UNIVERSITY OF CAPE COAST

SUFFICIENT CONDITIONS FOR EXPONENTIAL STABILITY AND INSTABILITY OF SOLUTIONS OF NONLINEAR DELAY ORDINARY DIFFERENTIAL EQUATIONS

BY

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DECLARATION

Candidate's Declaration

I hereby declare that this thesis is the result of my own original research and that no part of it has been presented for another degree in this university or elsewhere.

Candidate's Signature .... Date ....

Name: Adusei-Poku Afful

Supervisor's Declaration

I hereby declare that the preparation and presentation of the thesis were supervised in accordance with the guidelines on supervision of thesis laid down by the University of Cape Coast.

Supervisor's Signature .... Date ....

Name: Prof. Ernest Yankson

ABSTRACT

In this thesis, inequalities regarding the solutions of the nonlinear ordinary differential equation with multiple delays are obtained by means of Lyapunov functionals. These inequalities are then used to obtain sufficient conditions that guarantee exponential decay of solutions to zero of the multi delay nonlinear ordinary differential equation. In addition, a criterion for the instability of the zero solution is obtained. The results generalizes some results in the literature.

KEY WORDS

Exponential Stability

Instability

Lyapunov Functional

Multiple Delay

Nonlinear Delay Ordinary Differential Equation

Nonlinear Ordinary Differential Equation

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DEDICATION

To my family

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LIST OF ABBREVIATIONS

DE	Differential Equations
ODE	Ordinary Differential Equations
PDE	Partial Differential Equations

CHAPTER ONE

INTRODUCTION

This chapter is made up of the background of the study, the statement of the problem, the objectives of this study as well as the organization of the chapters in the thesis

Background to the Study

In science and engineering, mathematical models are developed to help in the understanding of physical phenomena. These models often yield an equation that contains some derivatives of an unknown function. Such an equation is called a differential equation.

Ince (1956) observed that the study of differential equations began in 1675, when Gottfried Wilhelm Von Leibniz wrote the equation

$$\int x dx = (\tfrac{1}{2})x^2.$$

However, according to Sasser (2005) the search for the general methods of integrating ordinary differential equations began when Isaac Newton put forth the following three "types" of differential equations known as "fluxional equations" in the 1670s:

$$\begin{aligned} \frac{dy}{dx} &= f(x), \\ \frac{dy}{dx} &= f(x, y), \\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} &= u. \end{aligned}$$

The first two equations contain only ordinary derivatives of one or more dependent variables, with respect to a single independent variable, and are known today as ordinary differential equations. The third equation involved the partial derivatives of one dependent variable and today is called partial differential equations.

The main concern of mathematicians during the 17th and 18th century focused primarily on integration of differential equations by means of elementary functions. Due to the works of several great mathematicians, all known elementary methods for solving first-order differential equations were practically found by the end of the 17th century. Many differential equations of second-order were derived, in the beginning of the 18th century, as models for problems in classical Mechanics. Also other phenomena led to differential equations of third order.

The period of initial discovery of general methods of integrating ordinary differential equations ended by 1775, a hundred years after Leibniz inaugurated the integral sign. For many problems the formal methods were not sufficient. Solutions with special properties were required, and thus, criteria guaranteeing the existence of such solutions became increasingly important. Boundary value problems led to ordinary differential equations, such as Bessel's equation, that prompted the study of Laguerre, Legendre, and Hermite polynomials. The study of these and other functions that are solutions of equations of hypergeometric type led in turn to modern numerical methods. Thus, by 1775, as more and more attention was given to analytical methods and problems of existence, the search for general methods of integrating ordinary differential equations ended.

In the middle of the 19th century, Joseph Liouville showed the impossibility of expressing the general solution of certain differential equations by a combination of elementary functions or Liouville functions. Consequently, a new approach to the study of differential equations had to be developed.

The qualitative theory of differential equations was born at the end of the 19th century with the works of Poincaré & Magini (1899) and Lyapunov

(1892). Its aim is to obtain the local and global behaviour of the solutions without having them explicitly. The main goal of the qualitative theory is the topological description of properties and configurations of solutions of differential systems in the whole space.

