UNIVERSITY OF CAPE COAST

DIDACTICAL CONCEPTUAL STRUCTURES IN EXTENDING THE TRIAD TO THE TETRAHEDRON EXEMPLIFIED IN THE TEACHING AND LEARNING OF EQUATIONS OF THE CIRCLE

CLEMENT AYAREBILLA ALI

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BY
CLEMENT AYAREBILLA ALI

Thesis presented to the Department of Mathematics and Information Communication Technology Education, of the Faculty of Science and Technology Education, College of Education Studies, University of Cape Coast in partial fulfilment of the requirements for award of Doctor of Philosophy Degree in Mathematics Education

SEPTEMBER 2019

DECLARATION

Candidate's Declaration

I hereby declare that this thesis is the result of my own original research and that no part of it has been presented for another degree in this University or elsewhere.

Candidate's Signature: ………………………………….... Date: ………………

Name: Clement Ayarebilla Ali

Supervisors' Declaration

We hereby declare that the preparation and presentation of the thesis were supervised in accordance with the guidelines on supervision of thesis laid down by the University of Cape Coast.

Principal Supervisor's Signature: …………………………. Date: ……………

Name: Prof. Ernest Kofi Davis

Co-Supervisor's Signature: …………………………………….Date: ……………

Name: Prof. Douglas Darko Agyei

ABSTRACT

The study explored didactical conceptual structures in extending the didactical triad to the tetrahedron in equations of the circle. Studies have shown that there are still inadequate interactions in the didactic triad and lack of knowledge of mathematics classrooms as indelible cultural forces. However, the didactic tetrahedron has helped students interact better in such situations. Therefore, the sequential explanatory mixed methods design was used in this study. This design adequately illuminated the interactions in the intersubjective didactic instructional models because meaning is based on one's experiences and socially situated. The research population was 1,500 senior high school Elective Mathematics students. Out of this number, 500 students were randomly sampled through the use of table of random number procedures. This was subsequently followed up by a purposive sample of 12 students whose responses were so interesting for the qualitative data. Having satisfied statistical assumptions, controlled internal and external threats to validity, established reasonable reliability of instruments and confounded possible covariates, the researcher analysed the quantitative results with probability values, estimated marginal means, effect sizes and statistical powers. The results and findings showed that there were steady improvements in interactions in the didactic tetrahedron. This was evident in students' scores in the tasks and equations of the circle. The interview transcripts confirmed and explained the reasons for these improvements. The researcher therefore, recommended among other things, that policy makers should adopt the didactic tetrahedron to enable students to fully interact during classroom discourse in order to enhance performance in elective mathematics.

KEY WORDS

Didactical conceptual structures

Didactical phenomenology

Exemplified equations of the circle

Extending the triad to the tetrahedron

Intersubjective theoretical-conceptual framework

Intersubjective didactical conceptual models

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DEDICATION

To my wife (Miss Phebe Anafo) and my children (Chalcoal, Chenani and Chantelle).

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LIST OF ACRONYMS

Acronym	**Meaning**
%	Percentage
2D	Two dimensional figure
3D	Three dimensional figure
1T	Task only or T
2T	Task and Technique only or T and t
3T	Task, Technique and Theory only or T, t and θ
4T	Task, Technique, Theory and Technology or T, t, θ and Θ
ANCOVA	Analysis of Covariance
ANOVA	Analysis of Variance
ATD	Anthropological Theory of Didactics
BCME	British Congress of Mathematics Education
CALM	Challenging, Active Learner-centred and Motivating
CAS	Computer Algebraic Software
CERME	Congress of European Research in Mathematics Education
CETL-MSOR	Continuing Excellence in the Teaching & Learning of Maths, Stats & OR
DGS	Dynamic Geometry Software
F-ratio	Ratio of between-effect to within effect
GES	Ghana Education Service
ICETIC	Innovative and Creative Education and Teaching International Conferences
ICME	International Congress on Mathematical Education
ICMI	International Commission on Mathematical Instruction
ICT	Information Communication Technology
IEEE	Institute of Electrical and Electronics Engineers

IGPME	International Group Psychology of Mathematics Education
JICA	Japan International Cooperation Agency
KMO	Kaiser-Meyer-Olkin
M	Mathematics in formal setting
m	mathematics in informal setting
MANCOVA	Multiple Analysis of Covariance
MANOVA	Multiple Analysis of Variance
MMLab	Mathematics Laboratory
MoE	Ministry of Education
NRC	National Research Council
RME	Realistic Mathematics Education
SAARMSTE	Southern African Association for Research in Mathematics, Science and Technology Education
SHS	Senior High School
SPSS	Statistical Package for Social Sciences
STEM	Science, Technology, Engineering and Mathematics
TDS	Theory of Didactical Situations
TPACK	Technological Pedagogical Content Knowledge
UCC	University of Cape Coast
UNESCO	United Nations Educational Scientific and Cultural Organization
USAID	United States Aid for International Development
WAEC	West African Examinations Council
ZPD	Zone of Proximal Development
ZDM	Zentralblatt für Didaktik der Mathematik (German)

CHAPTER ONE

INTRODUCTION

This study seeks to explore the didactical conceptual structures in extending the triad to the tetrahedron. The triad composes the roles of students, teachers and mathematics content (Østergaard, 2013). The roles of technologies extend the triad to tetrahedron (Wilson, 2015). This will help create necessary and sufficient conditions to advance didactical theories and practice. Technologies will also enhance interactions among students, teachers and mathematics content in teaching and learning equations of the circle, and by extension conic circles (Sinclair et al., 2016; Presmeg et al, 2016).

In the didactical mediation processes, the researcher undertook a baseline survey to establish the research problem. Thereafter, a mixed methods research design was used to test statistical significances with three layers of intersubjective didactical instructional models. These models are mathematisation and didactical phenomenology, instrumentations and instrumentalisations, and didactical situations and anthropological praxeologies. The models were followed up with ethnographic-phenomenology transcriptions to seek students' own experiences, skills and attitudes in the didactical tetrahedron. In this chapter, the background of the study with the state of didactics in the Ghanaian, didactical conceptual structures, didactical triad and tetrahedron, signs and symbols, tasks, artefacts, tools, instruments, technologies and didactical phenomenology have been presented. Also included are technologies, equations of the circle, possible covariates, statement of the problem, purpose of the study, objectives of the study, hypotheses, research questions, significance of the study, delimitations of the study and organization of the study (UCC, 2016).

Background to the Study

This background explains key issues in the state of didactics of mathematics in Ghana. The other issues are conceptualization, contextualization and mathematisation of didactical conceptual structures, tasks, signs, symbols, artefacts, tools, instruments and technologies. This will set the ground for the research problem.

The State of Didactics of Mathematics in Ghana

The Ministry of Education (MoE) through the Ghana Education Service (GES) always make efforts to overcome the challenges in classroom teaching and learning processes. The agency consistently changes or modifies the curriculum, improves teacher professionalism, upgrades courses for student-teachers', aligns educational theories with classroom practices and applies innovative instructional strategies. These help to improve upon the teaching and learning of school mathematics in Ghana. This is because with evolution of new theories and dynamics of the Ghanaian society, the quest to improve upon mathematics teaching and learning in senior high schools (SHS) is inevitable.

Every mathematics curriculum comes with its own unique approaches and implications in accordance with the new emerging educational theories and recommended best practices. For instance, the curriculum of 1987 focused more on classroom activity methods in the teaching and learning processes while those of 2007 and 2010 emphasized more on computers and other technology tools on mathematical problem-solving strategies (Fauzan, 2002; MoE, 2007; MoE, 2010). There is yet a new curriculum that will focus on task-based teaching and learning (MoE, 2019). However, studies (Davis, Seah, & Alan, 2009; Benning, & Agyei, 2016; Fletcher, 2016) in Ghana suggest that these changes have not significantly improved upon the teaching and learning of mathematics due to inadequate and/or little local content interactional-instructional models and low technological pedagogical content knowledge of teachers and students in the SHS classrooms.

Generally, there are several remote causes for the seemingly lack of significant improvements in the teaching and learning of didactics of mathematics in the Senior High School level and little appreciation of didactics. Didactics of mathematics is the science of the specific conditions of the diffusion of mathematical knowledge useful for the functioning of the human institutions. This definition widens the scope of didactics to not only narrow area of mathematics but the diffusion of mathematics in the society at large. However, because of the narrow view of didactics of mathematics in Ghana (Fletcher, 2016), little efforts are made to conceptualize and model phenomena suitable for classroom mathematics interactions. Other schools of thought even still link didactics of mathematics to rote learning and lecture methods (Straesser, 2007).

Secondly, the changes of curricular have always been carried out in the top-down model (Fauzan, 2002). In such a model, the initiatives to change the curriculum come from the state or state agencies responsible for education. In other case, groups of people who wield power and can influence the government of the day champion such changes. In such situations, there seem to lack of consultation and constitution of professional mathematics professors, educators, experts, teachers, researchers and associations to thoroughly examine the need for any changes. In such cases there exist little efforts to design teaching and learning models for effective interactions among the four didactical conceptual structures (Ali, Davis, & Agyei, 2017).

Thirdly, many curricula lack implementation strategies (Fauzan, 2002). They lack effective in-service training for serving teachers (Davis, &Chaiklin, 2015). Both in-service and preservice teachers lack pedagogical content knowledge in contemporary mathematics issues (Benning, & Agyei, 2016). As a result, some teachers who even participate in education and training frequently become dissatisfied, disgruntled and unconvinced to implement the new ideas. These could be attributable to low knowledge or even lack of interaction during teaching and learning (Ali, Davis, & Agyei, 2018).

Again, the implementation of each curriculum has never been evaluated properly. The only standards used by Ghanaian stakeholders to measure and evaluate the successes of the curricula implementations in the mathematics classroom are students' achievement tests administered. These achievement tests are in twofold---internal continuous assessment conducted by classroom teachers and external assessment conducted by WAEC. These two assessment techniques preclude vital information needed for the process of classroom curriculum implementation and instructions. This information includes but not limited to effective interaction strategies, methods and techniques conducted in the classroom. In most cases, students' learning procedures, best teaching and learning models, and teachers and students' mediation processes are limited to the items of the assessments. In such cases, the assessment approaches centre on only the students and not teachers, mathematics content and mediation tools. In other words, teachers exclude themselves from the learning outcomes.

But examinations' outcomes must necessarily reflect interactions among students, teachers, mathematics content and mediation tools (WAEC, 2015).

Also, the very centralized system of education in Ghana is a crucial factor in determining the successes of didactics of mathematics. In the Ghanaian education system, the major institution that manages issues of teaching and learning in the senior high schools is the Ministry of Education (MoE) through GES. All students, teachers and schools, irrespective of their geographical and socio-cultural areas use the same mathematics curriculum, textbooks and mediation tools. All students have to take the same national and international WAEC examinations with their counterparts from Nigeria, Liberia and The Gambia (MoE, 2010). These situations do not give much opportunity for students, teachers and school authorities to develop their own innovative strategies that suit their local conditions (Vygotsky, 1978; Davis, 2010). In some cases, GES, in collaboration with UNESCO, PIZA, British Council, JICA and USAID make interventions for teaching and learning (MoE, 2010). However, such interventions only run smoothly within the projects' gestations. The failures of such interventions automatically become unsupported and quickly abandoned as if they never existed. In such situations, the attempts to make continuous interactions among teachers, students, mathematics content and technologies become a mirage (Fausan, 2002).

Furthermore, the misuse and misrepresentation of the concept of technology in Ghana is a major hindrance to didactics. Generally, Technology is the sum of techniques, skills, methods, and processes used in the production of goods or services or in the accomplishment of objectives. It extends to a people's indigenous arts, culture, tradition, sculpture, basketry and workmanship. In classroom instructions, it includes materials and resources, namely mathematical set, rulers, calculators and graph sheets (Ali, & Davis, 2018). This knowledge is socially embedded in our complex material cultures and reflects the skilful, creative ways people negotiate and adapt within it (Nightingale, 2014). In Ghana, this concept has been narrowed down to applied science and discovery. But teachers and students think that until they use complex computer application software, a phenomenon cannot be wholly described as technology.

In addition, didactical technology includes the materials the students use to build the bridge model, in addition to the concepts and procedures they need for thinking, speaking and building. The teachers' interactions with the student through talking, gesturing and handling objects (mathematics book and resources) complete the classroom milieu (Svensson,& Johansen, 202017).This notion too has been frequently misrepresented, misused and narrowed down to ICT (MoE, 2010).This does not promote Ghanaian values, goals, missions and vision in mathematics.

Elsewhere, research (D'Amore, 2008; Mariotti, &Maracci, 2009; Lewis, 2015; Roth, 2016; Presmeg et al, 2016; Saenz-Ludlow, &Kadunz, 2016) shows that, the roles of didactical conceptual structures in the didactical tetrahedron have not still facilitated students' constructions of mathematical concepts as well as their problem-solving strategies much. Didactical technology-based representation tools still have little influences on students' mental constructions in both algebraic and geometric representations. Teachers' roles have not permeated their professional practices in didactics that could adequately address deeper conceptual understanding of didactics of mathematics. This does not provide epistemological interconnections among all fields of mathematics representations. Students and teachers still misconstrue technology tools for technical computer knowledge in hardware, software and computer programming.

Table 1 shows the performance of senior high school students in core mathematics. Even though the statistics were not in equations of the circle alone, it does give general picture of poor interactions in classroom teaching and learning of mathematics in Ghana.

Table 1: Performance of Core Mathematics between 2007 and 2020

Year	Number passed	Percentage passed
2007	33, 639	25.3%
2008	35,536	26.2%
2009	44,934	28.6%
2011	65,005	43.8%
2012	77,882	49.4%
2013	149,612	36.6%
2014	77,884	32.4%
2015	64,268	24.0%
2016	77,108	32.83%
2017	122,450	42.73%
2018	120,519	38.33%
2019	223,737	65.31%
2020	243,904	65.71%

Source: (Fletcher, 2016; WAEC, 2019, 2020)

The Table 1 seeks to review the granulating poor performance of students in mathematics in the senior high schools in Ghana in general. In Table 1, even though a particular problem has not been attributed to the poor performance, Fletcher (2016) attributed some of the factors of poor performance to weak computational skills, low self-confidence, limited use of learning resources in the classrooms, limited number of qualified mathematics teachers, weak content and/or pedagogical content knowledge, teaching to the test and *excessive use of didactic approaches that do not help students to grasp relevant concepts firmly*.

In adducing possible solutions, Fletcher (2016) ascribed to the use of learner centred, creativity/innovation (e.g. use technology appropriately) and *CALM* model. In CALM, 'C' stands for Challenging lessons with a range of activities, 'A' stands for Active lessons with involving engagement, 'L' stands for Learner-centred lessons for independent thinking skills and 'M' stands for Motivating lessons for interesting and fun activities (Fletcher, 2016). Even though very laudable and prudent model, it was clear that Ghanaian mathematics teachers excessively use didactics of mathematics. This is partly attributed to inappropriate use of technology tools and other learner-centred interaction models.

Therefore, the state of didactics of mathematics in Ghana can be summarised as follows:

1. Theories of mathematics didactics have not been factored into the classroom strategies. Didactics, didactical theories and models remained unnoticed and unattended to in the Ghanaian school mathematics (Fletcher, 2016).
2. There have been little efforts to intertwine and network theories and models that foster and promote didactics of mathematics in the Ghanaian classroom. Few attempts in didactics focus more on lecture and teacher-centred strategies (Fletcher, 2016).
3. Teaching and learning of mathematics are still largely mechanistic and empiricist. In the mechanistic, teaching and learning utilized recall and

memorization. In empiricist, teaching and learning follow rigid procedures and algorithms (Presmeg et al, 2016).

4. Teaching and learning processes are concentrated only on learning aims, outcomes and objectives. The processes that lead to achieving learning outcomes remained untold and unexploited. Most SHS students achieve the learning objectives through memorizing facts, concepts, computational formulae and guess work (MoE, 2010).
5. Changes, modifications and innovations of the Ghanaian curricular have not adequately addressed contemporary issues. This is partly due to lack of utilization of signs and symbols, amalgamation of tools and instruments, innovations of technologies and didactical conceptualization (MoE, 2018).

The didactical conceptual structures, therefore, seeks to develop and implement didactic conceptual models. These models will extend the triad to the tetrahedron using technology mediation tools. The technology tools become inalienable binding forces within three intersubjective inseparable theoretical frameworks for the teaching and learning of mathematics in SHS in Ghana.

Conceptualising Didactics

Originally, the word didactics came from the Greek word *ςιςασκειν* (*didάskin*), which means teaching. In contemporary times, didactics now serves as a major theory in teacher education and school syllabus development. For instance, in Finland, it is called *didaktiika*; in Russia, it is called *didaktika*; in Spain it is called *didάctica*, and in Dutch it is called *didactiek*. The term *Didakktik* in German stems from the German tradition of theorising classroom instruction (Arnold, 2012).

Again, while didactics is a discipline that is essentially concerned with the science of teaching and instruction, pedagogy is focused more specifically on the strategies, methods and various techniques associated with teaching and instruction. So, didactics is a more generalized term referring to the theory and practical applications behind the science of instruction(Leon-Henri, 2020).The main stages of evolution of didactics are pre-didactics, didactics-dialectics, classical didactics and digital age didactics (Tchoshanov, 2013).In this study, didactics does not connote the English pejorative meaning related to instructive matters or oversimplifying ways of teaching. It is a unique area of teaching and learning that seeks to draw relationships among the primary actors. The primary actors in the mathematics classroom are teachers, students, mathematics and materials or technologies

Didactics of Mathematics

According to Ruge and Hochmuth (2017), didactics of mathematics can be described as an interdisciplinary discipline ranging in between disciplines of mathematics and educational sciences. Didactics of mathematics therefore can be perceived as a scientific discipline that merges several theoretical influences, and at the same time driving the learners to achieve their vocational pursuits. As a bridge between mathematics and educational sciences, it provides learners the opportunity to apply education to mathematics phenomena (Ruge, &Hochmuth, 2017).

Again, didactics of mathematics can be conceptualised on the questions of ‘what is mathematics?’, ‘how is mathematics taught?’, and ‘why mathematics is practised (Dunphy, et al., 2014)?’ In seeking answers to these questions, Davis (2010),Clements, Keitel, Bishop, Kilpatrick and Leong

(2013), and Seah (2013) differentiate between Mathematics with an upper-case 'M' and mathematics with a lower-case 'm' in school mathematics. The 'M' is explained as the universal Mathematics in schools and other formal settings. The 'm' is regarded as the wider mathematical knowledge being practised in everyday life in a society (i.e. mathematics for all).

In furtherance to this notion of mathematics education, Davis (2010) and Clements et al (2013) adduce to particular kind of mathematics, where all students have the same opportunity to learn, and benefit from the learning. This notion perceives learning mathematics is in two folds. One fold is to prepare mathematically-functioning citizens of a society. Another fold is to prepare students to take up future mathematics careers. Therefore, in learning mathematics, conditions and context are crucial with regards to the socio-cultural phenomena, the needs of learners, and the goals of societies in which the students live (Davis, 2010; Sriraman, & English, 2010; Seah, 2013).

The third perspective of didactics of mathematics (conceptual understanding, procedural fluency, competence and skills) can be properly explained and promoted through engagements. Some of these processes of engagement are connecting, communicating, reasoning, arguing, justifying, representing, problem-solving and generalizing (Bishop et al, 2002). These processes are contextualized in the overarching concept of mathematisation. In mathematisation, students interpret and express their everyday experiences in mathematical form and analyze real world problems in mathematical ways. Therefore, mathematisation sets the pace for defining and explaining all mathematics phenomena. This helps to address a wide range of mathematical ideas to enable students to engage in their daily practices (Bishop, 1993).

Also, conceptual understanding as an aspect of proficiency in mathematics education revolves around comprehensive knowledge, skills and experiences in mathematical tools, symbols, signs, concepts, operations, tasks and relations. It is expected that students who have conceptual understanding will know more than just isolated facts, theorems and generalizations. They can integrate grasp of mathematical ideas, make connections between ideas, retain facts and then follow procedures (Dunphy, et al., 2014).

Procedural fluency is the skill in carrying out procedures flexibly, accurately, efficiently, succinctly and appropriately. This knowledge helps students to analyze similarities and differences between methods (written procedures, mental methods, concrete materials and technologies). This is enshrined in strategic competence and adaptive reasoning skills. Strategic competence is the ability to formulate, represent, solve mathematical problems, and form mental representations. The representations can be routine and/or non-routine problems, and mathematical relationships within problem-solving approaches. Adaptive reasoning is the capacity to follow logical thought, reflection, explanation, and justification by employing mathematical thoughts and concepts. Students can clarify their didactical reasoning using the concepts, procedures, reasons, and collaborations. They use physical and mental representations in achieving this goal of learning. And when students and teachers effectively conceptualize didactics of mathematics, the scope and contents of mathematics become meaningful, efficacious and rewarding (Dunphy, et al., 2014).

Conceptual Structures

The term ‘conceptual structure’ is related to syntactic and phonological structure on the one hand and, non-linguistic levels of representation (e.g. vision) on the other hand (Jackendoff, 1987). Jackendoff (1987) explains that conceptual structure is *decompositional* (i.e. decomposes meanings in terms of concepts), *conceptualist* (i.e. identifies meanings with concepts) and *localistic* (i.e. elaborates the ideas of location and movement) (Jackendoff, 1987). These three characteristics has enabled education sciences to build on the basic assumptions of its componential accounts in order to specify the internal structures of various classes of concepts and to subsequently determine the representational interactions (Moss, Tyler, & Taylor, 2014).

According to Richland, Stigler and Holyoak (2012), the term “conceptual understanding” has been given many meanings, which in turn has contributed to the difficulty in changing teacher practices. Conceptual understanding can be attainment of an expert-like fluency with the conceptual structure of a domain. This level of understanding allows learners to think generatively within that content area, enabling them to select appropriate procedures for each step when solving new problems, make predictions about the structure of solutions, and construct new understandings and problem-solving strategies. For the sake of clarity, the term ‘conceptual structure’ has been adapted to primarily focus on the goal of facilitating learners’ acquisition of the conceptual structure of mathematics.

In addition, all concepts are abstract entities and so have no physical properties. The first concepts, also called primary concepts, are formed on the basis of the sensory experiences of the outside world. Secondary concepts are abstracted from primary concepts, and tertiary or mathematics concepts are abstract and far removed from experiences. These notwithstanding, concepts have connections with other concepts, and each concept can be part of several hierarchies and relationships at different levels.

Dhankar (n.d.) opines that conceptual structures are complex networks of relationships among concepts (Dhankar, n.d.). The complex networks of relationships among concepts are called the conceptual structures. As the concepts are building blocks of understanding, the conceptual structures are basic tools of teaching and learning. If students do not have adequate conceptual structures, they cannot reconstruct new knowledge, cannot learn, and hence cannot generalize their knowledge and experiences. For example, a student who has only memorized algorithms for finding the centre and radius of the circle, in spoken and as well as written form, without having or connecting the notions of signs, artefacts, tasks, tools, instruments and technologies has formed inadequate conceptual structures (Dhankar, n.d.).

Didactical Conceptual Structures

In one context, didactical conceptual structures comprise several mathematical practices that are negotiated by the teacher and the students within broader social, scientific, and cultural contexts (Radford, 2014). This context opens didactics of mathematics up to two different domains. One domain is the field of practice where people engage in the activities connected to the teaching and learning of mathematics. This field delves into scientific enquiry on theorising about the field of practice. Another domain focuses more on relationships between the student, teacher, mathematical content and mediator. This view broadens the scope to include socio-cultural spectrum, networks of social and cultural practices and social institutions constituted by students and teachers in the classroom (Dunphy et al, 2014).

Didactical Tetrahedron

One way of conceptualising the didactical tetrahedron is to model the complexity of the mathematics classroom so interactive to generate increased interest in technologies (Ali, & Wilmot, 2016; Presmeg, Radford, Roth, & Kadunz, 2016; Presmeg et al, 2016). There are many theories and models that accommodate varied technologies. Particularly, the theories of realistic mathematics education, instrumental genesis and anthropological didactics are well grounded in the mediational roles of signs, symbols, artefacts, tools, instruments and technologies. These theories help facilitate and contextualize teaching and learning (Radford, 2008; Radford, & Sabena, 2015).

Also, Vygotskian conception of an instrumental act allows students to interact with mathematics artefacts and tasks. It is therefore essential to conceptualize the didactical tetrahedron in order to facilitate appropriate and effective mathematical learning (Matusov, 2015). Oerback (2008), Billington (2010), and Rezat and Strässer (2012) have re-conceptualized the didactical triad. The didactical triad consists of students, teachers and mathematics. However, the didactical triad can be extended to the didactical tetrahedron by adding technology tools or technologies. The addition of technologies brings new preconception of the didactical tetrahedron. This refocuses the didactical triad and makes it much robust.

Again, the didactical tetrahedron facilitates the roles and functions of the vertices and offers adequate and comprehensive representations of the classroom complexity. In utilizing this complex conception of the didactical instructional roles of the student, teacher, mathematics content and technology, the actions, activities and tasks must accommodate students' intentions, actions and interpretations. This can be achieved by synchronizing a variety of theoretical, conceptual and methodological frameworks. This will foster coherence and flexibility among the didactical conceptual structures (Van den Heuvel-Panhuizen, & Drijvers, 2012; Van den Heuvel-Panhuizen, Drijvers, Doorman, & Van Zanten, 2016).

Didactising Conceptual Structures

Conceptual structures are formalized knowledge representations of the triad (student, teacher and mathematics content). Didactical tetrahedron consists of the student, teacher, mathematics content and technologies and their interactions (Barzel et al, 2005). The conceptual structures require multifaceted mathematical competencies. These competencies include not just content knowledge, pedagogical knowledge and technological knowledge but also didactical intersections (Gros, 2016).

The pedagogical components are mathematical didactics of direct instructions and curriculum formulation. With emphasis on teaching, learning transformation of mathematical content knowledge are contextualized in the intersections of ontology, epistemology, and philosophy. Conceptual understanding of mathematical didactics mainly revolves around psychology and anthropology. In psychology, conceptual understanding change knowledge by diversifying the relationships of teaching. This differentiation provokes cognitive structures and increase relations of the conceptual structures in intersubjective theoretical frameworks (Sriraman, & English, 2010).

Also, mathematical didactics is the art of conceiving and conducting classroom conditions that determine mathematical knowledge shared for the didactical conceptual structures in the tetrahedron. Thus, learning starts as the set of modifications of behaviours, representations, specific beliefs, and sets of repertories. In this case, mathematical practices then set up conditions real projects, scientific experiments, concepts and methods. Students and teachers apply the coherence and contingencies in the theory of didactic situations to solve mathematics tasks (Novotna, & Sarrazy, 2010; Godino et al., 2012).

Coupled with didactical situations is the promulgation of the anthropological didactics. In anthropological didactics, conceptual structures create relationships within their socio-cultural environments in sequential and systematic orders (Sriraman, & English, 2010). Winsløw (2011) suggests that if conceptual structures are far from anthropology, the processes of didactising become weak. Therefore, students and teachers should employ repetitive teaching, various organisations, and different situations in mathematics instructions. If the relations among the conceptual structures are strong, students should be encouraged to mathematise, construct and didactise.

Broadly speaking, the main stages of evolution of didactics are pre-didactics, didactics-dialectics, classical didactics and digital age didactics. The pre-didactics stage began with Socratic dialogues and later transformed to the Socratic method of teaching of classical fine arts curricula. So, the two major blocs of trivium and quadrivium emerged. The didactics-dialectics stage began with the study of reading and further continued with dialectics. In dialectics, classroom is perceived as art of teaching. The classical or traditional didactics began with the transition from the art to the science of teaching and learning.

This was further developed in Didactica Magna in the didactical theory of teaching and learning.

The stage of technology or digital age didactics began with reconceptualisation and reconstruction of classical didactics in the era of ICT. The stages of pre-didactics and didactics-dialectics saw the emergence of the first teaching method (e.g., Socratic dialog), curriculum (e.g., classical Fine Arts) and dialectics. The stages of didactics-dialectics and classical didactics witnessed the emergence of didactics fields, theories, frameworks and models for teaching and learning. However, the emergence of digital age didactics ensure quality teaching and learning through ICT-based learning artefacts. These guide and develop students' competency and proficiency in didactics of mathematics (Tchoshanov, 2013).

Also, Chevallard's conceptualization of external didactic (outside the school), internal didactic (inside the school) and their interactions make it possible to extend the triad to the tetrahedron. Here, the mathematics content requires further analysis with the technologies. The technologies help pose fundamental epistemological questions on the various parts of equations of the circle. In this study, the common parts are concepts of dimensions, types of equations, centres and radii, and sample tasks (Martin, & Roitman, 2014). These provide specific cases of the didactical conceptual structures (Østergaard, 2013).

Furthermore, the trivium comprising the student, teacher and mathematics content and their interactions is called didactical triad (Østergaard, 2013). Schoenfeld as cited in Østergaard (2013) posits that because classrooms are cultural systems and mathematics classrooms are indelible functions of those cultural forces, society must now recognize the transforming effect of technologies. Therefore, extending the didactical triad to the didactical tetrahedron using the didactical instructional models is very laudable and must be embraced.

Mathematising Didactical Conceptual Structures

Realistic mathematics education espouses ways of mathematizing the didactical conceptual structures. In its original conception, Freudenthal (1973) thought of realistic mathematics education was not just a body of knowledge that must be transmitted in the form of human activity. The concept in contemporary didactics of mathematics means involving students actively in mathematisation acts. With appropriate guidance from teachers and other knowledgeable adults, students are given the opportunity to discover, innovate and reinvent mathematics in their own ways. This can be achieved in both horizontal and vertical mathematizations. The two mathematizations provide students the opportunity to identify relevant attributes, cultural objects, ideas and specific examples (Artigue, & Blomhøj, 2013; Davis, & Chaiklin, 2015).

Furthermore, in Bishop et al. (2002) and Dunphy, et al. (2014), horizontal mathematisation helps students to develop mathematical signs, symbols, artefacts, tools, instruments and into technologies. This helps them to solve problems situated in real-life contexts. Vertical mathematisation allows students to make connections between mathematical concepts, strategies and methods that already exist. The differentiation between mathematisation and mathematical processes by National Research Council--NRC (2009) positions mathematical processes as connection of general mathematical reasoning, representation, problem solving, connection, and communication mechanisms.

This helps to bridge the gaps between abstract mathematics and real situations. It also bridges the gaps mathematics phenomena and real world situations. This means mathematisation does not happen only when students are provided with technologies to create models of the situations and mathematical objects (numbers, shapes and actions). They equally mathematise during counting numbers, transforming shapes, representing interrelationships, drawing relationships and solving mathematics problems (NRC, 2009).

In addition, in mathematising the didactical conceptual structures, mathematics models have been categorized into conceptual structures, instructional designs and didactical relationships. In conceptual structures, specific activities are tailored to the content description, but in instructional designs, the processes of teaching and learning are dealt with the instructional elements. In didactics relationships, teaching and learning aims at bringing the conceptual structures and instructional elements in a single fold. This helps to achieve the learning outcomes and indicators. So, in mathematising the didactical conceptual structures, the researcher sought to actually bridge the gaps between many strategies, methods, techniques and models (Barbosa, Maldonado, & Ricarte, 2003).

Mathematising the didactical conceptual structures is also explained by different and varied conceptions. According to Bishop et al (2002) and Artigue, et al. (2005), a conception is explained as:

1. The different and/or multiple approaches (expressions and meanings) of a mathematical concept. Thus, a conception is employed to discriminate between different aspects of a mathematical concept in its definition and context.

2. The identifications of differences between the meanings students construct about a mathematical concept and the concept itself. Thus, a conception is employed to identify errors, mistakes, wrong notions and misconceptions about a concept.
3. The student's knowledge and skills expressed in different ways he/she conceives knowledge. Thus, a conception is employed to identify structures and build relationships within and between concepts.
4. The idea and belief about roles and functions of mathematics teaching and learning. Thus, a conception is employed to identify the roles and functions of teachers, students, mathematics content and technologies to the equations of the circle.

Therefore, in generating the didactical conceptual structures in the triad and extending the triad to the tetrahedron, conceptions are perceived and employed in the concepts and methodologies. These concepts and methodologies were portrayed and practised in the society, school and classroom cultures.

Finally, the didactical conceptual structures show the multiple and varied explanations of the relevance and significance of conceptual knowledge, conceptual processes and conceptual connections to specific mathematical concepts. The conceptual development of the didactics of mathematical also shows the conceptual mathematisation with mathematical mediators. In the end, mathematised constructs are the essential contexts. These essential are mathematised within and across the various facet or components of the didactical instructional models. This help to discover features, similarities, analogies, generalities, didactical inversions, objects and operations with the mediators (signs, symbols, tasks, artefacts, instruments, tools and technologies) (Bishop, et., 2002; Dunphy, et al., 2014).

Mathematising Signs and Symbols

Didactic cycle provides activities for designing and for analyzing most mathematics contents. These activities involve signs and symbols. Students employ also artefacts and tools to carry out mathematical tasks and then discover specific signs. The use of particular artefacts and tools bring out signs. Such artefacts and tools help students work in pairs or small groups. These promote social exchange with words, sketches, gestures (Wells, 2007). In mathematising signs and symbols, students identify, name, draw and write with signs and symbols. The teacher engages the student to discover different semiotic narratives, mimics, texts, drawings, discussions and discourses (Bartolini Bussi, & Mariotti; 2008).

Furthermore, signs and symbols in mathematisation vary with evolution processes. They originate from personal sense to mathematical meanings and from pure cultural artefacts, signs and symbols to mathematical ones. The first category of signs and symbols refer to the artefact signs. Students and teachers contextualize these artefact signs to generate the mathematics ideas. The second category is the pivot signs. The pivot signs are cultural heritage and constitute semiotic mediation processes that teachers orchestrate to explain complex processes. Students use the pivot signs to generate and derive activities. The third category of signs is the mathematics signs. Mathematics signs, in this context, refer to the mathematics meanings, statements, axioms, laws and generalizations (Presmeg et al., 2016).

In mathematising signs/symbols and tools, Vygotsky (1978, 1999) make further distinctions between signs/symbols and tools as follows:

1. Tools, like calculators and computers, are object-oriented materials. However, signs and symbols are means of social and intrapersonal interactions. These distinctions make it clear that signs and symbols are auxiliary means of solving psychological problems. These are recall, compare, construct, draw, and label concepts that are analogous to the invention and use of the tools.
2. Signs and symbols act as instruments of psychological activities analogous to the roles of tools in labour. However, some tools such as the mathematics calculator can function both as a tool and as a sign in. For instance, when students are drawing graphs, the calculator mediates the material activity as a tool. But if the students stop drawing, the calculator's graphs function as a sign to identify where the students would continue to the next sets of tasks.
3. In practice, all classroom mathematics activities involve the coordinated use of a variety of signs and symbols. These are embedded in communication, collaboration and problem solving. However, tools are just the materials used in mediating the construction of knowledge.

Mathematising Tasks

Tasks are operations undertaken within certain constraints and conditions. These operations are exercises students perform, interactions between students, teachers and mediators. Tasks extend to things that teachers use to demonstrate mathematics procedures interactively with students. Mathematising tasks means designing materials intended to promote complex mathematical activities (Watson, & Ohtani, 2013). In more succinct forms, mathematising tasks involve a wide range of activities involving repetitive exercises, constructing objects, exemplifying definitions used for solving single-stage and multistage problems. These activities help make decisions to carry out experiments and benefit from experiential learning (Wells, 2007).

In addition, mathematising tasks involve engaging technical and pedagogical skills, and sociocultural environments through mediations (Vygotsky, 1978; Davis, 2010; Davis, 2013). In other words, mathematised tasks compel teachers and other experienced adults to initiate actions and instructions tailored towards students' immediate environments. This help student to acquire knowledge through contacts and interactions with experiences (interpsychological plane). It also helps students to assimilate and internalize the knowledge through personal values (intrapsychological plane). So, both students and teachers transition their interactions and assimilate the mathematical concepts (Turuk, 2008).

Mathematising Artefacts

The notion of cognitive artefacts is based on the socio-historic school of Vygotsky (Vygotsky, 1978). These are on one hand, to create tools (external process), and on another hand, to improve logical reasoning and cognitive activities (internal process). This Vygotskian developmental of cognitive artefacts can offer adequate frameworks in the mathematics classroom. Vygotsky postulates two lines for the genesis of human mental activity. These lines are the natural line used for elementary mental functions and the social and cultural line used for the higher mental functions. Both the social and cultural lines give rise to the Vygotskian ZPD and internalization roles of artefacts (Bartolini Bussi, & Mariotti, 2008; Ali, & Davis, 2016). Mathematizing artefacts also assume a historical epistemological view. These views help students function and make meaning of practical and/or theoretical contexts. The artefacts also help recreate knowledge (Wells, 2007).

Again, there are three specific hierarchies of artefacts. These hierarchies are primary, secondary and tertiary artefacts. The primary artefacts are fundamental tools used in mediating teaching and learning. They play the roles of cultural interactions and design practical actions. For example, pens, pairs of compasses and abacuses are primary artefacts. In teaching and learning equations of the circle, the pens write, the compasses draw and the abacuses make place values). The secondary artefacts are the greater processing of the primary artefacts. They help speed up interactions, uses and modifications of the primary artefacts. The tertiary artefacts make further elaborations of the secondary artefacts. They propel teaching and learning to metacognitive dimensions. These levels help students to make independent, logical and practical learning (Fiorani, 2012).

Another perspective of mathematizing artefacts arise out of human dimensions. These dimensions emerge out of historical and environmental contexts of cultural experiences, behaviour patterns and artistic rituals. In this sense, the artefacts condense the signs of the cultural-historical orientation. These help to recreate and provide mediations during teaching and learning (Fiorani, 2012).

Mathematising Tools

Maschietto and Trouche (2011) distinguish between technical and psychological tools. They contend that technical tools are externally oriented, while psychological tools are internally oriented. In every mental activity, pupils in preschool levels reach higher levels through cultural tools. At the preschool levels, psychological tools are the only tools that provide artificial stimuli (semiotic tools). This helps pupils to produce internalize their thinking processes. However, students in the SHS levels use artefacts directed at the outward senses. Because this stage is complex, students mostly use technical tools to achieve knowledge that would otherwise have remained out of their reach (Bartolini Bussi, & Mariotti, 2008).

Also, mental activities at the SHS levels are supported and developed by psychological tools (oriented inward). The previous inventions and uses tools in the psychological domains still remain relevant and cardinal. Such tools help the students to remember, compare, analyze and report new inventions and uses of tools. It is therefore, important for students to possess the psychological tools too. These tools help them in making mnemonic techniques, algebraic symbols, schemes, diagrams, mechanical drawings, computer software and technology tools (Bishop, 1988; Ali, & Davis, 2016).

Mathematising Instruments

In mathematising instruments, Drijvers and Trouche (2010) distinguish between artifacts and instruments. Artifacts are bare tools that are available to students to perform certain kinds of activities. However, they may be meaningless objects to others so long as they do not know the kinds of tasks the objects support. It is only after the students have become aware of the objects and associate the objects to particular kinds of tasks and using the objects that for specific purposes that they become instruments. For instance, artifacts like calculators are not automatically instruments. The calculators are only instruments if they have been used to perform types of tasks.

Secondly, instruments are artefacts created and used by students to help and empower their activities in the processes of instrumentations and instrumentalisations. Thus, instrumentations are the actions given to the instruments to perform given tasks. These tasks influence the actions of students and knowledge in both psychological and material tools. In this case, the artefacts are general tools (e.g. mathematical sets, computers, or calculators) and the instruments are more specific tools. Specifically, rulers are used for drawing lines, calculator for computing and mathematics sets for constructing figures in equations of the circle (Sabra, Emprin, Connan, &Jourdain, 2014).

In addition, research (Maschietto, & Trouche, 2011) combines the Vygotskian and Piagetian perspectives to distinguish between artefacts and instruments. One differentiation opines that artefacts are materials or abstract objects that have been produced by a human activity. The aim is to create new activities in solving the types of tasks. For example, calculators are artefacts made by man. But instruments are the activities students and teachers build from the artefacts. The instruments link the activities to the given problems and situate the artefacts to target at knowledge and potentialities of the available artefacts. Another school of thought (Drijvers, & Trouche, 2010) advances that teachers and students build instruments in order to perform the types of tasks. The interactions require mental processes involving the didactical triad (teacher, student and mathematics content). The instrument then utilizes both the artifacts and the accompanying mental schemes developed by teachers and students to perform specific kinds of tasks. This helps students and teachers to develop mental schemes, problem-solving strategies and conceptual structures.

However, Drijvers and Trouche (2010) and Drijvers and Trouche (2010) lament the difficult transition from artifacts to instruments. One difficulty is that teachers and students already have sets of artifacts at their disposal (e.g. paper, pencil, rule, compass, calculator, algebraic software, geometric software, spreadsheet and word processing). But students and teachers fail to inculcate the instruments into teaching and learning of mathematics. It is always a challenge for students and teachers to identify the standard equation of the circle, study the types of circle equations, and solve for centres and radii. But each mathematical task usually requires the simultaneous utilization of several instruments. The second challenge is that the development of instruments is never an isolated process. Students and teachers usually face the same types of tasks. They simultaneously develop instruments in the same contexts (Drijvers, & Trouche, 2010).

Mathematising Technologies

Research alludes that conceptual structures cannot be easily grasped by only human senses and intelligence. They often require external representations of the mathematical objects. These external representations are signs, words, schemes, symbols, gestures, artefacts, tools, instruments and technologies. In most cases, external representations are full of static formulas and abstract generalizations. However, technology tools are mostly dynamic, graphical and interactive. They build relationships among key mathematical phenomena. Technologies also ensure that mathematics tasks are accurate, dynamical, constructed and verified (Presmeg et al, 2016).

Also, technologies are wide and manifold. They include the arithmetic operators or signs, geometric symbols, graphs and sketches. They extend to indigenous resources, mathematical sets, calculators, computer algebraic software (CAS), dynamic geometry software (DGS) and/or amalgamated innovative practices. In contemporary times, technologies include discovery, inventions, innovations, experimentation and visualization in mathematics contents (Dikovic, 2009). In more advance forms, technologies compass programming, methodological experimentations, technical knowledge, complex 3D extensions and MMLab systems (Presmeg et al, 2016).

Again, the necessity of having to combine coherently different instruments from the set of artefacts requires more complex objects. The complex stages of developments of the different instruments, and the different psychological tools means students and teachers should carefully innovate and design objects that suit their didactical goals. These complex phenomena can easily be exemplified and tackled with technologies (Olive, 2013). Technologies interactions and engagements are enormous for:

i. They allow mathematics to be explored effectively and constructively.
ii. They outsource processing powers out of reach of humans.
iii. They bring new representations and change mathematics instructions.
iv. They ensure total, holistic and comprehensive connectivity during interactions.
v. They bridge gaps between culture and school mathematics.
vi. They help engage in exploratory environments (Gros, 2016).

Didactical Phenomenology

Didactical phenomenology explains the various ways mathematics is presented and invented as students move from one worldview into another. The movement of one theoretical framework into another, one methodology into another and one analysis into another help students and teachers discover and construct new mathematics (Bishop et al, 2002). Supported with the Theory of Didactical Situations and Anthropological Theory of Didactics, didactical phenomenology recontextualises mathematics (Anh, 2006).

Also, studies (Billington, 2010; Viirman, 2014) show that mathematical phenomenology are the mathematical concepts, structures and ideas that relate to the phenomena for which they were created for. Didactical phenomenology helps teachers to place students to explore phenomenological activities. These activities are mathematising, axiomatising, formalising and schematising. The processes of mathematising are generalising and formalising. Formalising involves modelling, symbolising, schematising and defining mathematics concepts. Generalising is the reflective outcomes of learning, organising and structuring mathematics knowledge and skills. This is used to discover unknown concepts, relations and structures.

Secondly, didactical phenomenology is grounded in phenomenology. Phenomenology involves mathematical concepts, structures, ideas and relations. The relations are between mathematical thought and didactical phenomenon (Van den Heuvel-Panhuizen, & Drijvers, 2012). The didactical phenomena (*nooumenon*) build relations during the processes of organizing, organizing and constructing the phenomena. The didactical phenomenology also builds relations during the teaching and learning processes in the didactical conceptual structures. By abstracting concrete mathematisation and opportunities for students, the phenomenology constitutes the mental objects being mathematised. This relates mathematics tasks to real-life situations. This enables students to differentiate bare tasks from mathematical contexts (Van den Heuvel-Panhuizen et al, 2016).

Thirdly, didactical phenomenology is the study of relations between the phenomena that mathematical concepts represent and the concepts themselves. In didactical phenomenology, teachers interpret mathematical phenomena, reasoning and calculations. Teachers also suggest ways of identifying plausible instructional activities to support students' activities and whole-class discussions. This engages students in the progressive mathematisation and creates congenial classroom environments renegotiate experientially real problems. The phenomenological analysis focuses on the mathematical concepts, procedures and tools and develop further learning situations (Gravemeijer, 2008).

Again, didactical phenomenology of mathematical concepts, structures and ideas are the relations to the phenomena. This develops mathematical into categories (Freudenthal, 1983). The categories are procedural, conceptual, and dual texts. The procedural activities are mechanistic memorization of mathematical facts, operational procedures and applications of the teaching and learning processes. The conceptual activities establish conceptual understanding and applications in flexible (mental) calculations. Students design their own tasks, methods and worksheets. In the procedural strand, students and teachers are able to employ sophisticated and varied technologies to develop new techniques for teaching and learning. These technologies help them to interact with conceptual structures and enhance new didactical situations (Van den Heuvel-Panhuizen, & Drijvers, 2012).

Didactical Situations

There is no universal agreed upon definition of didactical situations in the teaching and learning of mathematics. One reason is that it is rather a new field of study developed as a scientific discipline for which researchers are still finding space to situate its meaning and context (Bishop et al, 2002). Another reason is that much of the activities within didactics of mathematics are subsumed by research in educational psychology (Johansson, 2006). This notwithstanding, didactical situations refocus on the scientific activities of describing, analysing and organizing activities in order to better understand mathematics in and outside classroom mathematics.

One school of thought perceives didactical situations in three domains--normative didactics, descriptive didactics, and meta-didactics. Normative didactics concerns the development and evaluation of the educational goals, choices of contents and methods, justifications and recommendations. Descriptive didactics conducts empirical studies of the actual teaching and learning with educational questions as well as extensive qualitative studies. Meta-didactics concerns and evaluates issues within the theories and philosophies of didactics of mathematics (Johansson, 2006).

Another school of thought partitions didactical situations into curriculum research and phenomenological approaches. These divisions were not developed independently from Germany and Sweden respectively. That is, the German didactical tradition of curriculum theory has elements of the Swedish didactical lineage to classroom pedagogy (Johansson, 2006). In this study, the didactical situations focus more on classroom interactions of the didactical instructional models. This helps to close gaps between educational didactic theories and classroom practices. The eventual aim is to expose the complex and multi-dimensional didactical tetrahedron to SHS mathematics students in Ghana (Ali, Davis, & Agyei, 2018).

3T and 4T Didactical Praxeologies

Didactical praxeologies involve both conceptual fields and didactical situations. These fields transform and complete the didactical conceptual structures from the triad to the tetrahedron. The fields transform the triad by engaging in problem-solving strategies with technologies (Sellers et al, 2007). Egodawatte (2011) proposes six phases of transforming the triad. The first phase, mobilisation of former is the phase of adapting the problem. The second, research is the phase of actions on the didactical situations. The third, local institutionalization is the phase of formulating essential mathematical objects. The fourth phase, institutionalization, is the phase of linking socio-cultural phenomena to the new mathematics knowledge. The fifth, familiarization reinvestment, is the phase of maintaining previous knowledge. The sixth, complexification of tasks or new problems is the phase of making new knowledge. While novel and effective, these phases require technologies to check errors and misconceptions (Egodawatte, 2011).

In addition, conceptual situations involve didactical dependences, didactical sequences, institutional teaching and learning and validation of students' knowledge. Here again, technologies synchronize social organizations, interactions in the triad and group management (whole class, small groups, and individual work). In this case, technologies realign conventional tasks with didactical assignments (Novotna, &Sarrazy, 2010).

Furthermore, 3T didactical praxeologies stand for the Task, Technique and Theory. This conceptualizes mathematical tasks, situations, experiences and knowledge in the didactical triad (Østergaard, 2013; Ali, Davis, & Agyei, 2017, 2018). With extension to technologies, we derive the 4T didactical praxeologies. The new extended 4T didactical praxeologies stand for the Task, Technique, Theory and Technology. These facets interdepend during interactions among the didactical conceptual structures. The didactical conceptual structures use the task to describe and analyze the didactical problem, the technique to solve the problem, the theory to guide the problem and the technology to explain the theory (Østergaard, 2013). At the end, the 4T didactical praxeologies formulate new situations and discourses, where the didactical conceptual structures interact meaningfully and effectively in the didactical instructional models (Marquet, & Coulibaly, 2011).

Equations of the Circle

Equations of the circle and by far conic sections comprise equations of ellipses, parabolas, and hyperbolas as seen in Figure 1.

