11) that $H : P_T \to P_T$ by the way that it was constructed in Lemma 2.3.

Theorem 2.4. *(Krasnosel'skii Theorem* [15]*) Let $\mathbb{M}$ be a closed convex nonempty subset of a Banach space $(\mathbb{B}, ||.||)$. Suppose that C and B map $\mathbb{M}$ into $\mathbb{B}$ such that*

   (i) *C is continuous and $C\mathbb{M}$ is included in a compact set,*
   (ii) *B is a contraction mapping.*
   (iii) *$x, y \in \mathbb{M}$, implies that $Cx + By \in \mathbb{M}$.*

Then there exists $z \in \mathbb{M}$ with $z = Cz + Bz$.

To apply Theorem 2.4 we need to construct two mappings of which one is a contraction and the other is compact. Therefore we express equation (2.11) as

$$(H\varphi)(t) = (B\varphi)(t) + (C\varphi)(t),$$

where $B, C : P_T \to P_T$ are given by

$$(B\varphi)(t) = -\int_{t-\tau}^{t} G(s)\varphi(s)ds, \tag{2.12}$$

and

$$
\begin{aligned}
(C\varphi)(t) \;=\; & K(t)(K^{-1}(T) - I)^{-1}\int_{t}^{t+T} K^{-1}(u)[A(u)\varphi(u - \tau) - \\
& G(u - \tau)\varphi(u - \tau) - \; G(u)\int_{u-\tau}^{u} G(s)\varphi(s)ds]du
\end{aligned}
\tag{2.13}
$$

respectively.

Lemma 2.5. *Suppose that the assumptions of Lemma 2.3 hold. C is continuous and the image of C is included in a compact set.*

Proof. Let $\varphi, \psi \in P_T$. Given $\epsilon > 0$, take $\delta = \epsilon/N$ with $N = rT(|A| + |G| + |G|^2\tau)$ where

$$r = \sup_{t\in[0,T]} \Big(\sup_{t\leq u\leq t+T} |[K(u)(K^{-1}(T) - I)K^{-1}(t)]^{-1}| \Big). \tag{2.14}$$

Now for $\|\varphi - \psi\| < \delta$, we have that

$$
\begin{aligned}
\|C\varphi(.) - C\psi(.)\| &\leq r \int_0^T \Big[|A|\|\varphi - \psi\| + |G|\|\varphi - \psi\| + |G|^2\tau\|\varphi - \psi\| \Big] du \\
&\leq N\|\varphi - \psi\| < \epsilon.
\end{aligned}
$$

This proves that C is continuous. To show that the image of C is contained in a compact set, we consider $D = \{\varphi \in P_T : \|\varphi\| \leq R\}$, where R is a fixed positive constant. Let $\varphi_n \in D$ where n is a positive integer. Thus,

$$
\begin{aligned}
\|C\varphi_n(.)\| &\leq r \int_0^T \Big[|A|R + |G|R + |G|^2\tau R \Big] du \\
&\leq rT \Big[|A|R + |G|R + |G|^2\tau R \Big] \\
&\leq L,
\end{aligned}
$$

for some positive constant L. We next we calculate $(C\varphi_n)'(t)$ and show that it is uniformly bounded. Using (2.3) we obtain by taking the derivative of (2.13) that

$$
\begin{aligned}
(C\varphi_n)'(t) &= K'(t)(K^{-1}(T) - I)^{-1} \int_t^{t+T} K^{-1}(u)[A(u)\varphi_n(u - \tau) \\
&\quad - G(u - \tau)\varphi_n(u - \tau) - G(u) \int_{u-\tau}^u G(s)\varphi_n(s)ds]du \\
&\quad + K(t)(K^{-1}(T) - I)^{-1}K^{-1}(t + T)[A(t)\varphi_n(t - \tau) \\
&\quad - G(t - \tau)\varphi_n(t - \tau) - G(t) \int_{t-\tau}^t G(s)\varphi_n(s)ds] \\
&\quad - K(t)(K^{-1}(T) - I)^{-1}K^{-1}(t)[A(t)\varphi_n(t - \tau) \\
&\quad - G(t - \tau)\varphi_n(t - \tau) - G(t) \int_{t-\tau}^t G(s)\varphi_n(s)ds] \\
&= G(t)(C\varphi_n)(t) + K(t)(K^{-1}(T) - I)^{-1}\Big[K^{-1}(t + T) - K^{-1}(t) \Big] \\
&\quad \times \Big(A(t)\varphi_n(t - \tau) - G(t - \tau)\varphi_n(t - \tau) \\
&\quad - G(t) \int_{t-\tau}^t G(s)\varphi_n(s)ds \Big).
\end{aligned}
$$