In the case of stability, Lyapunov (1892) introduced a completely different technique, known as Lyapunov's second method, to determine the stability behaviour of solutions of linear and nonlinear systems of differential equations. The major advantage of this method is that; basically, stability can be obtained without any prior knowledge of solutions. Earlier, this method was used only to establish simple theorems on stability. However, from the last 40 years his basic idea has been extensively exploited and effectively applied to entirely new problems in physics and engineering.

Today, this method is widely recognized as an excellent tool not only in the study of differential equations but also in the theory of control systems, dynamical systems, systems with time lag, power system analysis, time-varying nonlinear feedback systems, and so on. Its main characteristic is the construction of a scalar function, namely, the Lyapunov function.

Unfortunately, it is sometimes very difficult to find a proper Lyapunov function for a given system. Because the method yields stability information directly, that is, without solving the differential systems, it is also known as Lyapunov direct method.

Statement of the Problem

The study of the stability properties of differential equations have attracted the attention of several mathematicians lately. In particular, Wang (2004) obtained sufficient conditions for the zero solution of the equation

$$x'(t) = a(t)x(t) + b(t)x(t-h)$$

to be exponentially stable.

Also, Cable & Raffoul (2011) obtained sufficient conditions for the zero solution of the equation with multiple delays

$$x'(t) = a(t)x(t) + \sum_{i=1}^{n} b_i(t)x(t-h_i)$$

to be exponentially stable.

However, the results obtained by the above authors are for linear equations and do not apply to the equation

$$x'(t) = a(t)f(x(t)) + \sum_{i=1}^{n} b_i(t)f(x(t-h_i)),$$

which is the nonlinear version of the equations considered by the authors.

Purpose of the Study

The purpose of this study is to determine the sufficient conditions for exponential stability and instability of solutions of nonlinear delay ordinary differential equations.

Research Objectives

The study sought to achieve the following;

1. construct a suitable Lyapunov functional that yields results concerning the exponential stability of the zero solution of

$$x'(t) = a(t)f(x(t)) + \sum_{i=1}^{n} b_i(t)f(x(t-h_i))$$

2. obtain sufficient conditions for the exponential stability of the zero solution of

$$x'(t) = a(t)f(x(t)) + \sum_{i=1}^{n} b_i(t)f(x(t-h_i))$$

3. obtain an instability criteria of the zero solution of

$$x'(t) = a(t)f(x(t)) + \sum_{i=1}^{n} b_i(t)f(x(t-h_i)).$$

Significance of the Study

The results obtained in the study generalizes some results in the literature.

Delimitation

The study determined the sufficient conditions for exponential stability and instability of solutions of nonlinear delay ordinary differential equations. Results concerning exponential stability and instability of solutions of nonlinear ordinary differential equations was analyzed. The results obtained cannot be easily generalized for all ordinary differential equations.

Limitation

Even though there are other methods of obtaining stability properties of solutions of ordinary differential equations, the study was limited to Lyapunov's direct method. This was because the method allowed us to deduce inequalities that all solutions must satisfy and from which the exponential stability and instability is deduced. Also the study obtained results concerning nonlinear ordinary differential equations instead of linear equations.

Organisation of the Study

Chapter One of the study dealt with the background to the study. It gave a vivid history of the study of ordinary differential equations and the important role it plays in the modelling of physical phenomena. It further explained the qualitative properties of differential equations considered in this thesis. The Chapter also includes the problems statement and objectives as well as the organization of the study.

In Chapter Two, an extensive review of relevant related literature was carried out. It also includes a brief review of the relevant mathematical concepts. It draws extensively from the work of other researchers which are published in journals and scholarly articles.

The Chapter Three of the study dealt with the Methodology which included an overview of the tool used in the discussion of the stability properties of differential equations considered in this thesis.

In Chapter Four, results and discussion of major findings were done based on the objectives of the study. The Fifth Chapter of the study which is the final Chapter dealt with the summary of the results as well as the conclusions.

Chapter Summary

In this chapter, an introduction of the study is given by a brief background and the problem to be examined in this study. The objectives, significance, limitations and delimitations of the study are also stated. Then the structure of the study, that is how the study is organised is outlined.