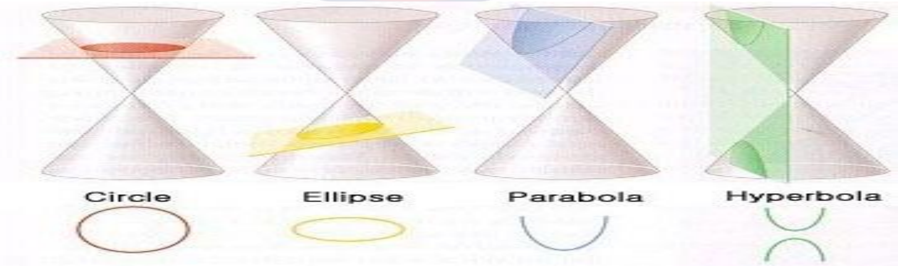

Figure 1: Two-Dimensional (2D) Conics (**Source**: Martin, & Roitman, 2014)

Figure 1 displays four sets of conic sections showing the cross sections and intersections of the plane with right circular cone. The plane cuts through one nappe of a cone and the intersection to form a circle if the plane is perpendicular to the axis. It forms an ellipse if the plane is not perpendicular to the axis. If the plane passes through the vertex of the cone, it produces atypical or degenerate conic sections which lack regular features associated with conic sections. This means the circle is considered a degenerate ellipse and a point is a degenerate circle. Therefore, conic sections can be determined by the discriminant, $B^2 - 4AC$ from the general equation, $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. If $B = 0$ and $B^2 - 4AC < 0$ (i.e. A and C have the same sign), then the equation is either an ellipse or a circle. It is a circle when 'A' and 'C' are the same (Martin, & Roitman, 2014).

Secondly, conic sections are conceptualized under geometric, algebraic and analytic themes. This property makes them suitable for applying the didactical conceptual structures in equations of the circle. This is because irrespective of the field, they always build relationships among concepts. Algebraically, because the circle is a degenerate ellipse, it is easier solving for the centre (h, k) and radius r > 0. Geometrically, a conic section has a set of points called locus satisfying a distance relationship between two points. The ellipse is the set of all points in the plane whose sum of distances from two fixed points (called the foci, plural of focus) is constant. With the two standard equations of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, 'a' $(a^2 = b^2 + c^2)$ represents the distance from the centre to vertex, 'b' represents the distance from the centre to the endpoint on the minor axis, and 'c' represents the distance between the centre and the focus. Here, the length of the major axis is 2a, the length of the minor axis is 2b, the centre is located midway between the foci, midway between the vertices, and midway between the endpoints on the minor axis.

The context of the ellipse depends on the parameter 'a', the term with the larger denominator, called the major axis. By translating the centre to the point (h, k), then we obtain the standard forms $\left(\frac{x-h}{a}\right)^2+\left(\frac{y-k}{b}\right)^2=1$ and $\left(\frac{x-h}{b}\right)^2+\left(\frac{y-k}{a}\right)^2=1$. If $a=b=r$, we obtain either $(x)^2+(y)^2=r^2$ (centred at the origin) or $(x-h)^2+(y-k)^2=r^2$ (centred at h, k) (Martin, & Roitman, 2014). Analytically, equations of conic sections relate two variables, x and y, which implicitly define y as one or more functions of x. In this case, a circle of radius, r, centred on the point (h, k) has a standard equation $(x-h)^2+(y-k)^2=r^2$. The intercepts can be found at where y = 0 and where x = 0 respectively, With an equation of the ellipse $\frac{x^2}{h^2}+\frac{y^2}{k^2}=1$, the equation of the ellipse is a circle centred on the origin (Ward, 2011).

Every equation of the circle is basically determined by its radius (r) and its centre $C(h, k)$. The equation $(x-h)^2+(y-k)^2=r^2$ describes the distance from (x, y) on the circle to (h, k) at the centre. If the standard equation is expanded, we obtain $x^2+y^2-2hx-2ky+h^2+k^2=r^2$, where the second-degree terms x^2+y^2, the linear terms $-2hx$ and $-2ky$ and the collection of h^2+k^2 are generalized into $x^2+y^2+2gx+2fy+c=0$. Here, the second-degree terms, the linear terms and the radius are compared and equated with $x^2 + y^2$, $2gx + 2fy$ and r^2 respectively to derive the new centre $C(-g, -f)$ and its radius $r = \sqrt{g^2 + f^2 - c}$ (Stitz, & Zeager, 2013; Whitney, 2015).

To solve for the centre and radius of the circle in standard form, the following steps are recommended by (Martin, & Roitman, 2014):

1. Group same variables together on one side and the constant on the other side.
2. Complete the square on both variables as needed.
3. Divide both sides by the coefficient of the squares because they should be the same.

Covariates

A covariate is a statistical variable that changes in a predictable way and can be used to predict the outcome of a study. Also, called concomitant or confound variables, two or covariates are variables that the researcher seeks to statistically control, minimize or statistically subtract their effects (Yu, 2015). Simple analysis of covariance (ANCOVA) and multiple analysis of covariance (MANCOVA) are two of the powerful techniques. A simple ANCOVA is a simple extension of ANOVA and has only one covariate as in gender (i.e. males and females).

In ANCOVA, the difference between the two groups in males and females adjusts or controls for the other independent variables. Because any number of covariates can always be included, the researcher added school to analyze MANCOVA. The covariates are necessary because they are divided into categories for the independent variables, they do not require construction of dummy variables, and the adjusted means are easier to obtain and report, since the covariates are rescaled automatically (Creswell, 2014).

ANCOVA and MANCOVA techniques are commonly used for analysis of quasi-experimental studies. Treatment groups may not be randomly assigned. But because the researcher wishes to statistically equate groups on one or more variables, controlling covariates is necessary (Baah-Korang, Gyan, McCarthy, & McCarthy, 2015). In this study, the pre-existing differences in the treatment groups gender, school, class level, programme of study, computer experiences, qualified mathematics teachers, mathematics resources and school management.

Statement of the Problem

Elective mathematics in the SHS level builds on the core mathematics. And the importance of learning elective mathematics is to help students to be able to apply their knowledge, develop critical thinking and apply analytical skills to problem-solving situations (MoE, 2010). The results of May/June performance in elective mathematics between 2007 and 2015 released by WAEC showed granulating patterns as seen in Table 2. It is sad to note that even though D7 and E8 are classified as pass marks or grades, students who fall in such categories of grades in Ghana hardly get admitted to pursue or mathematics-related careers. This makes the low performance of students in elective mathematics a much catastrophic issue as compared to low performance in core mathematics. It is now a matter of public knowledge that some institutions of higher learning are willing to organise remedial classes for students who obtain D7, E8 and even F9. Such opportunities are illusive for low performing students in elective mathematics. However, elective mathematics forms the core and basis of technological advancement. Stakeholders must therefore take a second look at the situation in Ghana

Table 2: Performance in Elective Mathematics between 2007- 2015

Year	Pass (A1 to C5)		Fail (D7 to E8)		Total
	Number	%	Number	%	
2007	13,685	36.5	23,817	63.5	37,502
2008	15,352	35.7	27,608	64.3	42,960
2009	17,862	35.7	32,189	64.3	50,051
2011	32,711	68.1	15,304	31.9	48,015
2012	44,185	75.2	14,546	24.8	58,731
2013	63,078	47.0	71,177	53.0	134,255
2014	15,484	20.5	60,135	79.5	75,619
2015	15,667	23.4	61,342	78.8	77,009

Source: WAEC, 2007-2015

A cursory look at Table 2 shows that more than 50% continuously perform below the average over the period. One reason assigned by the Chief Examiner's Reports is inability of students to find the equation of the circle. Students' techniques, strategies and solutions to equations of the circle were not lively and interesting (WAEC, 2012, 2016, 2017). However, studies suggest that integrating technologies in mathematics instructions could make mathematics innovative and thought provoking (Benning, & Agyei, 2016). As noted by WAEC (2016), students were unable to interact with the prescribed simple electronic calculators let alone using advance computers.

Again, another difficulty of teaching and learning mathematics has been acknowledged by the gaps between students' personal knowledge and school mathematical knowledge (Bartolini Bussi, & Mariotti, 2008; Gravemeijer, 2008). In Ghana, this problem is aggravated by the lack of independent knowledge to help students and teachers to reconstruct abstract and sophisticated mathematics concepts (Fletcher, 2016). This continuously widens the gaps between school mathematics and home mathematics. Didactical conceptual structures in the tetrahedron can help students and teachers to interact meaningfully and experientially. In this sense, classroom communications, strategies of solving tasks, building relationships become easier and practical (Presmeg et al., 2016).

Moreover, the quality of mathematics education in Ghana and other developing countries in the second-cycle sub-sector is dwindling. This is because classroom mathematics teaching and learning have still been dominated by conventional and traditional methods that impugn negative students' attitudes towards mathematics and its related careers (Fletcher, 2016). Fletcher (2016) attributes this gap to lack of effective strategies. In other words, there still exists low knowledge of technologies in didactics of mathematics to address, develop and implement effective teaching and learning of equations of the circle. As a result, students cannot demystify spatial knowledge and geometric concepts with confidence and creativity (Baah-Korang et al., 2015).

In addition to the poor quality of mathematics instructional strategies in the SHS levels in Ghana (Abreh, 2018), there exists seemingly lack of knowledge in designing mathematical tasks and problems with the didactical triad and extending the interactions to the didactical tetrahedron (Maschietto, & Trouche, 2011; Watson, &Ohtani, 2013; Fletcher, 2016). Students are not equipped with the knowledge, competencies and skills to identify mathematical tasks of socio-cultural orientations (Davis, 2013). Suffice me to state that there is even little or virtually no research in the didactical triad which starts with mathematics philosophies, theories and methodologies in the classroom interactions. Therefore, there are virtually limited strategies that utilize indigenous artefacts, and then transform the artefacts into technologies in Ghana (Ali, 2018).

Elsewhere, research (Bate, Day, & Macnish, 2013; Leikin, & Grossman, 2013; Dunphy et al., 2014) in didactics of mathematics does not still bridge the gaps between theory and practice. Researchers still stringently hold onto the two didactical divides (i.e. pedagogy and teaching practice, and pedagogical and mathematical knowledge). These ideological conflicts render didactics of mathematics underdeveloped and under-utilized. The school of pedagogy and practice has not been properly accentuated to school mathematics but just review theories and practices. In this school, nothing is known of signs, symbols artefacts, tools, instruments and technologies. The school of pedagogical and mathematical knowledge concentrates so much on mathematics theorems, algorithms and formulae (Østergaard, 2013). In Ghana, the problem is even worst. There is total lack of building didactical conceptual structures in teaching and learning of SHS mathematics. This situation does not create enabling environment to engage in relational mathematics, experientially experiences and socio-cultural transformation of indigenous signs, symbols and artefacts into tools, instruments and technologies. In the teaching and learning of equations of the circle, there are virtually no approaches that concurrently associate all algebraic, geometric and graphic properties of Equations of the Circle. However, the didactical tetrahedron dynamically associates algebraic, graphic and numeric representations (Ali, Davis, & Agyei, 2018).

Worst still, even research (Klette, 2007; Richland, Rezat, 2010; Meyer, 2012; Stigler, & Holyoak, 2012) has bemoaned the little elaborations on the roles and functions of students and teachers (who), subject matter content (what) and instructional methods or mediation tools (how) in the didactical tetrahedron. Despite the long history of the didactical triad, there are still low knowledge and practices of didactical tetrahedron. Teachers' methods and techniques have been poorly aligned with students' learning processes, activities, tasks, reflections and applications of technologies. This phenomenon partially attributes to the underdeveloped teachers' knowledge, use and awareness of the didactical tetrahedron. However, the didactical tetrahedron properly situate teaching and learning in indigenous and sociocultural settings (Ruthven, 2014).

Research in computer technology (Lagrange, & Psycharis, 2011; Olive, 2013; Leung, 2016) numerates several technology tools that offer multiple algebraic functionalities and geometric representations. However, in Ghana, there is still low appreciation and adoption of specific theoretical and conceptual frameworks that are purposely oriented towards these goals (Agyei, 2013: Davis et al., 2016). This makes it difficult to connect didactic contexts from realistic mathematics education, instrumental genesis and anthropological theory of didactics. The problem is worst when amalgamating computer algebra system (CAS) and dynamic geometry software (DGS). However, advance technologies explore and combine symbolic notations with dynamic manipulations of variable values. They connect dynamic geometry to symbolic environments and didactical epistemological. This helps to extend the triad to the tetrahedron (Clark-Wilson et al., 2015).

Students and teachers still use inappropriate tasks and techniques in representing 2D objects, dimensions, and shapes (Laborde, &Laborde, 2012; Leung, 2016). While there are widely available 2D objects in circles, research is still scanty on the applications of the phenomena in solving equations of the circle. This does not allow the didactical tetrahedron to situate the behaviours of complex systems in many interacting components of equations of the circle (Forsman, 2015; Wilson, 2015). In this case, technologies are not really conceptualized in didactics of mathematics (Tall, 2013; Radford, 2014). This inhibits the processes of mathematizations (Van Den Heuvel-Panhuizen et al, 2016). Therefore, Vygotsky's general genetic law of cultural development on the natural, social and psychological facets or instrumental genesis (Vygotsky 1971, Yasnitsky, 2011) does catalyze the extension of didactical praxeologies (Winsløw, 2012). Most importantly, the anthropological theory of didactics remains a mirage (Chevallard, 1998; Winslow, 2012; Østergaard, 2013).

A school of thought criticizes didactics of mathematics as being dominated by teacher-centred and lecture-based techniques. Such critics contend that didactics of mathematics promotes memorization of facts and development of superficial conceptual understanding (Forsman, 2015). However, Bishop (2002) and Dunphy et al. (2014) still show much growing interests in the area. They contend that didactics of mathematics is dynamic, objective, and coherent with social constructions and creativity. It is in line with this proposition that the researcher integrates three intersubjective conceptual frameworks and models to extend the triad to the tetrahedron in solving equations of the circle.

Purpose of the Study

The study sought to explore the didactical conceptual structures in extending the didactic triad to the didactical tetrahedron in solving equations of the circle.

Research Objectives

This research sought:

1. To use a baseline survey to explore the feasibility of extending the didactical triad to the didactical tetrahedron and implementing the didactical tetrahedron.
2. To ascertain the impact of the tetrahedron on students' performance in equations of the circle using intersubjective didactical instructional models.
3. To provide explanation of the statistical significances across the intersubjective didactical instructional models using qualitative sources.

Research Questions

1. How does the baseline survey ensure reliability and validity of the study? And what are the possible covariates that affect the interactions of the didactical tetrahedron and what are the dominant factors that determine the interactions of the tetrahedron?
2. What is the knowledge level of students in solving equations of the circle before the implementation of the didactical tetrahedron?
3. What is the knowledge level of students in solving equations of the circle after the implementation of the didactical tetrahedron?

4. What is the relationship between interactions in the didactical tetrahedron implemented and students' performance in solving equation of the circle?
5. How do the qualitative sources help explain statistical significance across the intersubjective didactical instructional models in students' performance in equations of the circle?

Research Hypotheses

The following hypotheses were posed to guide the research question 3:

H_{01}: There is no statistically significant difference in students' performance after interactions in the didactical tetrahedron.

H_{02}: There is no statistically significant difference in students' performance across the didactical instructional models after implementation of the didactical tetrahedron.

Significance of the Study

Didactical conceptual structures of the triad and tetrahedron are didactical instructional techniques grounded in classroom interactions. This boosts teaching and learning processes didactical fields (Van der Zalm, 2010). Through these didactical conceptualizations, teachers explore new mathematics dogmas, theoretical frameworks and methodologies in the classroom instructions. This study would open academia, researchers, teachers and educators up for new inventions and innovations of didactical instructional models. If properly understood, they will use the findings as the basis to develop mathematical knowledge, skills and understandings through conceptualization, mathematisation and digitisation. Particularly, the findings would enable teachers and students to relate didactical theories, conceptual frameworks and models. This would in turn, boost their knowledge, improve upon classroom practices, foster effective learning, and focus on building relations during instructions (Wilson, 2015).

Secondly, the understanding the didactical conceptual structures depend upon diverse experiences, knowledge, skills and competencies in technologies. Technologies describe, discuss, relate, formulate and implement mathematics tasks, applications and pedagogies (Zhou, & Xu, 2014). The findings would advance new emerging conceptualization and didactical strategies. This would help teachers and students build mutual relations and assume collective responsibility during mathematics instructions. The experiential and experimental techniques would ensure that mathematics teaching and learning is carried out devoid of formulas, generalizations, deductions and computer programming. In this respect, the study contributes to the following discussions:

1. Helping teachers and didacticians in mathematics education to acknowledge the ways technology tools contribute to developing didactical conceptual structures in mathematics knowledge, activities, practices and applications.
2. Providing curriculum enrichment programmes using algebraic and geometric technology tools for teaching, learning and communication in the didactical tetrahedron.
3. Baseline for developing validated, reliable and viable instructional models for advancing the course of didactical mediation,

mathematisation and digitisations with respects to didactical praxeology and anthropology models.

4. Providing further discussions on indigenous technology tools. These tools will improve upon skills, competencies and knowledge of teachers and students in simple practical work. In the end mathematics curricula can be aligned with available resources and laboratories in the school and community.
5. Providing mathematics proficiency, knowledge and skills in their interactions. The complex interplay of the relationships and roles among teachers, students, mathematics content and technology tools will boost emerging conceptualization, contextualization and phenomenological discourses.
6. Espousing didactical knowledge, skills, prowess and proprieties in mathematics research and classroom practices. These will provide informed choices of perspectives, theories and models in the intersubjective didactical theories. These interconnected theories will help inform theory, policy and professional practice.

Delimitations of the Study

The study was confined to the following seven areas:

1. *Purpose*: Extending and developing the didactical conceptual structures from the didactical triad to the tetrahedron with technologies in the teaching and learning of elective mathematics in senior high schools in Ghana.
2. *Technology mediational tools*: Using any interactive technology tools and innovative practices for algebraic and geometric enrichment tasks

and mathematics problems. The algebraic tools focus on manipulating the centres and radii of the circle and the geometric tools on diagrams and figures.

3. *Subjects of study*: Using only elective mathematics students in the senior high schools in Ghana who have been given the opportunity to study equations of the circles and other parts of conic sections as enshrined in the mathematics curriculum.
4. *Content of study*: Using the didactical triad and extending the triad to the didactical tetrahedron to establish statistically significant significances in the didactical instructional models.
5. *Research duration*: The gestation of the entire study covers 12 months of 90 minutes bi-weekly sessions with the 500 students and 12 teachers in the 2017/2018 academic year. The academic year commenced in September, 2017 and ended in August, 2018.
6. *Independent Variables*: These variables measured, manipulated or controlled the effects of the didactical instructional models. These variables were the four didactical conceptual structures in the tetrahedron (i.e. *treatment independent variables*) and demographic descriptions such as gender, class/form, name of school, management of school, course/programme of study, circle topics, mathematics resources and computer literacy (i.e. *attribute independent variables*) (Kahn, 2003).
7. *Dependent Variables*: These variables appeared, disappeared, varied, changed or modified as the independent variables measured, manipulated, introduced, removed, changed or controlled the

independent variables during the experiments. The dependent variables were test scores in mathematisation and didactical phenomenology, instrumentations and instrumentalisations, and didactical situations and anthropological praxeologies models. These were followed by the ethnographic-phenomenological interviews transcriptions (Kahn, 2003).

Organization of the Study

In this chapter, the researcher began by providing the state of didactics of mathematics in Ghana, highlighting the various conceptualizations, contextualisation and mathematizations used in didactical conceptual structures. In the statement of the problem, the researcher has discussed the problem of students' poor performance in mathematics in the context of didactics of mathematics in Ghana. This included the fact that didactics of mathematics is almost absent and/or improperly used. It was clear that no research in didactics of mathematics was undertaken in equations of the circle. The fact that WAEC examinations performance in equations of the circle was cited as being problematic to students in Ghana require alternative instructional models. The researcher's conceptualisation and contextualisation in the didactical conceptual structures, the purpose, objectives, hypotheses, research questions and significance of the findings as well as the delimitations of the study have also been discussed in Chapter One.

In Chapter Two, a review of literature in intersubjective didactical theories was used to illuminate the problem and previous research related to the study has been presented. These theories are realistic mathematics education, instrumental genesis and anthropological theory of didactics. Three layers of intersubjective didactical instructional models were carved out of the intersubjective didactical theories. These models were named as mathematisation and didactical phenomenology, instrumentations and instrumentalisations, and didactical situations and anthropological praxeologies. The literature also examined types of models, polygons of didactics, technology tools, mathematics laboratory tools, errors and misconceptions, equations of the circle and gender as covariates in mathematics (Grant, & Osanloo, 2014).

In Chapter Three, the researcher have been present the research methods planned for the study. This includes research design, the population, sample, sampling procedures, instruments of data collection, validity, reliability of instruments, data analysis and ethical issues (UCC, 2016). The implementation of the planned research methods discussed in Chapter Three has been presented in Chapter Four.

In Chapter Four, therefore, how the whole research procedure in Chapter Three have been presented. This started with the baseline study, the main study and the interview transcripts. The discussions used the baseline survey as a yardstick to justify the analysis of the main study. The results of the didactical instructional models were followed by interview transcripts in order to address the research problem.

In Chapter Five, the summary, conclusion and recommendations were drawn from discussions of the findings in Chapter Four. Some suggestions for further studies in didactics of mathematics have been presented.

CHAPTER TWO

LITERATURE REVIEW

In this chapter, the researcher has drawn on three intersubjective theoretical and conceptual frameworks to illuminate the research problem (Davis, 2010; Grant & Osanloo, 2014; Forsman, 2015). These theoretical-conceptual frameworks are realistic mathematics education for mathematisation and didactical phenomenology models, instrumental genesis for instrumentation and instrumentalisation models and anthropological theory of didactics for didactical situations and anthropological praxeologies of didactics models. As the study sought to explore the didactical conceptual structures in extending the didactical triad to the tetrahedron involving equations of the circle, these intersubjective theoretical frameworks were essential (Van den Heuvel-Panhuizen, & Drijvers, 2012).

Particularly, the realistic mathematics education helped to identify rich realistic situations in the teaching and learning processes and situations (Van Den Heuvel-Panhuizen et al, 2016), instrumental genesis helped to identify the socio-cultural signs, symbols, artefacts and tools (Matusov, 2015; Bartolini Bussi, & Mariotti, 2016; Roth, 2016), and anthropological theory of didactics situated the study within the 1T, 2T, 3T and 4T praxeologies(Grønbæk, &Winsløw, 2015; Jessen, Kjeldsen, & Winslow, 2015; Winslow et al, 2016). The review also discussed quantitative models, qualitative models, polygons of didactics, technology tools, mathematics laboratory tools, errors and misconceptions, equations of the circle and possible covariates. These areas helped the researcher to clearly identify the research gaps in literature and better positioned the intersubjective didactical instructional models to fill them.

Intersubjective Theoretical Perspectives

The intersubjective theoretical perspectives peruse the possibility of integrating knowledge from the three didactical theoretical frameworks, the three didactical conceptual frameworks/models, and four didactical conceptual structures (Grant, & Osanloo, 2014; Sinclair et al, 2016). Figure 2 presents a holistic intersubjective theoretical-conceptual framework to carry out the study.

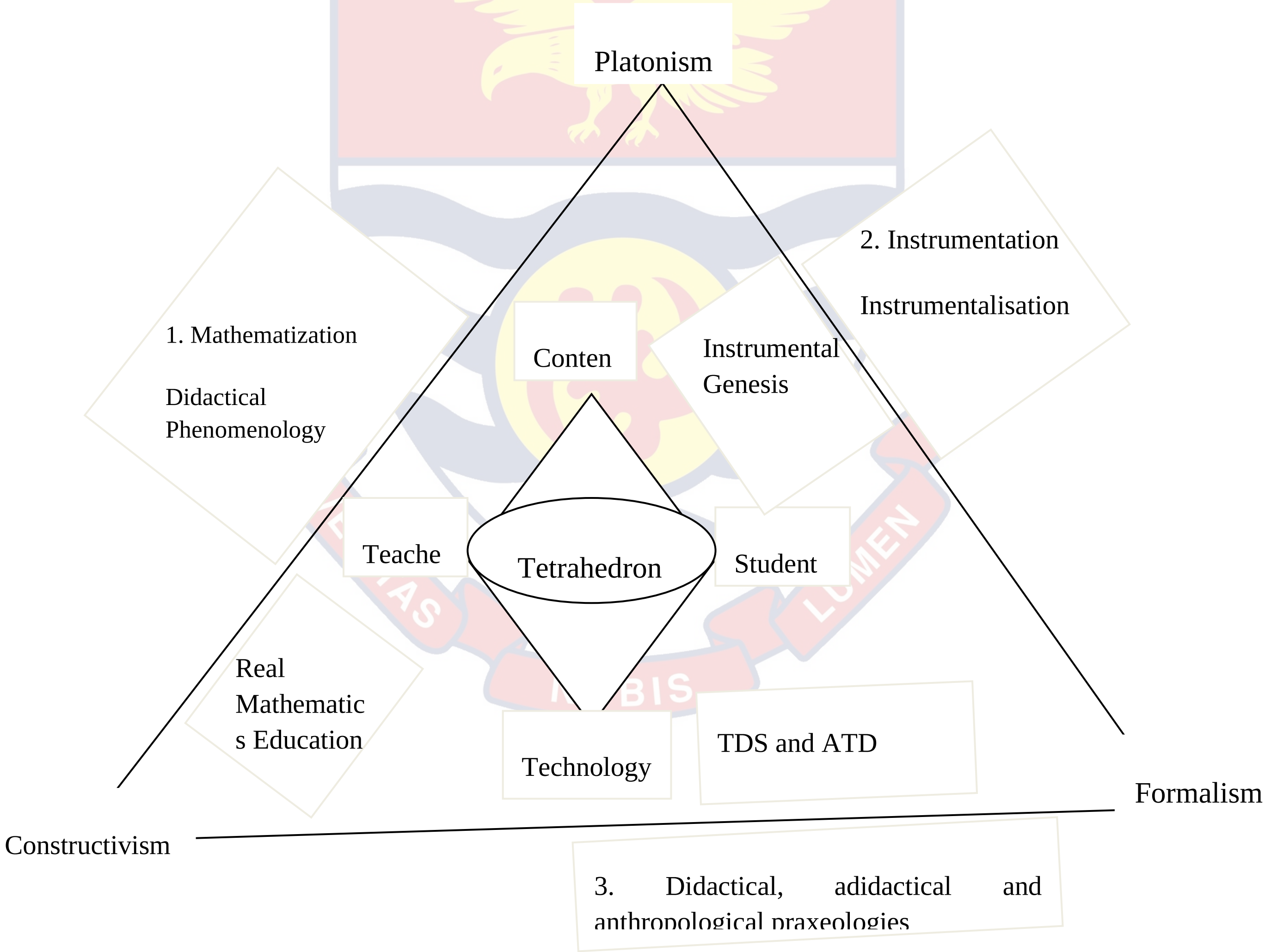

Figure 2: An intersubjective theoretical-and-conceptual framework (adopted from Grant, & Osanloo, 2014)

In Figure 2, the outer apex shows the didactical triad containing the three theoretical perspectives. Theses theoretical perspectives are Platonism, formalism and constructivism (Ernest, 1991; Dunphy et al., 2014; Saric, & Markic, 2015). Platonism or realism situates mathematical structures as being real, objective, static and unified knowledge that exists independent of individuals' knowledge to be discovered rather than created. The formalism or instrumentalism situates mathematical structures as existing in the exact definitions, facts, algorithms, rules, formulae, skills, lemmas, axioms and theorems in transforming teaching and learning of mathematics. The constructivism or anthropologitism positions mathematical structures in the tentative, intuitive, subjective and dynamic in constructing knowledge by problem-solving strategies (Maschietto, & Trouche, 2011; Østergaard, 2013; Dunphy et al, 2014; Travers, & Perry, 2014; Saric, & Markic, 2015; Bartolini Bussi, & Mariotti, 2016; Presmeg et al, 2016; Sinclair et al, 2016).

It is worthy to note the few disagreements among these perspectives. For instance, while the Platonists deny the human dimensions of mathematical constructions of knowledge and the constructivists fail to explain the universality of individually constructed mathematical knowledge. The debates, arguments, experimentations and discussions characterized in the mathematics classroom require merging the research paradigms in modelling the didactical conceptual structures. Teachers' conceptions or sets of beliefs about the nature of mathematics can form the bases of their choices of their philosophies in mathematics education. However, these philosophies must improve students' socio-cultural classroom climate for effective and efficient teaching and learning, co-constructive engagements and critical thinking.

Also, Barbosa, Maldonado and Ricarte (2003) outlines three main situated perspectives, namely, conceptual, instructional and didactic that drives the actual interactions within and between didactical conceptual structures. Conceptual perspectives outline the content description, instructional perspectives deal with the instructional techniques used to perform teaching and learning processes, and didactic perspectives relate conceptual structures to instructional elements in order to achieve the learning objectives previously established. Although each perspective addresses particular teaching and learning paradigm, they are intrinsically related, explicably integrated and covertly explained within. This prowess allows the three didactical conceptual structures (triad) to be extended to the tetrahedron.

In addition, the intersubjective theoretical perspective improves upon the old didactics that concentrated on only Platonist factual information and development of superficial conceptual understanding. In response to the limitations of these teacher-centred and lecture-based perspectives, new didactical research and practice have shifted the focus towards intersubjective experiential and situated models (Forsman, 2015). This requires much more participatory teaching and learning techniques devoid of overgeneralization and oversimplification but rather engage students in the construction, collaboration and presentation of real-world knowledge. Such a view is equally shared by postmodernism or humanism (Dunphy et al, 2014). In this context, the mathematics classroom is positioned as human-centred (i.e. part of human culture), as fallible knowledge (i.e. errors, mistakes, misconceptions and re-corrections), as different versions of proofs (i.e. depend on time and place) and as distinct variety of social-cultural objects (i.e. signs/symbols, artefacts, instruments and technologies).

In effect, the intersubjectivity of the study ensures that:

1. Mathematics knowledge would not be fixed and objective but rather be negotiated by teachers' and students' socio-cultural practices and engagements.
2. Mathematics knowledge is inextricably linked with equity and access, which is coherent with human, socially-constructed and creative views.
3. Mathematics knowledge would be promoted through processes of engaging, connecting, communicating, reasoning, arguing, justifying, representing, solving and generalizing.

Intersubjective Theoretical Frameworks

Intersubjectivity refers to shared understanding by various components of didactical structures. The philosophy of subjectivity holds the notion that meaning is necessarily coloured by one's experiences and biases. This meaning is based on one's position of reference and is socially mediated through interactions (Given, 2012). In other words, since knowing is not simply the product of individual minds in isolation, the intersubjective theories hold the common view that personal experiences are expressed in contexts (Orange, 2009). The main aim is to coordinate the joint interactions of individual component's contributions towards a common goal (Grant, & Osanloo, 2014).

Generally, theories come from a multitude of sources in each discipline, create and apply across fields. In this chapter, a plethora of educational theories could have been selected to form the theoretical frameworks. Some common educational theories are systems, developmental, cognitive, feminist, critical, self-efficacy, functionalist, relational, Marxist, gender, change, community of Inquiry, transformational and intersubjectivity (Saric, & Markic, 2015). The researcher's choices of the three intersubjectivity theories were particularly necessary to support the drive towards the didactical research problem. An intersubjective theory holds the view that personal experience always emerges, maintains itself and transforms in relational contexts. This is because intersubjectivity theories purposively conceptualized the research problem, since personal experiences helped in organizing principles and engagements throughout the research process (Orange, 2009). In this study, the most appropriate intersubjective theories were realistic mathematics education, instrumental genesis and anthropological didactics (Grant, & Osanloo, 2014).

Also, the conceptualization of the intersubjectivity theories in didactical conceptual structures was varied and multifaceted. First of all, they are sets of interrelated concepts, definitions and prepositions that restructure systematic phenomena for the purpose of explaining and/or predicting (Imenda, 2014; Saric, & Markic, 2015). In this case, the three didactical theories provided the best blueprints and guides for the models since:

i. They had a set of interrelated propositions, concepts and definitions.
ii. They had specified relationships and various interrelated concepts.
iii. They have occurrences of events in the specified relationships.
iv. They had common domains to which they are all applicable.
v. They had systems of deductions that generate laws and empirical studies which are confirmed or rejected through the research hypotheses.

vi. They had compatible observations and previously validated in empirical data or hypotheses and incorporated into new structure of ever-greater generalizability.

vii. They had powerful explanatory and predictive potentials and variables.

viii. They had precision, universality, falsification and verifications.

ix. They had operationalizable precision and replicability.

x. They had simplified terms, law of parsimony and data adequately.

Again, the conceptualization of the intersubjectivity theories in the didactical conceptual structures was carefully outlined within specific mathematics domain in order to explain how and why relationships and interrelationships in equations of the circle were systematically, sequentially and logically streamlined (Imenda, 2014; Forsman, 2015). For instance, the three theories have clear, precise and concise picture of events in the following major domains:

i. They possessed uniqueness–were distinguishable from others.

ii. They possessed conservatism–persisted until superior theories replace them.

iii. They possessed generalizability– the greater the area, the more powerful.

iv. They possessed fecundity – they were more fertile in collectively generating new models and hypotheses better than using only one of them.

v. They possessed parsimony–other things being equal, the fewer the assumptions the better the theories.

vi. They were internally consistent–they identified all the relationships and interactions in the didactical conceptual structures with adequate explanations.

vii. They possessed empirical riskiness– any empirical tests of the theories were risky, and refutations were possible for good theoretical synchrony.

viii. They possessed abstraction– they were independent of time and space, and achieved by adding more relationships in the didactical conceptual structures.

Moreover, the conceptualization of the intersubjectivity theories in the didactical conceptual structures was scientifically well-substantiated in the natural world, and the body of facts had been repeatedly confirmed through three stages of data collection (Creswell, 2014; Imenda, 2014; Forsman, 2015). Thus, the three theories provided factual and comprehensive explanation of the research and accepted facts in the following ways:

i. They provided explanations for the relationships among variables being tested in the quantitative quasi-experimental design.

ii. They served as lens for the inquiry and generated new inquiry during the qualitative interview transcripts.

iii. They employed many ways, associated with quantitative and qualitative data to help, consider, plan and incorporate different ideas in the mixed study.

Furthermore, the conceptualization of the intersubjectivity theories in the didactical conceptual structures provided formal language and sets of axioms derived from theorems. There were consistent with knowledge and practices in countable and/or uncountable equations of the circle (Chevallard, Bosch, & Kim, 2015) because they contained:

i. Mental conceptions, reflections and considerations.
ii. Coherent statements or sets of ideas, laws, principles and hypotheses that explained observed facts or phenomena.
iii. Underlying principles that give technical skills to its practices.
iv. Fields of studies that exhaustively described particular contents and constructs.
v. Hypotheses or conjectures in the research problem.
vi. Sets of axioms derived from all statements.
vii. Formal languages and sets of axioms consistent with the models.

In addition, the conceptualization of the intersubjectivity theories in the didactical conceptual structures provided guidance for the researcher's study questions and research for measuring selected questions in equations of the circle (Imenda, 2014). In this case, the three theories guided the research problem, the hypotheses, data collection, analysis and discussion. The theories helped check whether the findings agreed with empirical research and/or whether there were some discrepancies. In situations where discrepancies were observed, further questions in the interview guides explored whether or not there were alternative ways of answering the problem. This means the intersubjectivity theories built and expanded the discussions of didactical conceptual structures (Bartolini Bussi, & Mariotti, 2008; Saric, & Markic, 2015; Bartolini Bussi, & Mariotti, 2016) in the tetrahedron.

Lastly, the conceptualization of the intersubjectivity theories in the didactical conceptual structures successfully amalgamated the three theories (Corcoran, 2012; Dunphy et al, 2014; Grant, & Osanloo, 2014; Imenda, 2014; Saric, & Markic, 2015). They provided key pointers to concomitant conceptual models, modified behaviours cognitive structures, and selectively epitomized constructionism and sociocultural in the tetrahedron. The explanatory schemes achieved clarity, key issues and systematic constructions of knowledge in the social world (Radford, 2008; Radford, 2014). The concepts, systems, models, structures, beliefs, ideas and hypotheses guided statements about particular types of actions, events, activities and analyses of causes, consequences and processes explain the didactical philosophy, sociology and psychology of the didactical conceptual structures (Radford, & Sabena, 2015).

Theory of Realistic Mathematics Education

Realistic Mathematics Education (RME) was one of the intersubjective theoretical frameworks the researcher employed to study the didactical conceptual structures. RME is a domain-specific teaching, learning and instruction theory that provides rich realistic situations and contexts in the teaching and learning processes. The theory holds the view that mathematics must be connected to reality (Freudenthal, 1983, 1991). These situations serve as sources to the development of mathematical concepts, tasks, procedures and contexts (Van den Heuvel-Panhuizen, & Drijvers, 2012; Matusov, 2015).

In this theory, students offered a variety of problem situations to explore and construct knowledge from the real-world situations with either fantasy fairy tales or formal world of mathematics, so long as the problems were experientially real. Even the socio-constructivist dimensions embedded in

RME provides compatible and complementary collaborations, critical roles of classroom cultures and mathematics discourses (Bishop et al, 2002). In constructions and reconstructions, RME explains how four interrelated approaches of social problems could be transformed into realistic mathematics problems (Van Den Heuvel-Panhuizen et al, 2016).

1. Mathematisations

The two main criteria in RME are horizontal and vertical mathematisation and the four approaches are mechanistic, structuralist, empiricist and realistic. Both the criteria and approaches involved the conceptualizations, constructions and executions of RME in the didactical conceptual structures. Horizontal mathematisation ensured transfer of real world problems to mathematically stated problems, from the world of life to the world of signs and symbols for students, teachers, mathematics content and technologies to act (and suffer). In horizontal mathematisation, the four didactical conceptual structures were strongly and actively engaged in schematizing, formulating, visualizing, discovering recognizing and transferring real world problems and mathematical problems. Concurrently, the vertical components were responsible for mathematical processing and refurbishing of the real world problems and mathematics problems; shaping, reshaping and manipulating the signs, symbols, artefacts and tools comprehensively and reflectively; strongly representing, providing, refining, adjusting, combining and integrating relations, regularities and transitional models. The core aim was to formulate new mathematical concepts and generalize the results (Van Den Heuvel-Panhuizen et al, 2016).

Again, each criterion of mathematisation have to be coordinated by the four approaches, namely mechanistic, empiricist, structuralist and realistic (Van den Heuvel-Panhuizen, 2010; Van den Heuvel-Panhuizen, & Drijvers, 2011; Viirman, 2014; Van den Heuvel-Panhuizen et al, 2016). The mechanistic phase sets up the systems of logic, deductions, rules and formulas in the equations of the circle for the four conceptual structures without applications. So, no real phenomenon is incorporate into any kind of mathematisation and hence it is neither horizontal nor vertical. The empiricist sets up applications, methodologies, structures, interrelations and insights into the horizontal mathematisation phenomena more than vertical.

The environment is its key focus rather than mental manipulations. The structuralist organizes logical, closed deductive, procedural and algorithmic procedures on vertical mathematisation and not horizontal mathematisation. In the structuralist, signs/symbols, artefacts, instruments and technology tools provide an opportunity to students encounter real-world experiences but does not systematize and rationalize the experiences. It is the realistic that fully incorporates both vertical and horizontal mathematisation. Because of this novel quality, realistic support and incorporate both mathematizations in the didactical conceptual structures (Van den Heuvel-Panhuizen, & Drijvers, 2011; Van den Heuvel-Panhuizen et al, 2016).

2. Didactical phenomenology

In the discourses of the realistic approach, Freudenthal (1983) principle of didactical phenomenology creates and extends concept formation, model formation, applicability and practice in the didactical conceptual structures. The phenomenological activities are the axiomatising and formalizing processes (Billington, 2010). The processes of formalization involve modelling, symbolizing, schematizing, defining, generalizing, organizing and structuring knowledge and skills in creating regularities, relations and conceptual patterns. These processes organize and create phenomena that relate the teaching and learning processes to the didactical conceptual structures and abstracting the students' concrete and progressive mathematization. This connects mathematical tasks to real-life situations, reflects the various mathematical contexts (i.e. patterned problems, realistic contexts, authentic contexts and recreational or professional practices). In this way, teachers guide students to interpret mathematical phenomena, reasoning, computations and instructional techniques to support individual, small group and whole-class discussions. Students can also engage in and create congenial classroom environments to collectively obtain experientially real problems.

Again, the didactical phenomenology synchronizes mathematical concepts, structures, and ideas to transcribe students' accounts and transform mathematisation approaches into procedural, conceptual, and dual processes (Gravemeijer, 2008). The procedural activities are mechanistic and involve the memorization of mathematical facts, operational procedures, recognized problems and applications algorithms. The conceptual activities establish conceptual understanding and applications and flexible (mental) computations to make estimations. The dual activities synchronize the tasks and equations in order to develop new techniques that realistically interact across the various six basic principles of mathematisation.

Another area of mathematisation is the six principles of mathematization. The six principles of mathematisation are activity, level, interactivity, intertwinement, guidance and reality. The activity principle interprets the activities of students and teachers in both formal and informal ways. The interactivity principle critically reviews the mathematics tasks and devises solution paths along the higher levels of mathematical reasoning. The guidance principle restructures and reshapes classroom instructional strategies to facilitate discussions and applications to daily-life problem situations. The level principle structures informal situations and problem contexts into domain-specific situational knowledge and strategies in order to connect situations, transitions, relationships, contexts and procedures. This helps in discovering tackle new formal mathematical reasoning, reflection and appreciation. The intertwinement principle integrates networks of ideas, areas and themes in the mathematisation discourses. Even though each principle was important, it was the reality principle that essentially mathematised both horizontal and vertical facets. This principle helped the researcher to explore, investigate, solve and analyze experiential, contextual and real problems in the research (Artigue, & Blomhoj, 2013; Boon, Doorman, & Drijvers, 2013; Godino, Batanero, Canadas, & Contreras, 2014; Ndlovu, 2014).

Theory of Instrumental Genesis

Research (Vygotsky, 1971, 1978; Bishop et al, 2002) shows that the theory of instrumental genesis shares its origin from the Vygotskian approach. The theory of instrumental genesis holds the views that indigenous artefacts can be transformed into technology tools (Rabardel, 1995). The theory of instrumental genesis explores the interactions between human knowledge and technologically mediated tools in the conceptualizations, constructions and executions of knowledge (Meyer, 2012; Bartolini Bussi, & Mariotti, 2016). In this theory, signs, symbols, artefacts and tools play essential roles in transitioning and coordinating the didactical conceptual structures. The theory of instrumental genesis holds the view that social and cultural knowledge are inseparable and indispensable in explaining these roles and functions of signs, symbols, artefacts and tools (Jones, &Megeney, 2016). This theory further transforms the intersubjectivity discourses, constructions and mental activities enshrined in the research problem (Maschietto, & Bartolini Bussi, 2007; Maschietto, & Trouche, 2011).

Also, the theory of instrumental genesis helped students to internalize the cultural contexts of signs, symbols artefacts, instruments and technology tools (Dunphy et al., 2014). The key role of the theory in the involvement, appropriation and transformation fully developed mathematical objects (Nyamapfene, & Lynch, 2016). The multidisciplinary and interdisciplinary nature of the theory invites creative thinking, cognitive functions and cultural developments in the Vygotskian circle (Yasnitsky, 2011). The Vygotskian circle comprising clinical and special education; language, thinking and cultural philosophies, and affect, will and action link students' minds, emotions and phantasy to the signs, symbols, artefacts and tools (González Rey, 2011). This further creates cultural techniques and auxiliary cultural knowledge and practices in mathematics (Radford, & Sabena 2015; Presmeg, Radford, Roth, &Kadunz, 2016; Roth, 2016). In this way, unity is strengthened between cognition and emotion in one hand, and social interactions and subjective experiences on the other.

Again, the theory of instrumental genesis cognizes the development of student's ability to learn the socially relevant tools and link them to culturally based signs (e.g., symbols, words and number systems) in equations of the circle. The interactions with other students and teachers foster cultural mediation, social experiences and mental functioning. Consequently, such knowledge becomes the product of social experiences, socio-cultural practices and shared representations with the *dark rectangle*. The dark rectangle comprises external socio-cultural experiences, internal mental structures, social, cultural and contextual structures, and activity-based tasks. This synergy normally ensures that the viability of any newly co-constructed knowledge is a product of the four didactical conceptual structures (Doolittle, 2014).

In addition to the dark rectangle are the six cultural principles that shape students' interactions within the didactical conceptual structures (Doolittle, 2014; Dunphy et al, 2014). These principles are cultural-historical, situative, cognitive, constructive, constructionist and mediative (Sriraman, &Haverhals, 2010; Yasnitsky, 2011; Godino et al, 2014). The cultural-historical activity principle set up neat propositions to promote cultural diversity, identity, situations, communal activities, indigenous mediation and shared learning. The situative principles provides time, space, social and culture to gain insights in the social contexts, social practices, social engagements, classroom interactions and collaborations.

The cognitive principle helps students to relate mathematics knowledge to specific equations of the circle. The constructive principle helps in active constructions of knowledge through processes of assimilation and accommodation (Mariotti, & Maracci, 2009). The constructionist principle mediates between physical artefacts (i.e. calculators, computers and technology tools), symbolic resources (i.e. signs, symbols, natural language and words) and technical procedures (i.e. mathematical algorithms and procedures) in equations of the circle (Bartolini Bussi, & Mariotti, 2008).

The mediative principle digitizes the physical artefacts, the symbolic resources and the technical procedures employed to solve the problem. In doing so, the principle directly provides effective links between the mediators (i.e. technology tools), mathematics content (i.e. equations of the circle) and mediatees (i.e. students). This fosters the relations between teachers, students, and mathematical content and extended the relations to the didactical tetrahedron (Lewis, 2015; Bartolini Bussi, & Mariotti, 2016).

Instrumentations and instrumentalisations

Instrumentation and instrumentalisation models arise out of complex phenomena and socially situated human subjects (i.e. teachers and students) (Van den Heuvel-Panhuizen et al., 2016). In order to build utilization schemes necessary for advanced technology tools, the researcher required the instrumentation and instrumentalisation models. This model seeks to differentiate between artefacts and their uses for solving mathematics tasks, and instruments and their uses for measuring mathematics outcomes. In the context of this study, artefacts are the materials or symbolic objects designed for specific purposes, and instruments are schematic components used to identify and solve problems (Vu-Minh, Boileau, & Herbst, 2015).

Secondly, the instrumentation and instrumentalisation models emerge from the phenomenological discourses discussed earlier. In this model, they seek to shape mathematics tasks in the didactical conceptual structures in the tetrahedron (Leung, Chan, & Lopez-Real, 2006). This makes the transformation of the signs, symbols, artefacts, tools and other psychological constructs into technology tools much easier (Drijvers et al., 2010). In this regard, the artifacts can be formed by the instruments as psychological constructs. So, the didactical triad can be transformed to the tetrahedron with technology tools. The concentration is on technology tools as the sources of the phenomenological experiences and discourses (Trouche, 2014).

Furthermore, the operationalisations of instrumentations and instrumentalisations models conceptualize signs, symbols, artefacts, tools and instruments as technology tools (Trouche, 2014). In this study, the models help to employ and utilize calculators, computers, graph sheets and metre rules in solving tasks in equations of the circle(Maschietto, & Trouche, 2011; Lewis, 2015). The culturally and socially-oriented activities associated with the mathematics discourses, rules of actions, operations and algorithms (Gueudet, & Trouche, 2010; Radford, & Sabena, 2015) are guided and facilitated the ways teachers use the artefacts (i.e. instrumentalisation) and the ways students applied them (i.e. instrumentation).

Again, instrumentations and Instrumentalizations models analyze the new intermediary situations between the psychological objects and the socio-cultural contexts of the calculators, computers and non-material cognitive tools (Drijvers, & Trouche, 2008; Radford, & Sabena, 2015). So, these mediators were relevant in providing meaningful coordination, relationships and interactions in performing specific tasks in equations of the circle.

Criteria of instrumentations and instrumentalisations

In tandem with the RME models, the theory of instrumental genesis equally has two criteria, namely instrumentations and instrumentalisation, similar to the horizontal and vertical mathematizations. This helps to conceptualize, construct and execute tasks in the didactical conceptual structures. In instrumentation, students and teachers require knowledge and applications of the instruments and their links to utilization schemes. In instrumentalisation, students and teachers discover the mediators and their uses in building cognitive structures. The instrumentations require in equations of the circle emerge the schemes and techniques while the instrumentalisations transform the mediators during the interactions among the four didactical conceptual structures. In other words, instrumentations provide feedbacks from the situations, actions, discourses and activities in equations of the circle (Sabra, Emprin, Connan, &Jourdain, 2014; Radford, & Sabena, 2015).

In addition, instrumentations orchestrate the phenomenological experiences and discourses normally directed towards the students while instrumentalisations guide the stages of discovery and selection of relevant strategies, algorithms, solutions paths and directions (Gueudet, & Trouche, 2010; Fiorani, 2012). In this study, instrumentations of actions over the didactical conceptual structures and their associated usage emanated from teachers' analyses and observations of the technological tools (i.e. videos, interactive explanations, interactive exercises, pre-and-post tests and global tests). This ensured that the instrumented actions build didactical exploitations of the phenomena. On the other hand, instrumentalisations build actions for the mediators and associated usages (Trouche, 2014). In this study, the discourses from students to teachers' instrumentations, Instrumenting mathematics activities were the mathematical contents, textbooks, tasks and resources.