By noting that $K^{-1}(t + T) = e^{-BT}K^{-1}(t)$, we have

$$
K^{-1}(t + T) - K^{-1}(t) = (e^{-BT} - I)K^{-1}(t) = (K^{-1}(T) - I)K^{-1}(t).
$$

Using this in the last expression, we obtain

$$(C\varphi_n)'(t) = G(t)(C\varphi_n)(t)$$
$$+ \Big(A(t)\varphi_n(t-\tau) - G(t-\tau)\varphi_n(t-\tau)$$
$$-G(t)\int_{t-\tau}^{t} G(s)\varphi_n(s)ds\Big).$$

Thus,

$$\|(C\varphi_n)'\| \leq |G|L + |A|R + |G|R + |G|^2 R\tau.$$

Therefore, the sequence $C\varphi_n$ is uniformly bounded and equi-continuous. Hence by Arzela-Ascoli theorem $C(D)$ is compact. The proof is complete. $\square$

Lemma 2.6. *Suppose that*

$$|G|\tau < 1, \tag{2.15}$$

then B is a contraction.

Proof. Let B be defined by (2.12). Then for $\varphi, \psi \in P_T$ we have

$$\|B\varphi(.) - B\psi(.)\| = \sup_{t\in[0,T]} |B\varphi(t) - B\psi(t)|$$
$$\leq \tau|G|\|\varphi - \psi\|.$$

Hence B defines a contraction mapping with contraction constant $\tau|G|$. $\square$

Theorem 2.7. *Suppose the hypothesis of Lemma 2.6 holds. Let r be given by (2.14). Suppose further that (2.3) hold. Let J be a positive constant satisfying the inequality*

$$rT\Big[|A| + |G| + |G|^2\tau\Big]J + \tau|G|J \leq J. \tag{2.16}$$

Let $\mathbb{M} = \{\varphi \in P_T : \|\varphi\| \leq J\}$. Then (1.1) has a solution in $\mathbb{M}$.

Proof. Define $\mathbb{M} = \{\varphi \in P_T : \|\varphi\| \leq J\}$. By Lemma 2.5, we have that C is continuous and $C\mathbb{M}$ is contained in a compact set. Also, from Lemma 2.6, the mapping B is a contraction and it is clear that $C, B : P_T \to P_T$. We next show that if $\varphi, \psi \in \mathbb{M}$, we have $\|C\varphi + B\psi\| \leq J$. Let $\varphi, \psi \in \mathbb{M}$ with $\|\varphi\|, \|\psi\| \leq J$. Then

$$\|C\varphi(.) + B\psi(.)\| \leq$$
$$r\int_0^T \Big[|A|\|\varphi\| + |G|\|\varphi\| + |G|^2\tau\|\varphi\|\Big]du + \int_{t-\tau}^t |G|\|\psi\|ds$$
$$rT\Big[|A| + |G| + |G|^2\tau\Big]J + \tau|G|J \leq J.$$

We now see that all conditions of Krasnoselskii's theorem are satisfied. Thus there exists a fixed point z in $\mathbb{M}$ such that $z = Cz + Bz$. By Lemma 2.3, this fixed point is a solution of (1.1). Hence (1.1) has a T-periodic solution. $\square$

Theorem 2.8. *Suppose* (2.3) *holds. If*

$$\tau|G| + rT\Big[|A| + |G| + |G|^2\tau\Big] < 1, \tag{2.17}$$

then (1.1) *has a unique T-periodic solution.*

Proof. Let the mapping H be given by (2.11). For $\varphi, \psi \in P_T$, we have that

$$
\begin{aligned}
\|H\varphi(.) - H\psi(.)\| &\leq \int_{t-\tau}^{t} |G|\|\varphi - \psi\|ds + r\int_{0}^{T}\Big[|A| + |G| + |G|^2\tau\Big]\|\varphi - \psi\|ds \\
&\leq \Big(\tau|G| + rT[|A| + |G| + |G|^2\tau]\Big)\|\varphi - \psi\| \\
&< \|\varphi - \psi\|.
\end{aligned}
$$

Thus, H is a contraction. Thus by the contraction mapping principle, (1.1) has a unique T-periodic solution. This completes the proof. $\square$

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