CHAPTER TWO

LITERATURE REVIEW

Introduction

This chapter is divided into two sections. The first section deals with the review of existing literature which are obtained from the work of other researchers published in journals and scholarly articles which are related and also relevant and significant to the study. The second section consists of a review of basic concepts of ordinary differential equations.

Lyapunov's Stability

The study of behaviour of solutions of differential equations started in the latter part of the nineteenth century and became a subject of intense research since 1940. Early results include the work of the Russian mathematician Lyapunov (1892) in which he standardized the definition of stability to systems of ordinary differential equations of the form

$$x'(t) = f(x(t)).$$

In the century that followed, the use of Lyapunov functions to prove stability increased and is known alternatively as the "Direct method of Lyapunov" or "Lyapunov's second method".

La Salle & Lefschetz (1961) presented modest monograph "stability by Liapunov's direct method with applications", which is described as expounding the main lines of Lypunov's stability theory and of his direct method, and making them accessible to technical people with some mathematical equipments.

Hale (1977) in his book "Theory of Functional differential equations" studied the equation

$$x'(t) = ax(t) + bx(t-h)$$

where, a and b are constants and gave a stability region. For asymptotic stability, he requires $a < 0$.

Busenberg & Cooke (1984) derived some new sufficient conditions for uniform asymptotic stability of the zero solution of the linear non-autonomous delay equations. The equations considered includes scalar equations of the form

$$x'(t) = -c(t)x(t) + \sum_{i=1}^{n} b_i(t)x(t - T_i),$$

where, $c(t), b_i(t)$ are continuous for $t \geq 0$ and T_i is a positive number $(i = 1, 2, ..., n)$, and also systems of the form

$$x'(t) = B(t)x(t - T) - C(t)x(t),$$

where, $B(t)$ and $C(t)$ are $n \times n$ matrices.

The results are found by using the method of Lyapunov functionals. Again, Hatvani (1997), gave annulus arguments not requiring the boundedness of the derivatives of the functions involved and established Lyapunov type theorems for the attractivity, asymptotic stability, and partial stability properties of the zero solution of nonautonomous functional differential equations. Hatvani applied these results to the scalar equation

$$x'(t) = -c(t)x(t) + b(t)x(t - h) \qquad (c(t) \geq 0),$$

and the scalar equation with several delays

$$x'(t) = -c(t)x(t) + \sum_{i=1}^{n} b_i(t)x(t - h_i) \qquad (c(t) \geq 0),$$

as well as to the system

$$x'(t) = B(t)x(t - h) - C(t)x(t),$$

where, $B(t)$ and $C(t)$ are continuous matrix functions.

Knyazhishche & Shcheglov (1998) proved Lyapunov type theorems and applied the results to the scalar equation

$$x'(t) = b(t)x(t - r(t)),$$

where, $b(t)$ and $r(t)$ may be unbounded. In the paper, they gave new definition of the positive-definiteness of the Lyapunov functional involved in the stability and asymptotic stability investigation.

Moreover, Wang (2004) used Lyapunov functionals and obtained inequalities from which exponential stability was deduced on the zero solution of the constant delay equation

$$x'(t) = a(t)x(t) + b(t)x(t - h),$$

where, $a, b : R_+ \to R$ is continuous and $h > 0$ a constant.

Wang obtained sufficient conditions for asymptotic stability if $a(t) \leq 0$ and instability $a(t) \geq 0$. In the case that $a(t)$ and $b(t)$ are constant, Wang offered a region showing uniform asymptotic stability and instability of the zero solution of the equation which is different from the stability regions obtained by Hale's (1977).

Furthermore, Cable & Raffoul (2011) used Lyapunov functionals to obtain sufficient conditions that quarantee exponential decay of solutions to zero for the multi delay linear differential equation

$$x'(t) = a(t)x(t) + \sum_{i=1}^{n} b_i(t)x(t - h_i),$$

where, a, b are continuous with $0 < h_i \leq h^*$ for $i = 1, ..., n$, for some positive constant h^*. They also obtained a criterion for instability of the zero solution of the equation.

The highlight of the paper by Cable and Raffoul is that $a(t)$ is allowed to change signs. Cable and Raffoul compared their results to that of Wang (2004), by setting $n = 1$, and showed that their work improved the results obtained by Wang.