Kinds of instrumentations and instrumentalisations

Similar to the four approaches of RME are the four kinds of mediators, namely signs-symbols, artefacts-manipulatives, tools-instruments, and technologies/innovations. Just as the four approaches contextualize and operationalise RME in the didactical conceptual structures, the four kinds of mediators recontextualise and re-operationalise instrumental genesis (Roth, 2016; Ali, Davis, & Agyei, 2018). Signs-symbols are inactive in both instrumentations and instrumentalisations, artefacts-manipulatives are more active in instrumentalisations than instrumentations and tools-instruments are more prevalent in instrumentations than instrumentalisations. Technologies-innovations cut across instrumentations and instrumentalisations (Matusov, 2015). Teachers and students applied these pairs of mediators to attain mathematical achievements that would otherwise have remained out of their reach. In literature (Bartolini Bussi, & Mariotti, 2016; Roth, 2016), all mediators are psychological tools that drive both outward and inward learning processes to measure complex didactical conceptual structures. In this study, numbers, numerals, diagrams, figures and shapes were classified as signs and symbols of external representations that developed students' cognitive structures during instrumentalisations.

Again, the understanding of the roles of the mediators based on the socio-cultural contextual constructions of knowledge, students and teachers, through their experiences and behaviour patterns between the mediators and the external dimensions of reality (mathematics knowledge) improved their logical reasoning and internal cognitive thinking (Matusov, 2015; Roth, 2016). This was manifested in the psychological tests, where students steadily recalled, remembered, compared, reported and chose best and most appropriate methods, strategies and procedures of solving tasks in equations of the circle.

This improved their physical manipulations, interpersonal interactions, communication, collaborations and problem-solving skills. Tools-instruments equally functioned prominently in the constructions, reconstructions and uses of the tools and instruments (Fiorani, 2012). As pointed out by Bartolini Bussi, and Mariotti (2016), this equally contributed significantly in developing the cognitive learning processes in solving practical mathematics tasks, modifying the original forms of the mathematics tasks and developing further digitisation discourses for technologies-innovations.

Digitalisation of instrumentations and instrumentalisations

Digitisation of instrumentations and instrumentalisations further boosted the didactical relationships between teachers, students and mathematics content in the triad and transformed the signs-symbols, artefacts-manipulatives and tools-instruments into technologies/innovations. Alongside utilization schemes and situated abstractions, digitisation modernizes Vygotskian processes of internalizations and reshaping new the mathematics constructs (Haspekian, 2012).

In digitisation, the mediators that became instruments for students' mathematical activities equally became instruments for teachers' didactical activities (Maxwell, 2005; Billington, 2010). While the teachers instrumentalised the technology tools, the students instrumented the technology tools so that they benefited from the guidance, actions, set objectives and didactical configurations. In the end, the digitisation enhanced the representations, applications and developments in the didactical conceptual structures (Drijvers et al, 2010).

Theory of Didactical Situations

The theory of didactics situations holds the view that the art of conceiving and conducting conditions should determine teaching and learning. Didactical situations are systems of interactions that renegotiate previous personal knowledge on the basis of new mathematical phenomena (Brousseau, 1998). The previous mathematical phenomena originate from proofs, justifications, axioms, theories, definitions, relations and generalizations, and these are modified by sets of predetermined or personal knowledge. This requires different representations, specific beliefs, theories and new sets of proofs, justifications, axioms, theories, definitions, relations and generalizations (D'Amore, 2008). In this study, the theory of didactical situations (TDS) orchestrated mathematics practices, conditions and objects so that the concepts and methods, proofs and verifications, and problem-solving strategies were appropriate synchronized.

Again, in its original sense, TDS contextualizes the didactical triad of which teachers bring out students' changed behaviours and knowledge in the mathematics content (Brousseau, 1998; Bishop et al, 2002). By solving the mathematics tasks and problems, students' reactions and responses equally bring out teachers' knowledge and instructional strategies in the mathematics content. The mathematics content then in turn, helps teachers' to plan and derive relationships for the students to learn (Wilson, 2015). By extending the triads to the tetrahedron, the technology tools recontextualise, depersonalize and detemporalised each structures of the triad. This brings additional actions, recontextualisations and depersonalisations in the triad (Radford, 2008).

Consequently, the technology tools make judicious choices of mathematics tasks and provide concrete epistemological experiences from more general classroom teaching and learning instructions to specific ones. This helps students to gain deep mathematical knowledge and understanding (Winsløw, 2012). In this study, TDS offered the didactical conceptual models new perspectives and helped students to gain new knowledge, experiences, skills and practices. The students simply reformulated, reproved, reconstructed, reconceptualised and readapted innovative emerging learning situations.

Also, TDS differentiates between didactics and experimental designs from didactics and epistemological apriori analysis (Godino et al, 2012). The technology tools transform the didactical and adidactical situations within the remix of the three didactical contracts. These didactical contracts are macro-contract (i.e. concern teaching objectives), meso-contract (i.e. concern activities) and micro-contract (i.e. concern mathematical content) (Artigue, 2011). In this study, by modifying and recontextualising the triad, errors and misconceptions were exposed and corrected (Brown, 2008). Ernest (2018) opines that errors breed ignorance, uncertainty, chance and past knowledge. Having properly and systematically removed them, three essential obstacles were checked. These obstacles were ontogenic (i.e. developmental obstacles related to mental stages), didactical (i.e. instructional obstacles to choices of alternative instructional approaches), and epistemological (i.e. instructional obstacles to the constructions of concepts) (Brown, 2008).

Finally, TDS contextualizes the triad in constructivist epistemology, didactical situations, adidactical situations, didactical contracts, and didactical mediators (Billington, 2010). The teachers' roles changed, modified and extended the didactical triad. In this study, the didactical contracts constructed the new mathematical knowledge and provided alternative kinds of mathematical knowledge. The new and different kinds were facilitated and propelled by the technology tools. These helped the students to counterbalance learning environments, continuously adapt circumstances and new adidactical situations. These built implicit theories and frameworks, formulated new explicit mathematical theories and relations, validated new mathematics tasks, and institutionalized new knowledge and concepts (Godino et al, 2012).

Anthropological theory of didactics models

In conjunction with the theory of didactical situations, Chevallard (1998) theory of anthropological didactics (ATD) was reorganized and recontextualised to investigate human mathematical activities from students' socio-cultural and sociological settings. The activities required were tasks, techniques, theories and technologies (Artigue, & Winsløw, 2010; Ali, Davis, & Agyei, 2017; Winsløw, Gueudet, Hochmuth, &Nardi, 2018). These tasks, techniques, theories and technologies are called didactical praxeologies (Jablonka, & Bergsten, 2010; Grønbæk, &Winsløw, 2015; Jessen, Kjeldsen, & Winslow, 2015; Winslow et al., 2016). The tasks and techniques fully utilize the theories and technologies in the didactical contracts.

The sociological strategies analyze the different situations, basic principles of didactic transpositions, pre-existence mathematics knowledge, constraints of the didactic systems and scholarly mathematics mathematical processes and classroom practices. In this study, Maschietto and Trouche (2011), and Østergaard (2013) didactical transpositions of external didactic, internal didactic and interactive helped illuminate the problem. The external didactic transposition surveyed the outside the schools' noosphere, the internal didactic transposition concentrated on the inside the schools' mathematics practices, and the interactive modelled the classrooms' teaching and learning situations among the didactical conceptual structures. This reinforced the didactic moments that built strong and formidable two praxeological models in the didactical conceptual structures (Oerback, 2008; Winsløw, 2011; Sriraman, & English, 2010; Østergaard, 2013; Winslow et al, 2016).

Didactical and adidactical situations

Brousseau (1998) framework of didactical and adidactical models mimic a two-way contract between teachers and students in creating a meaningful didactical milieu. The didactical milieu comprises the mathematics content and technologies. The teachers' didactical situations, students' adidactical situations and the didactical milieu form a didactical triad. In order to ensure interactions in this triad, five phases of a didactical game are required. These phases are devolution, action, communication, validation and institutionalization. Devolution enables teachers formulate tasks, action enables students to take up the tasks, communication enables students transform the tasks into words and dialogue, validation equips students to test their solutions, and institutionalization bring together teachers, students, mathematics content and mediators (Bishop et. al, 2002; Kohanová, 2006; Petersen, 2010). In this study, the teachers reformulated the milieu, tasks, didactical contracts, experiments and psychological tests. Students' adidactical situations equally consummated their teachers. The milieu became the objects and materials used for the interactions (Florensa, Bosch, & Gascón, 2015).

Also, didactical and adidactical models operationalise the adidactic situations in which students construct mathematics without the influences of specific teachers' didactic conditions (D'Amore, 2008). Students' creations, organizations and usages of mathematics tasks in constructing and reconstructing mathematical knowledge produce and reproduce emerging sets of facts. Students' adidactic engagements draw mathematics knowledge from their own experiences, by interacting with mathematics content and mediators. By viewing mathematics in the phenomenological world, students make apriori hypotheses in intertwining complex interactions of assimilations and accommodations. The contradictions, difficulties and disequilibria experienced in assimilations and accommodations finetune their didactical situations (Kohanová, 2006; Winsløw, 2012; Østergaard, 2013; Jahnke, Norqvist, & Olsson, 2014). In this study, teachers' mathematics tasks provoked and arouse students' curiosity and motivations to construct mathematics knowledge.

Furthermore, Chevallard (2002) didactical and adidactical models conceptualize and contextualize knowledge and experiences in the original settings of students and teachers. The students' adidactical situations help them to construct the didactical situations with signs, symbols, artefacts, tools, instruments and technologies (Winslow, 2011). The technologies help extend the triad to the tetrahedron. Concurrently, teachers' didactical situations help them to arrange devolutionary tasks, relationships, contracts and mathematics contents (Kohanová, 2006). In this study, the teachers' didactical contracts steadily transferred the tasks to the students' adidactical situations in the following ways:

1. Teachers created enough conditions for the appropriations.
2. Teachers proposed new conditions with new solution paths.
3. Teachers ensured that students satisfied and completed their contractual conditions.
4. Teachers motivated students to continue the contract until otherwise stated.
5. If new learning did not occur, teachers reprimanded students with more re-engagements, reconstructions, probations, demotions and even complete withdrawals (Winslow, 2012).

These didactical and adidactical models provided the interplay between students and teachers, the milieu and the mathematics content in equations of the circle. The whole didactic system helped students to construct and reconstruct personal mathematics knowledge from their teachers' mathematics lessons. Teachers operationalised the didactical contracts with implicit instructions and guidelines. The mathematics content provided the expectations and obligations of the didactical and adidactical interactions. The technology tools models gave new phase lift to the three in solving equations of the circle (Winslow, 2009).

Anthropological praxeologies

In this study, the researcher ended the intersubjective models at the anthropological praxeologies. Anthropological praxeologies of didactics are inseparable from epistemological issues and pertain to the roles of human actions, reasoning, discourses and praxeologies (Chevallard, 1998). These praxeologies comprise the practical (know-how or praxis) comprising tasks and techniques and discursive (know-why or logos) comprising technologies and theories. However, praxes are not independent of the discourses (Chevallard, Bosch, & Kim, 2015). This is because they have meeting points during justifications and explanations of the chosen tasks, techniques, technologies and theories. In this context, technologies became the centre of the tasks, techniques and theories. Technologies also became vital in the processes of knowledge productions, explanations and justifications in extending the didactical triad (i.e. tasks, techniques and theories) to the tetrahedron (i.e. tasks, techniques, theories and technologies). This boosted the instructions and learning of tasks in equations of the circle especially the centres and radii (Florensa, Bosch, & Gascón, 2015; Ali, Davis, & Agyei, 2017).

Secondly, anthropological praxeologies models employ the techniques to solve the tasks and the technologies to solve theories (Garcia, Gascón, Higueras, & Bosch, 2006). As one progresses in the hierarchies of mathematical from pedagogical knowledge (i.e. didactical praxeologies) and students' learning (i.e. students' didactical praxeologies) to teachers' teaching (i.e. teacher's didactical praxeologies), there is the need to describe, analyze and extend the 3T-models (i.e. Tasks, Techniques and Theory) (Jahnke, Norqvist, & Olsson, 2014; Ali, Davis, & Agyei, 2017). In adding technologies, the anthropological praxeologies models generate unique types of tasks. The new emerging tasks justify, explain, connect and produce the different techniques required to coordinate, integrate and articulate mathematical theories and technologies (Artigue, & Winsløw, 2010; Østergaard, 2013).

Furthermore, anthropological praxeologies models reformulate the human practices (*prax*) and knowledge (*logos*) in interrelating, cognizing, reasoning and reassigning roles to the triad's 3T-models (T, τ, θ). This can be extended to the tetrahedron's 4T models $(T, \tau, \theta, \Theta)$, consisting of tasks (T), techniques (τ), technologies (θ) and theories (Θ) (Chevallard, Bosch, & Kim, 2015). In the tetrahedron's models, the tasks provide problems, instructions, courses of actions and blueprints to the triad, techniques reformulate and synchronize the axioms, theorems, relations, rules and generalizations to solve the tasks, theories pool the algorithms, explanations and arguments logically to strengthen and back up the techniques, and technologies provide programmes, algorithms and procedures to the whole triad (Jablonka, & Bergsten, 2010; Østergaard, 2013). In these models, the tetrahedron analyzed these relationships in the didactical conceptual structures. Students situated and solved the tasks, teachers' questions and experiments provided new methods and strategies, mathematics content contextualize the roles, obligations, and autonomy of teachers and students, and technologies systematically governed, funded and organized the triad (Artigue, & Winsløw, 2010).

Moreover, anthropological praxeologies models reconstituted the technical-practical block (i.e. tasks and techniques) and technological-theoretical block (i.e. technologies and theories) to solve mathematics problems (Jessen, Kjeldsen, & Winslow, 2015). The extended didactical praxeologies in the 4T-models explain research hypotheses in the independent and dependent variables. Teachers enacted the discourse hypotheses in the 3T-models as $[T / \tau / \theta]$ consisting of tasks (T), techniques (τ) and theories (θ). The relationships in the didactical triad mean the probability of T depends on $[\tau / \theta]$

, the probability of τ depends on $[T/\tau]$ and the probability of t depends on $[T/\theta]$. In extending the $[T/\tau/\theta]$to 4T-models $[T/\tau/\theta/\Theta]$, where (Θ) represents technologies, then the successes and/or failures of the whole of $[T/\tau/\theta]$ depends on Θ. In these models, the extended multiple-chain of conditional probabilities involving $[T/\tau/\theta/\Theta]$, segmented the practico-technical block or the praxis in part I as $[T/\tau]$ and technologico-theoretical block or the logos in part II as $[\theta/\Theta]$. This culminated in a mutually exclusive events of $praxis + \log os$ dialectically denoted by $\ell = I + II = [T/\tau] + [\theta/\Theta]$. It is therefore deduced that the effect of teaching and learning involving the didactical conceptual structures do not only rest on the triad but also on the tetrahedron (Chevallard, 2012; Otero, Gazzola, Llanos, & Arlego, 2016).

Quantitative and Qualitative Models

Walliman (2011) has categorized quantitative models into diagrammatic, physical and mathematical (or simulation) models means that no single models can adequately solve a research problem. According to Walliman (2011), diagrammatic models show interrelationships of the variables and make links between the variables. This simplifies complicated situations and interrelationships. The physical models are normally two-dimensional and three-dimensional representations of objects to show the effects of different inputs into a system. This helps to predict the resultant and combine outcomes. The mathematics models provide deterministic and stochastic dimensions of a research problem. The deterministic dimension helps to discover the predictability of the inputs within a closed system and the stochastic dimension unearths the chances or influences outside a system. In these intersubjective models, the purposes, the complexity of the real situations and the assumptions of the content and scope of the didactical conceptual structures ensured complete interactions in the didactical conceptual structures.

Secondly, the quantitative models arise from interval and ratio scales. This helps to make comparisons, relationships and forecasts. These models suggest several interpretations to a research problem, evidences and interpretations. If necessary, new relevant evidence and interpretations are required to cross examine the quality and sources of the previous evidence and interpretations. This ensures accuracy and consistency, logic and validity, and good conclusions (Walliman, 2011). In these intersubjective quantitative models, having reviewed the time-ordered, conceptually-ordered, role-ordered, partially-ordered, case-ordered and meta-ordered, the time-ordered displays helped record sequences of events in chronology of names, times and locations. The conceptually-ordered explained the abstract concepts related to the three intersubjective theories and their relationships between the variables' causes-and-effects. The role-ordered showed students' roles and relationships in the tetrahedron and the partially-ordered analyzed 'messy' situations, context charts or networks to make decisions. The case-ordered arranged cases in order of importance to compare and arrive at the best solutions and meta-displays amalgamated and contrasted data from each case to compare the data across the variables (Walliman, 2011; Kothari, &Garg, 2014).

On the other hand, qualitative models have been categorized into ethnography, phenomenology and ethnographic phenomenology. In the didactical conceptual structures, the ethnography studied the classroom cultures and behaviours of the students and teachers. The phenomenology examined the phenomena, events, interactions and commonalities through eye witnesses. The ethnographic phenomenology combined and integrated culture with phenomenological texts, documents, discourses and transcripts to observe patterns of behaviours over time (Morrell, & Carroll, 2010).

Kothari and Garg (2014), and Meyer (2015) have reformulated qualitative models into interrogative insertion, problem–solution discourse, membership categorization, semiotics, narrative analysis and discourse analysis. In these models, the interrogative insertions devised techniques to uncover the logic or lack of logic of the discourses, directions and arguments. The problem–solution discourses developed the interrogative insertions to closely report, instruct and uncover the sequences of arguments, responses, results and the conclusions. The membership categorization techniques analyzed the ways students expressed common views and related their views to the different socio-cultural phenomena in their schools and classes.

In addition, the intersubjective qualitative models examined the visual, audio-visual and written texts of participants in order to gain deep understanding of the conceptual terms. In these models, the meaning circle was compared with different features of the ellipse, the meanings of Centre of equations of the circle was embedded in the Systems of Foci, and the different signs and symbols (e.g. ‘signifiers’ as vehicles, ‘signified’ as particular instances, and ‘sign system’ as artefacts) were explained by the uses and applications of the metre rules, calculators and computers. In this way, the themes, conceptual structures and interactions from the students’ own accounts and experiences and subsequently explained them (Meyer, 2015).

Polygons of Didactics

Empirical studies (Barzel et. al, 2005) of polygons of didactics provide innovations in the mathematics instructions. Polygons of didactics have three layers. The first layer is the inner layer or interaction layer. This layer provides interactions between teachers, students, mathematics content and mediator. The second layer is the middle layer. This provides didactical designs in teaching, learning, social relations, and process-based assessments. The third layer is the institutional development. This layer provides tasks in the form of assignments, quizzes, tests, home works, tests and examinations (Barzel et al, 2005; Bartolini Bussi, & Mariotti, 2016). These models explored the first layer to construct the models with the tasks, techniques, technologies and theories (Chevallard, Bosch, & Kim, 2015). The sequences of 4T models provided an excellent baseline for studying the second and third layers. However, the strategies, conceptualizations and interrelations dispelled negative reasoning, counter knowledge, false theories and misconceptions in using the models.

In addition, the first layer of polygons of didactics provides four structures, sequences and activities (Barzel et al, 2005). In the first activities, technologies promulgate, promote, enhance and produce equations, sketches, diagrams, schema representations and graphs from the equations of the circle. In the second activities, technologies provide new tasks and report new mathematical formulations of the collective discourses. In the third activities, technologies provide the patterns, relationships, generalizations, formulas, axioms and graphs during mathematical discourses. In the fourth activities, technologies enhance the learning and understanding of higher dimensional knowledge and skills (D'Amore, 2008; Bartolini Bussi, & Mariotti, 2016).

In these models, technologies played crucial roles ininfluencing knowledge constructions, connections and extension of the didactical triad. This helped the students share collaborative or group works and transition from and into the didactical models. Even though in the polygons of didactics, the tetrahedron is insufficient as argued by D'Amore (2008) and Presmeg et al (2016), the intersubjective frameworks and models built strong foundations for research in higher didactical phenomena.

Technology Tools

Research in technology tools (Clark-Wilson, & Hoyles, 2017; Chartwell-Yorke, 2018) shows various conceptions of representing mathematical objects to obtain best solutions. However, technology tools applied to didactical praxeologies use the task-techniques for mathematical activities, practices and experiences and technologies-theories for the experimentations, implementations, practices and communications (Bartolini Bussi, & Mariotti, 2016). In the didactical praxeologies models, the relationships between theoretical and practical with the technology tools helped to construct and develop the didactical conceptual structures in the tetrahedron.

Another conception of technology tools (Konig, & Kramer, 2016; Benning, & Agyei, 2016) emanate from either technological pedagogical content knowledge (*TPACK)* or anthropological transpositions in realistic mathematics education. While *TPACK* takes a more generic approach in integrating technology tools into pedagogical technological knowledge and pedagogical content knowledge (Koehler,& Mishra, 2009), technology tools in didactical praxeologies conceptualise the ways teaching and learning change when specific technology tools are applied in different contexts (Chevallard, Bosch, & Kim, 2015). In the praxeologies models, the principles, conventions and techniques discover new and emerging principles and techniques in instrumental genesis and anthropological didactics. These help to convert signs, symbols and artefacts into didactic instruments and technologies (Chevallard, 2012). Nonetheless, the *TPACK* settings were adopted for the experiments to complement rather than compete with the didactical instructional technology tools.

Again, the Cabri-Geometers and Matlab provide new technological developments and advancement for teaching and learning equations of the circle (Sinclair et al, 2016). The theories and methodologies foster conjectures and proves, and representations and discourses in the mathematics classrooms (Chartwell-Yorke, 2018; Clark-Wilson et al, 2018). However, the technology tools in the didactical praxeologies models fostered new experiences, skills and competencies within the diversity of socio-cultural contexts (Maschietto,& Trouche, 2011; Presmeg et al, 2016). Particularly, it was observed that:

1. The combination of the different tools minimized the recurrent errors and misconceptions in equations of the circle.
2. The pooling of different tools performed specific mathematical tasks in equations of the circle.
3. The reduction of the technology tools into calculators, mathematical sets and metre rules equipped students and eased computations in equations of the circle.
4. The extension of the technology tools from the two-dimensional (2D) to the three-dimensional (3D) objects helped to construct and reconstruct all kinds of mathematics objects.

5. The 2D projections formed the conceptual knowledge of ellipses, and subsequent applications in designing, engineering, digitizing, and replicating 3D objects. This orchestrated the students' knowledge and understanding in normal and tangent equations of the circle (Maschietto, & Trouche, 2011).

The third conception of technology tools (Clark-Wilson, Hoyles, Saunders, & Noss, 2018) divides the tools into dynamic geometry environments, dynamic algebra environments or amalgamations. The unique features of those environments are navigations (students move around the screens or monitors, figures and coordinates), interactions (students click, hold and drag or manipulate objects), annotations (students construct points, lines, numerals, figures and diagrams), constructions (students model figures, diagrams and shapes), simulations (students relate figures, diagrams, models and outcomes), manipulations (students construct their own signs, symbols and artefacts) and innovations (students design tools, interactive diagrams and visual objects). However, the conspicuously missing features were sociocultural, indigenous and local contexts in the use and interpretations of the technology tools (Ali, & Davis, 2018).

Mathematics Laboratory Tools

The mathematics laboratory (MMLab) is a methodology based didactical models (Maschietto, & Trouche, 2011). The structured mediation activities and mediators in MMLab construct and reconstruct mathematical objects, structures and ideas. These propel students' knowledge and skills to interact more in the didactical conceptual structures. These knowledge and skills bridge the gaps between technology tools and psychological tools

(Vygotsky, 1978). By so doing, students are able to construct appropriate experiential mathematical concepts, constructs, meanings and applications. As initiated by a small group (Maschietto, & Trouche, 2011), the essence and significance of MMLab was to build instruments with local materials and socio-cultural signs, symbols, artefacts, tools and instruments. However, with time, mathematical didacticians have extended and widen the scope of MMLab to include multimedia models and virtual models. In this study, MMLab helped to organize the mathematics signs, symbols, artefacts, tools and instruments into technologies. This fostered the knowledge and skills in the didactical conceptual structures (Chartwell-Yorke, 2018).

In addition, there are two main fields of MMLab tools. One field raises public awareness and the other field studies the conditions necessary for introducing for technology tools (Maschietto, & Bartolini Bussi, 2007). In didactics of education, one field informally provides mathematical explorations, amusements and games, and another field performs the technology tools in blended exploratory guided discovery and experiential learning. In the didactical technology tools, the students experienced and/or encountered the following features:

1. time needed (more time in informal and less time in formal)
2. explorations required (free in informal and guided in formal)
3. aims set (enjoy novelty in informal and discover new knowledge)
4. postures (to stand in informal and to sit down in formal)
5. task design (handle and talk in informal and reflect and write in formal)
6. instruments (amazing and amusing ones intentionally directed towards didactical aims in formal) (Maschietto, & Trouche, 2011)

Particularly, in the didactical praxeologies models, the MMLab tools structured the technologies. This equipped students with competencies, skills and experiences to reconstruct and recontextualise the didactical conceptual structures in the tetrahedron. Eventually, solving equations of the circle became much easier and simpler (Sinclair et al, 2016).

In conclusion, it must be noted that MMLab tools are essentially socio-cultural mediators (Maschietto, & Bartolini Bussi, 2007; Maschietto, & Trouche, 2011; Sinclair et al, 2016). The signs, symbols and artefacts are used to restructure mathematics operations, algorithms and didactical cycles. This adds new impetus to students' group works, collective moments, utilization schemes and whole collective discussions. The therefore researcher used the MMLab settings to make representations of the mathematics signs, symbols, tasks, tools and instruments. The students then applied the settings to make constructions, reconstructions, contextualisations and representations in the equations of the circle (Dikovic, 2009; Maracci, & Mariotti, 2010).

Errors and Misconceptions

Teachers often become astonished and disappointed to discover that students fail to grasp basic fundamental conceptual structures in school mathematics. However, Ernest (2018) observes that committing errors and misconceptions is normal in mathematics. A common problem come from students who just strive to memorize theorems, formulas and algorithms and end up making these errors and misconceptions (McIntyre, 2007). But real and meaningful errors and misconceptions arise out of lack of conceptual understanding, inappropriate reasoning processes and conceptual generalizations (Sellers et al, 2007). Ernest (1991, 1994) traces the sources of these errors and misconceptions to the distinctions between students' schemas and new ideas that arise out of assimilation and accommodation.

In assimilation, some new, recognizably and familiar is encountered, and incorporated directly into an existing schema that is very much like the new idea. The new idea is interpreted or recognized from an existing concept or schema and this new idea contributes to the schemas by expanding existing concepts, and by forming new distinctions through differentiation. Sometimes the new idea may be quite different from existing schema, but that schema, though relevant, cannot adequately assimilate the new idea. Accommodation is therefore, necessary to reconstruct and reorganize the new schema. The reconstruction is likely to leave the previous knowledge intact, or part or subset or special case of the new may be modified schema. This means previous knowledge can never be erased (Ernest, 1994). In this study, because students likely interpreted the new concepts in negative ways, it was possible that the constructions and reconstructions of mathematics knowledge in equations of the circle created errors and misconceptions.

As a precursor from Veloo et al (2015), the common errors the researcher observed were simple lapses, lack of concentration, deviations from correct solutions and mistakes in solving mathematical problems algorithmically or procedurally. On one hand, because students likely conflicted with their own values, old beliefs, theories, meanings and explanations of the mathematical phenomena, misconceptions likely emerged from preconceptions, alternative conceptions, naive beliefs, naive theories, alternative beliefs, flawed conceptions and buggy algorithms.

Generally, in finding solutions to errors and misconceptions, the first set of finding solutions arise from the constructions and reconstructions of mathematics knowledge in the didactical conceptual structures. The students' misconceptions provided the researcher fora to confront misconceptions and refashion the didactical instructional models (Egodawatte, 2011). The experimental designs produced desired outcomes which were interpretable and logically constructed by the students. The teachers just anticipated the most common misconceptions and guided the students by using the didactical tetrahedron to address them (McIntyre, 2007).

The second sets of solutions required mathematics discourses in addressing the errors and misconceptions, namely:

1. Word use (i.e. words specific to the discourses)
2. Visual mediators (i.e. visual objects, diagrams and special symbols)
3. Narratives (i.e. sequences of utterances, relations, endorsements or rejections of concepts, constructs, theorems, definitions and equations)
4. Routines (i.e. repetitive patterns of discourses) (Viirman, 2014).

As suggested by Viirman (2014), the roles of teachers in the discourses were to discover the basic repetitives, patterned discourses, rules of actions and situations, and explorations. Having identified these, the teachers carried out construction (i.e. making new endorsable narratives), substantiation (i.e. deciding whether to endorse previous narratives) and recall (i.e. summoning past narratives). Students on the other hand, employed social actions to foster their conceptual understanding.

Equations of the Circle

The interplay of teachers, students, mathematics content and technologies preserve mathematical knowledge and improve upon teaching and learning of school mathematics (Ernest, 2018). In this light, teachers need to employ multidimensional strategies and methods in starting from standard and general equations of the circle. With a lot of blind alleys, incorrect solutions, reformulations of problems and solution paths, students can solve problems in diameter, normal and tangent equations of the circle (Stitz, & Zeager, 2013).

In Ghana, the commonest strategy solving the general equation of the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is to compare the various coefficients and constants to given circle equations (Ministry of Education, 2007; Asiedu, 2009; Hessse, 2011; Asare-Inkoom, 2012; Atteh, & Okpoti, 2013; Kabutey, 2016). For instance, coefficients of an equation like $x^2 + y^2 + 6x + 4y = 0$ is usually compared with the coefficients of $x^2 + y^2 + 2gx + 2fy + c = 0$ to arrive at the centre $C(-g, -f)$ and radius $r = \sqrt{g^2 + f^2 - c}$. In this example, $2gx = 6x$ or $g = 3$, $2fy = 4x$ or $f = 2$, $C(-g, -f) = C(-3, -2)$ and $r = \sqrt{g^2 + f^2 - c} = \sqrt{3^2 + 2^2 - 0} = \sqrt{13}$ *units*. Little attention paid to an alternative didactic transposition of the standard equation of the circle $(x-h)^2 + (y-k)^2 = r^2$ that directly arrive at the centre $C(h, k)$ and radius r.

Stitz and Zeager (2013), Whitney and Reno (2015), and Horsman (2018) propose that students rather employ concurrent completion of squares of the x and y variables. In so doing, if the centre is the origin, the standardized equation is $x^2 + y^2 = r^2$ with the radius, r and if the centre is a point $C(h, k)$, the standardized equation is $(x-h)^2 + (y-k)^2 = r^2$ with the radius, r. Clearly, the expansion of $(x-h)^2+(y-k)^2=r^2$ yield $x^2+y^2-2hx-2ky+h^2+k^2=r^2$. In this standardized expansion, the second-degree term x^2+y^2, the linear terms $-2hx$ and $-2ky$ and the collection of h^2+k^2 can be generalized to $x^2+y^2+2gx+2fy+c=0$. This strategy is much more meaningful and easier to learn.

Generally, the conceptual understanding of equations of the circle can be traced back to the equations of the ellipse. The standard equations of the ellipse at the origin are $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$. In these equations, 'a' represents distances from the centres to the major axes, 'b' represents distances from the centres to the minor axes, 'c' represents distances between the centres and the foci by relation $a^2=b^2+c^2$. Here again, the lengths of the major axes are $2a$, and the lengths of the minor axes are $2b$. By extension, the standard equations of the ellipse at the $C(h,k)$ are either $\left(\frac{x-h}{a}\right)^2+\left(\frac{y-k}{b}\right)^2=1$ or $\left(\frac{x-h}{b}\right)^2+\left(\frac{y-k}{a}\right)^2=1$. In the two scenarios, if the values of 'a' and 'b' are the same ($a=b$) and designated as r, then the standard equations become either $(x)^2+(y)^2=r^2$ (centre at the origin) or $(x-h)^2+(y-k)^2=r^2$ centre at the point $C(h,k)$ (Stitz, & Zeager, 2013).

If there are two (x_1,y_1) and (x_2,y_2) or more points, the didactical applications employ the midpoint strategies or methods. In this strategy, the endpoints of the diameter form line segments containing the centre $(h,k)=\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$ and radius $r=\frac{1}{2}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$. These can be simplified to $(y-y_1)(y-y_2)+(x-x_1)(x-x_2)=0$. But one of the points (x_1, y_1) is tangent to the standard equation. Hence, we can derive the tangent equation as $xx_1+yy_1=r^2$ and the normal equation as $\frac{y}{y_1}=\frac{x}{x_1}$ (Parsons, 2015; Horsman, 2018). Clearly, the processes leading to the centre and radius of the circle required technology tools. The technology tools did not only help the students and teachers sole the problems but also extended the triad to the tetrahedron. The technology tools brought more innovations, discoveries and conceptual dialogues in the teaching and learning processes.

Covariates

Generally, a covariate is any continuous variable that is expected to correlate with the outcome variable of interest. However, in modern times it is typically used to refer to variables that are not of direct or substantive interest in the study and only function as control variables (Creswell, 2016). In this study, random assignment could have generated unbiased causal estimates. This is because the baseline survey's treatments as control groups were equivalent over all possible covariates. However, random assignment was not implemented. Therefore, in order to reduce bias, the researcher rather reduced the effects of possible covariates determining the outcomes (Yu, 2015). This was particularly important in the selection of participants for the quasi-experimental study.

Also, the study involved more complex treatment assignment processes characterized by combinations of the researcher and third parties (heads and mathematics teachers) selections. Such complexity was likely to raise doubt about adequate statistical controls for selection. It was possible that covariates simultaneously correlated with both treatments and the potential outcomes (Cohen, Manion, & Morrison, 2011). So, the researcher deduced that the possible covariates could be gender, school, level, programme, school stream, ICT experience, geographical location and availability of qualified mathematics teachers.

Gender as a Covariate

Despite advances in gender equity in past decades, troubling patterns specific to mathematics still persists. Boys and girls who begin kindergarten education with similar mathematics proficiency and competencies encounter gender disparities in achievement and performance as they enter primary levels of education (Creswell, 2016; Cimpian, Lubienski, Timmer, Makowski, & Miller, 2016). For instance, Cimpian et al. (2016) found that, in the kindergarten, girls make up about 20% of students above the 99th percentile in mathematics. However, this drops to only 5% as they complete the primary school. In this study, the didactical instructional models went beyond simple statistically differences. The main aim was to interact freely across the didactical instructional models. Care was therefore needed to statistically reduce or remove gender as a pre-existing covariate that could influence the findings (Creswell, 2014).

Also, research findings in STEM in Ghana by Amponsah, and Ochonogor (2016) show that gender differences in mathematics performance still persist. While the findings suggest that some females perform equally well as their male counterparts, differences in achievement between males and females widens as students progress from primary to the university levels. It was revealed that even though constructivist approaches help reduce gender differences in mathematics achievement, Vygotskian social-constructivism perspective as the fundamental role of social interaction in students' cognitive development do not usually favour girls as boys.

Again, Baah-Korang, et al (2015) found low participation of females in elective mathematics studies in Ghana. Even though many girls who pursue careers in mathematics successfully, the absolute number of women in mathematics still in low. The education of girls in mathematics, whether core or elective, is generally not encouraged in the Ghanaian senior high schools. The participation of girls in elective mathematics is quite worrying such that male and female students must not study in the same environment and under the same conditions of learning. In view of these differences, the researcher confounded gender differences in order to eliminate any pre-existing conditions detrimental to any gender in the study.

School as a Covariate

The best way to measure the impacts of instructional interventions is to randomize schools to treatment groups that receive the intervention or control groups that do not and compare future outcomes for the groups. This is especially appropriate for evaluating schools' methods and strategies of teaching and learning. However, randomization was limited to statistical power or precision and so control groups were ignored. Another challenge of randomizing schools was how to identify with confidence intervention effects that are educationally meaningful. Therefore, an alternative was to use analysis of covariance to control for the characteristics of schools and/or students during the baseline study (Bloom, Richburg-Hayes, & Black, 2007; Bofah, &

Hannula, 2016). The baseline study took into consideration the students' past performance (pretest) as determined by WAEC classifications in Ghana.

Level as a Covariate

In Ghana, Davis, Carr, and Ampadu (2016) discovered that grade levels have effects on what students find important in their mathematics learning. These effects of grade levels enabled the researcher to engage students with more complex mathematics as they progress from one level to another in the six schools. Even though the data targeted only form three students, students of other levels took part in the exercises. This made levels clear potential covariates as prior mathematics knowledge varied from one level to another.

Programme as a Covariate

Tay, and Mensah-Wonkyi (2018) made a shocking revelation that most senior high school students are unable to construct, visualize and justify geometrical concepts due to the traditional approach of teaching and learning mathematics. Some programmes do not encourage their students to discuss, interact and explore the content collaboratively. However, *General Science* programmes build the exploration and visualization skills into their students. These make general science students more likely to acquire geometrical ideas, geometry reasoning and problem-solving skills than the other programmes. These made the school programmes potential covariates in the study.

Other Covariates

In addition to gender, school and class level, there were other demographic factors that affected the senior high school students' ability to interact in the didactical instructional models. Some of these were ICT experience, stream of school, geographical location and availability of professionally qualified teachers. Particularly, single-sex schools performed better than mixed, urban schools performed better than rural, and well-resourced schools performed better than poor schools (Kibriya et al., 2016).

Chapter Summary

In this chapter, it emerged that research in didactical conceptual structures required the fusion of manifold intersubjective theoretical perspectives, frameworks and models. This emanated from the debates, arguments, experimentations and discussions in the mathematics classroom; the overt and overt relations among the theories, attempts to depart from the old didactics involving only memorization and superficial conceptual understanding, and attempts to position contemporary mathematics classroom as human-centred shrouded with errors, misconceptions and social-cultural objects (Bartolini Bussi, & Mariotti, 2016; Sinclair et al, 2016).

Other researches in didactics concentrate on instrumental genesis and semiotic orchestrated, instrumental genesis and didactical situations, and didactical situations and anthropological didactics. These frameworks and models are deficient in illuminating and addressing the research problem. In particular, Østergaard (2013) work was limited to the triad, Dunphy et al. (2014) to semiotic mediation and Forsman (2015) to computer technology. However, the most interesting and thought provoking angle of these intersubjective didactical frameworks was the ability to realistically mathematise the classroom, transition into instrumentation and instrumentalisation, and then zoomed into 4T anthropological praxeologies (Grant, & Osanloo, 2014) to study the research problem.

The researcher also reviewed literature in quantitative and qualitative models, polygons of didactics, technology tools, mathematics laboratory tools, and errors and misconceptions, and equations of the circle. While many empirical researches have fused diagrammatic, physical and mathematical models as one (Walliman, 2011), nothing is known of deterministic models to construct and explain didactical conceptual structures in equations of the circle. Even though Walliman (2011) enumerated many quantitative models, this chapter has showed that it is only conceptually-ordered displays that are most suitable for modelling the research problem. These displays clearly addressed conceptual relationships, mathematics content, cognitive structures, and cause-and-effect phenomena of the didactical conceptual structures.

Moreover, in corroborating with the qualitative models, Morrell and Carroll (2010) classifications of ethnographic phenomenology texts, documents, discourses and transcripts, Kothari, and Garg (2014) and Meyer (2015) classifications of interrogative insertion, problem–solution discourse, membership categorization, semiotics, narrative analysis and discourse analysis helped the researcher to use the visual, audio-visual and written texts to gain deep understanding of the conceptual structures and contexts (Kumar, 2014).

In the polygons of didactics, Barzel et al. (2005) outer layer, inner layer and institutional developments were a bit verbose and presented a much more complicated system. The researcher perceived that polygons of didactics are additional structures to the vertices (Bartolini Bussi, & Mariotti, 2016). This helped the students to interact and construct sequences situations devoid of reject negative reasoning, false theories, errors and misconceptions. This stand was strongly supported by D'Amore (2008) fourth set of activities which are a set of activities with the technology tools in a new vertex to obtain tetrahedron.

In the technology tools, gaps existed in the generic instrumental genesis and/or technological pedagogical content knowledge. Although complementary, no clear cut criteria was established in converting signs, symbols and artefacts into didactic instruments and technologies. Since technology tools come from different experiences, skills and competencies (Chartwell-Yorke, 2018) within the diversity of socio-cultural and geographical contexts (Presmeg et al., 2016), the researcher gave more room to technologies that minimized recurrent errors and misconceptions, pooled technology tools, reduced digital tools and propelled students to extend 2D objects to 3D in equations of the circle (Maschietto, & Trouche, 2011).

There are still wide gaps between technology tools and psychological tools in constructing experiential knowledge in mathematics (Vygotsky, 1978). The literature in methodology-based MMLab tools (Maschietto, & Trouche, 2011) helped the students to construct and reconstruct knowledge in equations of the circle. With the MMLab tools, students can observe, imitate, perform, communicate and apply their mathematics knowledge and skills in the didactical conceptual structures. This helps students to conceptually informally and formally construct and exhibit mathematics prowess.

Many research works in didactics pay lip service to errors and misconceptions. As Ernest (2018) put it, *errors make mathematics and mathematics makes errors*. To undertake a research without envisaging errors and misconceptions is to start digging a foundation without making a feasibility study. Conceptual understanding is one of the main errors students encounter when constructing mathematics knowledge (Sellers et al., 2007). When well addressed, students can confront paradoxes and conflicts from their own preconceived notions and beliefs in solving equations of the circle.

Finally, wide research gaps exist in the ways teachers employ multidimensional strategies and methods for solving equations of the circle (Parsons, 2015). Literature concentrates on the general equation $x^2 + y^2 + 2gx + 2fy + c = 0$ to the neglect of the standard equation $(x-h)^2 + (y-k)^2 = r^2$ (Horsman, 2018). This research linked equations of the circle to equations of the ellipse. This helped students to grasps the origin and source of equations of the circle (Whitney, & Reno, 2015).

CHAPTER THREE

RESEARCH METHODS

In this chapter, the research design for the study has been presented. The mixed methods explanatory research design was explored in extending the didactical triad to the didactical tetrahedron. The design was employed to collect both quantitative and qualitative data from a cross-section of senior high school students (Creswell, 2014). This helped the researcher to address the research problem in Chapter One.

Literature suggests that pragmatist philosophies basically drive mixed methods designs. The pragmatic paradigm was deemed appropriate for this study because the research problem and questions informed the methodology for this research study (Davis, 2010). Following the pragmatist philosophies were the sequential explanatory mixed methods processes (Subedi, 2016). The researcher also presented the research population and sample size, sampling procedures, study area and participants, instruments of data collection procedures, validity and reliability of instruments, and instruments of data analysis.

The Research Design

Figure 3 shows the mixed methods research design that was used to carry out the study. The model was used for both the baseline survey and the main study.

Phase	Procedure	Product
Quantitative Data Collection	Cross sectional survey	Numeric data
Quantitative Data Analysis	Use of descriptive and inferential statistics	Meaningful measures
Connecting Quantitative and qualitative Phase	Selection of participants purposefully and interview questions development	Interview protocol
Qualitative Data Collection	In- depth interview	Textual data
Qualitative Data Analysis	Coding and thematic analysis Theme development cross thematic analysis	Codes and themes similar and different themes and categories cross thematic matrix
Integration of the Quantitative and Qualitative results	Interpretation and explanation of the quantitative and qualitative result	discussion implication future research

Figure 3: The Mixed Methods Research Design Processes (Subedi, 2016)

The mixed methods research design employed the pragmatist philosophies to explain inferences, interpretations, research questions, hypotheses and experiments. These were followed by qualitative transcripts and experiences of the models (Cohen, Manion, & Morrison, 2011; Naidu, 2013: Subedi, 2016). In this way, experimental designs can put the experimental treatments into interactions and subsequently employ the ethnographic-phenomenology engagements. The ethnographic-phenomenology engagements provided more complete and comprehensive understanding of the research problem rather than confining the interactions to only the experimental treatments (Creswell, 2014).

In addition, the pragmatist philosophies of the mixed methods research design systematically, scientifically and logically helped to carry out the hypotheses, research questions, data collection, data analysis, deductions and inferences (Creswell, 2012). The first process collected baseline data through questionnaire and interviews. This baseline data established the research problem and justified the kind of methodology to use. The second process explored didactical conceptual structures in the tetrahedron using the didactical instructional models in solving equations of the circle. The third process integrated the results of the interactions and interview transcripts. This helped to corroborate, explain and confirm the interactions. In this way, the techniques and procedures used for mixing the data provided better understanding of the merging, integrating and interactions processes than one. This equipped the researcher to follow up, combine and integrate the results of the baseline survey with main study (Harwell, 2011).

Secondly, even though literature is unclear whether both qualitative and quantitative data together constitute mixed methods or the concept should stand alone, the research employed the stand alone mixed methods research design (Naidu, 2013). This is because the pragmatist philosophies allow for flexible approaches to solve practical problems regardless of their objective truths or subjective perceptions (Harwell, 2011). It is best way of exploring data whose respondents are novice in the area of concern and lack the requisite knowledge and skills to provide complete and comprehensive data (Subedi, 2016).

Furthermore, according to Cohen, Manion, and Morrison (2011), the main purposes of data mixing are triangulation, complementarity, development, initiation and expansion. In this study, triangulation helped to control threats, complementarity helped to assess similarities and differences, development helped to refine instruments and expansion helped to add richness to findings. And even though any of the four main strategies of data-mixing, namely transformation, topology development, extreme case analysis, consolidation and merging were suitable, the researcher employed mainly transformation strategy to transform the baseline survey into the main study(Naidu, 2013). The interviews in the baseline survey helped to confirm the necessity of the research design.

Again, more specifically, the researcher employed the sequential explanatory mixed methods research design to explore the didactical conceptual structures. A mixed methods sequential explanatory design is one that employs quantitative data collection in a first phase. The purpose of the first phase is to explore the statistical significances of quantitative variables. After analysis of the quantitative results, the researcher employs qualitative data collection and analysis. The purpose of the second phase was to enhance, complement, confirm and corroborate the quantitative results (Creswell, 2014). In the first phase, the researcher collected the quantitative data through questionnaire and psychological tests. In the second phase, the researcher collected the qualitative data through interview guide. The purpose of the two sets of instruments was to check lapses and improve upon the lapses of the first instruments (Harwell, 2011).

Secondly, the mixed methods sequential explanatory started with quasi-experimental set up. In the quasi-experimental procedure, the research participants were in their intact schools, classes and course programmes. The researcher distributed the baseline survey questionnaire containing equations of the circle to all students pursuing elective mathematics. The baseline questionnaire was to test students' learning outcomes in equations of the circle without the didactical instructional models (as dependent variables). In this case, the quasi-experimental designs became necessary because the researcher did not randomly assign the students (Morrell, & Carroll, 2010).

However, the issues of causes-and-effects warrant deliberate controls and manipulations of experimental treatments. Therefore, elective mathematics students in each school were split into two groups of one using the normal existing conventional methods at baseline survey (i.e. control groups) and employing the didactical instructional models with the didactical tetrahedron (i.e. experimental groups). After the experimentations, only the results of the experimental groups were used to test the statistical significances in the didactical conceptual structures. If the results were high in each model, the null hypothesis was rejected. Thus, any improvements of learning outcomes were attributed to the new didactical instructional models (Walliman, 2011).

Also, literature has classified quasi-experimental designs in many ways. Campbell and Stanley (1963) classify quasi-experimental designs into time-series, equivalent time-samples, equivalent materials, non-equivalent control group, counterbalanced, separate-sample pretest-posttest, separate-sample pretest-posttest control group, multiple time-series, recurrent institutional cycle and regression-discontinuity analysis. Cohen, Manion and Morrison (2007) categorize quasi-experimental designs into one-group pretest-post-test, non-equivalent control group and time series.Koedinger et al (2008) group quasi-experimental designs into nonequivalent-group design, regression-discontinuity, proxy pretest, separate pre-post samples, double-pretest, switching-replications, nonequivalent dependent variable and regression point displacement. Creswell (2014) classify quasi-experimental designs as one-group pretest-post-test, time series, and non-equivalent control group.

In these few classifications, the non-equivalent control group design is the commonest. Particularly, the non-equivalent control group design redefined pre-existing variables in terms of time (i.e. before and after treatments) to produce between-subject designs (i.e. non-equivalent design) and within-subject designs (i.e. pre-post designs) from the independent variables. The between-subject designs compared two or more treatment conditions with components of the didactical instructional models. This balanced and equalised the groups, since the researcher could not randomly assign students across groups (Creswell, 2012).

The non-equivalent group designs are grouped into differential research, posttest-only research and pretest-posttest control group (Gravetter, 2008). However, both differential and posttest-only designs make no attempts to control or minimize assignment biases. Since there was a test (pretest) before administering the didactical tetrahedron, the researcher employed the pretest-posttest design to the non-equivalent groups. So, the experimental groups were measured twice: one before the treatment (pretest) and one after the treatment (posttest). In particular, the time-series pretest-posttest helped to minimize threats to internal validity. In this case, the three components/levels of intersubjective didactical instructional models helped the researcher to observe post-treatment trends (Creswell, 2013).

The main internal threats to internal validity associated with non-equivalent quasi-experimental designs are pretesting, blocking, covariates and matching. However, matching posed no threat due to the regression effects of the different groups. Gender and general equation as covariates reduced differences. The classes, programmes and schools already served as statistical blocks. The baseline survey reduced the effects of pretesting. Even though quasi-experiments are biased, sensitive, and inadequately control bias, the designs still generated statistical significances differences in the didactical instructional models.

The Study Area

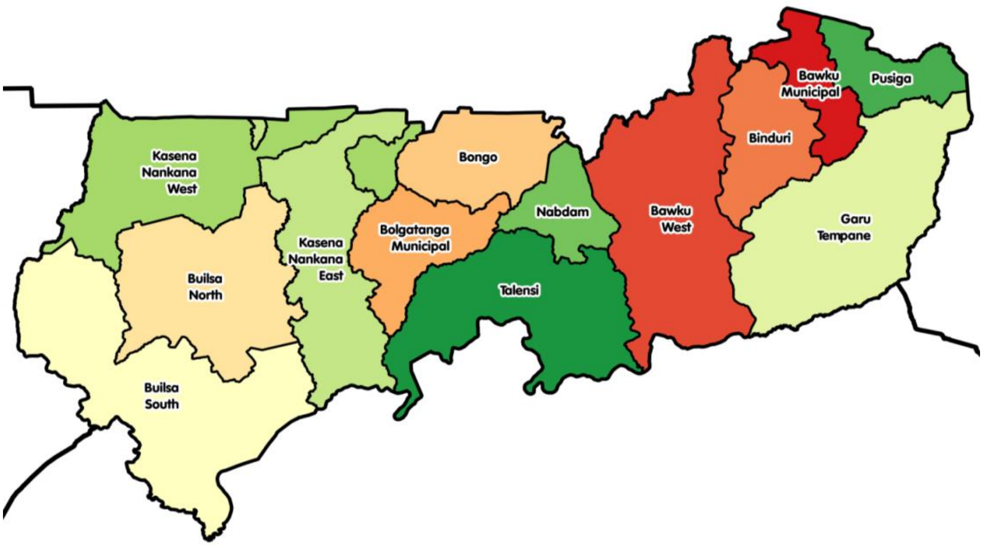

Figure 4: The Political Map of Upper East Region of Ghana (Ghana Statistical Service, 2012)

Figure 4 is the political map of the Upper East Region of Ghana. It has 13 administrative districts and Bolgatanga is the administrative capital. The Region borders two countries of West Africa-to the east is Togo and to the north is Burkina Faso. By the national boundaries, it is located to the east of Upper West and to the north of North-East Region. According to the Ghana Statistical Service (2012), Upper East Region is classified as one of the poorest regions in Ghana. According to the WAEC (2018), it is only Notre Dame Senior High School in Navrongo that found itself in the best 100 schools in Ghana. There is no other school in Upper East that boasts of 50% pass mark in Core Mathematics. Worst still, most schools consistently score below 20% pass rate in Elective Mathematics. This partly explains the researcher's choice of the region (MoE, 2010; WAEC, 2019).

Data available from the Ministry of Education in 2018 shows that there were about 25 senior high schools in the Upper East Region. However, the researcher selected only six senior high schools-three each of Bolgatanga and Kasena-Nankana municipalities. The use of the six schools was to give fair representations of students' performance in mathematics and the dire need for the interventions. This was demonstrated in the initial stages, where the results of the baseline survey and interview transcripts gave credence to this selection.

The senior high schools in the Bolgatanga Municipality were Zuarungu, Zamse and Bolgatanga Girls'. The senior high schools in the Kasena-Nankana were Navrongo, Awe and OLL Girls'. The selections of the schools ensured that students adequately represented the various demographic classifications. The demographic classifications are managements (i.e. public, government assisted and private), school rankings (i.e. Grade A, Grade B, Grade C and Grade D), streams (mixed and single-sex), specializations (i.e. General Science, General Arts, Business and Home Economics/Technical Skills), classes/forms (i.e. SHS1, SHS2 and SHS3), gender groups (i.e. boys and girls), students enrolments (i.e. less than 30, less than 40, less than 50, less than 60, and more than 60), mathematics teacher population (i.e. one, two, three and more than three), and school technologies (i.e. artefacts, tools, instruments and technologies). By G.E.S. classification, only five schools in region fall under Grade 'A'. The other grade schools are bedevilled with low enrolments in elective mathematics and few competently trained elective mathematics teachers. Over 95% of the senior high schools are run by government, and most of them lack basic instructional aids including technology tools (MoE, 2010).

Population of the Study

The table below represent the population distributions of the 6,500 students in the six schools in 2017. The researcher categorized the population according to the demographic information of the research instruments.

Table 3: Population Distribution of Elective Mathematics Students

School	Municipal	Grade	Stream	Number	Percentage (%)
Zuarungu	Bolgatanga	C	Mixed	200	13
Zamse	Bolgatanga	B	Mixed	350	23
Bolgatanga Girls'	Bolgatanga	A	Single	200	13
Navrongo	Navrongo	A	Mixed	350	24
OLL Girls'	Navrongo	D	Single	100	7
Awe	Navrongo	C	Mixed	300	20
Total				1,500	100

Source: Regional Directorate of Ghana Education Service (2017)

Table 3 shows that Grades 'C' and 'D' dominated the schools. Single schools and general science also had lower populations because Grades 'C' and 'D' do not admit many general science students. Such interventions were therefore required to increase these numbers. The target population of 1,500 comprised all enrolled students who were pursuing Elective Mathematics in the six senior high schools. This number cut across all the programmes in the schools. Analysis of students' performance in West African Senior School Certificate Examinations (WASSCE) shows that the region's performance was low as compared to other regions of Ghana (MoE, 2010). These students therefore required interventions to help the students in Elective Mathematics.

There were 35 elective mathematics teachers in the six schools. They were all orientated towards the preparation and development of effective and efficient experimental sessions. They were also ready to provide supports, coaching and mentoring skills to the students on didactically-supported classroom instructions. These instructions involve problem-based learning, instructional designing, school-based experiments and qualitative interviews in the didactical instructional models. However, many of these teachers were found wanting. It was therefore against the background of low students' mathematics performance in the region as compared to the other regions in Ghana and the demographic characteristics of the region that the researcher chose the region for the study. The population for this study therefore consisted of all senior high elective mathematics students and their teachers in the Upper East Region of Ghana (Davis, 2010; MoE, 2010).

Sampling Procedure

The researcher drew a sample of senior high school students and teachers from the six public senior high schools for the questionnaire survey. The six schools were selected because they are located in relatively urbanized municipalities as compared to the other schools in the Upper East Region of Ghana. This therefore gave the researcher a wider range of choices of other schools. Secondly, the researcher also minimized costs of accommodation and transportation during the entire three months of data collection, as the researcher's family already lives in Bolgatanga.

The selection of the students and teachers was based on the number of state public SHS. As public schools in Ghana are usually categorised by their performance, the researcher realised that all good public senior high schools are located in urban areas of Ghana. The researcher then grouped them according to their grades (i.e. Grade A, Grade B, Grade C and Grade D) as determined by the ministry of education. Based on the similarities and homogeneities of the areas, the researcher did not group the schools according to their methods of instructions. Using simple random sampling (table of random numbers) procedures the researcher randomly selected the students and teachers to represent the Grade A, Grade B, Grade C and Grade D senior high schools (MoE, 2010). Table 4 represents the summary of the sample distribution of Elective Mathematics students in the six schools.

Table 4: Sample Distribution of Elective Mathematics Students

School	Municipal	Grade	Stream	Number of students	Percentage (%)
Zuarungu	Bolgatanga	C	Mixed	50	10
Zamse	Bolgatanga	B	Mixed	90	18
Bolgatanga Girls'	Bolgatanga	A	Single	100	20
Navrongo	Navrongo	A	Mixed	150	30
OLL Girls'	Navrongo	D	Single	40	8
Awe	Navrongo	C	Mixed	70	14
Total				500	100

Source: Survey Data (Ali, 2018)

On Table 4, it can be seen that out of the 1,500 elective mathematics students, the researcher selected 500 students. Of the 500 students, 253 (51%) were males and 243 (49%) were females. On Table 4, 50 students were selected from Zuarungu, 90 from Zamse and 100 from Bolgatanga Girls' senior high schools in the Bolgatanga municipality, and 150 from Navrongo, 40 from OLL Girls' and 70 from Awe senior high schools in the Kasena-Nankana Municipality. The researcher, with the aid of elective mathematics teachers, assembled all the elective mathematics students to inform them about the study. Then students who were willing to take part were given numbers. Thereafter, the table of random number method in simple random sampling technique was used to select the samples in each of the six schools (see sampling procedures for experimental participants on page 119-122).

The choices of the samples were also informed by the students' participation, population and grades of schools. On Table 3 for instance, it can be observed that the highly populated schools were Zamse and Navrongo senior high schools (350 students each). However, 150 students were selected from Navrongo SHS and 90 from Zamse SHS because the students were much more willing to participate and actually came out in their numbers as compared to Zamse. The least populated school were Bolgatanga (200) and OLL Girls' (100). It therefore made sense to select fewer numbers from those schools.

The overall selections of the participating schools, teachers and students were done with permission from the regional education directorate, heads of schools, heads of mathematics departments, elective mathematics teachers and the elective mathematics students. The selection criteria were based on schools that offered Elective Mathematics across three or more programmes. This helped the researcher to gain deeper insights into the didactical instructional models in wider scope and content. Each of the six schools selected comprised of students who scored very high (80-100), high (70-79), medium (50-69) and low (below 50) in the baseline survey. This ensured that students fall across all cognitive levels (Pausigere, 2014; WAEC, 2015).

Now, the question that often plagues a mixed study is the selection of the sample size. Walliman (2011) contends that there is no clear-cut agreement on the correct sample size. Sample size selection depends on the purpose of the study and the nature of the population under scrutiny. In this study, the key factors that determined sample were the sample size, representativeness of the sample, access to the sample, and sampling strategy (Cohen, Manion, & Morrison, 2007). A large sample size does not automatically guarantee representativeness because of the different characteristics of the subsamples, and a small sample size does not also guarantee representativeness because overgeneralizations. In taking cues from Cohen, Manion and Morrison (2007), the researcher employed the simple random sampling because it is suitable for selecting a large sample size that reflects both the population and the amount of heterogeneity.

Secondly, the sample size was determined by the style of the research. According to Walliman (2011), a correlational research usually requires a sample size of not fewer than thirty, causal-comparative and experimental researches require a sample size of not fewer than fifteen, and a survey research requires not fewer than 100 cases. In qualitative research, the sample size is mainly constrained by high cost (Walliman, 2011). In this study, the baseline survey began with all the 6,500 mathematics students. Because the experimental phase was much more rigorous and robust, the researcher reduced the sample size to 500.

Again, the size was determined using tables of mathematical formulae. In the mathematical table, Walliman (2011) shows that the smaller the population, the larger the proportions drawn. So, as the population increases the sample size increases at a diminishing rate and remains constant at slightly more than 380 cases. Therefore, a research involving 100 participants require between 80 and 100 per cent, whilst a research involving 1,200 require just a sample of 25 per cent. In this study, a population of 1,500 elective mathematics students required 25% of the population (Walliman, 2011).

Also, the size of the sample took account of attrition and respondent mortality. In experimental designs such as this, participants were either bound to leave the research or failed to submit responses or exit the experimental processes. So, it was advisable to overestimate rather than to underestimate the size of the sample. This satisfied the required prediction and standard error. Because the researcher set the level of significance at ± 0.5 error, it is clear that the 500 out of 1,500 lied within the comfort zone (Creswell, 2014). And at the 95% confidence and 5% sampling error, a sample size of 500 students was approximately the same as the total population of 1,500 (Cohen, Manion, & Morrison, 2007).

Finally, the sampling strategy the researcher used for the intersubjective didactical instructional models was a deciding factor on the sample size. The researcher opted for both probability (i.e. simple random sampling) and non-probability sample (i.e. purposive sampling). In the simple random sampling strategy, the chances of the 500 students being selected were equally likely but these chances were clearly and unequivocally disclosed in the purposive sampling strategy. Thus, in the probability sampling, the inclusion or exclusion of a student from the sample was a matter of chance. In the non-probability sampling, many students were denied access to participate in the interviewing procedures (Cohen, Manion, & Morrison, 2011).

Sampling Procedures for Experimental Participants

The sampling procedures were carried out on three phases. In phases one and three, the researcher adopted the purposive sampling procedures to collect the baseline data. The purposive sampling technique, also called judgment sampling, is the deliberate choice of participants due to the qualities the participants possess. It is a non-random technique that does not need underlying theories or a set number of participants (Yu, 2015). So, the researcher decided the schools and the students who were capable and willing to provide the information by virtue of their knowledge and experiences in equations of the circle.

In the baseline survey (See Appendix J1), the researcher employed this procedure to select the six senior schools and the elective mathematics students. These six senior schools were selected because they are nearer the researcher's home. These schools also offered many programmes that include elective mathematics. The researcher also realised that equations of the circle is taught and learned in only elective mathematics classes. Even though the topic is normally taught in the second year, the researcher realized that many schools did not cover the topic. It was therefore, prudent to add the third-year students. In this phase, the regional directorate was instrumental in guiding and permitting me to get access to the schools (MoE, 2010).

Also, the school heads and mathematics teachers helped me to identify and select the groups of students who were relatively proficient and well-informed in solving equations of the circle. Because this phase was just a baseline survey, the researcher did not only consider knowledge, experience, availability and willingness of participants, but also the ability to translate these experiences and opinions in articulate, expressive and simple manner. Language and communication are cardinal in unearthing phenomenological experiences of participants (Etikan, Musa, & Alkassim, 2016).

In phase two of the quantitative data, the researcher adopted the simple random sampling procedures. Some common procedures are tossing dice, flipping coins, spinning wheels, drawing names, table of random numbers, and computer programme. However, after a list of the population had been constructed and the various random sampling procedures available, the researcher opted for the table of random numbers. This is because it is relatively easy to use, accessible, and truly random. The step by step procedures adopted were as follows:

1. The researcher assigned numbers to each student on the list. Because the populations differed from one school to another, the researcher numbered them either 0 to 99 for populations under 100 or 0 to 999 for populations under 1000. And then entered the table of random numbers.
2. Starting anywhere in the table, the researcher moved in any direction, preferably up and down. Since there were mostly 100 students on the list (0 through 999), each student was given an equal chance of being selected.

3. To do this, the researcher used three columns of digits from the tables. If the first three-digit number in the table is 218, participant number 218 on the population list was chosen for the sample. If the next three-digit number is 007, the participant assigned number 007 (or 7) was selected.
4. The researcher continued the process until all the 500 students were selected for the sample. If the same number came up more than once, it was simply discarded.
5. Sometimes the first digit in the population total was small. When this happened, many of the random numbers encountered in the table were not usable and therefore were being passed up. This was very common and did not constitute a sampling problem.
6. Also, the tables of random numbers came in different column groupings. Some came in columns of two digits and some three. These differences had no bearing on the principles of randomness.
7. It was imperative not to violate the random selection procedure. Once the list had been compiled and the process of selection had begun, the table of random numbers dictated the selections. The researcher did not alter this procedure (Creswell, 2014).

Still in phase two comprising the main study, the researcher went through another purposive sampling procedures to select only 12 elective mathematics students for the qualitative data. This was based solely on students whose responses were so interesting and needed further clarifications. There are various purposive sampling methods, namely maximum variation sampling (i.e. participants with greater knowledge and experiences), typical case sampling (i.e. participants with particular behaviours), extreme or deviant case sampling (i.e. participants with atypical cases), and critical case sampling (i.e. participants with importance). The others are total population sampling (i.e. taking the entire population), expert sampling (i.e. participants with expertise) and homogeneous sampling (i.e. participants with similar characteristics). The researcher adopted the homogenous purposive sampling procedures. This is because the assumption of the quantitative data rested on the homogeneity property of the students. This sampling procedure made it easier to focus on their precise similarity and relate their everyday methods of learning to the didactical instructional model (Etikan, Musa, & Alkassim, 2016).

Sampling Procedures for the Interview Participants

Interview procedures were conducted in both phases. The purposes of the interviews were to explain, corroborate and confirm the results of the hypotheses. The interview guides moved students away from just simply manipulating data to generating knowledge through conversations, experiences and constructions. The interviews were standardized in ethnographic, focus groups, semi-structured, unstructured, informal conversational and open-ended forms. They were conducted in both the baseline and the main study phases. The qualities of the interview guides accorded the participants to freely interact and delve deeper into the two phases (Creswell, 2012).

The interview procedures were neither subjective nor objective. They were rather intersubjective. As students discussed their interpretations of the didactical conceptual structures, they also relate the models to real life situations and everyday life's experiences, perceptions and understanding. In the interview items (see appendix), some items were dichotomous, rating scale, multiple choice and short-answer types. Whenever the students selected any option from the closed items, the researcher probed the students to justify and explain their answers (Cohen, Manion, & Morrison, 2011).

Other items were open-ended, value/opinion driven, construct-forming and interpretive information (Cohen, Manion, & Morrison, 2011). However, the main aim was to obtain answers that border on mutual trust, easiness and deepness, opaqueness and genuineness. The answers were paraphrased and transcribed to suit the research problem and objectives, research questions and hypotheses, and didactical conceptual structures. Such open items were contained in Appendix J3.

Again, because the interview items ranged from formal, semi-formal to completely informal, the researcher took on subordinate roles and allowed the students to take up the discourses and discussions among themselves. Occasionally, the researcher initiated 'two-person' conversations whenever detailed descriptions and explanations were required. Through direct verbal interactions, tape recording and paper transcription, the researcher assured the participants of absolute ensure anonymity, honesty and secrecy of their responses (Bishop, 2012). Creswell (2012) enumerates the precise qualities of the interview guide as Life world to students' lives, Meaningful to the subject matter, Qualitative to respondents' knowledge and experiences, open descriptions of the interview items, Specificity on the required items' actions and opinions, and focused on particular themes.

Instruments of Data Collection Procedures

The instruments of data collection procedures combined both quantitative and qualitative instruments. This enables the researcher to explain and describe the didactical situations in more comprehensive terms. The qualitative instruments enriched the quality and precision of the quantitative instruments. They also provided more comprehensive explanations (Tatar, 2013). The quantitative instruments were questionnaire and psychological tests. The questionnaire was used for the baseline survey and the psychological tests for the experimental design. The qualitative instruments were interview guides. The quantitative instruments sought to explore the didactical interactions among the didactical conceptual structures. The qualitative instruments sought to explain and corroborate the results of the quantitative ones. Because the interviews were verbal reports only, and as such, were subject to problems of bias, poor recall, and poor or inaccurate articulations, the questionnaire and psychological tests equally helped the researcher to structure the experiments (El-Demerdash, 2010).

Instruments of the baseline survey

A survey, unlike a census, is a representative sample of the potential group that the researcher is interested in, for reasons of practicality and cost-effectiveness. Surveys take many forms (El-Demerdash, 2010). In this study, the cross-sectional survey was carried out to explore and provide a snapshot of the research problem. There were five parts of the baseline survey instruments (See Appendix J1). These parts helped the researcher to identify causal relationships, show statistically significant differences and confound the six demographic variables. Because the survey instruments had inbuilt checks to internal and external validity, the sample was representative and produced findings which were generalisable to the wider population.

Tatar (2013) agrees that a baseline survey is efficient in random sampling techniques to generate findings which can be used to draw conclusions about the whole population. The baseline survey covered a wide geographically spread (See Figure 3) which ensured that all categories of students were included. It also ensured ethical considerations as it did not expose the students to possible invasions (UCC, 2016). It was flexibly combined with the main study instruments to produce richer data (Tatar, 2013).

According to Yu (2015), the four main purposes for baseline adjustment are to retain baseline values as outcomes with no assumptions about group differences, to retain baseline values as outcomes and assume group means are equal, to subtract baseline from post baseline responses and analyze differences, and to include baseline values as covariates. In this study, the researcher used the baseline as covariates. So, no statistical analyses sought to compare the outcomes between the baseline and the main study.

Even though a survey is dependent upon an accurate chosen sampling frame, is not so good at explaining reasons and an interview survey is only as good as the interviewer (Creswell, 2012), the researcher employed face-to-face interviews and questionnaires to offset these shortcomings. Because the selection of the method depends upon access to potential participants/ respondents, the literacy level of respondents, the subject matter, the motivation of the respondents and resources, the research decided to blend two instruments for the baseline, namely questionnaire and face-to-face interviews (Segal, 2009).

Pre-intervention questionnaire

The questionnaire was handed directly to all students pursuing elective mathematics in the six schools. The main drawback of the questionnaire was the bias in responding to some items. However, they had well validated and high coefficients of reliability than the face-to-face interview. Even though very convenient, the questionnaire was self-completed because of high literacy level of the respondents, the high expected response rate, the resources made available and topic and population of interest (Creswell, 2014).

The researcher prepared one set of questionnaire to be administered to only the elective mathematics students. The questionnaire consists of five main parts. The first part elicited demographic information of the students, the second part sought to examine components of mathematisation and didactical phenomenology, the third sought to transform signs, symbols and artefacts with instrumentations and instrumentalisations, the third part sought to transfer tools and instruments to technologies with didactical situations and anthropological praxeologies, and the final part sought to applied the 4T praxeologies to solving sampled equations of the circle. The questionnaire consisted of 58 items, items one through eight for the demographic information, whereas each of the other parts shared 10 items each (see Appendix K2) (Cohen, Manion, & Morrison, 2007).

Out of the 58 items that examined the didactical instructional models, five items were open-ended items. The closed ended items involved multiple-choice, yes/no and rating scales items. Some of the multiple-choice and yes/no items were followed by follow up questions (see Appendix K1). This enabled the researcher to get more insights from the students. The majority of the items were closed ended items since it is easy to score and code them for computer analysis (Davis, 2010).

The items covered mainly four areas of the didactical instructional models, namely mathematisation and didactical phenomenology, instrumentations and instrumentalisations, didactical situations and anthropological praxeologies, and equations of the circle. Each of the four main areas contained a mix of mathematics-based items and didactical-based items. Items that contained mathematisation and didactical phenomenology were socio-cultural, instrumentations and instrumentalisations were social-environmental, didactical situations and anthropological praxeologies were socio-cultural technologies whilst equations of the circle were culture-free. This mix ensured that students used artefacts from their own cultures and transformed them into modern day classroom technologies (Ruthven, 2014).

The researcher constructed the items based on the hypotheses and research questions, didactics of mathematics, didactical conceptual structures, didactical triad and didactical tetrahedron and equations of the circle (Bishop, 2012; Presmeg et al, 2016). However, a few of the items were adapted and modified from already prepared instruments (Ali, Davis & Agyei, 2018). To test for the validity of the questionnaires, the researcher pilot-tested the questionnaire by giving them to undergraduate students of the Department of Basic Education, University of Education, Winneba to complete the questionnaire.

The researcher used the responses received from the baseline survey to improve the instruments. The pilot test and the baseline survey gave the researcher an opportunity to further address other problems. Some of these problems were clarity of questions, clarity of options, difficulty levels of questions, clarity of instructions, required artefacts, tools and instruments, laboratories and professional teachers. The duration of the administration of the instruments to each participant was also noted. This enabled the researcher to also assess the suitability of the study. The researcher administered the questionnaires to the 500 students in the main study after explaining the purpose of the study and also responded to issues revolving round the questionnaire (Davis, 2010).

Baseline face-to-face interviews

Face-to-face or personal interviews even though very intensive, were the best way of achieving high quality data. Many questions were very sensitive, but not personal and some were very complex and the interviews were likely to be lengthy. In the face-to-face interviews, the questions were both quantitative and qualitative. The quantitative items were structured interview schedule. These items sought to examine students' knowledge in interactions with the didactical conceptual structures and equations of the circle. The qualitative items sought to assess the students' expressions, eloquence and willingness to participate after the experimental treatments (see Appendix J1). Even though expensive and time consuming, the face-to-face interviews helped to collect more complex information about students' interactions in the didactical conceptual structures. They were also justifiable in cases of students with special needs (Creswell, 2014).

Instruments for the Main Study

Unlike the baseline survey, psychological tests and group interviews were administered. These two instruments helped the researcher to measure the observed variables and explain the responses in details with the group interviews (Pausigere, 2014).

Psychological tests

A psychological test is an instrument designed to measure unobserved constructs or latent variables. Psychological tests are typically but not necessarily, a series of tasks or problems that the respondent has to solve. Psychological tests strongly resemble questionnaires but differ a bit. In a psychological test, the researcher asked for a respondent's maximum performance whereas in a questionnaire, the researcher asked for the respondent's participation. A useful psychological test must be both valid (i.e., there is evidence to support the specified interpretation of the test results) and reliable (i.e., internally consistent or give consistent results over time and across raters) (Creswell, 2014).

It is important that the students who were assumed equal, homogenous and similar on the measured constructs answered the same test items. The students also belonged to groups of schools, programmes and classes. These could influence the chance of correctly answering items. So, the tests were constructed for the specific elective mathematics students, and took into account these differences (See Appendix K2). If the test was invariant to some group differences (e.g. gender) in one school (Zuarungu), it did not automatically mean that it was also invariant in another school (Awe).

Van der Zalm (2010) agrees that a psychological test is one of the sources of data used for assessment. And, usually more than one test is required for a single research. It is used for treatment settings, particularly in school settings and classroom treatment outcomes. In this study, the tests in the intersubjective didactical models helped students to get the correct conceptions, operations, relationships and manipulations of the equations of the circle with high degree of internal reliability (Van der Zalm, 2010):

1. The researcher used the first psychological test (pretest) scores from the baseline survey to calculate the Cronbach's alpha coefficient. This coefficient ensured internal reliability of the instruments. It was greater or equal to 0.70 or $\alpha \geq 0.70$ and adjudged very reliable.
2. The researcher selected at least one elective mathematics teacher from each of the schools. Each teacher was asked to teach the students and conduct an exercise. The researcher inducted the didactical instructional models to each teacher. The teacher taught only the experimental class.
3. The researcher set 18 items in equations of the circle. Each item contained multiple-choice options. Students who selected any option were made to justify his/her answer on the space provided. The questions covered all areas of circle (see appendix K2 Part 'E').
4. The researcher administered the tests in each experimental group of the six schools. Participation of treatment groups was voluntary and the results were kept confidential (i.e. the researcher used only Arabic numbers for the purpose of identifications and matching schemes).

5. The students were entreated to eschew random guesses on the test items but not compelled, forced, lured or threatened to participate and complete the tests.
6. The researcher allowed the participating students sufficient length of period of two hours to complete the test and submit their answers.
7. All completed experiments and their tests were scored in time so that each student received his/her feedback as well as the number of correct answers obtained in each of the intersubjective instructional model (Meyer, 2015).

Moreover, the researcher ensured that the psychological tests addressed testing issues of publications, reporting and discriminating among the participating students. The test items were as correct as possible and the didactical conceptual structures were being assessed and measured with two parallel forms (i.e. pretest and posttest) (Meyer, 2015). Having obtained test scores from the psychological instruments, themes were generated from each of the didactical intersubjective models. These themes formed the basis for constructing the interview guide for the main study (Meyer, 2015).

Group interview guide

The researcher prepared one set of interview guide for all the 12 students of the six students. Each school had two students for the interview procedure. The researcher administered the interview guide to select the focus schools after the questionnaire survey and experimental treatments. The interview guide was made up of the same five parts as the questionnaire. Part 'A' solicited demographic information, part 'B' examined knowledge in mathematisation and didactical phenomenology, part 'C' examined their expressions in instrumentations and instrumentalisations, part ‘D’ examined their experiences in didactical situations and anthropological praxeologies, and part ‘E’ tested their excitement in solving sampled equations of the circle (see Appendix J3).

Some sets of students were administered in groups and the other set in individual face-to-face interviews. The focus group interviews were always administered first, followed by the individual face-to-face ones. The same interview guide was used for both groups to ensure consistency. In part ‘A’, the demographic information included gender, school, municipal, class/form, experience in ICT and programme and were considered as covariates.

Parts ‘B’ elicited information about how students experienced the didactical conceptual structures in both the triad and the tetrahedron. Part ‘E’ looked at students’ expressions, knowledge and procedures in solving equations of the circle using the didactical instructional models. The part ‘B’ of the didactical models (mathematisation and didactical phenomenology) required students to solve typical mathematical problems that require the meaning and interpretations of signs, symbols and artefacts. The implementation of the mathematisation and didactical phenomenology models requires the use of students’ knowledge of their culture practices. It is this knowledge that traced didactics of mathematics to culture and tradition. This is because good knowledge of culture and tradition set the pace for official school mathematics.

Like the test items in mathematisation and didactical phenomenology, the interview items in the third part followed the same pattern. Thus, in part ‘B’ students were asked to clarify the same problems that they solved with mathematisation and didactical phenomenology. This knowledge and experiences sought to link didactics of mathematics to local tools, artefacts and instruments. Knowledge of local tools and instruments laid good foundations for classroom technologies and innovations.

Part 'C' elicited information concerning the relationship between tools and instruments in one hand, and technologies and computers on the other hand. If the relationship was significant, then students could use tools, instruments, technologies and computers for solving the questions. The activities in equations of the circle enabled the researcher to explore these relationships in the classroom context. It enabled the researcher to explore how knowledge and experiences of local tools, artefacts and instruments could influence their conceptions and practices of the 4T praxeologies.

Part 'D' contained items on didactical situations and anthropological praxeologies. These items enabled the researcher to get information on students' experiences in the didactical tetrahedron. Each part of the interview guide was semi-structured. Even though the researcher predetermined answers, students were allowed to discuss their choices of options in detailed based on their phenomenological experiences. The researcher also clarified the contexts of the students' responses in line with didactics of mathematics. Part 'E' was the integration of parts two to four to solve sample equations of the circle. Each item in part five was solved with each of the intersubjective didactical instructional models.

Like the questionnaires, the researcher adopted some items. The researcher also piloted the interview concurrently with the questionnaire. Data from the pilot test were analysed to ascertain whether the questions brought out the themes and addressed the research problem, hypotheses and questions in

Chapter One. Also, items that were found to be ambiguous or sensitive were either modified or discarded. The duration for the administration of the instruments was also taken good care of. This enabled the researcher to modify the instrument in order to avoid lengthy interview time (Davis, 2010).

The researcher alone interviewed the students. The interviews took place in the premises of the six schools. Some schools even granted their serene offices to conduct the interviews. The other schools gave us their classrooms to conduct the interviews. The researcher negotiated and visited each school twice a month within the three-months. This gave students and helping teachers ample time to adequately prepare and obtain the tools and materials.

Validity of Instruments

Table 3 summarizes the main constructs in the study. The constructs sought to measure some key domains in the study.

Table 5: Constructs of the Research Instruments

Construct	Domain	Number
1	Demographic information	6
2	Didactical conceptual structures	4
3	Mathematisation and didactical phenomenology	10
4	Instrumentations and instrumentalisations	10
5	Didactical situations and anthropological praxeologies	10
6	equations of the circle	18
	Total	58

Source: Research Instrument (Ali, 2018)

Table 5 contains the constructs in the instruments. In other to obtain true students' interactions, the right number of points on the rating scales was critical. The use of the six demographic variables helped to control covariates and increased the response precision. The use of didactical conceptual structures helped to predict the interactions. The inclusion of the didactical instructional models helped to measure the interactions of the didactical conceptual structures. The use of the equations of the circle was to exemplify the interactions of the conceptual structures. More specifically, the use of multiple-choice scales in the didactical instructional models and equations of the circle lead respondents to make a definite choice. This increased tendency of respondents to respond negatively. The researcher therefore used a five-point scale for the instructions (Chan, & Idris, 2017).

Instruments in education are generally constructed to measure specific constructs, qualities or abilities (variables) of students. So, the extent to which the test measures the constructs, knowledge and experiences is its validity (Lovely Professional University—LPU, 2012). The purposes of addressing validity of survey and test instruments were to improve upon generalizability, replicability, controllability, predictability, context-freedom and randomization of samples. The purposes of addressing validity of interview guides were ensure honesty, depth, richness, triangulation and objectivity (Long, 2011).

The common validities the researcher addressed were content, construct, ecological, cultural, criterion-related and triangulation. Content validity is defined as the extent to which a test measures a representative sample of the subject matter and the behavioural changes under consideration. The researcher addressed content validity by ensuring that the items fairly and comprehensively covered all domains of the four Didactical Conceptual Structures, the intersubjective didactical instructional models and equations of the circle (LPU, 2012). For example, the researcher ensured that if students were supposed to solve 50 questions but they solved only 20, then the 20 represented all domains of the didactical instructional models. This is because all parts of the items contained similar items (see Appendix J2).

The construct validity of a test is defined as the extent to which the test measures the theoretical constructs. The researcher addressed construct validity by abstracting the operationalised forms of the concepts and the constructs of the didactical instructional models. For example, the researcher ensured students constructed the experiment and compared their results with the interview responses. The researcher then compared the outcomes of the two with theoretical constructs. In cases of conflicting interpretations, the researcher acknowledged the conflicts and interpreted the results as such. In the interview guide, the researcher ensured that the themes represented students' actual experiences and situations (Cohen, Manion, & Morrison, 2007).

Also, the researcher addressed the ecological validity in the tests by frequently isolating, controlling and manipulating the experimental contrived settings. In the interviews, the researcher deliberately manipulated the variables and social situations. The researcher was conscious of ethical tenets. Therefore, efforts were made in other not to violate non-traceability, anonymity and non-identifiability of the students (Chan, & Idris, 2017).

Again, cultural validity emerged as a result of cross-cultural, inter-cultural and comparative cultures. The researcher endeavoured to increase the degrees of sensitivity to the cultures, implementations and disseminations. Criterion-related validity is defined as the extent to which test performances are related to some other valued measures of performances (Creswell, 2016).

Furthermore, the researcher addressed criterion-related validity by comparing the results to external WAEC criterion. Two kinds of criterion-related are the predictive validity and concurrent validity. The predictive validity ensured that post test scores correlated highly with the pretest scores, and the concurrent validity ensured that psychological tests correlated highly with interview guides (Cohen, Manion, & Morrison, 2007).

In addition, the researcher triangulated the instruments by using questionnaire, tests and interviews. This yielded sufficient and unambiguous information. Particularly, time triangulation enabled the researcher to rectify omissions, and space triangulation addressed cross-cultures (LPU, 2012).

Reliability of Instruments

Internal consistency

The reliability of an instrument is concerned with the consistency, stability, and dependability of the scores (Yu, 2015). For this reason, the internal consistency was tested using Cronbach's alpha for each competency. If the alpha value is higher than 0.9, the internal consistency is excellent, and if it is at least higher than 0.7, the internal consistency is acceptable (Yu, 2015). Excellent internal consistency means that the survey items tend to pull together. In other words, a participant who answers a survey item positively is more likely to answer other items in the survey positively (Chan, & Idris, 2017). In this study, reliability was confined to the consistency, precision and accuracy and demonstration of respondents in similar contexts. The three principal types of reliability that militated against the researcher were stability, equivalence and internal consistency (Pallant, 2011).

The stability reliability measured consistency over time and over similar samples. This was achieved by the method of test and then retest. In this method, the students were given pretest during the baseline survey. After the experimental exercises, the students were given another test. The correlation coefficients of the two tests were calculated. The Cronbach alpha coefficient was the common measure in the stability reliability. It was over 0.70 in most parts of the didactical instructional models (Creswell, 2014).

The equivalence reliability was achieved through equivalent forms of instruments. The researcher used the same instruments during the pilot tests, baseline surveys and psychological tests (e.g., a control and experimental groups). The results of the main study were compared with the baseline and pilot tests. According to Chan and Idris (2017), satisfactory internal consistency ranges from 0.7 to 0.9. Many factors on this scale had a high rating reliability. The Cronbach alpha for mathematizing within tools, mathematizing within artefacts, mathematizing within instruments, mathematizing within technologies, mathematizing within contents, praxeologies in two structures and praxeologies in theories were mostly higher than .70 (Long, 2011)

In internal consistency, the researcher used the inter-rater reliability to compare the means and standard deviations between the groups. For example 12 elective mathematics teachers who part took in the data collection processes helped to ensured similarity of data. The items were divided into two halves by the split-half methods. Each part was scored by the researcher and the teachers separately. The Pearson product moment correlation was used to calculate the two halves (Caswell, 2011).

Exploratory factor analysis

The researcher also used exploratory factor analysis to increase reliability of the instruments. Exploratory factor analysis is a statistical method that increases the reliability of scale by identifying inappropriate items that can then be removed. It also identifies the dimensionality of constructs by examining relations between items and factors when the information of the dimensionality is limited (Yu, 2015).

In this study, the exploratory factor analysis was performed in the early stages of the baseline and pilot studies. Before performing the factor analysis, the research items were evaluated with both the mean of all responses and the standard deviations per item. If the mean of an item was close to either 1 or 5, eliminating it was inappropriate because it might decrease the standard of correlation among the rest of the items (Yu, 2015; Chan, & Idris, 2017). The normality in distribution was tested by examining skewedness and kurtosis before conducting prior to the exploratory factor analysis. Since the normality of the distribution was confirmed, the exploratory factor analysis was conducted with Statistical Package for Social Sciences (SPSS, version 22).

Seven factors with the highest commonalities—mathematising within tools, mathematizing within artefacts, mathematising within instruments, mathematising within technologies, mathematising within contents, praxeologies in two structures and praxeologies in theories—were used to determine the structural pattern of the baseline and pilot surveys along with a Scree plot and eigenvalues (see Appendices I1 and I3). The Scree tests plotted eigenvalues against the number of factors in order to best determine where a significant drop occurs within factor numbers (Yu, 2015). The factor solution was determined based on the number of eigenvalues greater than one.

Research (Yu, 2015) opines that 0.30 can be used as a factor loading criterion in exploratory factor analysis, five to ten participants per item, a sample size of 200 to 300 is good and a minimum sample size of 200 to achieve reliable results. The exploratory factor analysis process began with an initial analysis run to obtain eigenvalues for each factor, and Kaiser-Meyer-Olkin (KMO) measure of sampling adequacy test and Bartlett's Test of Sphericity to determine construct validity. In particular, the KMO test was used to verify the sampling adequacy for the analysis, and Bartlett's test of sphericity was used to determine if correlations between items were sufficiently large. In literature, Bartlett's Test of Sphericity should reach a statistical significance of less than .05 in order to show reliability (Pallant, 2011) (see Appendix I1-I3).

According to Yu (2015), the Scree plot and eigenvalues are accurate to determine how many factors should be retained when the sample greater than 200 and communalities (variance of the variables) are greater than 0.6 or when the questionnaire has more than 30 variables and communalities are greater than 0.7. The Varimax rotation was equally deemed to be the most statistically and conceptually appropriate measure. Yu (2015) indicates a factor loading of ±0.3 means the item is of minimal significance, ±0.4 indicates it is more important, and ±0.5 indicates the factor is significant. All baseline factors in the Varimax rotation were greater than .50 (see Appendix G3).

Covariates analysis

Similar to independent variables, covariates are complementary to the dependents variables. Variable are covariates if they are related to the dependent variables. According to Fan (2010), any variables that are measurable and considered to have statistical relationships with the dependent variables qualify as potential covariates. Covariates are thus possible predictor or explanatory variables of the dependent variables. In this study, all the ten independent variables were all tested for potential covariates. However, the six demographic variables were of primary interest. In this context, they were regarded as concomitant variables, auxiliary variables or secondary variables. Gender was one of the most important variables (See Appendix H1-H3). This is because some experimental units (i.e. students in the various schools) posed threats of heterogeneity. They interacted with the independent variables to obscure the true relationships between the didactical conceptual structures and the intersubjective didactical instructional models. The researcher therefore, needed to make efforts to control their effects. However, the four didactical conceptual structures remained pure independent variables (Pallant, 2011).

Item analysis

Item analysis is a method of reviewing test items, both qualitatively and statistically, to ensure that they all meet minimum quality-control criteria. After qualitative reviews during item development, the statistical analysis was conducted after items have been administered. This helped to identify items that still slipped through the item reviews, manifesting in problematic or misfit items. Such items were either unclear or bias against gender, schools and the other demographic information (MoE, 2016). The statistical analysis considered the correlation of each item with each scale (Yu, 2015). Specifically, the item-score to scale-score correlations determined whether an item belongs to the scale as assigned or should be eliminated. The scale-score was obtained by computing the arithmetic average of the scores of the items in that scale. According to Yu (2015), the values of item to scale correlations should be greater than 0.50. Scores lower than 0.50 do not share enough variance with the rest of the items in that scale as they do not measure the same constructs. Such items were deleted from the scale prior to the main study. This helped to understand whether items were assigned appropriately (Yu, 2015).

In Ghana, the commonest simple statistics of item analysis are item difficulty and item discrimination analysis. The item difficulty represents the proportion of students who answered the item correctly. Also known as the item p-value, the possible range is 0.0 to 1.0. Higher values indicate easier items as greater proportion of students answered correctly. The acceptable range is between 0.3 and 0.9. Discrimination represents how well items distinguish themselves between high-performing and low-performing students. The item-total correlation was used as a measure of item discrimination. The possible range for item discrimination was between –1.0 to 1.0, and a discrimination index below 0.0 signifies problems. Negative discriminations showed that high-performing pupils were getting the items wrong and low-performing pupils were getting right (Pallant, 2011; MoE, 2016). In this study, the acceptable discrimination index was greater than or equal to 0.25(see Appendix F1-F4).

Threats to internal validity

The Table 6 represents common threats to validity in an experimental and the controls to these threats. Because the study went through three stages, each stage came with its own threats and controls.

Table 6: Threats and Controls of Internal Validity

Features	Baseline Survey	Experiment	Interview
Randomization	No	Yes	No
Number of groups	One group	Two groups	Two groups
Number of interventions	One intervention	More interventions	More interventions
Number of times the dependent variables were measured	One	Four	Four
Controls	Students, teachers	Baseline survey, covariates	Students, teachers

Source: Cohen, Manion, & Morrison (2007)

In the Table 6, it was shown that an experimental research is one in which the researcher manipulates one or more variables and controls to measure any change in the other variables. In experimental research, we manipulate the independent variables and record the outcome on the dependent variables. This is to establish if there is a cause and effect relationship between the two variables. If a change is found and the independent variable is 'A' and the dependent variable is 'B', then we can conclude that that 'A caused B'. So, A is the cause and B is the effect (Seltman, 2015).

Addressing internal validity is important in experimental research because we often want to establish this cause-and-effect relationship. External validity refers to the extent to which the results of the study or experiments can be generalized to other settings, other people and other time (Creswell, 2014). Threats to internal validity compromise confidence in saying that a relationship exists between the independent and dependent variables. Threats to external validity compromise confidence in saying the findings are applicable to other groups. The common threats to internal validity are history, maturation, statistical regression, instrumentations and experimental mortality (Seltman, 2015). In this study, the commonest were history and maturation.

History was made up of events other than the experimental treatments occurred during the times between the baseline and pilot studies, and psychological tests. Maturation affected changes in observations independent of the experimental treatments.History and maturation were both threats to internal validity that involved time. History in this context referred to the period of time between the first and last experiments. These were unintended events that occurred during the treatment periods. Maturation was referred to the things associated with aging and fatigue (Seltman, 2015; Creswell, 2016).

The threats to history and maturation were controlled by making sure that there was a reasonable time period between the baseline survey and the psychological test. Randomization also controlled for history and maturation because the passage of time did not affect memory and fatigue. Controlling history and maturation automatically reduced experimental mortality. So, the participants were willing to stay and participate in the data collection processes (Caswell, 2011; Long, 2011).

External validity

The common threats to external validity include Hawthorne effect, interaction effects and ecological effects. In Hawthorne effect, participants perceived their roles as guinea pigs and became reluctant to either take part or put up their best. In interaction effect, participants likely took part in the baseline, survey and psychological tests from one or more schools. In ecological validity, behaviours observed in some schools, classes or programme contexts were different and threaten generalization of results (Creswell, 2014).The researcher minimized these threats by carefully selecting the elective mathematics from the populations after the baseline survey. The students also represented various environments and cultures. Even though some schools had low elective mathematics enrolments (low generality), there were incentives in the form of pens, pencils and mathematical sets to encourage them to participate (Pausigere, 2014). In the interviews procedures, the common threats were socially situated and unique to particular students. So, the most viable ways was to address the selection effects, setting effects, history effects, and construct effects (Caswell, 2011).

Data Processing and Analysis

Table 7: Data Analysis Plan

Level/Research Question(RQ)	Tools	Type of data	Statistic(s)
One/RQ1	Reliability, item analysis, factor extraction, covariates, Transcripts	Quantitative and qualitative	Cronbach's alpha/ Friedman's test, eigenvalues, marginal means and statistical powers
Two/RQ2/RQ3/RQ4	ANOVA, MANOVA, ANCOVA, MANCOVA	Quantitative	F-statistics, p-values, effect sizes, marginal means, statistical powers, Lambda Wilks
Three/RQ5	Transcripts	Qualitative	Phenomenological experiences, and box plots

Source: Data Processing and Analysis (Ali, 2018)

On Table 7, the data analysis was done at three levels. The first level involved analysis of the baseline survey. This contained both the questionnaire and the interview guide both of which contain quantitative and qualitative data. So the researcher employed tools such as reliability, item analysis, factor extraction, covariates, and transcripts. The second level which was the main study utilized tools because the psychological test items were administered after the students had gone through the intervention with the tetrahedron. The tools were ANOVA, MANOVA, ANCOVA, MANCOVA because the interactions among the didactical conceptual structures were mainly quantitative. And so the main statistics were F-statistics, p-values, Lambda Wilks, marginal means, effect sizes and statistical powers (Creswell, 2014).

This was followed by the third level with research question five. This analysis explained and confirmed the results of the didactical instructional models after the intervention. The tools were the interview transcripts because the data was mainly qualitative. And the statistics were drawn from students' own Phenomenological experiences, and subsequently tabulated box plots. The box plots primarily compared the similarities and differences in the participants' responses (Caswell, 2011). The following subheadings give more details on the choice of Table 7.

Analysis before intervention with the didactical tetrahedron

As the study employed a mixed methods approach, both quantitative and qualitative approaches were employed in the data analysis before the intervention. As this baseline survey was primarily set up to examine students' prior interactions in the didactical tetrahedron, the researcher used Cronbach's alpha coefficients, exploratory factor analysis, item analysis and covariates (Davis, 2010). Internal consistency is the degree to which the items that make up the scales are all measuring the same underlying attribute. The Cronbach's coefficient alpha provides an indication of the average correlation among all of the items that make up the scales.

The minimum Cronbach's alpha was set up at 0.70. This was supported by the inter-item correlation matrix coefficients. According to Pallant (2005), when there is small number of items in the scale (fewer than ten),the better way is to calculate and report the inter-item matrix correlation. The inter-item matrix correlation ranges between .2 and .4.However, when there are a large number of items, the best way is to add the inter-total statistics. These statistics give an indication of the degree to which each item correlates with the total score. Low values (less than .3) indicate that the item is measuring something different from the scale as a whole (Creswell, 2014; Creswell, 2016).

Exploratory factor analysis was used to extract the most significant factors and scales in the didactical instructional models. Exploratory factor analysis is a statistical method employed to increase the reliability of the scales. This method identifies inappropriate items that can be removed and the dimensionality of constructs. According to (Pallant, 2005), factor analysis is in two parts. The first is to explore for commonalities, dominant Eigen-values and Scree plots, and the second part is to rotate the extracted factors. In the first part, the researcher examined the existence of relationships between items and factors in order to reduce a large number of related variables to a more manageable number, prior to using them in multivariate analysis of variance (Davis, 2010).

In the second part, the researcher rotated the extracted factors. The rotation was necessary to help interpret the patterns of factors. According to (Pallant, 2005), Varimax method is common because it minimizes the number of variables that have high loadings on each factor. Therefore, in the *Rotated Component Matrix***,** the loadings of each of the variables on the dominant factors were selected. The highest loading variables on each of the component helped to identify the nature of the underlying latent variable represented by each component (Creswell, 2016).

Covariates were also determined in advance. The commonest covariates were gender, school, level, programme, management, ICT experience and teacher professionalism. They reduced the errors and improved the statistical significances (Subedi, 2016). Covariates that were significantly correlated with the dependent variables were removed or confounded to improve upon the models' explanatory powers. Research (Chan, & Idris, 2017) shows that covariates with a Pearson's correlation coefficient greater than or equal to .20 and p-value less than or equal to .10 are deemed to be significantly correlated. school, level, programme and gender have been shown in prior studies to have impact on mathematics performance (Pallant, 2011; Subedi, 2016).

The knowledge of covariates had an impact on the ability to perform the psychological tests (Baah-Korang et al, 2015). In order to measure any non-significant differences in the didactical instructional models due to covariates, data were collected by asking each participant about their prior strategies and methods of solving problems in equations of the circle. The data were then run on SPSS software against the various components of the didactical instructional models. After this, the potential covariates were analyzed separately in the baseline survey and merged into the discussion.

Analysis of the Main Study with the Didactical Tetrahedron

In the main study, the researcher ensured that assumptions of independent random sampling, measurement of the variables, absence of multicollinearity, normality, homogeneity of variance, and relationship between covariates and dependent variables pertaining to t-tests, ANOVA and MANOVA were satisfied. In few cases, there were no statistical significances. However, with use of the ANCCOVA and MANCOVA, the researcher was able to reduce the effects of some covariates and attain statistical significances (Creswell, 2012). Particularly, it was detected in the baseline study that school, level, programme and gender were the four main covariates. As opined by MoE (2011), gender and school in turn affect other demographic variables such as students' class levels, programmes, locations and school resources.

The results of ANCOVA and MANCOVA were analyzed by the contributions of the covariates from the adjusted means (Howell, 2013). After satisfying the conditions, the researcher first used the independent-samples t-test to compare the mean scores of the experimental groups across the didactical instructional models. In the independent-samples t-tests, probability values equal to or less than .05 indicated statistically significant differences and greater than .05 showed otherwise (Pallant, 2011). However, because when two groups are independent and the variances are equal, F-statistic is the same as t^2-statistic, the researcher mostly used F-statistic in the analysis (Howell, 2013).

ANOVA was also used to compare the mean scores of the experimental groups in equations of the circle. The ANOVA used the F-ratio to compare the variance in scores between the different groups with the variability within each of the groups. Large F-ratios indicated more variability between the groups (caused by interactions of the independent didactical conceptual structures) than there is within each group (the error terms). Like the independent-samples t-test, significant F-test indicated that we reject the null hypothesis and followed the analysis up with post-hoc tests or use multiple comparisons. In cases where some components of the models were not statistically significant, the researcher, as usual, used only ANCOVA and MANCOVA analysis to confound the covariates and/or reduce their effects (Creswell, 2014).