In the case where there are time varying delays, Burton (2003) obtained asymptotic stability of the zero solution of the equation

$$x' = b(t)x(t) - a(t)x(t - h(t)),$$

where, $b(t) = 0$ and when the delay is constant, $h(t) = h$ for all t, and used both Lyapunov functionals and fixed point theory for the purpose of comparing both methods.

In the next section, basic concepts of ordinary differential equation is reviewed. The mathematical concepts are reviewed from the book "Ordinary Differential Equations" by Nagy (2019).

Basic Concepts of Differential Equations

A differential equation is an equation, where the unknown is a function and both the function and its derivatives may appear in the equation. For example

$$m\frac{d^2x}{dt^2}(t) = f\Big(t, x(t), \frac{dx}{dt}(t)\Big), \tag{2.1}$$

where, the unknown is $x(t)$.

$$\frac{\partial T}{\partial t}(t,x) = k\Big(\frac{\partial^2 T}{\partial x^2}(t,x) + \frac{\partial^2 T}{\partial y^2}(t,x) + \frac{\partial^2 T}{\partial z^2}(t,x)\Big), \tag{2.2}$$

where, k is a positive constant.

Equation (2.1) is an example of ordinary differential equations (ODEs), this is because the unknown function depends on a single independent variable, t. Equation (2.2) is an example of partial differential equations

(PDEs), this is because the unknown function depends on two or more independent variables, t, x, y, and z, and their partial derivatives appear in the equations.

Order of differential equations

The order of a differential equation is the highest derivative order that appears in the equation. For instance, Equation (2.1) is a second order and Equation (2.2) is first order in time and second order in space variables.

This thesis focuses on ordinary differential equations (ODEs) and therefore basic concepts of ODEs are provided in the section that follows.

Ordinary Differential Equations

An ordinary differential equation (ODE) is an equation that involves some ordinary derivatives of an unknown function. For example

$$\frac{dy}{dt} = t^2 \tag{2.3}$$

is an ODE where, $y(t)$ is the unknown function.

First order ordinary differential equation

The general first order ODE in R^n, $n \geq 1$, is given by

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0. \tag{2.4}$$

on a domain $D \subset R \times R^n$, where, $(t_0, y_0) \in D$ and $f(t, y)$ is a function from $D \subset R \times R^n$ into R^n.

Equation (2.4) is usually referred to as non-autonomous differential system whereas, a differential system of the form

$$y' = f(y). \tag{2.5}$$

in which the right hand side does not involve the independent variable ’t’ is called autonomous system.

Solution of an ordinary differential equation

A function $y(t)$ is said to be a solution of Equation (2.4) on an interval I if $t_0 \in I$, $(t, y(t)) \in D$ for $t \in I$, $y(t)$ is differentiable for $t \in I$ and satisfies Equation (2.4) for $t \in I$.

Linear ordinary differential equations

The Equation (2.4) is linear if and only if the source function f is linear on its second argument.

Consider the linear ODE,

$$y' = a(t)y + b(t). \tag{2.6}$$

The Equation (2.6) has constant coefficients if and only if both a and b above are constants. Otherwise the equation has variable coefficients.

For example, consider the first order linear ODE,

$$y' = 2y + 3. \tag{2.7}$$

On the right-hand side, the function $f(t, y) = 2y + 3$, where $a(t) = 2$ and $b(t) = 3$. Since these coefficients do not depend on t, Equation (2.7) is a constant coefficient equation.

Also, consider the ODE,

$$y' = -\frac{2}{t}y + 4t. \tag{2.8}$$

In this case, the right-hand side is given by the function $f(t, y) = -\frac{2}{t} + 4t$, where $a(t) = -\frac{2}{t}$ and $b(t) = 4t$. Since the coefficients are non-constant functions of t, Equation (2.8) is a variable coefficients equation.

Nonlinear ordinary differential equations

An ordinary differential equation is called nonlinear if and only if the function f is nonlinear in the second argument.

For example

$$y'(t) = \frac{t^2}{y^3(t)} \tag{2.9}$$

is nonlinear, since the function $f(t, y) = \frac{t^2}{y^3}$ is nonlinear in the second argument.