In the ANCOVA and MANCOVA, the power tests were first computed. In the data, power was not a major statistic because the sample size was larger than 100. The next statistic was the effect sizes. These were calculated from the partial eta squares. This is important because every small difference between groups in large samples become statistically significant (Long, 2011). According to Caswell (2011), partial eta squared values determine the relative magnitudes of the differences in the dependent variables. An eta squared of 0.01 means small effect, 0.06 means moderate effect and 0.14 means large effect. Apart from effect sizes, the researcher used the F-values, p-values and confidence intervals. These statistics helped to determine the statistical significances of the intersubjective didactical instructional models with respect to the didactical conceptual structures (Pallant, 2005). In some cases, the researcher first confounded covariates in the ANCOVA and MANCOVA to improve upon interactions between the independent and dependent variables. And if confounding took place, the marginal means differences were used to support the reduction of group differences.

The MANOVA test statistics are many and varied. The common ones in SPSS are Wilks' Lambda, Lawley-Hotelling Trace, Pillai's Trace and Roy's Largest Root. Whenever the hypothesis degree of freedom is exactly one or greater than one, all statistics still lead to the same results. In all these cases, the multivariate tests are statistically significant and we can conclude that the effects of the didactical conceptual structures on the didactical instructional models are statistically significant, after controlling for the effects of the covariate(s) (Davis et al, 2016).

However, when results are different, then Wilks' Lambda, Lawley's trace, and Roy's largest root are preferred to Pillai's trace. But because Pillai's trace is more powerful and robust in heterogeneous populations, it is preferred in such few circumstances. The Roy's is the least preferred since it estimates only the lower bound probability (Seltman, 2015). Pallant (2005) agrees that Wilks' Lambda is the commonly and generally used. However, if the sample size is small, unequal or violates assumptions, then Pillai's trace is more robust. In situations of only two groups, the F-tests for Wilks' Lambda, Hotelling's Trace and Pillai's Trace are identical.

Analysis of interview transcripts in support of the didactical tetrahedron

In order to give more details about the distribution of responses on each of the models, a further analysis of the items was provided through the use of interview transcripts. Participants' responses to the open-ended items on their experiences about didactical instructional models were analysed by reading through their responses to the various open-ended items thoroughly. This enabled the researcher to have adequate understanding of the data (Cohen, Manion, & Morrison, 2011). The researcher then identified trends that emerged from the participants' responses and then grouped their responses according to the components of the didactical instructional models. This was done by organising, transcribing, exploring, coding and presenting the data with boxplots and narrative discussions (Davis, 2010).

Lastly, the Boxplot was used to analyse the interview transcripts. These boxplots were used in the coded interview transcripts. The main statistics are minimum/maximum, inter-quartile range, median and outlier in their responses.In a Boxplot, the $25th$ percentile is described as the 25% and the $75th$ percentile as the 75%. The middle 50% falls between the $25th$ and the $75th$ percentiles. The distance between the $75th$ percentile and $25th$ percentile is the inter-quartile range. It measures the spread or differences in the data.

The line inside the box is the median or the $50th$ percentile. This line measures the similarities of the responses. Also, if the notches in the boxplots did not overlap, then it was concluded that with 95% confidence, there were statistical significances in the true medians. The whisker is the line that goes out from the box to the whisker boundaries. Any point outside the whisker boundaries is an outlier (Galarnyk, 2018).

In other to validate the findings of the qualitative data analysis, the researcher triangulated the information with the various schools, teachers and students (for instance, compare teachers' response on students' interactions evidence from the scores in the psychological test papers) (Creswell, 2013).

Ethical Procedures

The researcher followed a number of ethics procedures before the administration of the research instruments. Before the researcher embarked on the data collection, Academic Board of the Departments of Mathematics and ICT, and Science organized a proposal defence between 16th and 29th September, 2017. After the proposal defence, the Board through the University' Institutional Review Board approved the project and directed the researcher to contact the Supervisors and the Department of Mathematics and

ICT for an introductory letter (See Appendix A). The researcher sent the introductory letter to the regional directorate of the Upper East Region. The Directorate approved the project and copied to the University and the six senior high schools (See Appendix B).The Regional Director ensured only qualified elective mathematics tutors helped the researcher to carry out the project. This ensured that research participants are adequately protected. It also ensured that the researcher carried out the project in a minimal cost.

In each of the six senior high schools the researcher selected, each participant signed a consent form (See Appendix C). This was to ensure that the research participants' rights were projected. This was also to hold the researcher liable for research breaches against informed consent, anonymity, confidentiality, privacy, plagiarism and fraud. In some cases, some students demanded the researcher's identification numbers which was always available. The researcher also informed students of intended disseminations and publications of the study in workshops, seminars, conferences and journal publications. The data collection processes which started in earnest in October 2017 and ended in June, 2018 will be analyzed in the next chapter.

CHAPTER FOUR

RESULTS AND DISCUSSION

In this chapter, the results have been displayed in three parts. The first part displayed the results of the baseline survey. The baseline survey set the pace for the analysis. The baseline was divided into qualitative and quantitative. The qualitative baseline survey transcribed the interviews. This set the pace selecting appropriate didactical conceptual structures. The quantitative survey explored reliability measures, tests item analysis, factor extractions and multivariate tests. The reliability measures helped to establish the problem and justify the study. The factor extractions explored the most important variables in the questionnaire.

The multivariate tests selected two or more pairs of important independent variables. The MANOA tests included possible covariates in the models. The second part showed the results of the four main hypotheses in the didactical instructional models and the follow up research questions in the interview transcripts. The research questions analyzed students' knowledge, skills and experiences in extending the didactical triad to the tetrahedron. The transcripts sought to support, corroborate and confirm the results of the models and the baseline transcripts. The third part showed discussions of the findings. These discussions were partitioned according to the order of the models and the research questions. The discussions compared the findings of the study with literature.

Research Question 1: How does the baseline survey ensure reliability and validity of the study? And what are the possible covariates that affect the interactions of the didactical tetrahedron and what are the dominant factors that determine the interactions of the tetrahedron?

In this researcher question, the first part sought to measure reliability of the instruments. These were measured by Friedman's Chi-Square tests (internal consistency), Cronbach's alpha coefficient and item analysis. The second part of the research sought to ascertain the possible covariates and select dominant factors by exploratory factor analysis. The aim of the exploratory factor analysis is to obtain a new set of distinct variables, fewer than the original number of items. The third sought to ensure validity of the study especially the qualitative aspect. These helped to reduce high correlations and threats to internal validity (Pallant, 2011).

Reliability/Internal Consistency Analysis

Table 8 is the Friedman's test used to examine the internal consistency.

Table 8: Reliability Test of the Demographic Independent Variables

Source		Sum of Squares	df.	Mean Square	Friedman's Chi-Square	Sig.
Between People		453.393	494	.918		
Within People	Between Items	3019.277[a]	9	335.475	2017.138	.000
	Residual	3649.023	4446	.821		
	Total	6668.300	4455	1.497		
Total		7121.693	4949	1.439		

Source: Research data (Ali, 2017)

In the demographic independent variables, the Cronbach alpha coefficient for the independent variables was as low as .345. However, Table 1 shows that the internal consistency was statistically significant $[F(9,494) = 2017.138, p = 0.000]$. Pallant (2011) and Creswell (2014) suggest that the mean inter-item correlations can be used to report the consistency of the items between .2 and .4.The constructions of the instruments for the demographic variables were therefore internally consistent and reliable (see Appendix F1). However, students' gender did not only correlate high but positive most of the time. Out of the four didactical conceptual structures too, interactions with technologies had the highest number of inter-item correlations coefficients 0.20 and 0.40 (see Appendix F1). Therefore, the instruments were internally consistent and the variables were appropriate. However, the interactions were very low. The low interactions require further rigorous and robust statistical analysis. This will not only reveal which variable(s) influence the interactions more but also the reasons for the interactions.

Table 9 is the ANOVA test set up to test for the statistically significances of didactical instructional models. The models were intersubjective and sought to integrate the important parts for students learning. Unlike Table 8, the inter-total statistics were used to measure both internal consistency and item discrimination with the total score.

Table 9: Reliability Analysis of the Didactical Instructional Models

Source		Sum of Squares	df	Mean Square	F	Sig
Between		5974.933	495	12.071		
Within	Between Items	4303.832	29	148.408	118.573	.000
	Residual	17967.035	14355	1.252		
	Total	22270.867	14384	1.548		
Total		28245.800	14879	1.898		
Grand Mean = 2.52						

Source: Research data (Ali, 2017)

The overall Cronbach alpha coefficient for the didactical instructional models was .897. This Cronbach's alpha coefficient is above .70. So, the scales were considered internally consistent, reliable and statistically significant $[F(29,495) = 118.573, p = 0.000]$. Even though statistically significant and high Cronbach alpha coefficient, it was incumbent to add the inter-total statistics. These statistics helped to explain the degree to which each item correlates with the total score in the models. The impacts of removing any of the components (i.e. see *Cronbach Alpha if Item Deleted* on Appendix F2) in the main study did not make any statistically differences in the models. This is because the overall Cronbach alpha coefficient was still more than .70. Also, because the scales were well validated, removing any item from the baseline would have further impacted on the interactions of the didactical conceptual structures. This confirmed that the didactical instructional models were appropriate potential dependent variables in the study.

Table 10 describes the tasks in sample tasks in equations of the circle. The sample tasks were drawn from types of circle equations, ways of solving for the centre and radius of a circle and applications.

Table 10: Reliability of Sample Tasks in Equations of the Equations

		Sum of Squares	df	Mean Square	F	Sig.
Between People		1506.418	494	3.049		
Within People	Between Items	422.743	17	24.867	35.552	.000
	Residual	5874.034	8398	.699		
	Total	6296.778	8415	.748		
Total		7803.196	8909	.876		
Grand Mean = 1.79						

Source: Research data (Ali, 2017)

In the sample tasks, the Cronbach alpha coefficient was .780 and the Friedman's test was statistically significant $[F(17,495) = 35.552, p = 0.000]$. Also, the inter-total statistics (see Appendix F3) show that all the items in the *Cronbach Alpha If Items Deleted* column were less than the overall Cronbach alpha of .780. Therefore, this means the sample tasks were internally consistent and should be retained in the study. The statistical significance was also due to the excellent constructions and administration of the baseline instruments. This ensured total and complete interactions among the didactical conceptual structures in solving equations of the circle. In order words, every students took participate in the study as a result of the excellent experiences, interesting and most enjoyable test items the researcher chose for the study.

Test item analysis

The *Corrected Item-Total Correlation* was used to measure item discrimination as the corrected point biserial correlation (see Appendix F2). It was shown that all items correlated well except artefacts and innovations. Since artefacts loaded high in the exploratory factor analysis (see Table 10), it was necessary that we deleted only innovations. This increased the overall Cronbach's alpha coefficient to .897. Indeed, the low value of the *Alpha if Item Deleted* (see Appendix F2) confirms that innovations should be deleted.

On the other hand, low point-biserial values meant that students who got the item incorrect also scored high on the overall test, while students who got the item correct scored low on the test overall (Yu, 2015). Therefore, the items such as innovations, artefacts and tasks with low point-biserial values needed further examinations. The researcher refined the wording, presentation and content of these items. However, innovations were completely removed from scoring and future testing because it felt below 0.25 (see Appendix F2).

Again, the item difficulty (p-value) was analyzed by the *Mean Column* (see Appendix F4). The scale ranged from 1 to 5 with an average difficulty of 2.517. The item difficulties vary between 1.97 and 4.33. This means that the mixture was not very high, many items were very difficult (i.e. had low p-values) and a few very cheap. Once again, the items on innovations (4.33) and artefacts (4.10) were so cheap and needed complete overhaul. Therefore, the researcher revised and restructured the items and deleted innovations from the study.

Analysis of One-Way Covariates

On Table 11, it was revealed that the dominant dependent variables were mathematizing within tools, mathematising within instruments, mathematising within contents, praxeologies in two structures, mathematising within technologies, mathematising within artefacts and mathematising within signs and symbols. There were ten independent variables in this study, namely the six attributes (demographic variable) and the four treatment(didactical conceptual structures) independent variables (see Appendix F1). Each of these six attributes was used as a covariate to reduce or control their effects in the didactical interactions. It was evident that the marginal means and statistical powers of the six proposed covariates were quite high and really affected the interactions (see Appendix H). In fact, in the exploratory factor analysis, it emerged that the six covariates actually had high correlations between .2 and .4 (see Appendix G) in order to confirm their potentials for being covariates.

Gender as a Covariate

It was shown that gender affected interactions in the didactical tetrahedron and was justified to be covariate. It was therefore tested and removed from the interactions with tools, instruments, contents, praxeologies, technologies, artefacts, and signs and symbols. Although the t-test alone showed that the students' gender did not significantly differ (see Table 15), this could not have been a better analysis. This is because the significance test was only used for making inferences about the population. However, the most important issue was whether the students themselves differ in gender. In this case, gender did not attain statistical significance in solving equations of the circle. Indeed, it was after confounding gender, that the interactions of the didactical tetrahedron across the didactical instructional models attained statistical significances.

School as a Covariate

School was also detected as a covariate. When tested against the dominant factors, students' interactions were statistically significant in most factors $(p \leq .05)$ with over 90% power except in 2T praxeologies and technologies $(p > .05)$. However, school became a covariate because some dependent variables were not statistically significant. It was therefore necessary to confound school in order to attain total statistical significances or reduce the effect of the covariate (see Appendix H2).

Level as a Covariate

Students' forms were included as a covariate. When tested against the dominant factors, students' interactions were statistically significant in four factors $(p \leq .05)$ with over 90% power. However, school became a covariate because two didactical praxeologies and technologies were not statistically significant. It was therefore important to control class level in other to improve upon statistical significances (see Appendix H3).

Programme as a Covariate

Students' programmes were included as a covariate. Out of the seven dominant factors, students' interactions were statistically significant in five factors $(p \leq .05)$ with over 80% power. However, artefacts and technologies were not statistically significant. It was therefore important to control programmes in order to reduce error variances (see Appendix H4).

Other Demographic Information as Covariates

In addition to gender, school, level and programme, there were other demographic factors that affected students' interactions in the didactical instructional models. Some of these were managements of schools, residential statuses of students, ICT experience and streams of schools. Though potential covariates, the researcher limited the discussion to the two most important covariates, namely gender and school. This was to avoid making the discussion verbose, repetitive and ambiguous (Kibriya et al, 2016).

Analysis of Two-Way and Three-Way ANCOVA

On the analysis of two-way ANCONVA, students and teachers' interactions in the didactical tetrahedron were generally statistically significant (see Appendix H5) after controlling gender. The only exception was the interactions in 2T only praxeologies and technologies. This was not different from the three-way ANCOVA interactions involving students, teachers and contents' interactions (see Appendix H6), and involving students, teachers and technologies (see Appendix H7). It was therefore necessary to reduce the effects of gender in the interactions in the tetrahedron across the didactical instructional models. This helped to improve upon students and teachers' interactions. Having dealt with gender, the second worrisome covariate was school. On Tables 4 to 9, it was revealed that school also had an effect on the interactions. Having controlled the effects, almost all components in the models attained statistical significances too.

Analysis of MANOVA and MANCOVA

In the multivariate tests with gender and school as covariates, the interactions of students and mathematics content improved significantly. Both gender and school was also statistically significant (see Appendix H6). However, with the inclusion of technology tools, only school and students interactions were statistically significant (see Appendix H7) across the models. This showed how deep and perverse gender and school jointly affected interactions in the didactical tetrahedron and needed special attention. The control of the two covariates actually reduced error variances.

In comparing Pillai's Trace, Wilks' Lambda, Hotelling's Trace and Roy's Largest Root coefficients in the MANCOVA tables, the Wilks' Lambda statistic was more statistically significant than the others. It actually explained over 90% of the covariates than the other three (Seltman, 2015). The Partial Eta effect sizes were quite large but still had impacts because the sample size was large (Caswell, 2011) (see Appendix H7).

Exploratory Factor Analysis

Table 10 shows the initial exploratory factor analysis conducted on 30 items across the didactical instructional models using SPSS version 20. The table displays only the commonalities of the factors. The rest of the statistics can be found in Appendix G.

Table 11: Initial Communalities in the Didactical Instructional Models

Components	Initial	Extraction
Mathematising within Signs and Symbols	1.000	.600
Mathematising within Tools	1.000	.719
Mathematising within Artefacts	1.000	.619
Mathematising within Instruments	1.000	.677
Mathematising within Technologies	1.000	.639
Mathematising within Contents	1.000	.674
Praxeologies in Two Structures	1.000	.659

Source: Research data (Ali, 2017)

In the pre-intervention or baseline study, the researcher extracted seven dominant factors. These were tools (.719), instruments (.677), contents (.674), 2T praxeologies (.659), technologies (.639), artefacts (.619) and, signs and symbols (.600). These seven factors were used to determine the pattern of the structure in the 58-item didactical instructional models in the tetrahedron.

In the initial analysis to obtain eigenvalues for each factor, the $KMO = .919$ was above Kaiser's threshold of 0.6 and the sampling adequacy from the Barlett's test of sphericity was statistically significant $[\chi^2 = .919, p < .000]$. This indicated that correlations between items were sufficiently large for the exploratory factor analysis.

Also, the Scree plot clearly shows that there were seven dominant factors in this study (see Appendix G2). These factors from the initial 58-item structure explained only 53.54% of the variance in the pattern of interactions in the didactical tetrahedron. The percentages explained by each factor were tools (15.991), instruments (10.220), contents (7.711) and 2T praxeologies (5.855). The rest were technologies (5.351), artefacts (4.467) and, signs and symbols (3.946) (See Appendix G1). Based on the results of this initial exploratory factor analysis, there were ten items which loaded on factor 1, four on 2, three on 3, two on 4, and one each in 5, 6 and 7. There was neither cross loading nor negative loading (see Appendix G3).

The researcher could therefore, labelled the new latent variables as instrumentations (Factor 1), 4T praxeologies (Factors 2 and 3), and mathematizations (Factors 4 to 7). In other words, many subsequent discussions concentrated on mathematization, instrumentations and 4T praxeologies to determine the interactions in the tetrahedron across the didactical instructional models in solving equations of the circle.

Table 12 shows the initial exploratory factor analysis conducted on 18 items of the didactical instructional models using SPSS. It displays the commonalities of the factors.

Table 12: Initial Communalities in Equations of the Circle

Components	Initial	Extraction
Circle Shape	1.000	.864
Circle Angle	1.000	.631
Circle Dimension	1.000	.917
Three Basic Circle Parts	1.000	.890
Circle Standard Equation	1.000	.914
Circle Radius-Centre Equation	1.000	.909
Circle General Equation	1.000	.747
Circle Diameter Equation	1.000	.666
Circle Standard Tangent Equation	1.000	.687
Circle Standard Normal Equation	1.000	.728
Circle at Centre (*0,0*) with no Constant *c*	1.000	.779
Circle at Centre (*h,0*) with No Constant *c*	1.000	.629
Circle at Centre (*0,k*) with No Constant *c*	1.000	.714
Circle at Centre (-*h*,-*k*) with No Constant *c*	1.000	.606
Circle at Centre (-*h*,-*k*) with Non-Unit Coefficients	1.000	.741
Circle at Centre (*h*, *k*) with Non-Unit Coefficients	1.000	.769

Source: Research data (Ali, 2017)

In the baseline study in sample equations and tasks of the circle, five dominant factors were extracted. There were circle dimension (.917), standard equation (.914), radius-centre circle (.909), three parts of the circle (.890) and circle shape (.864). These five factors the structure of the 18-item conceptualizations in equations and tasks of the circle. All other items cloud around these five dominant factors to determine the interactions of the didactical conceptual structures.

In this initial analysis of the baseline study, the Kaiser-Meyer Olkin Measure, $KMO = .811$ was above Kaiser's threshold of 0.6. This shows that the sample was adequate for the analysis. Barlett's test of Sphericity was statistically significant $[\chi^2 = 1.417x10^3, p < .000]$. This shows that correlations between the items were sufficiently large for factor extraction. The Scree plot shows horizontal asymptote of the 18 components after five factors. This confirms that five factor interacted more and should be selected for the didactical instructional (see Appendix G4).

In the eigenvalues greater than one, as the Scree plot clearly illustrates in Appendix G4, the initial 18-item structure explained 44.44% of the variance among the items. The percentages explained by each dominant factor were circle dimension (21.702), standard equation (9.323), radius-centre circle (6.824), three parts of the circle (5.995) and circle shape (5.707) (see Appendix G5). Based on these initial dominant factors, many factors were cross loading.

According to Pallant (2005), if there are cross loadings on the dominant factors, one needs to direct oblinim (pattern matrix) (see Appendix G6). The rotated pattern matrix shows that four items load on factor one, two factors load on factor two, three items on factor three, four items on factor four and the rest load on one item each. Clearly, the first five factors had high loadings and predict the structure of the baseline survey. These high loadings give ample credence to strong interactions in the equations and tasks of the circle.

Validity of the Interactions in the Didactical Conceptual Structures

In order to ensure validity of the instruments and analyze the data appropriately, the researcher interviewed the students on the methods and strategies of their everyday classroom interactions. This formed another basis to explore the interactions among fours didactical conceptual structures.

Figure 5 shows the box plot of the structured and semi-structured interview guide. The frequency of interactions were divided into 'most often', 'very often', 'often', 'scarce' and 'never'. These divisions sought to simplify the results of the baseline interactions.

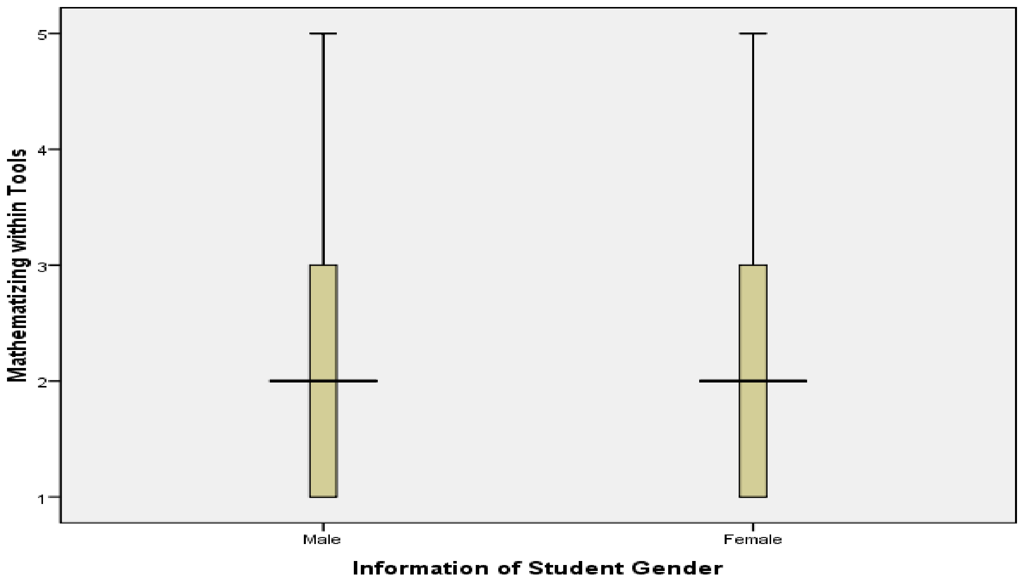

Figure 5: Initial gender interactions in mathematisations (Research data, 2017)

On Figure 5, it can be observed that the frequency of interactions was not normal. The whiskers were directed to 4 (scarce) and 5 (never).However, the responses were very similar and uniform at 2 (often). This means the interactions among the didactical conceptual structures were initially poor and required immediate interventions. The following baseline transcripts validated this claim.

Students' interactions in the baseline survey

In the transcripts in students' interactions, the students contended that they *scarcely* assimilated the teacher's demonstrations, procedures, sequences and algorithms in mathematics problems, tasks and practices. Neither did they experience a wide variety of mathematics textbooks, curriculum and syllabi that enhance understanding. This means there were no opportunities to apply mathematics to daily life. They could not create innovative techniques and strategies for learning the mathematics content.

Worst still, the students *never* provided any routine procedures with the variety of signs, symbols, artefacts, tools, instruments, computers or calculators or other technologies. They *never* assessed and/or evaluated their teacher's lesson objectives, learning outcomes, oral or written exercises and test or examination items. However, in order that they equipped themselves with mathematics vocabulary, mental structures and personal library, they badly need the knowledge and use of the variety of signs, symbols, artefacts, tools, instruments and technologies. These objects could have helped them to create their reflective journals and case cards. This would have helped them to demonstrate understanding and transform their mathematics thinking over time. A sample of some students' transcripts suggested more dissatisfactory interactions:

Student 'B': Sometimes if we do not know the topics we consult our friends to find the information. Some teachers we contact sometimes intimidate and scare us. And if we do not question them well they teachers quickly tell us to do research on the issues concerned.

Student 'C': Not all that much that we understand all the procedures. Our teachers use the prescribed methods of teaching but we do not understand how they arrive at the solutions.

Student 'E': We usually compete in academic sense to get good positions and improved performance. However, we do not discuss our challenges with our own mates.

Teachers' interactions in the baseline survey

In this baseline transcripts in teachers' interactions, the responses of most students showed that teachers *scarcely* created open, informal, congenial, democratic and free atmospheres. They neither provided sources of information, materials and resources to students. They failed to develop instructional signs, symbols, artefacts, tools, instruments and technologies in the classroom except those provided by the Ghana Education Service. They *never* associated signs, symbols, artefacts, tools, instruments and technologies with teaching objectives, strategies, methods and outcomes. *Scarcely* did they evaluate their learning outcomes with signs, symbols, artefacts, tools, instruments, computers or calculators or other technologies.

Again, the teachers *never* monitored each student's participation and progress in the mathematics classroom. This by extension means that the teachers did little remediation and timely feedbacks, using both formal and informal assessment strategies. In teaching, the *scarcely* connected mathematics content with signs, symbols, artefacts, instruments, tools and technologies. Apart from syllabi provided by the Ghana Education Service, the teachers *scarcely* brought a variety of books, syllabi and course outlines on daily basis. Therefore, apart from the routine activities, little innovation was brought to stimulate discussions of mathematics tasks, problems and solutions. These made the applications of signs, symbols, artefacts, tools, instruments and technologies very difficult. Some students made the following comments relative to their interactions with teachers:

Student 'A': It is not always all the time that the teacher addresses our problems. Scarcely do we see the teachers to help us solve our problems.

Student 'C': They use textbooks like AKI-Ola and Akrong series to interact with us. They mostly just introduce the topic and give us assignments and examinations.

Student 'E': The teachers do not believe in some of the mathematics textbooks. The teachers do not utilize signs and symbols in equations of the circle. We lack some of the assets to help us interact with the teacher.

Mathematics content's interactions in the baseline survey

The baseline transcripts in mathematics content showed that most students ***scarcely*** interact with the mathematics content. Interactions in the mathematic content are vital in establishing and maintaining orderly, sequentially, logically and systematic patterns of instructions, unified themes or topics for teaching and learning. These help students to achieve their learning purposes, goals, objectives, tasks and directions. These also help teachers to correctly apply the appropriate techniques, methods and procedures. Signs, symbols, artefacts, tools, instruments and technologies equally provide orderly, sequentially and logically cue. In the end, the students would comfortably solve mathematics tasks.

Again, students contended that the mathematics content ***never*** provided diverse solution paths, orderly mathematics problems and worked out examples to the teachers. The mathematics contents only contained topics, tasks, story problems and questions, assessment and evaluation procedures. However, mathematics contents provide excellent instructional techniques and strategies. Some of these techniques require knowledge and understanding of specific signs, symbols, artefacts, tools, instruments and technologies. The baseline study showed that these knowledge and skills were woefully inadequate as portrayed in the following transcripts.

Student 'A': The mathematics content in equations of the circle ok. However, some solutions to the questions are incorrect and some textbooks have typing errors.

Student B': The mathematics textbooks provide us with only formulas and procedures. The worked examples provide no strategies and techniques. Worst still, the textbooks provide no alternative strategies and techniques of solving the problems and tasks.

Student 'C': The textbooks are not in the order in which the syllabi have been structured. The topics have not been orderly arranged and many contents have scarce explanations even with the difficult language/

Student 'D': Some textbooks work tasks and obtain different solutions as we know and learn. Worst still, the worked answers are sometimes incorrect.

Student 'E': Some textbooks have been orderly arranged. However, we do not understand the contents without being guided and taught by our teachers. This is because some of the contents are unfamiliar to learn without teachers.

Technology tools' interactions in the baseline survey

The baseline transcripts in technologies interactions showed that most students *scarcely* knew the signs, symbols, artefacts, tools, instruments and technologies in mathematics. Even though students encountered signs, symbols, artefacts, tools, instruments and technologies, they *scarcely* employed these objects to work mathematics problems. In some cases, they *never* appreciated that signs, symbols, artefacts, tools, instruments and technologies. Furthermore, they *never* related any technologies to the signs, symbols, artefacts, tools and instruments. However, relating these objects to technologies ensures smooth, logical and coherent transference. This knowledge also challenges and provides clues, innuendos and technical support during teaching and learning. For further clarity, the transcripts were summarized as follows:

Student 'B': Even though we do not have access to the computer, the calculators are very good. The calculators help during examinations to finish computations in time.

Student 'D': Some of them are good. However, the scientific calculators and graph sheets are not standard, and their outputs do not usually correspond to one another.

Student 'E': The signs and symbols are not transferable due to lack of linkages. The calculator does not work alone too.

On giving their impressions, feelings and opinions about the didactical interactions, the students ubiquitously stated as follows:

Student 'A': We have not been able to solve some tasks due to inadequate and even lack of knowledge and skills in these interactions.

Student 'B': The didactical relationships are good and must be recommended. However, the arrangements of the mathematics contents confuse us and inhibit our potentials to solve tasks and equations of the circle orderly ad sequentially.

The analysis of the transcripts in the baseline survey supported and corroborated with the results of the Cronbach's alpha reliability tests, exploratory factor analysis, covariate analysis and interview transcripts. Both outcomes helped to establish the problem and devise techniques to analyze the main study. The conclusions drawn were that the instruments were valid and reliable, seven variables were dominant and two covariates were instrumental. In the forthcoming results, the researcher will present the main analysis win the seven dominant factors with the covariates.

Research question 2: What is the knowledge level of students in solving equations of the circle before the implementation of the didactical tetrahedron?

The researcher used the new latent variables, namely instrumentations, 4T praxeologies and mathematizations to explore this research questions. These variables were the main variables extracted in the exploratory factor analysis. These variables best determined the interactions in the tetrahedron across the didactical instructional models in solving equations of the circle. Table 12 shows ANOVA among students' interactions in the didactical tetrahedron across mathematisation models before implementation of the didactical tetrahedron.

The components that were analysed in the table are signs/symbols, artefacts, tasks, instruments, tools and technologies. The main essence was not just to compare the differences among these components but also to test students' knowledge in each of the components. It afforded an opportunity to the researcher to select the best and most appropriate components inclusion in the subsequent experimentations in the second part of the data collection.

Table 13: ANOVA of Students' Knowledge before Implementation of the Tetrahedron across Mathematisations

Components		Sum of Squares	df	Mean Square	F	Sig.	Eta squared
Signs/Symbols	Between-Groups	43.870	4	10.967	7.653	.000	.06
	Within Groups	703.614	491	1.433			
	Total	747.484	495				
Artefacts	Between-Groups	3.605	4	.901	.554	.696	.004
	Within Groups	798.554	491	1.626			
	Total	802.159	495				
Tasks	Between-Groups	20.255	4	5.064	2.679	.031	.02
	Within Groups	927.985	491	1.890			
	Total	948.240	495				
Instruments	Between-Groups	25.209	4	6.302	3.888	.004	.03
	Within Groups	795.816	491	1.621			
	Total	821.024	495				
Tools	Between-Groups	34.154	4	8.539	4.814	.001	.04
	Within Groups	870.957	491	1.774			
	Total	905.111	495				
Technologies	Between-Groups	15.365	4	3.841	1.860	.116	.01
	Within Groups	1013.875	491	2.065			
	Total	1029.240	495				

Source: Research data (Ali, 2017)

Table 13 shows the knowledge level of students in solving equations of the circle before the implementation of the didactical tetrahedron across mathematization. A one-way between-groups ANOVA was conducted to explore the students' knowledge before the implementation of the tetrahedron. The students were divided into five groups according to the number of students (Group 1: less than forty; Group 2: less than fifty; Group 3: less than sixty; Group 4: sixty and more; Group 5: others). There was a statistically significant difference at the $(p < .05)$in students' interactions across four components except artefacts [F(4, 495)=.554, P=.696] and technologies [F(4, 495)=1.860, P=.116] for the five class groups.

Despite four reaching statistical significance, the actual difference in mean scores between the groups were quite small. The effect size, calculated using eta squared, ranged between .004 and .06. Post-hoc comparisons using the Tukey HSD test indicated small mean scores for the groups. The mean scores did not differ significantly among all the groups in artefacts and technologies (see Appendix J1). Therefore, there was no interaction with respect to artefacts and technologies before the implementations of the didactical tetrahedron.

Table 14 shows the knowledge level of students in solving equations of the circle before the implementation of the didactical tetrahedron across mathematization. It sought to illuminate how the successive combinations of the four didactical conceptual structures helped to implementing the didactical tetrahedron.

Table 14: ANOVA of Students' knowledge before implementation of the tetrahedron across instrumentations

Structures	Groups	Sum of Squares	df	Mean Square	F	Sig.	Eta squared
One Structure	Between Groups	5.482	4	1.371	.947	.437	.007
	Within Groups	710.768	491	1.448			
	Total	716.250	495				
Two Structures	Between Groups	10.173	4	2.543	1.897	.110	.015
	Within Groups	658.260	491	1.341			
	Total	668.433	495				
Three Structures	Between Groups	3.908	4	.977	.665	.616	.005
	Within Groups	720.834	491	1.468			
	Total	724.742	495				
Four Structures	Between Groups	10.673	4	2.668	1.584	.177	.013
	Within Groups	826.873	491	1.684			
	Total	837.546	495				

Source: Research data (Ali, 2017)

On Table 14, a one-way between-groups ANOVA was conducted to explore the students' knowledge before the implementation of the tetrahedron across instrumentations. The students were divided into five groups according to the number of students in the elective mathematics class (Group 1 represented less than forty; Group 2 less than fifty; Group 3 less than sixty; Group 4 sixty and more; Group 5 others). Generally, there was no statistically significant difference $(p > .05)$ in students' interactions across all the four components. The effect size, calculated using eta squared, ranged between .005 to .015. Post-hoc comparisons using the Tukey HSD test indicated small mean scores for the groups. The mean scores did not differ significantly among all the groups in artefacts and technologies (see Appendix J2). Therefore, there was ample evidence to conclude that no interactions existed across instrumentations before the implementations of the didactical tetrahedron.

Research Question 3: What is the knowledge level of students in solving equations of the circle after the implementation of the didactical tetrahedron? The following two hypotheses were posed to guide this research question 3:

H_{01}: There is no statistically significant difference in students' performance after interactions in the didactical tetrahedron.

H_{02}: There is no statistically significant difference in students' performance across the didactical instructional models after implementation of the didactical tetrahedron.

The researcher sought to use ANCCOVA and MANCOVA to answer research hypotheses 1 and 2 respectively. In this case, the contributions of the covariates from the adjusted means were paramount. However, before employing the ANCCOVA and MANCOVA, another exploratory factor extraction was done (see Appendix I4). This time, the communality values were quite higher than the values before the intervention. Ten variables had less than .30 and were deleted in order to reveal the true structure of the factors (Yu, 2015). Factors that also cross-loaded were deleted. Unlike the pre-intervention survey, ten items loaded for factor 1, five items for factor 2, three items for factor 3, two items each in factors 4 and 5, and one item each for factors 6 and 7. The final communalities showed remarkable increases as compared to the initial factor extraction (See Appendix I). This is because the intersubjective didactical instructional models in improved the structure of the items in the baseline survey.

Finally, the seven-item structure was found to explain 84.95% of the variance in the pattern of relationships among the items as shown in Appendix I1 and I2. The Varimax factor correlation between the main factors revealed high correlations of about .80 or more (See Appendix I3).In the final seven-factor structure, it was revealed that there were as many as eight items that cross-loaded. In fact, Yu (2015) suggested deleting those items will yield better interpretation of the factor structure because these items are not statistically significant. Care needs to be taken when deciding to delete such items. Some dominant factors can be cross-loaded but cannot be deleted. However, minor items were readily deleted in order to simplify the analysis. Such items will not found space in the results and discussion of the t-test, ANCOVA, MANCOVA and interview transcripts.

Table 15: Independent Samples T-Test of Students' Knowledge after Implementations of the Tetrahedron

	Levene's Test for Equality		t-test for Equality of Means				
	F	Sig.	t	df	Sig. (2-tailed)	95% Confidence Interval of the Difference Lower	Upper
Signs/Symbols	7.417	.007	-3.235	494	.001	-.569	-.139
Artefacts	.909	.341	.353	494	.724	-.184	.265
Tools	.811	.368	1.344	494	.180	-.075	.402
Instruments	8.327	.004	2.438	494	.015	.054	.507
Technologies	.093	.761	-1.473	494	.141	-.445	.064

Source: Research data (Ali, 2018)

Table 15 shows an independent-sample t-test that was conducted to compare students' knowledge in the didactical instructional models in the tetrahedron implemented using gender. There were significant differences in artefacts [$t(494)=.353$, $p=.724$], tools [t(494)=1.344, p = .180] and technologies [t(494)= -1.473, p = .141]. The magnitude of the differences in the means were very small (eta squared<.001). However, there were no significant differences in signs/symbols [t(494)=-3.235, p=0.001] and instruments (p=.015). The magnitude of the differences in the means was equally very small. This outcome yielded almost a split decision on the statistical significances across the didactical instructional models after the tetrahedron was implemented. And this certainly required a much more robust statistical tool like ANCOVA and MANCOVA. ANCOVA and MANCOVA did not only detect the discrepancies in the statistical differences but also did confound the covariates that affected the results. The ANCOVA and MANCOVA analyses have dealt in details the results of the didactical instructional models across the three intersubjective frameworks.

Across Mathematisation and Didactical Phenomenology Models

Table 16 shows a one-way between-group ANCOVA. This table compared between-subjects effects of the didactical tetrahedron across mathematisation and phenomenology. This table begins the analysis of the second part of the data collected. The table contains the dependent variable, the independent variables and the covariate. The main statistics used for the analysis are the significance and Partial Eta squared coefficients.

Table 16: One-way ANCOVA of Students' Knowledge after Implementations of the Tetrahedron across Mathematisations

Source	Type III Sum of Squares	df	Mean Square	F	Sig.	Partial Eta Squared
Corrected Model	39.248[a]	7	5.607	3.863	.000	.053
Intercept	48.545	1	48.545	33.449	.000	.064
Mathematics Content	15.211	3	5.070	3.494	.016	.021
Gender	1.151	1	1.151	.793	.374	.002
Mathematics Content * Gender	12.867	3	4.289	2.955	.032	.018
Error	708.236	488	1.451			
Total	2668.000	496				
Corrected Total	747.484	495				

a. R Squared = .053 (Adjusted R Squared = .039)

Source: Research data (Ali, 2018)

In Table 16, the independent variables were the components of mathematisation and phenomenology (signs and symbols). Gender groups were used as covariates. After adjusting for gender differences, there were statistically significant differences in the mathematics content [F(3,488)=3.494, p=.016, partial eta squared=.021]. The interaction effects between mathematic content and gender was also statistically significant [F(3,488)=2.955, p=.032, partial eta sq.=.018]. However, gender itself was not statistically significant $[F(1,488) = 0..783, p = .0374, partial\ eta\ sq. = .002]$ and had no independent effects. Because students were not randomly assigned to the groups, confounding gender only significantly reduced the differences of group means.

Also, the partial eta squared coefficients described how much of the variances in mathematisation and didactical phenomenology were explained by the significant didactical conceptual structures. The small partial eta squared values revealed that there were no strong relationships between the didactical conceptual structures in mathematisation and phenomenology. This notwithstanding, the estimated marginal adjusted the mean scores and statistically removed their effect by 1.49.

Table 17 shows the MANOVA between-groups tests. These tests compared the effects of the didactical conceptual structures on mathematisation and phenomenology after implementation of the didactical tetrahedron.

Table 17: MANOVA Tests of Students' Knowledge after Implementations of the Tetrahedron across Mathematisations

Source	Type III Sum of Squares	df	Mean Square	F	Sig.	Partial Eta Squared
Corrected Model	87.078[a]	29	3.003	2.119	.001	.116
Intercept	9.822	1	9.822	6.931	.009	.015
Students Interactions	14.486	4	3.622	2.555	.038	.021
Gender	3.873	1	3.873	2.733	.099	.006
School	.038	1	.038	.027	.870	.000
Gender * School	.124	1	.124	.088	.767	.000
Technology Interactions	4.802	4	1.201	.847	.496	.007
Students Interactions * Technology Interactions * Gender * School	18.768	18	1.043	.736	.775	.028
Error	660.405	466	1.417			
Total	2668.000	496				
Corrected Total	747.484	495				

a. R Squared = .116 (Adjusted R Squared = .062)

Source: Research data (Ali, 2018)

The results on Table 17 show that the effects of students' interactions in mathematisation and phenomenology were statistically significant. It was revealed that after adjusting for gender and school as joint covariates, there were generally significant differences in mathematisation and didactical phenomenology models with respect to students' interactions $[F(4,466) = 2.555, p = .038, partial\ eta\ sq. = .021]$.

However, their interaction effects[F(1,466)=.735, p=.775, partial eta sq.=.28]and each of the covariates, gender [F(1,466)=2.733, p=.099, partial eta sq.=.006] and school[F(1,466)=.027, p=.870, partial eta sq.=.000] were not statistically significant. This notwithstanding, gender and school statistically reduced group differences. In addition, the partial eta squared values were equally small (0.038 for gender) or (0.000 for school). The estimated marginal means showed that gender and school did controlled and statistically removed the effects of the covariates.

Across Instrumentations and Instrumentalisations Models

Table 18 shows results of one-way between-groups ANCOVA. These results compared the between-subjects effects of the didactical conceptual structures on the instrumentations and instrumentalisations models.

Table 18: One-Way ANCOVA of Students' Knowledge after Implementations of the Tetrahedron across Instrumentations

Source	Type III Sum of Squares	df	Mean Square	F	Sig.	Partial Eta Squared
Corrected Model	17.204[a]	8	2.150	1.486	.159	.024
Intercept	11.568	1	11.568	7.995	.005	.016
Gender	6.303	1	6.303	4.356	.037	.009
Technology Interactions	11.371	3	3.790	2.620	.050	.016
Technology * Gender	11.584	3	3.861	2.669	.047	.016
Error	704.633	487	1.447			
Total	3477.000	496				
Corrected Total	721.837	495				

a. R Squared = .024 (Adjusted R Squared = .008)

Source: Research data (Ali, 2018)

On Table 18, the independent variable was technology interactions and the dependent variables were the instrumentations and instrumentalisations models, and the covariate was gender groups. After adjusting for gender differences, there were statistically significant differences in the interactions in instrumentations and instrumentalisations models [F(3,487)=2.620, p=.050, partial eta sq.=.016]. The interaction effects [F(3,487)=2.669, p=.047, partial eta sq.=.016] as well as gender effects [F(3,487)=4.356, p=.037, partial eta sq.=.008] were equally statistically different.

Indeed, gender actually directly affected the scores in mathematisation and didactical phenomenology models and required the confounding. Even though the partial eta squared values showed small effects, the estimated marginal means showed that gender controlled the error variance by 1.49.

Table 19 shows two-way between-groups MANCOVA tests. The tests compared the effects of the didactical conceptual structures on instrumentations and instrumentalisations.

Table 19: MANCOVA Tests of Students' Knowledge after Implementations of the Tetrahedron across Instrumentations

Source	Type III Sum of Squares	df	Mean Square	F	Sig.	Partial Eta Squared
Corrected Model	80.405[a]	23	3.496	2.103	.002	.093
Intercept	22.116	1	22.116	13.305	.000	.027
Students Interactions	11.095	4	2.774	1.669	.015	.014
Teachers Interactions	6.333	3	2.111	1.270	.028	.008
School	.352	1	.352	.212	.646	.000
Gender	.424	1	.424	.255	.614	.001
Students Interactions * Teachers Interactions * School * Gender	16.223	13	1.248	.751	.712	.020
Error	782.920	471	1.662			
Total	4030.000	495				
Corrected Total	863.325	494				

a. R Squared = .093 (Adjusted R Squared = .049)

Source: Research data (Ali, 2018)

On Table 19, the independent variables were students and teachers' interactions and the dependent variables were the instrumentations and instrumentalisations models. Students' gender and school were the covariates. After adjusting for the two covariates, there were significantly statistical differences in students' and teachers' instrumentations and instrumentalisations effects [F(4,471)=1.669, p=.015, partial eta sq.=.014]and[F(4,471)=1.270, p=.028, partial eta sq.=.008] respectively. However, their interaction effects [F(4,471)=.7751, p=.712, partial eta sq.=.020], the covariates, gender [F(4,471)=.255, p=.614, partial eta sq.=.001] and school [F(4,471)=.212, p=.0646, partial eta sq.=.00] were not statistically significant. Despite these, the partial eta squared values explained by gender (0.014) and school (0.000) showed very small effects. The covariates statistically reduced the differences. This was manifested by the contribution of the estimated marginal means.

Across Didactical Situations and Anthropological Praxeologies Models

Table 20 shows one-way between-groups ANCOVA tests. The tests compared the between-subjects effects of the didactical conceptual structures on didactical situations and anthropological praxeologies models.

Table 20: One-way ANCOVA of Students' Knowledge after Implementations of the Tetrahedron across Praxeologies

Source	Type III Sum of Squares	df	Mean Square	F	Sig.	Partial Eta Squared
Corrected Model	26.305[a]	8	3.288	1.977	.048	.031
Intercept	4.987	1	4.987	2.998	.034	.006
School	12.757	1	12.757	7.668	.006	.016
4T Praxeologies	11.701	3	3.900	2.345	.042	.014
4T praxeologies * school	11.804	3	3.935	2.365	.034	.014
Error	810.138	487	1.664			
Total	3730.000	496				
Corrected Total	836.444	495				

a. R Squared = .031 (Adjusted R Squared = .016)

Source: Research data (Ali, 2018)

On Table 20, the independent variable was school and the dependent variables were didactical situations and anthropological praxeologies models. The models were administered after the experiments. It was revealed that there were statistically significant differences [F(3,487)=2.365, p=.034, partial eta sq.=.014]. Not only did the interaction effects became statistically significant [F(3,487)=2.365, p=.042, partial eta sq.=.014], but also school [F(3,488)=7.668, p=.006, partial eta sq.=.016]. This means there were differences among the schools, and these differences were accounted for by the covariates as supported by the estimated marginal means.

Table 21 shows between-groups MANCOVA tests. The tests compared the effects of the didactical conceptual structures in didactical situations and anthropological praxeologies models.

Table 21: MANCOVA Tests of Students' Knowledge after Implementations of the Tetrahedron across Praxeologies

Source	Type III Sum of Squares	df	Mean Square	F	Sig.	Partial Eta Squared
Corrected Model	96.403[a]	27	3.570	2.146	.001	.110
Intercept	35.613	1	35.613	21.400	.000	.044
Students Interactions	10.280	4	2.570	1.544	.018	.013
Gender	1.846	1	1.846	1.109	.293	.002
School	.607	1	.607	.365	.546	.001
Content Interactions	17.701	3	5.900	3.546	.015	.022
Students * Content * Gender *School	36.033	18	2.002	1.203	.254	.044
Error	778.815	468	1.664			
Total	3692.000	496				

Source: Research data (Ali, 2018)

The results on Table 21 were obtained after the experiments. Students' gender and school were detected as covariates. After adjusting for these covariates, there were significantly statistical differences in students' interactions $[F(4,468)=1.544, p=.018, partial\ eta\ sq.=.013]$ and mathematics contents [F(3,468)=3.346, p=.015, partial eta sq.=.022]. However, the interaction effects [F(1,468)=1.202, p=.254, partial eta sq.=.044], gender [F(1,468)=1.109, p=.293, partial eta sq.=.002]and school[F(1,468)=.365, p=.0546, partial eta sq.=.001] were not statistically significantly. Even though gender (0.002) and school (0.001) showed small effect sizes, they statistically reduced the differences. This was demonstrated in the estimated marginal means.

Across tasks in equations of the circle

Research Question 4: What is the relationship between interactions in the didactical tetrahedron implemented and students' performance in solving equation of the circle?

Tables22to 24 sought to test the effects of the didactical tetrahedron on solving sample tasks in equations of the circle. In this way, students would have physically applied the models to solve particular problems in equations of the circle.

Table 22: ANCOVA Tests of Relationships between Interactions in the Didactical Tetrahedron Implemented and Students' Performance

Source	Type III Sum of Squares	df	Mean Square	F	Sig.	Partial Eta Squared
Corrected Model	47.382[a]	23	2.060	2.412	.000	.105
Intercept	23.187	1	23.187	27.152	.000	.055
Students Interactions	2.334	4	.583	.683	.0604	.025
Teachers Interactions	10.645	3	3.548	4.155	.006	.026
School	6.567	1	6.567	7.690	.006	.016
Gender	2.113	1	2.113	2.474	.116	.005
Students * Teachers * School* Gender	31.508	13	2.424	2.838	.001	.073
Error	402.222	471	.854			
Total	2486.000	495				

Source: Research data (Ali, 2018)

On Table 22, there were significantly statistical differences in students' interactions [F(4,471)=.683, p=.060, partial eta sq.=.015], teachers' interactions [F(3,471)=4.155, p=.006, partial eta sq.=.026], Interaction effects [F(1,471)=2.838, p=.006, partial eta sq.=.073],and school[F(1,471)=7.690, p=.006, partial eta sq.=.016] were statistically significant. However, gender was not statistically significant [F(1,471)=2.474, p=.116, partial eta sq.=.005].

Even though small the partial eta squared values were recorded in gender (0.005) and school (0.016), the results of the estimated marginal means showed statistical reductions.

Table 23 shows the results of MANCOVA tests. These tests were conducted to compare the effects of the four didactical conceptual structures on sampled equations of the circle (see Appendix J15).

Table 23: MANOVA Tests of Relationships between Interactions in the Didactical Tetrahedron Implemented and Students' Performance

Effect		Value	F	Hypothesis df	Error df	Sig.	Partial Eta Squared
Intercept	Wilks' Lambda	.903	21.703[a]	2.000	403.000	.000	.097
Gender	Wilks' Lambda	.998	.322[a]	2.000	403.000	.725	.002
School	Wilks' Lambda	.994	1.121[a]	2.000	403.000	.327	.006
Students Interactions	Wilks' Lambda	.977	1. 200[a]	8.000	808.000	.296	.012
Teachers Interactions	Wilks' Lambda	.996	.865[a]	2.000	403.000	.422	.004
Content Interactions	Wilks' Lambda	.957	2. 982[a]	6.000	808.000	.007	.022
Technology Interactions	Wilks' Lambda	.993	. 691[a]	4.000	808.000	.598	.003
All* Gender * School	Wilks' Lambda	.641	1. 376[a]	146.000	808.000	.004	.200

Source: Research data (Ali, 2018)

After adjusting for the two covariates as on Table 23, there were not significantly statistical differences in any of the four didactical conceptual structures [Students: Wilks' Lambda=.977, $F(8,808)=1.200$, $p>.05$, multivariate partial eta squared=.012; Teachers: Wilks' Lambda=.996, $F(2,403)=0.865$, $p>.05$, multivariate partial eta squared=.004; Content: Wilks' Lambda=.957, $F(6,806)=2.982$, $p>.05$, multivariate partial eta squared=.022; Technologies: Wilks' Lambda=.993, $F(4,806)=0.691$, $p>.05$, multivariate partial eta squared=.003]. Also, both gender and school were not statistically significant [Wilks' Lambda=.998, *F*(2,406)=.322, *p*>.05, multivariate partial eta squared=.002; Wilks' Lambda=.994, *F*(2,403)=1.121, *p*>.05, multivariate partial eta squared=.006].However, the interaction effects were statistically significant [Wilks' Lambda=.641, *F*(146,806)=1.376, *p*<.05, multivariate partial eta squared=.200] (See Appendix H). Even though, the multivariate partial eta squared values showed small effect sizes, the removal of gender and school statistically reduced the differences. This was supported by the estimated marginal means.

To confirm the significant effects of the didactical tetrahedron across didactical instructional models in the equations of the circle, the MANCOVA was displayed Table 20. According to Wilks' lambda (W) statistic, 1% increases in the didactical instructional models increases the rate of interactions of the students, teachers, mathematics content and technology by 0.994%, 0.996%, 0.957 and 0.993 respectively. These outcomes suggest positive significant effects of the didactical tetrahedron on students' performance in equations of the equations.

Table 24 shows the overall MANCOVA tests involving all the four interacting didactical conceptual structures on the tetrahedron. These tests sought to compare the effects of the four didactical conceptual structures across all the three didactical instructional models.

Table 24: MANCOVA Tests of Relationships between Interactions in the Didactical Tetrahedron Implemented and Students' Performance

Effect		Value	F	Hypothesis df	Error df	Sig.	Partial Eta Squared
Intercept	Wilks; Lambda	.855	26.835[a]	3.000	475.000	.000	.145
	Roy's Largest Root	.169	26.835[a]	3.000	475.000	.000	.145
Gender	Wilks' Lambda	.944	9.310[a]	3.000	475.000	.000	.056
	Roy's Largest Root	.059	9.310[a]	3.000	475.000	.000	.056
School	Wilks' Lambda	.993	1.047[a]	3.000	475.000	.371	.007
	Roy's Largest Root	.007	1.047[a]	3.000	475.000	.371	.007
Students Interactions	Wilks' lambda	.917	3.478	12.000	1.257E3	.000	.028
	Roy's Largest Root	.061	7.317[b]	4.000	477.000	.000	.058
Teachers Interactions	Wilks' Lambda	.976	.953	12.000	1.257E3	.492	.008
Content Interactions	Wilks' Lambda	.974	1.393	9.000	1.156E3	.186	.009
	Roy's Largest Root	.019	2.948[b]	3.000	477.000	.032	.018
Technology Interactions	Wilks' Lambda	.981	.755	12.000	1.431E3	.698	.006
	Roy's Largest Root	.011	1.350[b]	4.000	477.000	.250	.011

Source: Research data (Ali, 2018)

On Table 24, the independent variables were the four didactical conceptual structures and the dependent variables were the three didactical instructional models. Students' gender and school were held as the covariates. After adjusting for the two covariates, there were significantly statistical differences in students' interactions [Wilks' Lambda=.917, F=3.478, p<.05, multivariate partial eta squared=.028]. However, teachers' interactions[Wilks' Lambda=.974, F=1.393, p=.492, multivariate partial eta squared=.008], mathematics content [Wilks' Lambda=.974, F=1.393, p=*.186*, multivariate partial eta squared=.009], technology tools [Wilks' Lambda=.981, F=.755, p=.698, multivariate partial eta squared=.006], and their interaction effects [Wilks' Lambda=.565, F=1.154, p=.077, multivariate partial eta squared=.173] were not statistically significant. On the covariates, gender was statistically significant [Wilks' Lambda=.944, F=9.310, p=.000, multivariate partial eta squared=.056] while the school was not [Wilks' Lambda=.993, F=91.047, p=.371, multivariate partial eta squared=.007]. Even though the multivariate partial eta squared values showed small effects, the results of the estimated marginal means statistically controlled the effects.

To confirm the significant effects of the didactical instructional models in the tasks of the circle, the MANCOVA was displayed on Table 19. According to Wilks' lambda, 1% increases in the didactical instructional models increases the rate of interactions of the students, teachers, mathematics content and technology by 0.917%, 0.976%, 0.974 and 0.981 respectively. These suggest positive significant effects of the didactical instructional models in the sampled tasks in equations of the circle.

Qualitative Sources in the Study

Research question 5: How do the qualitative sources help explain statistical significance across the intersubjective didactical instructional models in students' performance in solving equations of the circle?

Interview transcripts were used to show students' experiences and interactions in the tetrahedron. The transcripts were either written down in whole or paraphrased to answer the research question. The research question answered the interactions across each of the three intersubjective didactical instructional models. The responses were compared with the pre-intervention outcomes and the statistical significances.

Interactions across mathematisations and didactical phenomenology

The transcripts of mathematisation and didactical phenomenology were based on the realistic mathematics education theory. This Netherlands tradition theory offers both pedagogical and didactical philosophies for teaching and learning. There were two important connections of the dialogue to the reality of mathematics. The first connection was mathematics as a human activity. This moved mathematics dialogues very closed to the students' daily life situations I order to solve experientially real mathematics problems. The second connection was premised on mathematics as environmental activity. This started the principles of guided reinvention and progressive mathematizations.

On the conceptual meaning realistic mathematics education, the students produced varied but similar contextual understanding. For instance, student 'A said a sign is an identity given to an object, student 'B' explained the sign as something used to represent or replace a long sentence, and student 'C' advanced that it is letter that represents a constant. Upon further deliberations, student 'A' mentioned the equal to, addition, square and constants signs and student 'B' minus, square root and multiplication signs. These responses were quite encouraging as compared to the responses during the baseline survey. The most interesting responses of the conversations were on the signs and symbols they mostly preferred to use:

1. Student 'A': *I like the addition sign most because it makes calculations of the general equation easier.*
2. Student 'B': *I prefer x and y because they are mostly known by teachers, textbooks and computers.*

Similarly, on the meaning of a tool in mathematics, student 'A' associated a tool to a device that helps in doing something without being drawn part of the process so that one obtains an obtain accurate measurements or values, and student 'B' likened a tool to an instrument used in measuring quantities or aids in calculations.

Unlike signs and symbols, the commonest tools in the Ghanaian senior high school were the metre rule, protractor, pair of dividers, pair of compass, set squares, calculator, eraser and pencil. It is no wonder that the exploratory factor analysis extracted tools as the most dominant factors. Despite the long list of tools the students enumerated, they mostly used metre rule, pair of compass, protractor and pencil in measuring and solving equations of the circle. Even though most of these tools have been prescribed by the Ministry of Education of Ghana for use in the senior high schools (MoE, 2018), appropriateness and effectiveness of use in equations of the circle was still a problem.

Figure 6 shows the interview transcripts in mathematisation and didactical phenomenology models. This sought to explain why school was a potential covariate. In the figure, the number of students who mathematised the signs, symbols, artefacts and tools in solving tasks in equations of the circle were displayed.

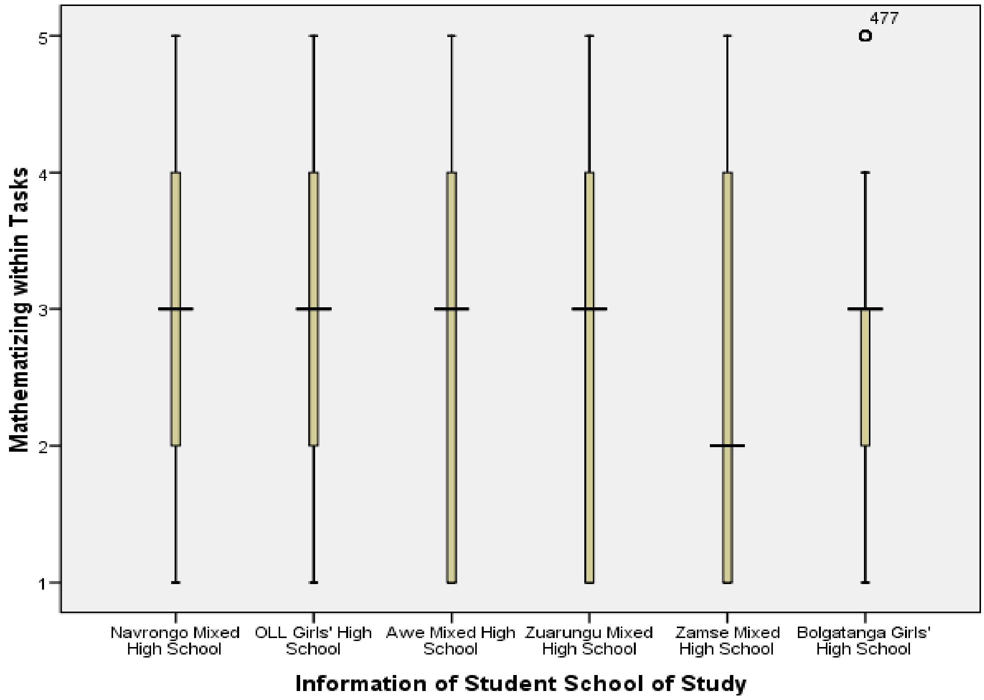

Figure 6: School interactions in mathematisation and didactical phenomenology

Source: Research data (Ali, 2018)

The boxplots on Figure 6 shows the students' patterns of interactions in the schools and variability in scores within each school. This establishes visual inspections of the differences between the schools. It was observed that mathematizing across the schools were highly varied in Zamse and Bolgatanga Girls' Senior High Schools and less varied in Awe and Zuarungu Senior High Schools. But variability was relatively uniform in Navrongo and O.L.L. Girls' Senior High Schools. There was an outlier in Bolgatanga Girls' Senior High School. Even though there were differences in mathematisation and didactical phenomenology, the signs, symbols, artefacts and instruments helped students to solve problems in equations of the circle. These results corroborated the results of the MANOVA (see Tables 1 and 2). It was therefore revealed that signs, symbols, artefacts and tools tasks were not only science of knowledge but also art and skill of applying knowledge in the intersubjective models.

Secondly, the students' dialogues in the interactions with the signs, symbols, artefacts and tools helped students a lot. The students were able to define, organize and represent the mathematics problems in both psychological and cultural ways. As the students progressed from signs, artefacts and tools to instruments, they encountered higher psychological, cultural and classroom roles. The physical and psychological objects helped the students to mathematise the given problem from their cultural viewpoints. Mathematisation and didactical phenomenology drove the whole contents and structure of the psychological operations. This process altered the ways the students came to discover the contextual phenomena in the equations of the circle. It was therefore clear that the signs, symbols, artefacts and tools altered and transformed their socio-cultural roles and functions. These roles and functions support the Vygotsky's genetic law of cultural developments on the social and the individual levels (Radford, & Sabena, 2015).

Again, the dialogues showed that the relations in the mathematisation and didactical phenomenology were not inherent/natural but rather established through in the articulations. In fact, the students related the sign-symbolic notations to the tools and instruments. This helped the students to explore for the centre and radius of the equation of the circle. In the graphs and the physical phenomena, the students interacted with their teachers, mathematics contents and technology tools. These interview transcripts actually enhanced understanding of the mathematisation and didactical phenomenology models.

Interactions across instrumentations and instrumentalisations models

The interview transcripts in the instrumentations and instrumentalisations models depend upon the mathematisation and phenomenological dialogues(Imenda, 2014). These dialogues provided evidence for the transition into the instrumentations and instrumentalisations models. The dialogues in the instruments provide sound base for the teaching and learning of mathematics involving technologies and innovations. The interview transcripts in the instrumentations and instrumentalisations models were based on the theory of instrumental genesis.

The theory of instrumental genesis regards instrumentations as actions by which someone require instruments to influence knowledge acquisition (Clark-Wilson et al, 2015).These actions and influences are premised on Vygotsky's work of situating human activity in the environment and culture. The actions and influences of instrumentations and instrumentalisations ultimately altered the entire interactions and structure of the students' mental functions. The new instrumental acts in the instrumented technologies contributed in reshaping the students' environment and culture.

Again, the students defined technology in mathematics both as knowledge in science used to make work easier and faster, and the application of mathematical knowledge to solve problems. Unlike signs and symbols, the students' commonest technologies were mathematical sets, calculators, computers and smart phones. These technology tools were mostly acquired for learning mathematics and were the basis of their technological innovations. The boxplots in Figure 7 shows the interview transcripts in the instrumentations and instrumentalisations models. The scores were continuous and the interactions were categorical.

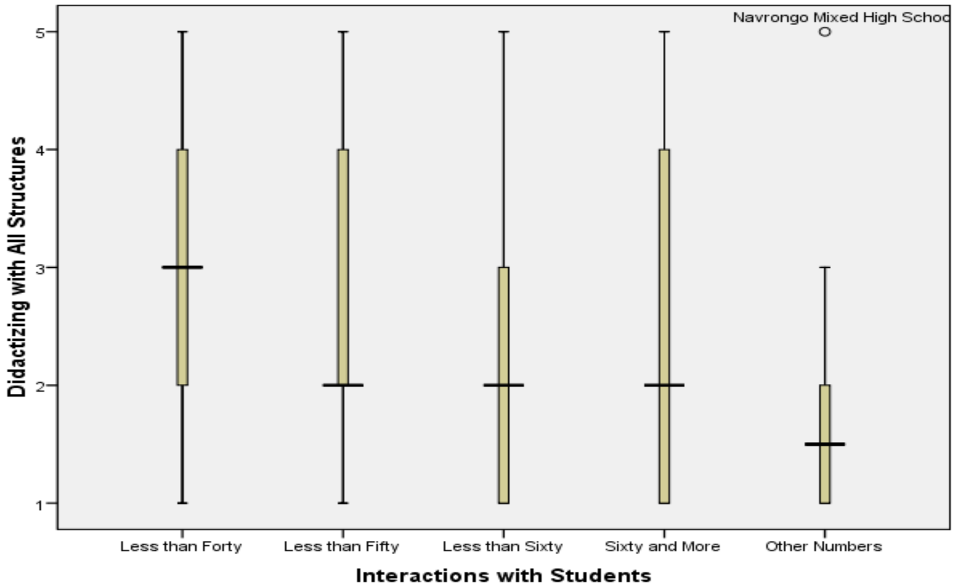

Figure 7: Students' interactions across instrumentations and instrumentalisations
Source: Research data (Ali, 2018)

In Figure 7, the boxplots provided the students' patterns of interactions. The variability in scores within each group was the visual inspections of the differences between the groups. It was observed that instrumentations and instrumentalisations across the groups were high in less than fifty and less than seventy but relatively uniform in less than forty and less than sixty. There was an outlier in Navrongo Senior High School. Even though there were differences in students' interactions in instrumentations and instrumentalisations, all groups of students agreed that the models provided them with the knowledge and its applications. Once knowledge, it sufficed one to conclude that the instrumentations and instrumentalisations models plausible and effective in solving problems in equations of the circle. The patterns of responses were equally quite remarkable as compared to the baseline survey.

In addition, the students outlined computers, calculators, mathematical set and smart mobile phones that have extended their scientific knowledge, methods and skills in solving problems in equations of the circle. The mathematics teachers interacted with the students in using calculators, computers and other technological tools. The mathematical discourses and generated a lot of debates, contemporary issues and consensuses that made effective use of the instrumentations and instrumentalisations. These processes helped to acquire new knowledge, solve new problems, and engage in creative and critical thinking (Presmeg et al, 2016). In the instrumentation and instrumentalisation models, the interactions among students, teachers, mathematical content and technologies became more interesting. Students discovered not just the types of tools but also the ways of these tools were being used.

The tools were positioned well to interact in the physical, social, psychological and socio-cultural aspects. The instrumentations and instrumentalisations in allowed the students to come out with absolute truths, regardless of the operations. On the other hand, cultural genesis of the transcripts tackled the Vygotsky's (1978) strand by constructing and contextualizing knowledge. These processes were products of individual student's social interactions, cultural contexts and interconnections with teachers, peers, mathematics content and technologies (Forsman, 2015).

Equally important was technology use in instrumental genesis in the processes of instrumentations and instrumentalisations by the teachers, students and mathematics content. The processes involved techniques and mental schemes that evolve, develop and apply technologies. The instrumentations and instrumentalisations models helped students to transition signs and symbols, artefacts, tools and instruments into technologies (Ali, Davis & Agyei, 2018). The students' dialogues guided the ways these tools were being transitioned. The interplay was being shaped and remodelled by the signs and symbols, artefacts, tools and instruments. This was determined by the teachers, students and mathematics content through the process of instrumental genesis. In other words, the technologies were essential in promoting and propelling the socio-cultural views and reshaping scientific knowledge and skills in Vygotsky's (1978) perspectives.

Furthermore, the dialogues showed the rapid psychological development and transformation of the Vygotsky's instrumental theory. In this sense, the students integrated their psychology learning processes and the genesis of psychic functions and roles effectively (Grant, &Osanloo, 2014). This improved upon their understanding of the relationships, roles and practical applications of signs and symbols in technologies. It also incorporated Vygotsky's (1999) theory of instrumental genesis incorporated psychology sociability of the students to tackle the practical applications. This helped built relationships with teachers, mathematics content and technologies to cognitively develop these practical applications. This means the metacognitive processes in incorporating psychological and pedagogical knowledge and skills was exemplified in equations of the circle.

By far, the fourth Vygotsky's concept of 'artificial development' enabled the students to develop analytical strategies for solving problems-based sociocultural tasks. By forming individual, social and cultural study groups, students determined the parameters in selecting and solving the tasks.

The fifth Vygotsky's conceptualization incorporated cooperative learning, guided learning, socio-cognitive conflict and knowledge construction. The final one made simple the applications of modern calculators, tablets and smart mobile phones in teaching and learning equations of the circle (Vygotsky, 1978). Unfortunately, only teachers were allowed to operate the smart phones. Indicated in the seven main factors in the baseline survey, few of these components were really extracted (See Table 10). This can be traced to the limited use of ICT in mathematics curriculum delivery in Ghana (Davis et al, 2016). Only calculators are allowed in examinations but smart phones are totally prohibited (WAEC, 2015).

Interactions across anthropological praxeologies

The transcripts in the didactical situations and anthropological praxeology models focus on the students' mathematical knowledge as both human activities and social interactions (Chevallard, Bosch, & Kim, 2015). This knowledge is embedded in the didactical situations. While the anthropological praxeologies of didactics analyze the viability and efficiency of utilizing technology tools, the amalgamation of the two ensures that mathematics knowledge is retained within socio-cultural settings of the classroom. Therefore, in ordering, sequencing and progressing through the task (1T), the task-technique (2T), the task-technique-theory (3T) and the task-technique-theory-technology (4T), mathematisation sets the pace for didactical phenomenology. The two aligned components can easily be transitioned into instrumentations and instrumentalisations. In soliciting and intertwining their didactical phenomenological contexts, the students mostly interacted with small groups. These groups allowed for candid sharing of common ideas based on individual's comparative advantages.

On reasons for choosing particular mathematics content for the didactical situations and anthropological praxeologies, simplicity, precision, detailed explanation and accuracy were the driving forces. In the psychological tests, the blank-space, multi-choice and short-answer tasks provided strong feeling of understanding of the content. On how they interacted with teachers, mathematics content and technologies to solve the tasks, student 'A' emphatically stated *we usually solve the simple tasks alone. However, we refer challenging tasks to our mates and teachers* but student 'B' minced no words in acknowledging the crucial roles of the three didactical conceptual structures whenever they got stacked!

The boxplots in Figure 8 show the interview transcripts in the didactical situations and anthropological praxeologies. In the interactions, it can be shown that there were much more improvements as compared to the baseline survey. The figure also explains the results of the ANCOVA and MANCOVA in the didactical situations and anthropological praxeologies.

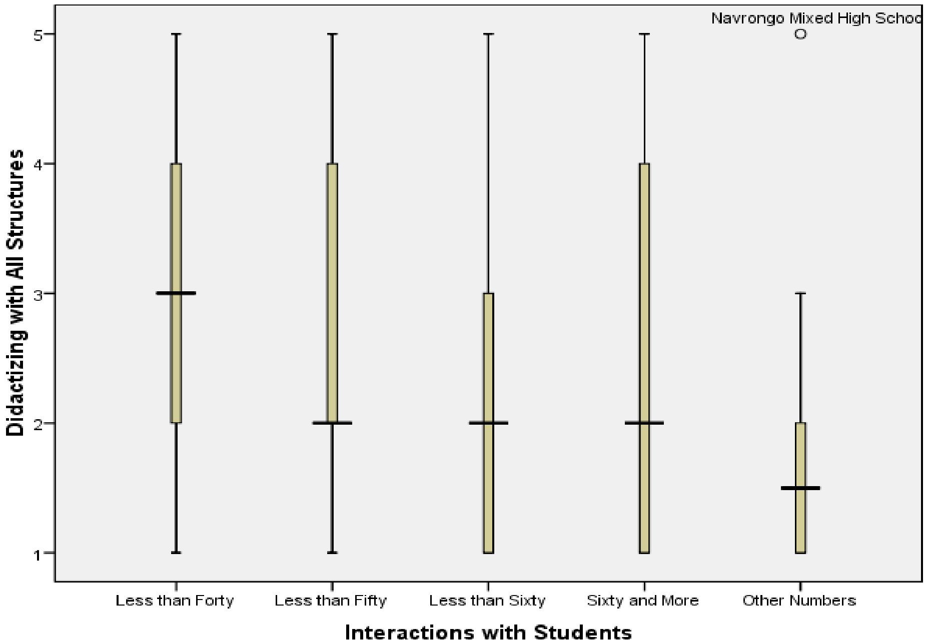

Figure 8: Students' interactions across anthropological praxeologies

Source: Research data (Ali, 2018)

The boxplots in Figure 8 showed the transcriptions of the number of cases in students' interactions categories of didactical situations and anthropological praxeologies on didactising all structures. The boxplots suggest that all categories of students interacted well in solving equations of the circle except Navrongo Senior High School that produced an outlier. But the differences were more pronounced among the less than forty and very longer in the sixty or more groups.

However, it must be noted that apart from the less than forty, the differences between the groups were quite small and supported the statistically significance differences of the between-groups ANCOVA and MANCOVA results (See Appendix H1-H8). In other words, the didactical praxeologies were the best model to modify and integrate new tasks, techniques, theories and technologies. As students begin to relate with teachers, mathematics content and technologies, it is didactical praxeologies that help them to move from technical-practical block into technological-theoretical block (Drijvers, & Trouche, 2005; Chevallard, Bosch, & Kim, 2015).

Students' interactions in equations of the circle

The transcripts in exemplified equations of the circle explored the students' algebraic and geometric knowledge in six key equations of the circle. These were general, centre-radius, diameter, standard, tangent and normal equations in solving tasks exemplified in equations of the circle. These areas provided proof of knowledge, skills, understanding and utilization of the didactical conceptual structures in the tetrahedron (Grant, &Osanloo, 2014).

In the interview transcripts, almost all the students referred to both standard and general equations of the circle as $x^2 + y^2 + 2gx + 2fy + c = 0$ even though a few alluded to differences in the centres and radii. In using the standard equation to solve for the centres, $C(0,0)$, $C(h,0)$, $C(0,k)$, $C(-h,0)$, $C(0,-k)$, $C(h,k)$, $C(-h,k)$, $C(h,-k)$ and $C(-h,-k)$, some students came out with the approximate solutions (See Appendix E). Figure 7 is sample worksheet in comparing the standard and general equations of the circle.

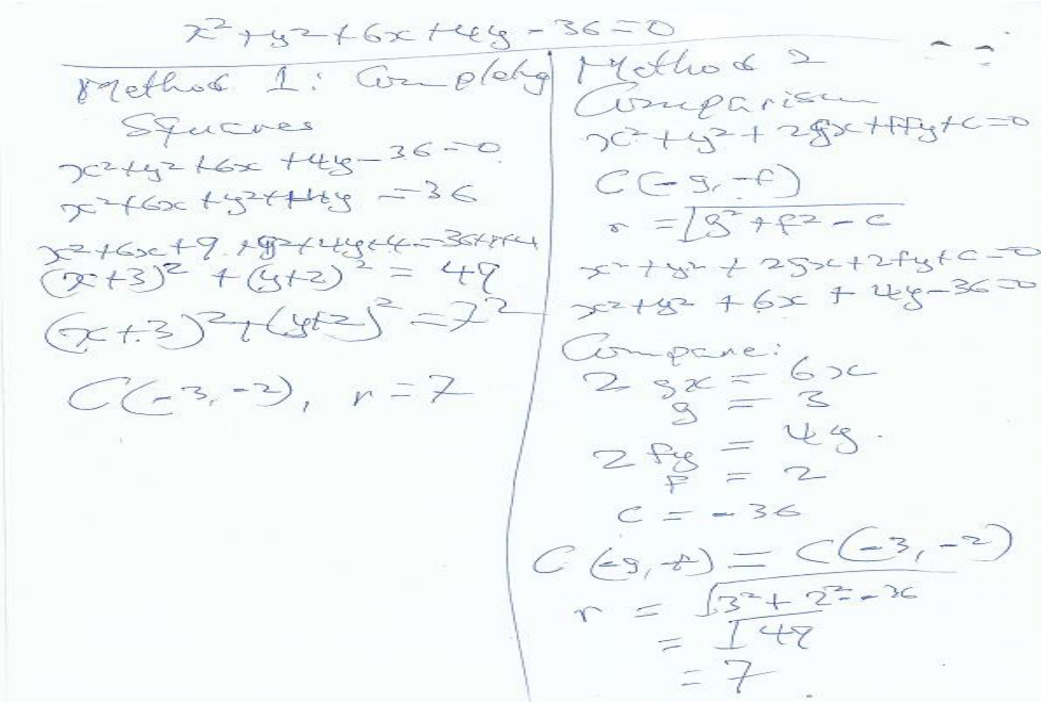

Figure 9: Sample worksheet in general and standard equations

Source: Research data (Ali, 2018)

After one student solved the tasks on Figure 9, majority of them were simply overwhelmed and perplexed with the most captivating and interesting interrelationships and interconnections of the solutions! The standard equations seemed simple, easy and short! Subsequently, students who solved an equation like $x^2+y^2+8x=0$ whose centre is $C(-4,0)$ and the radius is 4 units could easily solved an equation like $x^2+y^2+6y=0$, whose centre is definitely $C(0,-3)$ and the radius is 3 units (Horsman, 2018). Combining the two examples, it was easy to interact and come out with a solution to the equation $x^2+y^2+8x+6y=0$, whose centre is $C(-4,-3)$ and the radius is 5 units. The reverse of this becomes the equation $x^2+y^2-8x-6y=0$, whose centre is $C(4,3)$ and the radius is also 5 units. By extension, the equations $3x^2+3y^2+36x+12y+84=0$ and $3x^2+3y^2-36x-12y+84=0$ had the centres $C(-6,-2)$ and $C(6,2)$ respectively but the same radius $2\sqrt{3}$ units (Whitney, 2015).

These solutions were easily obtained through active mathematisation followed by the didactical phenomenology, the instrumentations and instrumentalisations, and the didactical situations and anthropological praxeologies. In complete interactive didactical conceptual structures, students' interactions were much improved as compared to the baseline. It was relational for a student to conclude that irrespective of a particular type of equation, the centre and the radius remain cardinal. However, while the centre of the circle is affected by the sign and size of the coefficients of x and of y, the radius is always positive (Selma, 2015).

Discussion of key findings

In this section, the discussions have been presented in the order of the research questions.

Discussion of Research Question 1

Overall, the baseline intervention was successful, and the researcher retained many of the strategies. There were however, a few a few deviations to enable the researcher obtain more efficient results during the experimental processes. Most specifically, the findings gave the researcher an opportunity to finetune the test items to set for the respondents. Items that were most illuminating were preserved in the main study, and repeating them showed that the didactical instructional model have effected changes in the didactical conceptual structures. By condensing the items into only the seven dominant factors in the factor analysis, the data collection processes did not only take less time to execute, but ensured active participation. These contributed to acceptable statistics in the test items in the main study.

The data collection process also helped to finetune the strategy for the study. It was clear that collecting qualitative data from all the 500 students of the six schools was not feasible. Since it was required that the researcher visited all the students, the pre-intervention gave the researcher directions on how to visit every school two or three times and select two to three students. Thus, the study would have taken substantially longer time and more cost. Even the time each interview participant took was shortened. With the help from the regional directorate and the mathematics teachers, the best days to visit the schools and the students were ascertained. Particularly, the researcher was directed by the regional director not to engage the students during active instructional hours. Therefore, public holidays, mock examinations days, and sports and games days were exempted. These contributed to the high reliability and validity of the data.

Also important was how to collect the quantitative data from each school at the same days and times. This ensured that the test items administered to the experimental groups in the various schools did not leak. It ensured that the number of students selected were not less than the number of students taking part in the tests as was a problem at baseline, where much data had to be discarded. With the permission of the heads and students, photographs of students were taken and number of students was counted to ensure high recovery and data quality.

In addition, the major ingredient in the baseline instruments was the questionnaire for pretesting. The baseline instruments were pretested during the training in each school. Since the survey was conducted across six schools, training was conducted consecutively so that changes in the school could be made to the main study questionnaires. A feedback obtained from the pilot survey was also used to make modifications in the main study. This focused on clarity, length and ease of administration. It helped in deleting, adding, modifying, rearranging and clarifying questions. At the end, the questionnaires were pretested again with minimal revisions required due to the fact that approximately 90% of the questions were derived the baseline instruments.

During reporting, the post intervention was closely matched to the structure of the baseline report in order to ensure easy and simple comparisons. More important in this stage was the selection and inclusion of the covariates. Even though the research methods clearly illuminated various covariates, the baseline made it easier to identify gender and school as the main covariates. And even though the quantitative results portrayed a few departures, the followed up interview transcripts showed closed resemblances to the baseline. It helped to incorporate the qualitative data into the quantitative by following the order of the didactical instructional models. This served as a concrete foundation, better and quality interpretations and discussions of the findings.

Discussion of Research Question 2

The new latent variables the researcher used were instrumentations, 4T praxeologies and mathematizations. Having extracted these variables, they become the main variables for study (Yu, 2015). In the interactions in the tetrahedron, students steadily migrated across the didactical instructional models in solving equations of the circle. In the one-way between-groups ANOVA, the students' knowledge before the implementation of the tetrahedron was quite limited. However, as the students progressed from one level of interaction to another, they became quite proficient and efficient. This explains why they achieved the statistically significance across four components (see Table 12). Despite reaching statistical significance, the actual differences in mean scores were quite small as supported by the effect sizes through eta squared values and Tukey HSD test. However, the mean scores did not differ significantly in artefacts and technologies (see Appendix J1). This was traced to the exclusion of the didactical tetrahedron. Unlike Østergaard (2013) who is contended with the didactical triad, these results shows that the didactical tetrahedron is much superior (Ruge, &Hochmuth, 2017).

This picture was blurrier in students' knowledge/performance in the didactical tetrahedron before the implementation of the tetrahedron across instrumentations. In this area, no components attained statistically significant. This is a clear case that technologies are necessary tools and instruments for learning equations of the circle (Trgalova, Clark-Wilson, &Weigand, 2016). Not attaining statistical significance $(p > .05)$ means that students could not interact well. And not interacting well was traced to the cause of the poor performance. The statistics of eta squared values and Tukey HSD comparisons supported the non-significance (Pallant, 2011). For better interactions and alternative performance, the didactical tetrahedron can perform the magic and needs to be revisited (Bishop et al., 2002). The mean scores did not differ significantly among any group (see Appendix J2). Therefore, there was ample evidence to conclude that no interactions existed the didactical tetrahedron was the main cause.

Discussion of Research Question 3

This question has been discussed in accordance with the outcomes of the didactical tetrahedron implemented. The discussion was based on the three didactical instructional models across the mathematisations, instrumentations and praxeologies.

Discussions across mathematisations

Analysis of the mathematisations and didactical phenomenology models provided new mathematical concepts, structures and ideas in relation to the phenomena and contexts in learning and solving tasks in equations of the circle. This reality of didactical phenomenology placed students strategically in applying, explaining and relating knowledge. It played key roles in students' concept formations (Radford, & Sabena, 2015). The students could describe authentic settings of the mathematics tasks during the experiments, exercises and tests. They established connections between reality and mathematics, and created further constructs relating to mathematical realities, situations and contexts. This explained the statistically significance differences [F(4,466)=2.555, p=0.038, partial eta squared=0.021].

The models really depicted the ways students presented, invented, innovated, published and performed the tasks. The models also transformed, translated and transitioned students into new mathematizations and didactical phenomenology (Saric, &Markic, 2015). This spectrum of mathematisations and didactical phenomenology did not exist prior to the main study. Even though the interaction effects were not statistically significant in some few cases, the confounding of the covariates statistically reduced the nonsignificant differences. In deed the estimated marginal means showed that gender and school did controlled and statistically removed the effects in mathematizations and didactical phenomenology (See Appendix F2).

In Freudenthal's mathematising and didactical phenomenology activities, ten components mathematically set up problem in horizontal. This was then processed and refurbished the real-world tasks by the vertical (Anh, 2006). This really engendered the students to translate and transform the horizontally mathematised tasks from signs and symbols to instruments and technologies. The mechanical roles of vertical mathematizations in the horizontal interactions broadened specific mathematics to general contexts. In this ways, student easily conceptualized signs, symbols, artefacts and tools (Van Den Heuvel-Panhuizen et al, 2016). For instance, *a sign is an identity given to an object, it is something used to represent or replace a long sentence, and it is a letter that represents a constant* were contexts of general scope.

The didactical conceptual structures schematized, formulated and visualized the general contexts by discovering and recognizing relationships and regularities in the didactical phenomenology. This transferred real world tasks to mathematical tasks and to the phenomenon in equations of the circle (Doolittle, 2014). For instance, in outlining signs and symbols commonly explored to solve tasks in equations of the circle, the students stated equal to, addition, squares and constants, and addition, equation to, minus, square root and multiplication, and mostly preferred the addition, x and y signs as they commonly computed equations of the circle and mostly known by teachers, mathematics content and technologies. In fact, the students' mathematizing activities in strong vertical components represented and connected these signs and symbols (Dunphy et al., 2014). This did not only integrate different classes of signs and symbols but also formulate new mathematical concepts of signs and symbols.

Also, the paramount approaches were the four different approaches, namely mechanistic, structuralist, empiricist and realistic encountered in mathematisation and didactical phenomenology during the processes of horizontal and vertical mathematization. The mechanistic provided systems of rules to the students to verify and apply. Similar previous tasks were adequately incorporated into the applications, methodology, structure, interrelatedness and insights. In this way, students viewed tools as devices that help one in doing something without being drawn part of the processes, devices that help one to obtain accurate measurements or values, instruments used in measuring quantities and aids used for calculations.

The structuralist phenomenology provided organized and closed deductive systems that stressed vertical mathematization. In this way, students applied their metre rule, protractor, divider, pair of compasses, set square, calculator, eraser and pencil. The empiricist phenomenology narrowed down the structuralist to only classroom tools, namely metre rule, calculator and pair of compasses. These tools helped them to acquire useful experiences in non-routine forms of learning (Van den Heuvel-Panhuizen, & Drijvers, 2012). Students preferred these three instruments because there were easy to use, well known and mostly acquired by students, parents/guardians, schools and municipalities. The realistic components fully incorporated both vertical and horizontal mathematizations in utilizing and incorporating the tools in solving the tasks (Van Den Heuvel-Panhuizen et al, 2016).

Discussions across instrumentations

The instrumented signs, tools, instruments and technologies discovered, transitioned and built genesis. These were generated from instrumentations and instrumentalisations and incorporated into the schemes to stabilize, corroborate and explain the statistically significant differences (Haspekian, 2011). For instance, after adjusting for the two covariates, the interactions in instrumentations and instrumentalisations revealed statistically significant differences. Even though the interaction effects were not statistically significant, the effects of instrumentations and instrumentalisations statistically improved the interactions. These were attributed to the estimated marginal means that significantly reduced the effects of the covariates to effectively and efficiently improve students and teachers interactions.

Again, the instrumentations and instrumentalisations analyzed concurrently served as bridges for instruments and technologies. The students defined and explained technology in mathematics as knowledge in science used to make work easier and faster or it is the application of mathematical knowledge to solve problems (Boileau, & Herbst, 2015). However, the contextual meanings of technology were mission. They likened technologies to materials or symbolic objects designed to achieve specific knowledge and this is instrumentalisations. But by developing the schemes is called instrumentations (Bartolini Bussi, & Mariotti, 2016). Now, the schemes are related to the phenomenological experiences of the students. In the Ghanaian senior high schools, technologies and ICT have been offered to students to develop these schemes (MoE, 2007, 2010). The technologies are deployed to develop these cognitive schemes. Therefore, the tasks-mediated mathematical sets, calculators and computers bridged the sociocultural dimensions of the didactical schemes shared by students, teachers, mathematics content and technologies (Matusov, 2015).

Secondly, in the Vygotsky's approach to the cognitive interpretations and functioning of instrumentations and instrumentalisations (Presmeg et al, 2016), the teaching and learning of mathematics should be played within the genesis of human mental activities. This is easily accomplished by bridging the gaps in the socio-cultural line for the higher concepts to take place. The socio-cultural lines are everyday teaching and learning situations. They can be formal or informal. However, students should develop higher mental cognitive skills through the social-cultural interactions to lead to local technologies. For instance, students preferred the instruments and technologies because the schools always acquire them for learning mathematics.

But findings of the phenomenological dialogues shows that there are differences between differentiated psychological, technological and technologies. These concepts are derived from cultural signs and symbols. If well developed, they can help students to conceptualize, formulate and solve mathematics tasks, especially equations of the circle. After all, Roth (2016) agrees that the didactical cycle of teaching and learning develop from cultural signs and symbols, promoted through social exchanges. As students engage through social and cultural constructs, they gain knowledge from their collective discourses (Presmeg et al., 2016).

Discussions across anthropological praxeologies

The didactical situations and the anthropological praxeologies contain tasks, techniques, theories and technologies. This is used in describing, explaining and justifying the didactical instructional models (Chevallard, 2013). The engagements also build around sociological and socio-cultural levels of schematic conceptual structures and relationships (Florensa, Bosch, &Gascón, 2015). In the build up to didactical situations, the first students utilized mathematisation components, they bridged the gaps between mathematisation and instrumentations, and then finally manipulated 1T, 2T, 3T and 4T praxeologies.

Unlike the baseline survey, all the steps and strategies precedent to didactical situations and anthropological praxeologies were embedded (Otero, Gazzola, Llanos, &Arlego, 2016). There statistically significant differences in the didactical situations and anthropological praxeologies models in the main effects, the interaction effects, and the covariates. These statistical significances proved that the students independently and successfully transitioned and utilized the didactical situations and anthropological praxeologies. It was not only the work of the covariates but also the roles of the phenomenological discourses.

The didactical triad was equally extended to the didactical tetrahedron by adding technologies. These helped students to conceptualize and didactise teaching and learning of equations of the circle in the socio-cultural systems (Winslow et al, 2016). In the conceptual evolution of didactics from pre-didactics, didactics-dialectics, classical didactics and now digital age didactics (Tchoshanov, 2013), the dialogues exemplified didactics-dialectics and classical didactics. Unlike the baseline survey transcripts, the digital age didactics reconceptualised and reformulated the constructs and concepts. In the didactical situations, the students contended that they preferred minimum of thirty and a maximum of forty students in groups and the reasons for such preferences were based on sharing ideas and taking comparative advantages of one another.

In addition, Chevallard's didactic transpositions of external didactics, internal didactics and learning situations equally played roles in accomplishing the tasks. The four didactical conceptual structures mainly negotiated between students and teachers with the textbooks. As students began to enumerate the kinds of textbooks in the baseline survey, namely Concise, AKI-OLA, Nyansapoo, Akrong, PowerPoint, Action and A-plus, it was observed that they were mainly content-bound. However, they contended that these books and pamphlets were simple, accurate, precise, concise and unambiguous in solving tasks in equations of the circle. After the experimental treatments, they progressed well in the tasks, techniques, theories and technologies or the four-tuple (T, τ, θ, Θ) (Winslow et al., 2016).

Quite apart, the four-tuple (T, τ, θ, Θ) can initiate activities that motivated and probe students to learn (Chevallard, Bosch, & Kim, 2015). To solve tasks, students have to adequately make the explanations and arguments with the techniques. Then they furnish the explanations and arguments with the technologies. Then they realize that they embedded definitions, algorithms and axioms with the theories (Østergaard, 2013). In effect, they integrate horizontal praxeologies as practice block and vertical praxeologies as theory block. The processes involved in transforming the horizontal to the vertical are the techniques, arguments, debates and dialogues. This transformation was lacking in the baseline survey. This is because the students went straight into solving the tasks without making phenomenological descriptions. Even where tasks are provided in blank spaces, multi choice objective and short tasks, there must strategies and methods to ensure strong feelings of understanding and self-satisfaction in interacting with the tasks (Winslow et al., 2016).

Lastly, the dialogues in the interview transcripts clearly differentiated didactics of mathematics from pedagogy of mathematics. In the didactics of mathematics, students locate the didactical situations with the given tasks. They then devise strategies of understanding the problem. In making the strategies, they have to consult teachers and mathematics books. In consulting these two, they realize that technologies are essential in making the interactions and communications. If students constrain themselves to the teachers and the mathematics books, they cannot strongly present and model their own learning situations.

It is therefore important that pedagogical phenomena are generated from generic practices, discourses, strategies and regularities in the didactical conceptual structures in the construction of new knowledge. As a results, students can always proceed from theoricism (organizing the mathematics content to follow logical constructions of concepts), technicism (executing all four conceptual structures) and dialogical constructivism (Florensa, Bosch, &Gascón, 2015). This process is very essential in developing deep learning and solving tasks in equations of the circle.

Discussion of research questions 4 and 5

In using the didactical tetrahedron to solve the tasks in equations of the circle, the students utilised the concept of a circle as sets of points (called loci) that satisfy distance relationships between two points in the plane. The findings revealed that the standard equation $(x-h)^2+(y-k)^2=r^2$ was the best circle equation to use to achieve the statistically significant differences (Parsons, 2015; Horsman, 2018). In Table 9, it was shown that the four didactical conceptual structures attained statistically significances. These significances were achieved after the baseline survey. Even the few tasks that showed no statistical significances had improved after statistically reducing, controlling and removing the effects of the covariates on the equations of the circle. In the baseline survey, students' interactions did not achieve statistical significances (See Appendix H). It was therefore, clear that the relationship between interactions in the didactical tetrahedron implemented and students' performance in solving equation of the circle was a massive success. The following are ample evidence that there was generally positive relationship between the didactical tetrahedron implemented and the students' performance in equations of the circle.

First and foremost, in the interview transcripts, the equations of the circle were grouped into localization of conceptual structures, real-world conceptual structures, multiple-choice and word problems, and extra-mathematical knowledge. The strategy for solving the equation, $x^2+y^2+6x+4y=0$ was the general equation of the circle. In this strategy, students compared $x^2+y^2+6x+4y=0$ with $x^2+y^2+2gx+2fy+c=0$ for the centre $C(-g,-f)$ and radius $r=\sqrt{g^2+f^2-c}$. However, in the standard equation of the circle $(x-h)^2+(y-k)^2=r^2$ for the centre $C(h,k)$ and radius

r students made double-completion of squares involving the variables x and y (Stitz, & Zeager, 2013). As in Figure 7, the solution is simpler and shorter.

This also stemmed from the fact that equations of the circle are special limited cases of equations of the ellipse. Since, every equation of the circle is basically determined by its radius r and its centre $C(h, k)$, then the distance from $P(x, y)$ on the circumference to $C(h, k)$ at the centre is the radius r (Whitney, & Reno, 2015). It was discovered that if the standard equation of the circle is expanded with the centre $C(h, k)$, then the polynomial became $x^2 + y^2 - 2hx - 2ky + h^2 + k^2 = r^2$, where the second-degree terms $x^2 + y^2$, the linear terms $-2hx$ and $-2ky$ and the collection of $h^2 + k^2$ can be summarized into one general equation $x^2 + y^2 + 2gx + 2fy + c = 0$. Here, the second-terms, the linear terms and the radius were compared with $x^2 + y^2$, $2gx + 2fy$ and r^2 respectively to arrive at the new centre $C(-g, -f)$ and radius $r = \sqrt{g^2 + f^2 - c}$ (Horsman, 2018).

Another didactical conceptual application was based on the midpoint strategy or method as seen in Appendix K2 of items 48 to 50. In this strategy, the circle was divided into two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ as the endpoints of the diameter in a line segment containing the centre. We all know that half of the diameter is the radius, for which the centre is $(h, k) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ and the radius is $r = \frac{1}{2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. If $P_1(x_1, y_1)$ is tangent to the standard equation, then the tangent equation of the circle was derived as $xx_1 + yy_1 = r^2$ and the normal as $\frac{y}{y_1} = \frac{x}{x_1}$ (Stitz, &Zeager,

2013; Whitney, 2015). The technologies helped students to reduce these complex phenomena into geometric, algebraic and analytical contexts. They then factorized the equations and obtained their solutions.

Chapter Summary

The baseline survey paved ways for better constructions of the research instruments for the main study. It also provided clues for covariate confounding. Particularly, it was discovered that gender and school had effects on the findings. After confounding, the three intersubjective didactical instructional models were generally statistically significant. This was evident in the estimated marginal means. The summary, conclusion and recommendations have been presented in Chapter Five.

CHAPTER FIVE

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

The study explored didactical conceptual structures in extending the didactical triad to the tetrahedron in solving equations of the circle. Empirical studies and literature showed that there are still inadequate interactions in the didactic triad as compared to the didactic tetrahedron and didactical conceptual structures interact better in the didactic tetrahedron than the didactic triad. The sequential explanatory mixed methods design was used in this study. This design adequately illuminated the interactions in the intersubjective didactic instructional models for the purposes of interpreting, explaining, confirming and corroborating quantitative and qualitative results and findings. Two baseline research questions, six research questions were posed to guide the whole study. These were as follows:

1. How does the baseline survey ensure reliability and validity of the study? And what are the possible covariates that affect the interactions of the didactical tetrahedron and what are the dominant factors that determine the interactions of the tetrahedron?
2. What is the knowledge level of students in solving equations of the circle before the implementation of the didactical tetrahedron?
3. What is the knowledge level of students in solving equations of the circle after the implementation of the didactical tetrahedron?
4. What is the relationship between interactions in the didactical tetrahedron implemented and students' performance in solving equation of the circle?

5. How do the qualitative sources help explain statistical significance across the intersubjective didactical instructional models in students' performance in equations of the circle?

In this chapter, the researcher has discussed the summary of findings in the research methods, baseline survey and the main study.

Summary

Psychological test questions were administered and responded to by 500 senior high schools, comprising 51% male and 49% female in the Upper East Region of Ghana. The distributions of the senior high schools selected can be found on Table 3 of Chapter Three. The 500 students were selected through the simple random sampling procedure (table of random numbers). This was then followed by six focus group interviews with 12 students (two each from the schools). These six schools and 12 students selected consisted of the four-grade classifications of the school performance in the Ghana Education Service to represent the categories of Excellent, Good, Average and below average (See Table 3). Consent form was sought from the regional education directorate before the administration of the research instruments. Consent was also sought from all participating students and teachers before the data collection processes.

The data gathered from the closed ended items in the psychological tests were analyzed quantitatively through the use of ANCOVA and MANCOVA, whilst the open ended items were analyzed qualitatively through the interview transcripts. The main findings from analysis of the results were summarised in the subsequent pages in this Chapter. This will be done in the order of the didactical instructional models and the followed up research questions. The Chapter will end with the implications drawn from the conclusions of the results, as well as limitations of the study and suggestions for future research.

The didactical instructional models took into considerations the influences of the covariates. After the covariates have been statistically removed in the ANCOVA and MANCOVA, normal ANOVA and MANOVA techniques were performed on the corrected or adjusted scores. These increased the powers or sensitivities of the F-tests to detect statistical differences. The references and contributions of the covariates and adjusted means were added to the between-groups' F-tests to describe and explain the statistically differences in the ANCOVA and MANCOVA results. The contributions of the covariates justified and inferred by the contributions of effect sizes, power statistics and estimated marginal means.

The one-way between-groups ANCOVA conducted in mathematisation and phenomenology models generally showed statistical significances after adjusting for gender differences. Interactions of mathematics content and technology attained much significance as compared to the interactions teachers and technologies. However, having confounded gender, all didactical conceptual structures attained statistical significances. This means that gender significantly reduced the differences. As supported and explained by the partial eta squared values and the estimated marginal means, gender is a major influence in the didactical conceptual structures in the tetrahedron.

In addition, the two-way between-groups ANCOVA conducted showed significant differences after adjusting for gender and school. Even though there were no statistical differences in the main effects and the two covariates, there was ample evidence to show that the didactical conceptual structures showed better interactions in mathematics and didactical phenomenology. The utilization of the ANCOVA and MANCOVA tests provided new mathematical concepts, structures and ideas for students apply, explain and relate mathematics knowledge with socio-cultural signs, symbols and artefacts. The students' ability to interact with signs, symbols and artefacts provided opportunities and possibilities to further construct mathematics with tools, instruments and technologies. With or without gender and school, students present, invent, innovate and solve mathematics tasks by employing mathematisation and didactical phenomenology.

The one-way between-groups ANCOVA in instrumentations and instrumentalisations showed statistically significant differences. Where few non-significant differences were observed, the inclusion of the covariates ensured that the main effects, the interaction effects and the covariates all reached statistical significances. Indeed, the covariates statistically controlled, reduced and removed the error variances by 1.49 in instrumentations. This means the instrumented tools, instruments and technologies relatively stabilized, integrated, corroborated and explained the mathematisation and didactical phenomenology models. Therefore, for the didactical conceptual structures to interact effectively in mathematics it is incumbent that mathematics tools, instruments and technologies are provided. The transcripts justified instrumentations in no uncertain terms.

In this models, the one-way between-groups ANCOVA revealed statistically significant differences in all the didactical conceptual structures. After removing the covariates, there were much better interactions among the conceptual structures interaction and the covariates. In fact, school was the main covariate as it directly controlled and statistically removed the confounding effects. Because of its socio-cultural nature, it combined both mathematisation and didactical phenomenology, and the instrumentations and instrumentalisations models to manipulate the 1T, 2T, 3T and 4T praxeologies. This helped students to improve upon their interactions.

In the MANCOVA tests in the four didactical conceptual structures with the covariates, there were generally statistically significant in the main components, the interactions and gender. Even though school was not significant, the multivariate estimated marginal means showed significant reduction of the error variances by 1.49. This result was not different from the way and manner the students physically expressed their feelings and experiences in the models.

The MANCOVA results in sampled equations of the circle, adjusting for the covariates, initially revealed no significantly statistical differences in the tasks in equations of the circle. However, after controlling for the covariates, the interactions were significant. This clearly shows gender and school still pose a lot of teaching and learning challenges in didactics of mathematics. What was even more revealing is the fact that without technologies, the didactical conceptual structures in the triad failed to reach statistical significances with and without the covariates. It was therefore the inclusion of technologies that those significant differences. The discourses in the solving the tasks indicated that students badly required technologies to solve the questions. It was therefore prudent to extend the didactical triad to the tetrahedron in order to give students holistic experiences in equations of the circle. This notwithstanding, the covariates statistically reduced, controlled and removed the effects.

The interview transcripts in the mathematisation and didactical phenomenology closely examined students' daily life situations. This was followed by experientially real mathematics problems and applications of the models. On the conceptual meanings of the signs and symbols, students co-constructed and jointly reproduced varied contextual understanding. These constructions were related to the algebraic expressions and geometric meanings. They further enumerated mathematics tools that transformed and translated the signs and symbols. After this, students utilized the tools in the processes of measuring, computing and manipulating the tasks in equations of the circle. Therefore, it was clear that the mathematisations and didactical phenomenology completely changed and modified students' understanding of signs and symbols.

Also, the instrumentations and instrumentalisations, technologies had been transformed and translated from mathematisation and phenomenological dialogues. These provided significant influences and impeccable knowledge in the ways the students solved the questions. The students' mathematical sets, calculators and computers provided both knowledge and applications.

In addition, the transcripts in the didactical situations boosted their social interactions in solving tasks in equations of the circle. At the same time, the anthropological praxeologies of didactics analyzed the effectiveness and efficiency in ordering, sequencing and progressing through 1T, 2T, 3T and eventually 4T. This provided simplicity, precision, detailed explanation and accuracy in the discourses. This confirmed and corroborated the statistically significance differences. It was therefore worthwhile to intersubject the didactical praxeologies models into new tasks, techniques, theories and technologies among the didactical conceptual structures.

In the exemplified equations of the circle, students applied the didactical models to improve upon their knowledge, skills, understanding and utilization of the interactions in the tetrahedron. Students comfortably and without hesitation differentiated between the standard and general equations of the circle, solved for the centres and radii, and related and connected their tasks to other tasks. The results showed that students interacted uniformly and optimally in solving the tasks in equations of the circle. Therefore, the extension from the didactical triad to the tetrahedron modified and transitioned the interactions much better.

Conclusions

In this section, conclusions from what the study found in relationship to the problem in Chapter One will be presented. Conclusions based on the research hypotheses and research questions will be presented on the themes of the didactical instructional models. These are mathematisation and didactical phenomenology, instrumentations and instrumentalisations, didactical situations and anthropological praxeologies, and equations of the circle.

The findings showed that the utilization of the mathematisation and didactical phenomenology provided new mathematical concepts, structures and ideas in relation to the phenomena and contexts. In learning and solving tasks, it can be concluded that students must consider didactical phenomenology in applying, explaining and relating a variety of signs, symbols, artefacts and tools. These helped to establish connections between reality and mathematics during constructions and measurements of mathematical situations and contexts. The statistically significance differences suggested that the didactical conceptual structures interacted well in most components of these models.

Secondly, the mathematisation and didactical phenomenology best followed the Netherlands didactical tradition. The constructs were both formative and summative processes. The students evaluated proficiency and make their decisions. And the particular engagements with the signs, symbols and tools were found in the socio-cultural dimensions, representations and procedures. The signs, symbols and tools as manifested in the calculators, computers, graph sheets and mathematical sets offered opportunities to students to make the didactical phenomenological choices. These choices could be developed without new tasks, techniques, theories and technologies.

Also, the theory of realistic mathematics education helped students to conceptualize mathematics both as a body of knowledge to be transmitted and as a form of human activity. The human activities were performed in both vertical and horizontal mathematizations. The roles of students were to develop the mathematical signs, symbols and tools to solve the tasks in the horizontal components. This helped them to make connections between the mathematical skills, concepts, strategies, methods and theories. With the signs, symbols and tools, the students mathematised reasoning, representations, connections and communications in the vertical components. These two components of mathematizations allowed students to formulate real situations in equations of the circle. In the tetrahedron, students excellently organized and solved tasks with the signs, symbols and tools.

Coupled with horizontal and vertical mathematisation were the utilizations of the mechanistic, structuralist, empiricist and realistic didactical phenomenology in constructing and shaping mathematical knowledge. The mechanistic dialogue provided systems of signs, symbols and tools. The structuralist dialogue provided deductive systems and generalizations of the signs, symbols and tools. The empiricist dialogue provided actual classroom applications and utilizations of the signs, symbols and tools. The realistic dialogue provided comprehensive and holistic phenomenology. The transcripts in mathematisation and didactical phenomenology connected the signs, symbols and tools to their daily life situations.

Largely, mathematisation and didactical phenomenology activities helped students to translate, transform and mathematise the tasks with the signs, symbols and tools. This dimension helped to shape, reshape and manipulate the equations of the circle mathematics through comprehensive vertical mathematization. Students associated the general contexts of signs and symbols. For instance, in transcripts, a student explained *a sign is an identity given to an object, it is something used to represent or replace a long sentence, and it is a letter that represents a constant*. However, when the students reformulated and discovered the didactical phenomenology, they were able to transfer real world tasks to mathematical tasks. So, knowing they discovered that there are other signs and symbols such as 'the equal to', 'addition', 'squares', 'constants', 'addition', 'equation to', 'minus', 'square root', 'multiplications', and 'x' and 'y'.

The findings showed that instrumentation and instrumentalisation provided new cultural and social interactions in transforming the signs, symbols, artefacts and tools to instruments and technologies. As a result, the hypothesis revealed statistical significances. The frameworks of instrumentations and instrumentalisations successfully helped to analyze the instruments and technologies. The mathematical sets, calculators and computers bridged the sociocultural dimensions of the didactical schemes to the social schemes of constructing knowledge. This Vygotskian interpretation helped students to project from the signs, symbols, artefacts and tools to higher mental cognitive skills. In the instruments and technologies, students created concrete activities associated with sign-symbolic structures. This helped them to construct advance symbols, manipulations, sketches, diagrams and drawings in equations of the circle.

Coupled with instrumentation and instrumentalisation was the successfully applications and utilization of the theory of instrumental genesis itself. As students metaphorically transitioned through signs and symbols, artefacts, tools, instruments and technologies to the dialogues, they shaped, remodelled and reconceptualised the signs and symbols, artefacts and tools with new socio-cultural perspectives. These socio-cultural acts were redeveloped and reshaped in any of the local and social situations.

Lastly, the dialogues showed that the students transformed and transitioned the Vygotsky's instrumental theory and the genesis of functions. The roles and responsibilities of the Didactical Conceptual Structures in didactics of mathematics helped to boost cognitive metacognitive processes in equations of the circle. The students developed analytical strategies that incorporated cooperative, guided, peer and experiential learning. They precisely, accurately and contextually explained technology as *knowledge in science used to make work easier and faster and the application of mathematical knowledge to solve problems*. They outlined the instruments and technologies as *calculators, mathematical set, computers, graph sheets, mathematical tables and even smart mobile phones*. And they preferred computers because *their school always acquire computers for learning mathematics and the mathematical sets as he basic technology instruments because they initiate all processes involved*.

The findings showed that didactical situations and anthropological praxeologies boosted their practices, discourses and strategies. They established interrelations and statistically significant differences in the progression from the 1T, 2T and 3T to the 4T praxeologies. In the anthropological praxeologies (T, τ, θ, Θ) in the tasks, techniques, technologies and theories, the students could describe, justify and explain the phenomenology from the socio-cultural sign-symbolic structures.

Again, the students reinvented activities, practices and experiences in the radius-centre, diameter, tangent, normal, standard and general equations of the circle. For instance, the students started with the general equation techniques of $x^2 + y^2 + 2gx + 2fy + c = 0$ with the centre $C(-g, -f)$ and radius $r = \sqrt{g^2 + f^2 - c}$, to solve equations with the standard equation techniques of $(x-h)^2 + (y-k)^2 = r^2$ with the centre $C(h, k)$ and radius r.

It was concluded that a circle is a typical or degenerate coordinate-free conic section (Stitz, & Zeager, 2013). The standard equations is ($(x)^2 + (y)^2 = r^2$) with the centre at the origin or $(x-h)^2 + (y-k)^2 = r^2$) with the centre at the point (h, k). The applications of the standard equations yielded better statistically significant differences in the tetrahedron than in the triad. The students could code the parts of the standard equation into a computer or calculator and solved them. However, they were not able to do same for the other types of circle equations. This made the standard equation much more effective than the other types of equations. Even one novelty was to use the standard equation, $(x-h)^2+(y-k)^2=r^2$ to particularly derive the general equation. It was discovered that when the standard is expanded, then the terms can be compared to the general equation. In this case, the second-degree terms (x^2+y^2), the linear terms ($-2hx$ and $-2ky$) and the constants (h^2+k^2) are equivalent to the terms x^2+y^2, $2gx+2fy$ and r^2 in $x^2+y^2+2gx+2fy+c=0$.

Another novelty and significant giant strides the students made was using the midpoint strategy or method to solve equations of the circle. For instance, given any two endpoints $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, then the midpoint of the diameter is the centre, $(h, k)=\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ and the radius, $r=\frac{1}{2}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$. By extension, the tangent equation becomes $xx_1+yy_1=r^2$ and the normal equation becomes $\frac{y}{y_1}=\frac{x}{x_1}$. These further enabled the students to create robust didactical interactions with teachers, mathematics content and technologies.

Recommendations

The recommendations combine both the implications and contributions of the study to theory, methodology, research, practice, policy and equations of the circle in didactics of mathematics.

In order to impact theory, it was recommended that the hypothesis drawn from the intersubjective theories must always demonstrate statistical significances. The integration of the qualitative and quantitative phases has broadened the conceptualization of didactics of mathematics. The smooth flow and flexible strides within and between the theories, and the subsequent transcripts gives credence to enlarged didactical theory mathematical didactics. Even within each theory, carefully chosen components were selected to form conceptual frameworks. Therefore, wider and broader networks of didactical interactions make enormous contributions to mathematical knowledge.

Didactical conceptual structures could be also expanded to larger study areas, higher number of schools and larger sample sizes. More expanded collaborative theories and conceptual framework can diversify the number of didactical conceptual structures not just in the tetrahedron but to higher polygons of didactics. Expanded theories can elucidate many key components and relationships amongst students, teachers, mathematics content, technologies and mathematics organizations to infuse more focus to didactics.

In order to impact on methodology, it was recommended that mixed methods research design be considered by graduate and other researchers. The design helped to explore, explain, describe and organize a range of suitable theoretical and conceptual frameworks. The design can help researchers to combine sequential and concurrent methods of data collection. In both the survey and experimental stages, one can use two designs in collecting and analyzing sequentially or concurrently.

Also, the baseline survey results before the treatments and experimental design processes can foster redesign of research instruments in the second stage. While combining sequential and concurrent, one can establish initial survey results and improve upon them in an experimental design. This seeming congruence between the testable hypotheses and the initial baseline survey bridges gaps in the methodology. It was therefore recommended that considerably short period of time should be adopted in such designs. However, there is a debate on the length of time required for full data collection in sequential explanatory mixed methods (Creswell, 2012).

In other to boost research it was recommended that experts greatly organize research fora and colloquia to extend the didactical triad. It is best methods and techniques that can propel the teaching and learning of Equations in equations of the circle in Ghana. Researchers can synchronize several didactical connections in setting socio-culturally related contexts. It was therefore, recommended that the didactical relationships that emerged from the findings should be used as catalysis for spearheading didactical pedagogies, philosophies and instructions in all mathematical domains.

Again, the findings of this study were particularly applicable and useable to researchers who are seeking further expansions and replications in multiple connections. It was therefore recommended that multiple constructs be used to bring better efficient models than this three-phase model.

Lastly, the implication of the design is the integration of a survey, psychological tests and interview transcripts. The goals were to confirm, support and explain the stages of the research. In this way, the study has contributed to the research utility and fruitfulness of data integration through the mixed methods research in the context of mathematical didactics.

In order to strengthen classroom instructional practice and boost students' performance in mathematics, it was recommended that school heads and teachers practically implement the didactical tetrahedron to the fullest. Teaching and learning of didactics of mathematical should focus more efforts on increasing classroom interactions through the use of locally-manufactured technologies. Based on the students' transcripts, locally manufactured technologies seem to have been prominently evidenced since they require less funding and less operational challenges..

Also, beyond orientations of metre rules, mathematical sets and calculators the findings have implications on the manufacturing and usage of locally artefacts and improvised technologies. Local artefacts facilitated the interactions better than foreign ones. Locally manufactured, produced and designed technologies can enforce interactions not just among school-community practices but also between socio-cultural synergies. As frequent interactions produced improved results and academic performance, creating locally manufactured and designed technologies can actually increase interactivity, performance and participation.

In other to influence policy, it was recommended that the state and its agencies give students and teachers independent powers to implement school-based didactical tetrahedron. In building connections and relationships between students, teachers, mathematical content and technologies, students can interact better in the mathematics classroom. It was therefore, recommended that stakeholders develop conceptual activities in mathematical so as to enrich conceptual activities, methods and strategies. Also, there must be strategies to compel active engagements of students, teachers, mathematics content and technologies in the teaching and learning of mathematics.

In other to improve upon students' performance in equations of the circle, it was recommended that authors, publishers and editors strengthen their knowledge and practices on the didactical tetrahedron. The results shows that the stronger the mathematics content, the better the interactions in the didactical tetrahedron. In this study, it emerged that algebraic/geometric relationships were improved with using the technologies. Algebra and geometry domains were integrated in building didactical relationships. Students' algebraic/geometric in centre/radius interactions and relationships were strengthened. These helped students to transition from algebraic equations to geometric graphs.

Also, the results of this study have implications for senior high mathematics content in focussing on topic sequencing and technology-related instructional strategies. However, the mathematics contents rarely afford students the opportunities to play their roles and functions in the didactics of mathematics. The models can also assess students in both formative and summative. Using these chains of models, students can develop explicit connections in the concepts.

Suggestions for Further Research

The qualitative stage nearly jeopardized the quality and expectations of the didactical tetrahedron. This might have been caused by poorly specified models. Although this study enhances and improves upon didactical knowledge and practices in the tetrahedron, a few issues remain to be addressed. Both the quantitative and qualitative results indicated that some didactical conceptual structures showed stronger statistical significances than others, but no single conceptual structure dominated in the attainments of the statistical significances.

Even the representation of one region of Ghana requires replication of the study in the other regions. This will fully overcome all the threats to validity and enhance the generalizability of the results. And although the students in the interview stage satisfied the selection criteria and openly rendered complete detailed dialogues, they may slightly differ from all the 500 randomly selected students. Replication is therefore necessary to validate this study.

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APPENDICES

APPENDIX A

COPY OF INTRODUCTORY LETTER FROM UNIVERSITY OF CAPE COAST

UNIVERSITY OF CAPE COAST
COLLEGE OF EDUCATION STUDIES
FACULTY OF SCIENCE AND TECHNOLOGY EDUCATION
DEPARTMENT OF MATHEMATICS AND I.C.T EDUCATION

Telephone: 0332096951
Telex: 2552, UCC, GH
Telegrams & Cables: University, Cape Coast
Email: dmicte@ucc.edu.gh

University Post Office
Cape Coast, Ghana

Your Ref:

Our Ref: DMICTE/P.3/V.1/010

Date: 14th December, 2017

The Head
Department of Mathematics Education
University of Education, Winneba
Winneba

Dear Sir/Madam,

RESEARCH VISIT

The bearer of this letter, Mr. Clement Ayarebilla Ali, with registration number ED/DME/13/0004 is a PhD (Mathematics Education) student of the Department of Mathematics and ICT Education, College of Education Studies, University of Cape Coast.

As part of the requirements for the award of a doctorate degree, he is required to undertake a research visit to your school with the purpose of collecting data on the topic "**DIDACTIC CONCEPTUAL STRUCTURES IN EXTENDING THE TRIAD TO THE TETRAHEDRON WITH APPLICATIONS IN CONIC SECTIONS**".

I would be grateful if you could give him the necessary assistance he may need.

Thanks for your usual support

Yours faithfully,

Prof. Ernest K. Davis
SUPERVISOR

APPENDIX B

COPY OF CONTENT LETTER FROM UPPER EAST REGIONAL DIRECTOR

GHANA EDUCATION SERVICE

In case of reply, the number and date of this letter should be quoted

Our Ref. REO/2/Vol.12/117

REPUBLIC OF GHANA

REGIONAL EDUCATION OFFICE
P.O. BOX 110
BOLGATANGA – U.E.R.

26th January, 2018

RE: RESEARCH VISIT

We write to inform you that permission has been granted to Mr. Clement Ayarebilla Ali, a PhD (Mathematics Education) student of the Department of Mathematics and ICT, Education College of Education Studies, University of Cape Coast to undertake a Research Visit to your school with the purpose of collecting data on the topic: "DIDACTIC CONCEPTUAL STRUCTURES IN EXTENDING THE TRIAD TO THE TETRAHEDRON WITH APPLICATIONS IN CONIC SECTIONS".

We have informed him to interact with the students after contact hours.

Kindly grant him the needed assistance.

Thank you.

PATRICIA AYIKO
REGIONAL DIRECTOR OF EDUCATION (UER)

DISTRIBUTION:
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4. ZUARUNGU SENIOR HIGH SCHOOL, **BOLGATANGA**
5. ZAMSE SENIOR HIGH/TECHNICAL SCHOOL, **BOLGATANGA**
6. BOLGATANGA GIRLS' SENIOR HIGH SCHOOL, **BOLGATANGA**

Cc:- Mr. Clement Ayarebilla A.
UCC
Cape cCoast.

APPENDIX C

COPY OF CONSENT LETTER TO REGIONAL DIRECTOR OF EDUCATION

UNIVERSITY OF EDUCATION, WINNEBA
FACULTY OF EDUCATIONAL STUDIES
DEPARTMENT OF BASIC EDUCATION
P.O. BOX 25, WINNEBA, GHANA

Our Ref: **Date**: September 19, 2017

Your Ref:

TO:

THE REGIONAL DIRECTOR OF EDUCATION

UPPER EAST REGION

BOLGATANGA

Dear Sir/Madam,

CONSENT FORM TO CONDUCT PHD RESEARCH IN CONIC SECTIONS

I wish to seek your consent to undertake PhD study on your students about ***Didactical Conceptual Structures in Extending the Triad to the Tetrahedron with Applications in Conic Sections***. I am asking for your students' participations to assist in the completion of the questionnaire and computer laboratory experiments in Elective Mathematics class. I ask that you read this letter and the attached questionnaire and clarify any issue you may have before agreeing to this study. I can be reached by the telephone numbers +233208554016; +233244122385; ayarebilla@yahoo.com; acali@uew.edu.gh.

i). *The study*: The purpose of this study is to explore students' ***Conceptual Structures in Conic Sections***. If you agree to have your students participate in this study, your students will be asked to perform one laboratory computer experiment with either GeoGebra or Matlab software and complete two exercises with instructional models. Your students will also rate the instructional models with the

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APPENDIX D

COPY OF CONSENT FORM TO MATHEMATICS STUDENTS

QUALITATIVE INTERVIEW GUIDE ON CONCEPTUAL STRUCTURES IN CONIC CIRCLES

UNIVERSITY OF EDUCATION, WINNEBA

FACULTY OF EDUCATIONAL STUDIES

DEPARTMENT OF BASIC EDUCATION

Our Ref: ***Date***: February 8, 2018

Your Ref:

STUDENT

GHANA EDUCATION SERVICE

UPPER EAST REGION

Dear Master/Miss,

QUESTIONNAIRE ON THE TEACHING AND LEARNING OF CONIC CIRCLES

I wish to make a follow up on the PhD study on the topic, '***DIDACTICAL CONCEPTUAL STRUCTURES IN EXTENDING THE TRIAD TO THE TETRAHEDRON WITH APPLICATIONS IN CONIC SECTIONS (CIRCLES)'.*** I would therefore, be very grateful if you could answer the following questions to enable me carry out the research. The responses are meant for research only, and would therefore be treated with the necessary confidentiality.

Thank you very much.

Best regards,

Clement Ayarebilla Ali.

ZuSHS, Bolga

→ Isaac Asaah

→ Jennifer Apabum

→ Benedict Ndego

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APPENDIX E

A SAMPLE OF STUDENT'S WORKSHEET IN TASKS OF EQUATIONS OF THE CIRCLE

$x^2+y^2+6x+4y-36=0$

Method 1: Completing Squares

$x^2+y^2+6x+4y-36=0$

$x^2+6x+y^2+4y=36$

$x^2+6x+9+y^2+4y+4=36+9+4$

$(x+3)^2+(y+2)^2=49$

$(x+3)^2+(y+2)^2=7^2$

$C(-3,-2),\ r=7$

Method 2 Comparison

$x^2+y^2+2gx+2fy+c=0$

$C(-g,-f)$

$r=\sqrt{g^2+f^2-c}$

$x^2+y^2+2gx+2fy+c=0$

$x^2+y^2+6x+4y-36=0$

Compare:

$2gx=6x$

$g=3$

$2fy=4y$

$f=2$

$c=-36$

$C(-g,-f)=C(-3,-2)$

$r=\sqrt{3^2+2^2--36}$

$=\sqrt{49}$

$=7$

APPENDIX F

INTER-ITEM MATRIX CORRELATIONS

APPENDIX F1

INTER-ITEM CORRELATION MATRIX OF INDEPENDENT VARIABLES

Variables	Gender	Level	School	Management	Residence	Course	Content interactions	Technologies interactions	Teachers interactions	Students interactions
Gender	1.000	.089	.374	.265	.046	-.077	-.134	.035	.025	.033
Level	.089	1.000	-.154	.057	-.136	-.071	.208	.110	-.096	-.027
School	.374	-.154	1.000	-.194	-.014	-.097	-.099	-.051	-.003	-.083
Management	.265	.057	-.194	1.000	.398	-.052	-.109	.049	.061	-.120
Residence	.046	-.136	-.014	.398	1.000	.081	.044	.092	.153	.115
Course	-.077	-.071	-.097	-.052	.081	1.000	.064	.183	-.031	-.076
Content	-.134	.208	-.099	-.109	.044	.064	1.000	.249	.208	.153
Technologies	.035	.110	-.051	.049	.092	.183	.249	1.000	.381	.112
Teachers	.025	-.096	-.003	.061	.153	-.031	.208	.381	1.000	.297
Students	.033	-.027	-.083	-.120	.115	-.076	.153	.112	.297	1.000

APPENDIX F2

ITEM-TOTAL STATISTICS OF THE DIDACTICAL INSTRUCTIONAL MODELS

Components .897	Scale Mean if Item Deleted	Scale Variance if Item Deleted	Corrected Item-Total Correlation	Squared Multiple Correlation	Cronbach's Alpha if Item Deleted
Mathematizing within Signs and Symbols	73.54	345.619	.328	.192	.895
Mathematizing within Tasks	72.80	345.099	.294	.185	.896
Mathematizing within Artefacts	71.41	355.252	.109	.183	.899
Mathematizing within Instruments	73.46	342.778	.371	.253	.895
Mathematizing within Tools	73.05	345.164	.301	.179	.896
Mathematizing within Innovations	71.17	352.299	.190	.225	.898
Mathematizing within Teachers	73.41	342.356	.421	.287	.894
Mathematizing within Students	73.54	346.298	.315	.205	.896
Mathematizing within Contents	73.36	343.241	.391	.273	.894
Mathematizing within Technologies	72.08	340.949	.358	.247	.895
Transitioning across Didactic Tasks	73.34	340.404	.480	.327	.893
Transitioning across Didactic Techniques	73.19	336.237	.552	.470	.891
Transitioning across Didactic Theories	73.15	334.813	.585	.467	.891
Transitioning across Didactic Technologies	72.98	335.624	.515	.374	.892
Transitioning across Teachers	73.20	334.443	.579	.427	.891
Transitioning across Students	73.20	334.428	.554	.418	.891
Transitioning across Contents	73.10	334.659	.553	.426	.891
Transitioning across All Technologies	72.92	333.856	.554	.417	.891
Transitioning across All Elements	72.98	333.888	.548	.404	.891
Transitioning across All Structures	72.87	331.485	.609	.455	.890
Didactizing with One Structure	73.38	340.430	.456	.361	.893
Didactizing with Two Structures	73.33	338.780	.514	.447	.892
Didactizing with Three Structures	73.38	341.364	.431	.294	.893
Didactizing with Four Structures	73.03	337.181	.487	.325	.892
Didactizing with Tasks	72.75	338.012	.451	.321	.893
Didactizing with Techniques	72.90	338.154	.470	.335	.893
Didactizing with Theories	73.09	337.241	.486	.374	.892
Didactizing with Technologies	73.12	334.692	.527	.395	.892
Didactizing with All Elements	72.85	336.108	.517	.353	.892
Didactizing with All Structures	73.05	336.796	.519	.381	.892

APPENDIX F3

ITEM-TOTAL STATISTICS OF THE EQUATIONS AND TASKS OF THE CIRCLE

Components	Scale Mean if Item Deleted	Scale Variance if Item Deleted	Corrected Item-Total Correlation	Squared Multiple Correlation	Cronbach's Alpha if Item Deleted
Conceptualizing in Circle Shape	30.66	49.841	.298	.142	.764
Conceptualizing in Circle Angle	31.15	50.691	.456	.276	.757
Conceptualizing in Circle Dimension	30.34	49.614	.212	.117	.776
Conceptualizing in Circle Three Basic Parts	30.41	50.773	.272	.140	.766
Conceptualizing in Circle Standard Equation	30.15	51.533	.255	.097	.766
Conceptualizing in Circle Radius-Centre Equation	30.39	50.562	.261	.105	.767
Conceptualizing in Circle General Equation	30.82	49.517	.383	.222	.758
Conceptualizing in Circle Diameter Equation	30.43	50.068	.376	.213	.759
Circle Standard Tangent Equation	30.30	49.315	.342	.199	.761
Circle Standard Normal Equation	30.25	49.419	.340	.238	.761
Circle at Centre (0,0) with no Constant c	30.55	49.920	.325	.145	.762
Circle at Centre (h,0) with No Constant c	30.41	48.185	.385	.238	.757
Circle at Centre (0,k) with No Constant c	30.56	48.741	.392	.265	.757
Circle at Centre (-h,-k) with No Constant c	30.44	49.186	.421	.281	.755
Circle at Centre (h,k) with No Constant c	30.45	48.625	.423	.252	.754
Circle at Centre (-h,-k) with Constant c	30.40	49.002	.404	.280	.756
Circle at Centre (-h,-k) with Non-Unit Coefficients	30.47	48.707	.422	.405	.755
Circle at Centre (h,k) with Non-Unit Coefficients	30.58	49.475	.416	.410	.756

F4

ITEM DIFFICULTY ANALYSIS OF THE DIDACTICAL INSTRUCTIONAL MODELS

Components	Mean	Std. Deviation	N
Mathematizing within Signs and Symbols	1.97	1.229	496
Mathematizing within Tasks	2.70	1.384	496
Mathematizing within Artefacts	4.10	1.273	496
Mathematizing within Instruments	2.04	1.288	496
Mathematizing within Tools	2.46	1.352	496
Mathematizing within Innovations	4.33	1.183	496
Mathematizing within Teachers	2.09	1.178	496
Mathematizing within Students	1.97	1.223	496
Mathematizing within Contents	2.15	1.202	496
Mathematizing within Technologies	3.43	1.442	496
Transitioning across Didactic Tasks	2.16	1.150	496
Transitioning across Didactic Techniques	2.32	1.207	496
Transitioning across Didactic Theories	2.36	1.208	496
Transitioning across Didactic Technologies	2.53	1.312	496
Transitioning across Teachers	2.30	1.235	496
Transitioning across Students	2.30	1.286	496
Transitioning across Contents	2.41	1.277	496
Transitioning across All Technologies	2.58	1.311	496
Transitioning across All Elements	2.53	1.322	496
Transitioning across All Structures	2.64	1.305	496
Didactising with One Structure	2.12	1.203	496
Didactising with Two Structures	2.17	1.162	496
Didactising with Three Structures	2.12	1.210	496
Didactising with Four Structures	2.47	1.301	496
Didactising with Tasks	2.75	1.345	496
Didactising with Techniques	2.60	1.291	496
Didactising with Theories	2.42	1.300	496
Didactising with Technologies	2.38	1.330	496
Didactising with All Elements	2.66	1.284	496
Didactising with All Structures	2.46	1.246	496

APPENDIX G

EXPLORATORY FACTOR ANALYSIS

APPENDIX G1

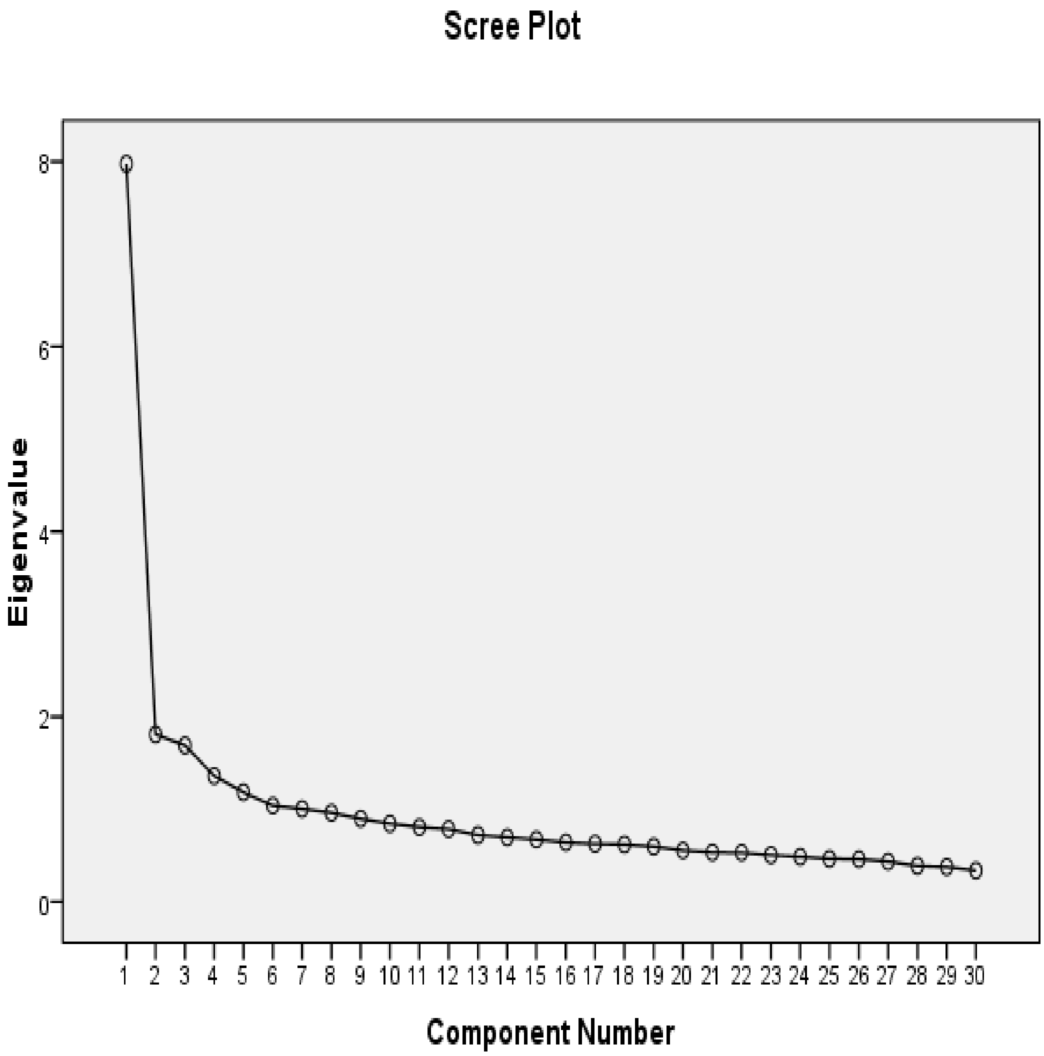
TOTAL VARIANCE EXPLAINED IN COMPONENTS OF THE DIDACTICAL MODELS

	Initial Eigenvalues			Extraction Sums of Squared Loadings			Rotation Sums of Squared Loadings		
Component	Total	% of Variance	Cumulative %	Total	% of Variance	Cumulative %	Total	% of Variance	Cumulative %
1	7.976	26.586	26.586	7.976	26.586	26.586	4.797	15.991	15.991
2	1.809	6.030	32.616	1.809	6.030	32.616	3.066	10.220	26.211
3	1.690	5.633	38.249	1.690	5.633	38.249	2.313	7.711	33.922
4	1.360	4.534	42.783	1.360	4.534	42.783	1.756	5.855	39.776
5	1.184	3.947	46.730	1.184	3.947	46.730	1.605	5.351	45.127
6	1.039	3.462	50.192	1.039	3.462	50.192	1.340	4.467	49.594
7	1.004	3.348	53.539	1.004	3.348	53.539	1.184	3.946	53.539
8	.962	3.207	56.746						
9	.897	2.989	59.735						
10	.844	2.813	62.548						
11	.806	2.688	65.236						
12	.785	2.618	67.853						
13	.721	2.402	70.255						
14	.697	2.322	72.578						
15	.673	2.242	74.820						
16	.641	2.136	76.956						
17	.627	2.091	79.046						
18	.620	2.066	81.112						
19	.597	1.990	83.102						
20	.555	1.850	84.952						
21	.533	1.778	86.729						
22	.530	1.768	88.497						
23	.504	1.681	90.178						
24	.485	1.617	91.795						
25	.464	1.546	93.341						
26	.460	1.533	94.874						
27	.435	1.449	96.323						
28	.388	1.293	97.616						
29	.377	1.256	98.872						
30	.338	1.128	100.000						

Extraction Method: Principal Component Analysis.

APPENDIX G2

THE SCREE PLOT OF THE EXPLORATORY FACTOR ANALYSIS IN THE DIDACTICAL MODELS



APPENDIX G3

INITIAL ROTATED COMPONENT MATRIX BY EXPLORATORY FACTOR ANALYSIS

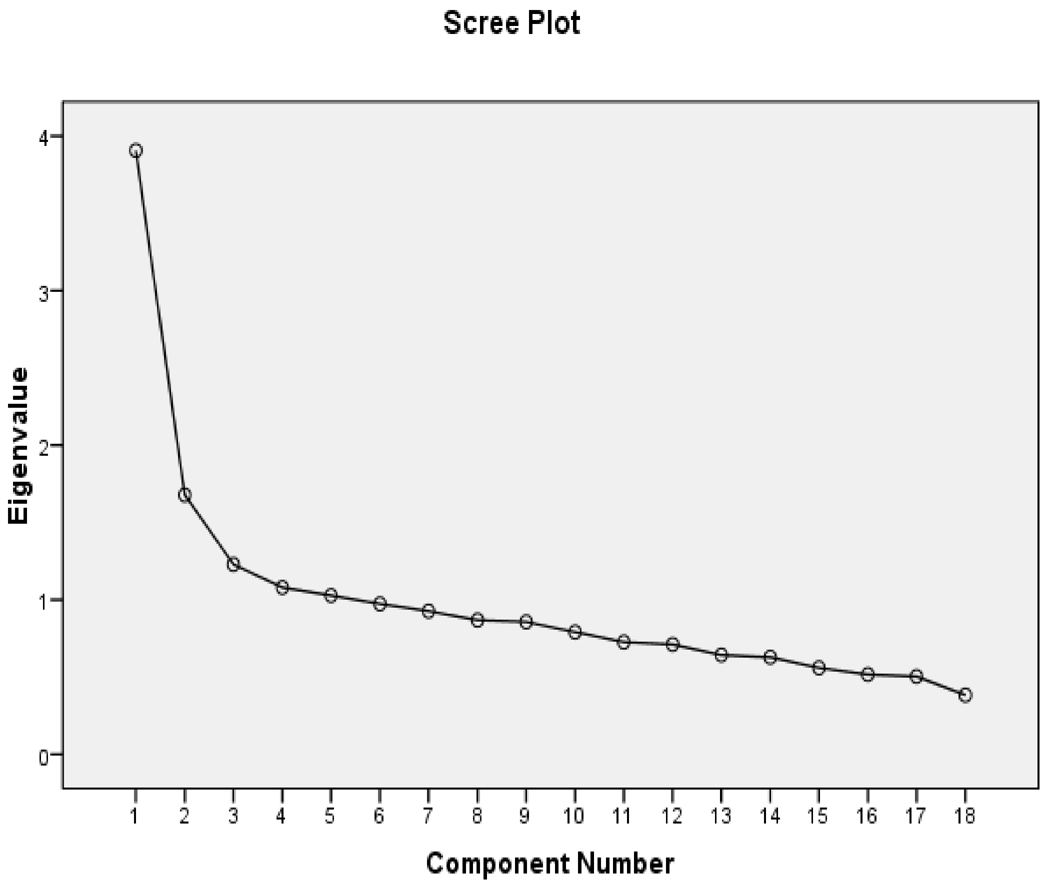
Components	Component 1	2	3	4	5	6	7
Instrumentations in Mathematics Contents	.694						
Instrumentations in Didactic Techniques	.692						
Instrumentations in Signs	.680						
Instrumentations in Didactic Theories	.668						
Instrumentalizations in Teachers	.646						
Instrumentations in All Elements	.611						
Instrumentations in All Structures	.611						
Instrumentations in Didactic Technologies	.608						
Instrumentations in Students	.604						
Instrumentations in Didactic Tasks	.539						
Praxeologies in Techniques		.704					
Praxeologies in Tasks		.681					
Praxeologies in Theories		.648					
Praxeologies in Technologies		.587					
Praxeologies in Two Structures			.730				
Praxeologies in One Structure			.679				
Praxeologies in Three Structures			.551				
Mathematizing in Instruments				.773			
Mathematizing in Students				.519			
Mathematizing in Technologies					.787		
Mathematizing in Artefacts					.780		
Mathematizing in Tools						.804	
Mathematizing in Signs and Symbols							
Mathematizing in Contents							.694

Extraction Method: Principal Component Analysis.
Rotation Method: Varimax with Kaiser Normalization.

a. Rotation converged in 8 iterations.

APPENDIX G4

SCREE PLOT IN EXPLORATORY FACTOR ANALYSIS IN CIRCLE EQUATIONS



APPENDIX G5

TOTAL VARIANCE EXPLAINED IN SAMPLE TASKS AND EQUATIONS

Component	Initial Eigenvalues			Extraction Sums of Squared Loadings			Rotation Sums of Squared Loadings		
	Total	% of Variance	Cumulative %	Total	% of Variance	Cumulative %	Total	% of Variance	Cumulative %
1	3.906	21.702	21.702	3.906	21.702	21.702	2.003	11.126	11.126
2	1.678	9.323	31.025	1.678	9.323	31.025	1.891	10.507	21.633
3	1.228	6.824	37.849	1.228	6.824	37.849	1.530	8.498	30.131
4	1.079	5.995	43.844	1.079	5.995	43.844	1.429	7.938	38.069
5	1.027	5.707	49.550	1.027	5.707	49.550	1.146	6.368	44.437
6	.973	5.404	54.955	.973	5.404	54.955	1.114	6.189	50.626
7	.926	5.146	60.101	.926	5.146	60.101	1.079	5.994	56.620
8	.868	4.822	64.923	.868	4.822	64.923	1.075	5.971	62.591
9	.856	4.757	69.680	.856	4.757	69.680	1.036	5.755	68.347
10	.790	4.390	74.070	.790	4.390	74.070	1.030	5.724	74.070
11	.726	4.032	78.102						
12	.711	3.947	82.050						
13	.642	3.567	85.616						
14	.628	3.487	89.103						
15	.559	3.105	92.208						
16	.516	2.867	95.076						
17	.504	2.797	97.873						
18	.383	2.127	100.000						

Extraction Method: Principal Component Analysis.

APPENDIX G6

DIRECT OBLINIM PATTERN MATRIX ROTATION OF TASKS AND EQUATIONS

	Component									
	1	2	3	4	5	6	7	8	9	10
Circle at Centre (h, k) with Non-Unit Coefficients	.909									
Circle at Centre (-h,-k) with Non-Unit Coefficients	.880									
Circle at Centre (-h,-k) with Constant c	.490									
Circle Standard Tangent Equation		.791								
Circle Standard Normal Equation		.788								
Circle at Centre (0,k) with No Constant c			.804							
Circle at Centre (h,0) with No Constant c			.721							
Circle at Centre (-h,-k) with No Constant c			.542							
Circle General Equation				.815						
Circle Diameter Equation				.534					.380	
Circle Angle				.419		.329				
Circle at Centre (h, k) with No Constant c				.366						
Circle Dimension					.948					
Circle Three Basic Parts						.964				
Circle Radius-Centre Equation							.924			
Circle at Centre (0,0) with no Constant c								-.812		
Circle Shape									.916	
Circle Standard Equation										-.948

APPENDIX H

COVARIATES' MARGINAL MEANS AND STATISTICAL POWERS

Covariates (Base=.20)	Marginal means	Power corrected model
Gender	1.49	.970
School	3.40	.981
Level /experience	2.93	.988
Programme	1.65	.981
Management	1.08	.978
Residential status	1.03	.979

APPENDIX H1

GENDER AS A COVARIATE (ESTIMATED MARGINAL MEANS: 1.49)

Source	Type III Sum of Squares	df	Mean Square	F	Sig.	Partial Eta Squared	Noncent. Parameter	Observed Power[b]
Corrected Model	79.430[a]	36	2.206	1.224	.179	.088	44.071	.970
Intercept	156.095	1	156.095	86.607	.000	.159	86.607	1.000
Gender	2.572	1	2.572	1.427	.233	.003	1.427	.222
Students Interactions	11.288	4	2.822	1.566	.182	.013	6.263	.484
Teachers Interactions	2.376	4	.594	.330	.858	.003	1.319	.125
Content Interactions	1.155	3	.385	.214	.887	.001	.641	.090
Students Interactions * Teachers Interactions	7.358	5	1.472	.817	.538	.009	4.083	.294
Students Interactions * Content Interactions	11.127	8	1.391	.772	.628	.013	6.173	.361
Teachers Interactions * Content Interactions	2.906	3	.969	.537	.657	.004	1.612	.161
Students Interactions * Teachers Interactions * Content Interactions	6.365	6	1.061	.589	.740	.008	3.531	.236
Error	825.471	458	1.802					
Total	3897.000	495						
Corrected Total	904.901	494						

a. R Squared = .088 (Adjusted R Squared = .016)

b. Computed using alpha = .05

APPENDIX H2

SENIOR HIGH SCHOOL AS A COVARIATE (MARGINAL MEAN IS 3.40)

Source	Type III Sum of Squares	df	Mean Square	F	Sig.	Partial Eta Squared	Noncent. Parameter	Observed Power[b]
Corrected Model	88.090[a]	39	2.259	1.258	.142	.097	49.070	.981
Intercept	179.345	1	179.345	99.903	.000	.180	99.903	1.000
School Name	2.368	1	2.368	1.319	.251	.003	1.319	.209
Students Interactions	21.052	4	5.263	2.932	.021	.025	11.727	.787
Teachers Interactions	4.202	4	1.051	.585	.674	.005	2.341	.194
Technology Interactions	8.736	4	2.184	1.217	.303	.011	4.867	.382
Students * Teachers Interactions	8.177	5	1.635	.911	.474	.010	4.555	.327
Students * Technology	14.955	10	1.495	.833	.597	.018	8.330	.444
Teachers Interactions * Technology	11.241	6	1.873	1.044	.396	.014	6.262	.414
Students * Teachers * Technology Interactions	6.212	5	1.242	.692	.630	.008	3.460	.251
Error	816.811	455	1.795					
Total	3897.000	495						
Corrected Total	904.901	494						

a. R Squared = .097 (Adjusted R Squared = .020)

b. Computed using alpha = .05

APPENDIX H3

LEVEL AS A COVARIATE (MARGINAL MEAN IS 2.93)

Source	Type III Sum of Squares	df	Mean Square	F	Sig.	Partial Eta Squared	Noncent. Parameter	Observed Power[b]
Corrected Model	93.444[a]	39	2.396	1.343	.086	.103	52.396	.988
Intercept	63.846	1	63.846	35.800	.000	.073	35.800	1.000
Class Level	7.722	1	7.722	4.330	.038	.009	4.330	.546
Students Interactions	20.724	4	5.181	2.905	.021	.025	11.621	.783
Teachers Interactions	4.463	4	1.116	.626	.644	.005	2.502	.205
Technology Interactions	9.481	4	2.370	1.329	.258	.012	5.316	.416
Students Interactions * Teachers Interactions	7.844	5	1.569	.880	.494	.010	4.399	.316
Students Interactions * Technology Interactions	15.414	10	1.541	.864	.567	.019	8.643	.461
Teachers Interactions * Technology Interactions	10.745	6	1.791	1.004	.422	.013	6.025	.399
Students Interactions * Teachers Interactions * Technology Interactions	5.978	5	1.196	.670	.646	.007	3.352	.244
Error	811.457	455	1.783					
Total	3897.000	495						
Corrected Total	904.901	494						

a. R Squared = .103 (Adjusted R Squared = .026)

b. Computed using alpha = .05

APPENDIX H4

PROGRAMME AS A COVARIATE (MARGINAL MEAN IS 1.65)

Source	Type III Sum of Squares	df	Mean Square	F	Sig.	Partial Eta Squared	Noncent. Parameter	Observed Power[b]
Corrected Model	87.529[a]	39	2.244	1.249	.150	.097	48.724	.981
Intercept	154.619	1	154.619	86.071	.000	.159	86.071	1.000
Programme	1.807	1	1.807	1.006	.316	.002	1.006	.170
Students Interactions	20.815	4	5.204	2.897	.022	.025	11.587	.782
Teachers Interactions	3.996	4	.999	.556	.695	.005	2.224	.186
Technology Interactions	9.484	4	2.371	1.320	.262	.011	5.279	.413
Students Interactions * Teachers Interactions	7.629	5	1.526	.849	.515	.009	4.247	.306
Students Interactions * Technology Interactions	15.793	10	1.579	.879	.553	.019	8.792	.469
Teachers Interactions * Technology Interactions	10.783	6	1.797	1.000	.424	.013	6.002	.398
Students Interactions * Teachers Interactions * Technology Interactions	6.270	5	1.254	.698	.625	.008	3.490	.253
Error	817.372	455	1.796					
Total	3897.000	495						
Corrected Total	904.901	494						

a. R Squared = .097 (Adjusted R Squared = .019)

b. Computed using alpha = .05

APPENDIX H5

MANAGEMENT AS A COVARIATE (ESTIMATED: 1.08)

Source	Type III Sum of Squares	df	Mean Square	F	Sig.	Partial Eta Squared	Noncent. Parameter	Observed Power[b]
Corrected Model	86.078[a]	39	2.207	1.226	.170	.095	47.831	.978
Intercept	114.616	1	114.616	63.689	.000	.123	63.689	1.000
Management	.356	1	.356	.198	.657	.000	.198	.073
Students Interactions	21.231	4	5.308	2.949	.020	.025	11.798	.790
Teachers Interactions	4.243	4	1.061	.589	.670	.005	2.358	.195
Technology Interactions	10.043	4	2.511	1.395	.235	.012	5.581	.435
Students Interactions * Teachers Interactions	8.281	5	1.656	.920	.467	.010	4.602	.330
Students Interactions * Technology Interactions	15.485	10	1.549	.860	.571	.019	8.605	.459
Teachers Interactions * Technology Interactions	11.257	6	1.876	1.043	.397	.014	6.255	.414
Students Interactions * Teachers Interactions * Technology Interactions	5.555	5	1.111	.617	.687	.007	3.087	.226
Error	818.823	455	1.800					
Total	3897.000	495						
Corrected Total	904.901	494						

a. R Squared = .095 (Adjusted R Squared = .018)

b. Computed using alpha = .05

APPENDIX H6

RESIDENTIAL STATUS AS A COVARIATE (ESTIMATED MARGINAL MEAN: 1.03)

Source	Type III Sum of Squares	df	Mean Square	F	Sig.	Partial Eta Squared	Noncent. Parameter	Observed Power[b]
Corrected Model	86.744[a]	39	2.224	1.237	.160	.096	48.241	.979
Intercept	93.543	1	93.543	52.022	.000	.103	52.022	1.000
Residence	1.021	1	1.021	.568	.451	.001	.568	.117
Students Interactions	21.668	4	5.417	3.012	.018	.026	12.050	.800
Teachers Interactions	4.193	4	1.048	.583	.675	.005	2.332	.193
Technology Interactions	10.330	4	2.582	1.436	.221	.012	5.745	.447
Students Interactions * Teachers Interactions	8.712	5	1.742	.969	.436	.011	4.845	.348
Students Interactions * Technology Interactions	15.603	10	1.560	.868	.564	.019	8.677	.463
Teachers Interactions * Technology Interactions	11.675	6	1.946	1.082	.372	.014	6.493	.429
Students Interactions * Teachers Interactions * Technology Interactions	5.744	5	1.149	.639	.670	.007	3.194	.233
Error	818.157	455	1.798					
Total	3897.000	495						
Corrected Total	904.901	494						

a. R Squared = .096 (Adjusted R Squared = .018)

b. Computed using alpha = .05

APPENDIX H7

MULTIVARIATE TESTS FOR COVARIATES

Effect		Value	F	Hypothesis df	Error df	Sig.	Partial Eta Squared	Noncent. Parameter	Observed Power[b]
Intercept	Pillai's Trace	.280	61.798[a]	3.000	477.000	.000	.280	185.393	1.000
	Wilks' Lambda	.720	61.798[a]	3.000	477.000	.000	.280	185.393	1.000
	Hotelling's Trace	.389	61.798[a]	3.000	477.000	.000	.280	185.393	1.000
	Roy's Largest Root	.389	61.798[a]	3.000	477.000	.000	.280	185.393	1.000
Gender (Man. Mean=1.49)	Pillai's Trace	.014	2.262[a]	3.000	477.000	.080	.014	6.787	.571
	Wilks' Lambda	.986	2.262[a]	3.000	477.000	.080	.014	6.787	.571
	Hotelling's Trace	.014	2.262[a]	3.000	477.000	.080	.014	6.787	.571
	Roy's Largest Root	.014	2.262[a]	3.000	477.000	.080	.014	6.787	.571
School (Man. Mean=3.40)	Pillai's Trace	.029	4.788[a]	3.000	477.000	.003	.029	14.363	.902
	Wilks' Lambda	.971	4.788[a]	3.000	477.000	.003	.029	14.363	.902
	Hotelling's Trace	.030	4.788[a]	3.000	477.000	.003	.029	14.363	.902
	Roy's Largest Root	.030	4.788[a]	3.000	477.000	.003	.029	14.363	.902
Teachers Interactions	Pillai's Trace	.035	1.399	12.000	1.437E3	.159	.012	16.786	.780
	Wilks' Lambda	.966	1.404	12.000	1.262E3	.157	.012	14.851	.714
	Hotelling's Trace	.036	1.409	12.000	1.427E3	.155	.012	16.910	.784
	Roy's Largest Root	.030	3.628[c]	4.000	479.000	.006	.029	14.512	.876
Students Interactions	Pillai's Trace	.074	3.008	12.000	1.437E3	.000	.025	36.099	.993
	Wilks' Lambda	.928	3.027	12.000	1.262E3	.000	.025	31.982	.984
	Hotelling's Trace	.077	3.039	12.000	1.427E3	.000	.025	36.462	.993
	Roy's Largest Root	.054	6.455[c]	4.000	479.000	.000	.051	25.820	.991
Teachers * Students Interactions	Pillai's Trace	.033	1.064	15.000	1.437E3	.386	.011	15.964	.707
	Wilks' Lambda	.967	1.065	15.000	1.317E3	.385	.011	14.691	.660
	Hotelling's Trace	.034	1.065	15.000	1.427E3	.385	.011	15.979	.707
	Roy's Largest Root	.024	2.339[c]	5.000	479.000	.041	.024	11.697	.751

APPENDIX H8

MULTIVARIATE TESTS FOR COVARIATES

Effect		Value	F	Hypothesis df	Error df	Sig.	Partial Eta Squared	Noncent. Parameter	Observed Power[b]
Intercept	Pillai's Trace	.289	61.185[a]	3.000	452.000	.000	.289	183.556	1.000
	Wilks' Lambda	.711	61.185[a]	3.000	452.000	.000	.289	183.556	1.000
	Hotelling's Trace	.406	61.185[a]	3.000	452.000	.000	.289	183.556	1.000
	Roy's Largest Root	.406	61.185[a]	3.000	452.000	.000	.289	183.556	1.000
Gender	Pillai's Trace	.010	1.572[a]	3.000	452.000	.195	.010	4.717	.414
	Wilks' Lambda	.990	1.572[a]	3.000	452.000	.195	.010	4.717	.414
	Hotelling's Trace	.010	1.572[a]	3.000	452.000	.195	.010	4.717	.414
	Roy's Largest Root	.010	1.572[a]	3.000	452.000	.195	.010	4.717	.414
School Name	Pillai's Trace	.031	4.867[a]	3.000	452.000	.002	.031	14.601	.907
	Wilks' Lambda	.969	4.867[a]	3.000	452.000	.002	.031	14.601	.907
	Hotelling's Trace	.032	4.867[a]	3.000	452.000	.002	.031	14.601	.907
	Roy's Largest Root	.032	4.867[a]	3.000	452.000	.002	.031	14.601	.907
Teachers Interactions	Pillai's Trace	.034	1.299	12.000	1.362E3	.213	.011	15.586	.741
	Wilks' Lambda	.966	1.301	12.000	1.196E3	.211	.011	13.761	.672
	Hotelling's Trace	.035	1.303	12.000	1.352E3	.210	.011	15.636	.742
	Roy's Largest Root	.027	3.091[c]	4.000	454.000	.016	.027	12.362	.811
Students Interactions	Pillai's Trace	.043	1.667	12.000	1.362E3	.068	.014	20.002	.863
	Wilks' Lambda	.957	1.671	12.000	1.196E3	.068	.015	17.671	.805
	Hotelling's Trace	.045	1.674	12.000	1.352E3	.067	.015	20.091	.865
	Roy's Largest Root	.034	3.805[c]	4.000	454.000	.005	.032	15.221	.893
Technology Interactions	Pillai's Trace	.034	1.303	12.000	1.362E3	.210	.011	15.641	.743
	Wilks' Lambda	.966	1.306	12.000	1.196E3	.209	.011	13.810	.674
	Hotelling's Trace	.035	1.308	12.000	1.352E3	.207	.011	15.693	.744
	Roy's Largest Root	.027	3.066[c]	4.000	454.000	.016	.026	12.264	.807
Teachers * Students Interactions	Pillai's Trace	.034	1.028	15.000	1.362E3	.422	.011	15.423	.687
	Wilks' Lambda	.967	1.028	15.000	1.248E3	.422	.011	14.187	.640
	Hotelling's Trace	.034	1.028	15.000	1.352E3	.422	.011	15.425	.687
	Roy's Largest Root	.023	2.116[c]	5.000	454.000	.062	.023	10.581	.700
Teachers * Technology	Pillai's Trace	.046	1.187	18.000	1.362E3	.264	.015	21.358	.824
	Wilks' Lambda	.954	1.183	18.000	1.279E3	.267	.015	20.074	.792
	Hotelling's Trace	.047	1.180	18.000	1.352E3	.269	.015	21.240	.822
	Roy's Largest Root	.023	1.739[c]	6.000	454.000	.110	.022	10.432	.659
Students * Technology Interactions	Pillai's Trace	.079	1.220	30.000	1.362E3	.192	.026	36.615	.954
	Wilks' Lambda	.923	1.221	30.000	1.327E3	.191	.026	35.837	.948
	Hotelling's Trace	.081	1.222	30.000	1.352E3	.191	.026	36.657	.954
	Roy's Largest Root	.046	2.106[c]	10.000	454.000	.023	.044	21.061	.902
Teachers * Students * Technology Interactions	Pillai's Trace	.034	1.051	15.000	1.362E3	.399	.011	15.765	.700
	Wilks' Lambda	.966	1.051	15.000	1.248E3	.399	.011	14.500	.652
	Hotelling's Trace	.035	1.051	15.000	1.352E3	.399	.012	15.763	.700
	Roy's Largest Root	.024	2.175[c]	5.000	454.000	.056	.023	10.877	.714

APPENDIX I

FINAL EXPLORATORY FACTOR ANALYSIS

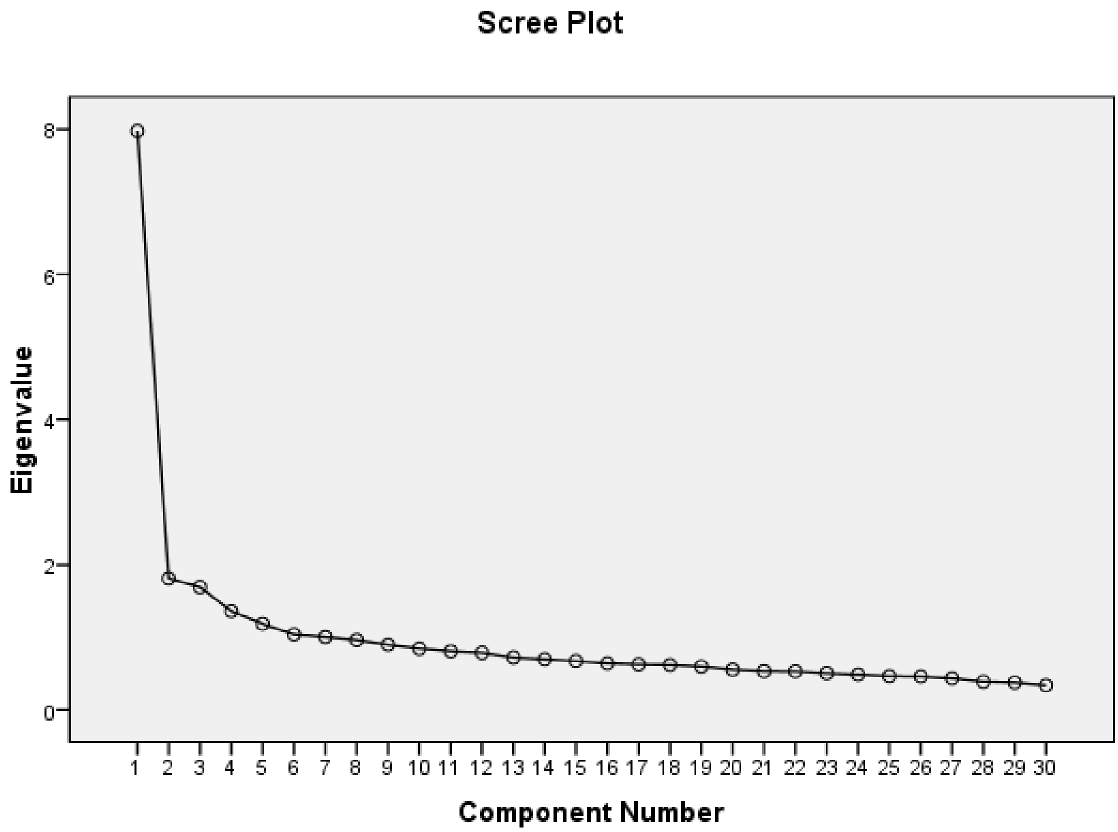
APPENDIX I1: TOTAL VARIANCE EXPLAINED

	Initial Eigenvalues			Extraction Sums of Squared Loadings			Rotation Sums of Squared Loadings		
Component	Total	% of Variance	Cumulative %	Total	% of Variance	Cumulative %	Total	% of Variance	Cumulative %
1	7.976	26.586	26.586	7.976	26.586	26.586	2.601	8.669	8.669
2	1.809	6.030	32.616	1.809	6.030	32.616	2.129	7.098	15.767
3	1.690	5.633	38.249	1.690	5.633	38.249	1.577	5.255	21.022
4	1.360	4.534	42.783	1.360	4.534	42.783	1.395	4.650	25.672
5	1.184	3.947	46.730	1.184	3.947	46.730	1.302	4.340	30.012
6	1.039	3.462	50.192	1.039	3.462	50.192	1.264	4.214	34.226
7	1.004	3.348	53.539	1.004	3.348	53.539	1.197	3.991	38.216
8	.962	3.207	56.746	.962	3.207	56.746	1.126	3.754	41.970
9	.897	2.989	59.735	.897	2.989	59.735	1.120	3.734	45.704
10	.844	2.813	62.548	.844	2.813	62.548	1.120	3.733	49.437
11	.806	2.688	65.236	.806	2.688	65.236	1.112	3.706	53.144
12	.785	2.618	67.853	.785	2.618	67.853	1.101	3.670	56.814
13	.721	2.402	70.255	.721	2.402	70.255	1.083	3.609	60.423
14	.697	2.322	72.578	.697	2.322	72.578	1.077	3.588	64.012
15	.673	2.242	74.820	.673	2.242	74.820	1.071	3.571	67.583
16	.641	2.136	76.956	.641	2.136	76.956	1.052	3.505	71.088
17	.627	2.091	79.046	.627	2.091	79.046	1.051	3.504	74.592
18	.620	2.066	81.112	.620	2.066	81.112	1.050	3.501	78.094
19	.597	1.990	83.102	.597	1.990	83.102	1.033	3.443	81.536
20	.555	1.850	84.952	.555	1.850	84.952	1.025	3.415	84.952
21	.533	1.778	86.729						
22	.530	1.768	88.497						
23	.504	1.681	90.178						
24	.485	1.617	91.795						
25	.464	1.546	93.341						
26	.460	1.533	94.874						
27	.435	1.449	96.323						
28	.388	1.293	97.616						
29	.377	1.256	98.872						
30	.338	1.128	100.000						

Extraction Method: Principal Component Analysis.

APPENDIX I2

FINAL SCREE PLOT



APPENDIX I3

FINAL ROTATED MATRIX

	Component																			
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Transitioning Students	.682		.337																	
Transitioning Teachers	.681	.394																		
Transitioning Contents	.622											.324								
Transitioning All Structures	.616							.300												
Transitioning All Technologies	.527																			
Transitioning Didactic Theories		.737																		
Transitioning Didactic Technologies		.676																		
Transitioning Didactic Techniques		.674											.362							
Didactizing One Structure			.858																	
Didactizing Two Structures			.656								.338									
Didactizing Theories				.762											.362					
Didactizing Technologies				.654		.310														
Didactizing Tasks					.770															
Didactizing All Elements					.619	.502														
Didactizing All Structures						.815														
Didactizing Three Structures							.839													
Mathematizing Contents	.308						.491		.399	.344										
Mathematizing Technologies								.890												

Mathematizing Tasks									.911											
Mathematizing Instruments										.858										
Didactizing Four Structures											.845									
Transitioning All Elements												.814								
Transitioning Didactic Tasks													.882							
Mathematizing Artefacts														.941						
Didactizing Techniques															.833					
Mathematizing Teachers																.884				
Mathematizing Students																	.954			
Mathematizing Signs and Symbols																		.913		
Mathematizing Innovations																			.921	
Mathematizing Tools																				.936

APPENDIX I4

FINAL COMMUNALITIES OF THE EXPLORATORY FACTOR ANALYSIS

Components	Initial	Extraction
Mathematising within Signs and Symbols	1.000	.930
Mathematising within Tasks	1.000	.918
Mathematising within Artefacts	1.000	.942
Mathematising within Instruments	1.000	.875
Mathematising within Tools	1.000	.957
Mathematising within Innovations	1.000	.941
Mathematising within Teachers	1.000	.937
Mathematising within Students	1.000	.985
Mathematising within Contents	1.000	.843
Mathematising within Technologies	1.000	.909
Transitioning across Didactic Tasks	1.000	.939
Transitioning across Didactic Techniques	1.000	.759
Transitioning across Didactic Theories	1.000	.746
Transitioning across Didactic Technologies	1.000	.711
Transitioning across Teachers	1.000	.766
Transitioning across Students	1.000	.732
Transitioning across Contents	1.000	.718
Transitioning across All Technologies	1.000	.763
Transitioning across All Elements	1.000	.902
Transitioning across All Structures	1.000	.724
Didactising with One Structure	1.000	.851
Didactising with Two Structures	1.000	.775
Didactising with Three Structures	1.000	.880
Didactising with Four Structures	1.000	.905
Didactising with Tasks	1.000	.833
Didactising with Techniques	1.000	.882
Didactising with Theories	1.000	.840
Didactising with Technologies	1.000	.827
Didactising with All Elements	1.000	.830
Didactising with All Structures	1.000	.868

Extraction Method: Principal Component Analysis.

APPENDIX J

ANALYSIS OF INTERACTIONS BEFORE IMPLEMENTATIONS

APPENDIX J1

TUKEY HSD POST HOC MULTIPLE COMPARISON BEFORE IMPLEMENTATION OF TETRAHEDRON ACROSS MATHEMATIZATIONS

Dependent Variable	(I) Interactions with Students	(J) Interactions with Students	Mean Difference (I-J)	Std. Error	Sig.	95% Confidence Interval Lower Bound	Upper Bound
Artefacts	Less than Forty	Less than Fifty	.122	.184	.964	-.38	.63
		Less than Sixty	.196	.153	.700	-.22	.61
		Sixty and More	.204	.199	.845	-.34	.75
		Other Numbers	-.045	.326	1.000	-.94	.85
	Less than Fifty	Less than Forty	-.122	.184	.964	-.63	.38
		Less than Sixty	.074	.160	.990	-.36	.51
		Sixty and More	.082	.205	.995	-.48	.64
		Other Numbers	-.167	.329	.987	-1.07	.73
	Less than Sixty	Less than Forty	-.196	.153	.700	-.61	.22
		Less than Fifty	-.074	.160	.990	-.51	.36
		Sixty and More	.007	.177	1.000	-.48	.49
		Other Numbers	-.241	.313	.939	-1.10	.62
	Sixty and More	Less than Forty	-.204	.199	.845	-.75	.34
		Less than Fifty	-.082	.205	.995	-.64	.48
		Less than Sixty	-.007	.177	1.000	-.49	.48
		Other Numbers	-.248	.338	.948	-1.17	.68
	Other Numbers	Less than Forty	.045	.326	1.000	-.85	.94
		Less than Fifty	.167	.329	.987	-.73	1.07
		Less than Sixty	.241	.313	.939	-.62	1.10
		Sixty and More	.248	.338	.948	-.68	1.17
Technologies	Less than Forty	Less than Fifty	-.077	.207	.996	-.64	.49
		Less than Sixty	.304	.172	.395	-.17	.77
		Sixty and More	-.025	.225	1.000	-.64	.59
		Other Numbers	-.188	.367	.986	-1.19	.82
	Less than Fifty	Less than Forty	.077	.207	.996	-.49	.64
		Less than Sixty	.381	.180	.216	-.11	.87
		Sixty and More	.052	.231	.999	-.58	.68
		Other Numbers	-.111	.371	.998	-1.13	.90
	Less than Sixty	Less than Forty	-.304	.172	.395	-.77	.17
		Less than Fifty	-.381	.180	.216	-.87	.11
		Sixty and More	-.328	.200	.470	-.88	.22
		Other Numbers	-.492	.352	.631	-1.46	.47
	Sixty and More	Less than Forty	.025	.225	1.000	-.59	.64
		Less than Fifty	-.052	.231	.999	-.68	.58
		Less than Sixty	.328	.200	.470	-.22	.88
		Other Numbers	-.163	.381	.993	-1.21	.88
	Other Numbers	Less than Forty	.188	.367	.986	-.82	1.19
		Less than Fifty	.111	.371	.998	-.90	1.13
		Less than Sixty	.492	.352	.631	-.47	1.46
		Sixty and More	.163	.381	.993	-.88	1.21

Dependent Variable	(I) Interactions with Students	(J) Interactions with Students	Mean Difference (I-J)	Std. Error	Sig.	95% Confidence Interval	
						Lower Bound	Upper Bound
Artefacts	Less than Forty	Less than Fifty	.122	.184	.964	-.38	.63
		Less than Sixty	.196	.153	.700	-.22	.61
		Sixty and More	.204	.199	.845	-.34	.75
		Other Numbers	-.045	.326	1.000	-.94	.85
	Less than Fifty	Less than Forty	-.122	.184	.964	-.63	.38
		Less than Sixty	.074	.160	.990	-.36	.51
		Sixty and More	.082	.205	.995	-.48	.64
		Other Numbers	-.167	.329	.987	-1.07	.73
	Less than Sixty	Less than Forty	-.196	.153	.700	-.61	.22
		Less than Fifty	-.074	.160	.990	-.51	.36
		Sixty and More	.007	.177	1.000	-.48	.49
		Other Numbers	-.241	.313	.939	-1.10	.62
	Sixty and More	Less than Forty	-.204	.199	.845	-.75	.34
		Less than Fifty	-.082	.205	.995	-.64	.48
		Less than Sixty	-.007	.177	1.000	-.49	.48
		Other Numbers	-.248	.338	.948	-1.17	.68
	Other Numbers	Less than Forty	.045	.326	1.000	-.85	.94
		Less than Fifty	.167	.329	.987	-.73	1.07
		Less than Sixty	.241	.313	.939	-.62	1.10
		Sixty and More	.248	.338	.948	-.68	1.17
Technologies	Less than Forty	Less than Fifty	-.077	.207	.996	-.64	.49
		Less than Sixty	.304	.172	.395	-.17	.77
		Sixty and More	-.025	.225	1.000	-.64	.59
		Other Numbers	-.188	.367	.986	-1.19	.82
	Less than Fifty	Less than Forty	.077	.207	.996	-.49	.64
		Less than Sixty	.381	.180	.216	-.11	.87
		Sixty and More	.052	.231	.999	-.58	.68
		Other Numbers	-.111	.371	.998	-1.13	.90
	Less than Sixty	Less than Forty	-.304	.172	.395	-.77	.17
		Less than Fifty	-.381	.180	.216	-.87	.11
		Sixty and More	-.328	.200	.470	-.88	.22
		Other Numbers	-.492	.352	.631	-1.46	.47
	Sixty and More	Less than Forty	.025	.225	1.000	-.59	.64
		Less than Fifty	-.052	.231	.999	-.68	.58
		Less than Sixty	.328	.200	.470	-.22	.88
		Other Numbers	-.163	.381	.993	-1.21	.88
	Other Numbers	Less than Forty	.188	.367	.986	-.82	1.19
		Less than Fifty	.111	.371	.998	-.90	1.13
		Less than Sixty	.492	.352	.631	-.47	1.46
		Sixty and More	.163	.381	.993	-.88	1.21

*. The mean difference is significant at the 0.05 level.

APPENDIX J2

TUKEY HSD POST HOC MULTIPLE COMPARISON BEFORE IMPLEMENTATION OF TETRAHEDRON ACROSS INSTRUMENTATIONS

	(I) Interactions with Students	(J) Interactions with Students	Mean Difference (I-J)	Std. Error	Sig.	95% Confidence Interval Lower Bound	95% Confidence Interval Upper Bound
	Less than Sixty	Less than Forty	-.325	.139	.133	-.70	.05
		Less than Fifty	-.150	.145	.839	-.55	.25
		Sixty and More	-.020	.161	1.000	-.46	.42
		Other Numbers	.250	.284	.905	-.53	1.03
	Sixty and More	Less than Forty	-.305	.181	.444	-.80	.19
		Less than Fifty	-.130	.186	.956	-.64	.38
		Less than Sixty	.020	.161	1.000	-.42	.46
		Other Numbers	.270	.307	.905	-.57	1.11
	Other Numbers	Less than Forty	-.574	.296	.297	-1.38	.24
		Less than Fifty	-.400	.299	.668	-1.22	.42
		Less than Sixty	-.250	.284	.905	-1.03	.53
		Sixty and More	-.270	.307	.905	-1.11	.57
Three Structures	Less than Forty	Less than Fifty	.061	.175	.997	-.42	.54
		Less than Sixty	.153	.145	.830	-.24	.55
		Sixty and More	-.085	.189	.991	-.60	.43
		Other Numbers	.194	.310	.971	-.65	1.04
	Less than Fifty	Less than Forty	-.061	.175	.997	-.54	.42
		Less than Sixty	.092	.152	.974	-.32	.51
		Sixty and More	-.146	.195	.944	-.68	.39
		Other Numbers	.133	.313	.993	-.72	.99
	Less than Sixty	Less than Forty	-.153	.145	.830	-.55	.24
		Less than Fifty	-.092	.152	.974	-.51	.32
		Sixty and More	-.238	.168	.620	-.70	.22
		Other Numbers	.041	.297	1.000	-.77	.86
	Sixty and More	Less than Forty	.085	.189	.991	-.43	.60
		Less than Fifty	.146	.195	.944	-.39	.68
		Less than Sixty	.238	.168	.620	-.22	.70
		Other Numbers	.279	.321	.908	-.60	1.16
	Other Numbers	Less than Forty	-.194	.310	.971	-1.04	.65
		Less than Fifty	-.133	.313	.993	-.99	.72
		Less than Sixty	-.041	.297	1.000	-.86	.77
		Sixty and More	-.279	.321	.908	-1.16	.60
Four Structures	Less than Forty	Less than Fifty	.175	.187	.882	-.34	.69
		Less than Sixty	.382	.155	.102	-.04	.81
		Sixty and More	.238	.203	.766	-.32	.79
		Other Numbers	.209	.332	.970	-.70	1.12
	Less than Fifty	Less than Forty	-.175	.187	.882	-.69	.34
		Less than Sixty	.206	.163	.712	-.24	.65
		Sixty and More	.063	.209	.998	-.51	.63
		Other Numbers	.033	.335	1.000	-.88	.95
	Less than Sixty	Less than Forty	-.382	.155	.102	-.81	.04
		Less than Fifty	-.206	.163	.712	-.65	.24
		Sixty and More	-.143	.180	.932	-.64	.35
		Other Numbers	-.173	.318	.983	-1.04	.70
	Sixty and More	Less than Forty	-.238	.203	.766	-.79	.32
		Less than Fifty	-.063	.209	.998	-.63	.51

APPENDIX K

RESEARCH INSTRUMENTS

APPENDIX K1

PRE-INTERVENTION INSTRUMENTS

PART A: DEMOGRAPHIC INFORMATION AND DIDACTICAL CONCEPTUAL STRUCTURES

Questions	Options: Tick on only one of these options		
1. Gender	Male	Female	Other
2. Class/Form	SHS One	SHS Two	SHS Three
3. Name of SHS			
4. Management of SHS	Government	Religious	Private
5. Residential status	Boarding status	Day status	Other
6. Programme	General Science	Business	Technical/Home Economics
7. Number of topics studied in Circles	One	Two	Three or More
8. Number of mathematics resources	Two	Three	Four or More
9. Number of mathematics teachers	Two	Three	More than three
10. Number of mathematics students			

PART B: ISSUES OF DIDACTICAL RELATIONSHIPS OF THE TEACHER

Questions	Options: Tick on only one box				
Conceptual Structures: Tick only one box	*Most often*	*Very often*	*Often*	*Scarcely*	*Never*
11. The Teacher establishes and maintains an atmosphere of order, respect, rapport and courtesy for himself/herself.					
12. The Teacher creates and ensures an open, informal, congenial, democratic and free atmosphere for himself.					
13. The Teacher interacts, discusses and addresses his/her needs and challenges by himself/herself.					
14. The Teacher provides prompt feedbacks and mark students' exercises					
15. The Teachers provides sources of information, materials and resources to students.					
16. The Teacher communicates with confidence and enthusiasm, communicates to the levels of students, uses appropriate verbal, non-verbal, oral and written signs, and projects voice.					
17. The Teacher monitors each student's participation and progress, remediates and gives immediate feedbacks, uses both formal and informal assessment strategies, and bases evaluation and assessment on goals and objectives.					
18. The Teacher exhibits knowledge of the subject matter in Circles, sets specific, measurable, achievable, realistic and time-bound objectives, and guides the student to realize his/her potentials.					
19. The Teacher connects mathematics content in Circles with signs, symbols, artefacts, instruments, tools and technologies to link the content to knowledge, skills and competencies.					
20. The Teacher brings variety of books, similar topics, and aligns topics to each other and redesigns topics in Circles.					
21. The Teacher plans lessons, sets examples, derives test questions from course outline in mathematics books in Circles.					
22. The Teacher sets examination questions from mathematics content in Circles, mixes logical and sequential order of Circle topics, and scores with standard markings schemes.					
23. The Teacher develops instructional signs, symbols, artefacts, tools, instruments and technologies.					
24. The Teacher stimulates discussions of mathematics problems and solutions in Circles with signs, symbols, artefacts, tools, instruments and technologies.					
25. The Teacher adequately associates signs, symbols, artefacts, tools, instruments and technologies with teaching objectives, strategies, methods and outcomes in Circles.					
26. The Teacher applies signs, symbols, artefacts, tools, instruments and technologies to present lessons, work mathematics problems and provide solutions in Circles.					
27. The Teacher assesses and evaluates learning outcomes with signs, symbols, artefacts, tools, instruments, computers or calculators or other technologies in circles.					

a. *Interview: How do you assess the relationship between the teacher and students, mathematics content and* signs, *symbols, artefacts, tools, instruments or technologies?*

PART C: ISSUES OF DIDACTICAL RELATIONSHIPS OF THE STUDENT

Questions	Options: Tick on only one box				
Conceptual Structures: Tick only one box	*Most often*	*Very often*	*Often*	*Scarcely*	*Never*
28. The Student establishes and maintains order, respect, rapport and courtesy for himself/herself.					
29. The Student creates and ensures an open, healthy, informal, congenial and competitive atmosphere for himself/herself.					
30. The Student interacts, discusses and addresses his/her needs and challenges by himself/herself.					
31. The Student attentively assimilates the teacher's demonstrations, procedures, sequences and algorithms in mathematics problems, tasks and practices.					
32. The Student engages in frequent, focused and well directed discussions with the teacher.					
33. The Student assesses and evaluates the teacher's lesson objectives, learning outcomes, oral or written exercises and test or examination scores.					
34. The student challenges and compels the teacher to search, research and assess more and better strategies and methods					
35. The student experiences and accesses a wide variety of mathematics textbooks, curriculum and syllabi that enhance understanding in Circles and apply to daily life.					
36. The student understands and assimilates the mathematics content in Circles and builds relationships between new topics and already existing topics.					
37. The student creates and innovates best techniques and strategies of learning the mathematics content in Circles, and apply innovations in solving problems in daily life.					
38. The student creates own story problems, relates stories to Circles, and provides best strategies and techniques of solving the problems in Circles.					
39. The student compiles and equips his/her vocabulary, mental structures and personal library with a variety of signs, symbols, artefacts, tools, instruments and technologies.					
40. The student creates reflective journals and case cards to use a variety of signs, symbols, artefacts, tools, instruments and technologies to demonstrate understanding and transformation of thinking over time.					
41. The student follows and provides routine procedures of solving Circle problems with a variety of signs, symbols, artefacts, tools, instruments and computers or calculators or other technologies.					
42. The student provides short programming codes, personal algorithms and pneumonias for solving Circle problems.					

b. Interviewer: How do you assess the relationship between the students and teachers, mathematics content and signs, symbols, artefacts, tools, instruments and technologies?

PART D: ISSUES OF DIDACTICAL RELATIONSHIPS OF THE MATHEMATICS CONTENT

Questions	Options: Tick on only one box				
Conceptual Structures: Tick only one box	*Most often*	*Very often*	*Ofte n*	*Scarcely*	*Never*
43. The mathematics content establishes and maintains order, sequence, logical and unified themes or topics.					
44. The mathematics content ensures open, free, self-explanatory and comprehensive themes or topics.					
45. The mathematics content discusses and addresses problems and challenges in the questions.					
46. The mathematics content in Circles provides learning purposes, goals, objectives, tasks and directions to the teacher.					
47. The mathematics content provides adequate and robust instructional techniques and strategies to the teacher.					
48. The mathematics content contains adequate and diverse topics, story problems and questions to the teacher.					
49. The mathematics content provides diverse solution paths, orderly mathematics problems and worked out examples to the teacher.					
50. The mathematics content provides learning purposes, goals, objectives, tasks and directions to the student					
51. The mathematics content provides learning paths, tasks, activities and skill practices to the student.					
52. The mathematics content contains adequate topics, mathematics problems, story problems and questions, assessment and evaluation procedures, problems and solutions to the student.					
53. The mathematics content correctly applies techniques, methods and procedures to the student.					
54. The mathematics content provides specific signs, symbols, artefacts, tools, instruments and technologies to cover specific topics, themes, syllabi and curriculum.					
55. The mathematics content provides orderly, sequentially and logically topics with signs, symbols, artefacts, tools, instruments and technologies.					
56. The mathematics content enumerates the relationships, formulas and generalizations with signs, symbols, artefacts, tools, instruments and technologies.					
57. The mathematics content in Circles provides class exercises, test items, examination questions and answers or solutions with signs, symbols, artefacts, tools, instruments and technologies.					

c. Interviewer: How do you assess the relationship between the mathematics content and teacher, student, and technology tool? Explain your answer?

PART E: ISSUES OF DIDACTICAL RELATIONSHIPS OF TECHNOLOGIES

Questions	**Options: Tick on only one box**				
Conceptual Structures: Tick only one box	***Most often***	***Very often***	***Ofte n***	***Scarcel y***	***Never***
58. The signs, symbols, artefacts, tools, instruments and technologies are arranged in order and relate to one another.					
59. The signs, symbols, artefacts, tools, instruments and technologies ensure smooth, logical and coherent transference from one to another.					
60. The signs, symbols, artefacts, tools, instruments and technologies interact with each other, and address needs and challenges.					
61. The signs, symbols, artefacts, tools, instruments and technologies provide sequences and logical procedures to the teacher.					
62. The signs, symbols, artefacts, tools, instruments and technologies provide story problems and worked out solutions to the teacher.					
63. The signs, symbols, artefacts, tools, instruments and technologies challenge and provide clues, innuendos and technical support to the teacher.					
64. The signs, symbols, artefacts, tools, instruments and technologies assess and evaluate strategies and techniques to the teacher.					
65. The signs, symbols, artefacts, tools, instruments and technologies provide easier, cheaper and shorter procedures and techniques to the student.					
66. The signs, symbols, artefacts, tools, instruments and technologies provides interactive, task-oriented and friendly environment to the student.					
67. The signs, symbols, artefacts, tools, instruments and technologies provide challenging, rigorous, and disciplined routines activities and examples to the student.					
68. The signs, symbols, artefacts, tools, instruments and technologies evaluate and assess learning tasks and outcomes to the student.					
67. The signs, symbols, artefacts, tools, instruments and technologies provide goals and objectives to the mathematics content in Circles.					
68. The signs, symbols, artefacts, tools, instruments and technologies provide inherent themes, quality activities, and coherent procedures to the mathematics content.					
69. The signs, symbols, artefacts, tools, instruments and technologies provide interactive tools and procedures to the mathematics content.					
70. The signs, symbols, artefacts, tools, instruments and technologies assess and evaluate curriculum, syllabus and topics to the mathematics content.					

d. *How do you assess the relationship between the teacher and student, mathematics content and* signs, symbols, artefacts, tools, instruments and technologies?

e. *Give any impression, feeling or opinion you may have about the didactical relationships with your teachers, students, mathematics content in Circles and signs, symbols, artefacts, tools, instruments and technologies.*

THE END!

THANK YOU

Appendix K2: Psychological Test Instruments

Part A: General Demographic Information and Didactical Conceptual Structures

Demographic Variables	*Options: Tick on only one of these options*		
11. Gender	Male	Female	Other
12. Class/Form	SHS One	SHS Two	SHS Three
13. Name of SHS			
14. Management of SHS	Government	Religious	Private
15. Residential status	Boarding status	Day status	Other
16. Programme	General Science	Business	Technical/Home Economics
17. Number of topics studied in Circles	One	Two	Three or More
18. Number of mathematics resources	Two	Three	Four or More
19. Number of mathematics teachers	Two	Three	More than three
20. Number of mathematics students			

Part B: Mathematisation and didactical phenomenology

Mathematizations in Instrumental Genesis	**Options: Tick on only one box**				
Read and Tick only one option	***Most often***	***Very often***	***Often***	***Scarcely***	***Never***
11. Mathematising within Symbols and Signs----how do you use word arithmetic operators, equality and/or inequality signs, symbols and mimics?					
12. Mathematising within Tasks----how do you use repetitive exercises, objects, definitions, story problems, experiments, instructions and values?					
13. Mathematising within Artefacts---how do you use indigenous baskets, hats, calabashes, mats, designs and objects in class activities?					
14. Mathematising within Instruments--- how do you use mathematical sets, computers, calculators, rulers or computer software to solve mathematics?					
15. Mathematising within Tools--- how do you use mental activities, manipulatives, graph sheets and mnemonics to solve mathematics problems?					
16. Mathematising within Technologies--- how do you use cameras, videos and tape recordings to solve mathematics problems?					
17. Mathematising within Teachers---how do you involve mathematics teachers to solve mathematics problems?					
18. Mathematising within Students---how do you participate in discussions with your fellow students?					
19. Mathematising within mathematics content--- how do you acquire mathematics books, pamphlets and learning materials to solve mathematics problems?					
20. Mathematising within signs, symbols, artefacts, tools, instruments and technologies--- how do apply any available electronic and non-electronic materials to solve mathematics problems?					

Part C: Instrumentations and instrumentalisations

Transitioning Elements and Conceptual Structures	**Options: Tick on only one box**				
Read and Tick only one option	**Most often**	**Very often**	**Often**	**Scarcely**	**Never**
21. ***Transitioning across Didactic Tasks***---how do you relate the tasks with teachers, students, mathematics content and technologies in mathematics discussions, problem solving, exercises, assignments, quizzes, tests, home works and examinations?					
22.***Transitioning across Didactic Techniques***------how do you relate the tasks with teachers, students, mathematics content and technologies in mathematics discussions, problem solving, exercises, assignments, quizzes, tests, home works and examinations?					
23. ***Transitioning across Didactic Theory***------ how do you relate the tasks with teachers, students, mathematics content and technologies in mathematics discussions, problem solving, exercises, assignments, quizzes, tests, home works and examinations?					
24. ***Transitioning across Didactic Technologies***------how do you relate the tasks with teachers, students, mathematics content and technologies in mathematics discussions, problem solving, exercises, assignments, quizzes, tests, home works and examinations?					
25. ***Transitioning across Teachers***--- how do you relate the teachers with students, mathematics content and technologies in mathematics discussions, problem solving, exercises, assignments, quizzes, tests, home works and examinations?					

26. ***Transitioning across Students---*** how do you relate the students with teachers, mathematics content and technologies in mathematics discussions, problem solving, exercises, assignments, quizzes, tests, home works and examinations?					
27. ***Transitioning across Mathematics Content---*** how do you relate the mathematics content with teachers, students, and technologies in mathematics discussions, problem solving, exercises, assignments, quizzes, tests, home works and examinations?					
28. ***Transitioning across Signs/Technologies---*** how do you relate the signs and technologies teachers, students, and mathematics content in mathematics discussions, problem solving, exercises, assignments, quizzes, tests, home works and examinations?					
29. ***Transitioning across Mathematics Elements---*** how do you relate tasks, techniques, theories and technologies with teachers, students, mathematics content and technologies o solve mathematics discussions, problem solving, exercises, assignments, quizzes, tests, home works and examinations?					
30. ***Transitioning across All Conceptual Structures---*** how do you relate the teachers, students, mathematics content and technologies with tasks, techniques, theories and technologies to solve mathematics problems, exercises, tests and examinations?					

Part D: Didactical Situations and Anthropological Praxeologies

Didactising Conceptual Structures	**Options: Tick on only one box**				
Read and Tick only one option	**Most often**	**Very often**	**Ofte n**	**Scarcel y**	**Never**
31. ***Didactising with 1T Structures***--- how do you organize your schedules to learn mathematics					
32. ***Didactising with 2T Structures***--- how do you involve teachers and mates to solve mathematics problems?					
33. ***Didactising with 3T Structures***--- how do you involve teachers, mates and mathematics books or materials to solve mathematics problems?					
34. ***Didactising with 4T Structures***---how do you involve teachers, mates, mathematics books and technologies to solve mathematics problems?					
35. ***Didactising with Tasks***---how do you combine and integrate questions, story problems and exercises to solve problems?					
36. ***Didactising with Techniques***---how do you use and apply methods, strategies, algorithms and patterns?					
37. ***Didactising with Theories***--- how do you formulas, inductions, deductions, proofs and laws to solve mathematics problems?					
38. ***Digitising with Technologies***----how do you use interactives, computers, calculators, or diagrams and graphs to solve mathematics problems?					
39. ***Didactising Teachers, Students, Mathematics Content and Technologies with tasks, techniques, theories and technologies***--- how do you involve both the elements and the structures to solve mathematics problems?					
40. ***Didactising*Tasks, Techniques, Theories and Technologies with *Teachers, Students, Mathematics Content and Technologies***---how do you combine and integrate all the four elements with your teachers, mates, mathematics topics and signs/technologies to solve mathematics problems?					

Part E: Conceptualizing equations of the circle

Conceptualizing Circle Problems	**(Tick only the correct option or supply your own answer)**				
Read, Solve and Tick only one option	**A**	**B**	**C**	**D**	**Other**
41.***Circle Shape***: What is the shape of a circle?	Spherical	Oval	Round	Shapeless	
42 ***Circle Angle***: How do relate the total angle of a circle?	360º	180º	90º	0º	
43. ***CircleDimension***: What is dimension of a circle?	One	Two	Three	Dimensionless	
44. ***Basic Parts of a CircleEquation***: What are three basic parts of a circle equation?	Circumference Diameter Centre	Diameter Radius Centre	Tangents Normal Chord	Sector Secant Major	
45. ***Circle Standard Equation at (0,0)***: What form of circle equation is $x^2+y^2=r^2$?	Diameter Form	General Form	Centre-Radius Form	Standard Form	
46. ***Circle Radius-Centre Equation at C(h, k):*** What form of circle equation is $(x-h)^2+(y-k)^2=r^2$?	Diameter Form	General Form	Centre-Radius Form	Standard Form	
47. ***Circle General Equation***: What form of circle equation is $x^2+y^2+2gx+2fy+c=0$?	Diameter Form	General Form	Centre-Radius Form	Standard Form	
48. ***Circle Diameter Equation***: What form of circle equation is $(y-y_1)(y-y_2)+(x-x_1)(x-x_2)=0$?	Diameter Form	General Form	Centre-Radius Form	Standard Form	
49. ***Circle Tangent Equation***: What form of circle equation is $xx_1+yy_1=r^2$?	Standard Normal	Standard Tangent	General Normal	General Tangent	
50. ***Circle Normal Equation***: What form of circle equation is $x/x_1=y/y_1$?	Standard Normal	Standard Tangent	General Normal	General Tangent	
51. ***Circle Origin Solution***: Solve for the centre C(h, k) and radius (r) of $x^2+y^2=0$.	C(3,3) r=3	C(2,2) r=2	C(1,1) r=1	C(0,0) r=0	
52. ***Circle x-intercept Solution***: Solve for the centre C(h, k) and radius (r) of $x^2+y^2+6x=0$.	C(3,0) r=3	C(-3,0) r=3	C(0,3) r=3	C(0,-3) r=3	
53. ***Circle y-intercept Solution***: Solve for the centre C(h, k) and radius (r) of $x^2+y^2+4x=0$.	C(2,0) r=2	C(-2,0) r=2	C(0,2) r=2	C(0,-2) r=2	
54 ***Circle Positive Coefficients***: Solve for the centre C(h, k) and radius (r) of $x^2+y^2+6x+4y=0$.	C(3,2) $r=\sqrt{13}$	C(-3,2) $r=\sqrt{13}$	C(3,-2) $r=\sqrt{13}$	C(-3,-2) $r=\sqrt{13}$	
55. ***Circle Negative Coefficients***: Solve for the centre C(h, k) and radius (r) of $x^2+y^2-6x-4y=0$.	C(3,2) $r=\sqrt{13}$	C(-3,2) $r=\sqrt{13}$	C(3,-2) $r=\sqrt{13}$	C(-3,-2) $r=\sqrt{13}$	
56. ***Circle with 'c'***: Solve for the centre C(h, k) and radius (r) of $x^2+y^2+6x+4y-36=0$.	C(3,2) $r=\sqrt{13}$	C(-3,2) $r=\sqrt{13}$	C(3,-2) $r=\sqrt{13}$	C(-3,-2) $r=\sqrt{13}$	
57. ***Circle Non-Unit Coefficients***: Solve for the centre C(h, k) and radius (r) of $3x^2+3y^2-36x-12y+84=0$.	C(6,2) $r=2\sqrt{6}$	C(-6,2) $r=2\sqrt{6}$	C(6,-2) $r=2\sqrt{6}$	C(-6,-2) $r=2\sqrt{6}$	
58. ***Circle Non-Unit Coefficients***: Solve for the centre C(h, k) and radius (r) of $3x^2+3y^2+36x+12y+84=0$.	C(6,2) $r=2\sqrt{6}$	C(-6,2) $r=2\sqrt{6}$	C(6,-2) $r=2\sqrt{6}$	C(-6,-2) $r=2\sqrt{6}$	

THE END

THANK YOU

Appendix K3: Interview Guide for the Main Study

Part B: Mathematizations and Didactical Phenomenology

1. What is a sign or symbol in mathematics?

2. Outline any signs and symbols you use in solving equations of the circle.

3. Which sign or symbol do you prefer most and what is your reason for liking that sign or symbol?

Part C: Instrumentations and Instrumentalisations

4. What is a tool in mathematics?

5. Outline any tools you use in solving equations of the circle.

6. Which tool(s) do you prefer most and what is your reason for liking that tool(s)?

7. What is a technology in mathematics?

8. Outline any technologies you use in solving equations of the circle.

9. Which technologies do you prefer most and what is your reason?

Part D: Didactical situations and Anthropological Praxeologies

10. How many students do you normally study mathematics with?

11. Why do you prefer to study with your colleagues or mates in relation to gender, intelligence, former school, district of origin or living, having resources, and so on?

12. What are mathematics textbooks or pamphlets do you use to solve equations of the circle?

13. Which of the books or pamphlets do you prefer most and what are the reasons (in relation to simplicity, adequate explanations, worked answers, attraction of quality or other influences?

14. What are the tasks in the mathematics textbooks or pamphlets (in relation to objectives, short answer, completing blank spaces, or story telling?

15. What kind of tasks do you prefer? Give any reasons for your choice.

16. How do you solve the mathematics tasks--- is it you alone; you and teacher; you, your teacher and your mates or you, your teacher, your mates and your books?

17. Concluding general questions:
- Give me a general view of experiences in the didactical instructional models.
- Think of an Equation of the Circle problem and write down steps to solve it.
- Describe the reasons you will now use the new didactical instructional models.
- How would you transform signs, symbols and artefacts to technologies?
- How would you transform tools and instructions to technologies?
- How would you transform tasks, techniques and theories to technologies?

Part E: Conceptualizing equations of the circle

18. What is the standard and general equation?

19. Do the standard and the general equations of the circle relate with respect to their centres and radii?

20. If the equation of the circle is $x^2 + y^2 + 8x = 0$, what is the centre and how is the centre related to the radius?

21. If the equation of the circle is $x^2 + y^2 + 6y = 0$, what is the centre and how is the centre related to the radius?

22. If the equation of the circle is $x^2 + y^2 + 8x + 6y = 0$, what is the centre and how is the centre related to the radius?

23. If the equation of the circle is $x^2 + y^2 - 8x - 6y = 0$, will the centre be the same as 18 and will the radius be the same as 18? Briefly explain your answers.

24. If the equations of the circle $3x^2 + 3y^2 + 36x + 12y + 84 = 0$ and $3x^2 + 3y^2 - 36x - 12y + 84 = 0$, will the centres be the same and will the radii be the same? Briefly explain your answers.

THE END!

THANK YOU