Delay ordinary differential equations

When modelling a system using a differential equation where the fundamental assumption is that the time rate at time t, $y'(t)$, depends only on the current status at time t, $f(t, y(t))$ resulting in the differential Equation (2.4).

In applications, this assumption and the initial condition should be improved so the situation can be modeled more accurately and better results derived. One improvement of Equation (2.4) is to assume that the time rate depends not only on the current state, but also on the state in the past; that is, the past history will contribute to the future development, or, there is a time-delay effect.

Differential equations incorporating delay effect, or using information from the past, are called delay differential equations. They include finite delay differential equations and infinite delay differential equations.

Consider the delay ordinary differential equation below,

$$\frac{dy}{dt} = f\Big(t, y(t), y(t-\tau)\Big), \quad \tau > 0 \tag{2.10}$$

with

$$y(t) = \phi_0(t), \qquad t_0 - \tau \leq t \leq t_0. \tag{2.11}$$

Here, $\phi_0 : R \to R^n$ is a known function, usually taken to be continuous. $\phi_0(t)$ is called the initial function for , t_0 the initial instant and $[t_0 - \tau, t_0]$ the initial set.

Qualitative Properties of Differential Equations

Differential equations are required tools in scientific modelling of physical systems which found their applications in almost every area of human efforts from agricultural sciences, engineering, medical science, social sciences to physical sciences. Among the earlier work on differential equations, the works of Euler and Lagrange stand out. They first worked on the theory of small oscillations and consequently, the theory of linear system of ordinary differential equations.

In the study of theory of differential equations, the knowledge of two different streams should be known. They include;

1. An effort to get a definite or one of the definite types, either in closed forms, which is rarely possible or else by some method of approximation. This is referred to as the Quantitative theory.

2. An effort to abandon all attempts to obtain an exact or approximate solution, one strives to obtain information about the whole class of solution. This is called the Qualitative theory. The most important qualitative properties of solution of differential equations are stability and boundedness.

Stability of differential equations

Stability has a great role in the study of differential systems. The mathematical models or equations that describe physical phenomena are in most cases differential systems of the form $x' = f(t, x)$, with the initial data $x(t_0) = x_0$. Since the initial data, which often results from all types of measurements, may have errors, it is important to know the extent to which small disturbances in the initial data affect the desired behaviour of the solutions of given systems.

If by making a sufficiently small change in the initial data, a substantial deviation is observed in the corresponding solutions, then the solution obtained from the given initial data is unacceptable because it does not describe the required phenomena even approximately.

The idea of investigating the conditions that will not allow the solutions to remarkably deviate from the desired behaviour is therefore vital.

The area of mathematics that deals with such problems relating to the behaviour of the solutions is usually referred to as stability theory (Ahmad & Rao, 1999).

Definition 1 (Exponential stability)

The zero solution of Equation (2.4) is exponentially stable if for a positive constant λ, any solution $x(t, t_0, \varphi)$ satisfies

$$\| x \| \leq K(|\varphi|, t_0)e^{-\lambda(t-t_0)}, \tag{2.12}$$

for all $t \geq t_0$, where, $K(|\varphi|, t_0)$ is a positive constant depending on t_0 and φ, with φ being an initial given function.

Chapter Summary

In this chapter, review was done on relevant literature and basic concepts of ordinary differential equations as well as stability of differential equations. From the literature review, there is evidence that numerous studies on exponential stability and instability of differential equations by the use of Lyapunov direct method have been done by earlier researchers.

However, there seems to be a gap in the literature that needs to be filled and this study will ensure that these gaps are addressed.

It was discovered that few works has been documented in determining the exponential stability and instability of nonlinear differential equations as in linear differential equations.

CHAPTER THREE

METHODOLOGY

Introduction

In this chapter, the method that was used in achieving the research objectives is discussed.

Lyapunov's Method

According to Parks (1992), Lyapunov in 1892, dealt with stability by two distinct methods. The First method pre-supposes an explicit solution known and this is applicable to some restricted but important cases. The second method, which is also called the Direct method, is of great generality and power and, above all, does not require the knowledge of the solutions themselves.

The application of the Lyapunov's method lies in constructing a scalar function (say V - some time energy liked function) and its derivatives V' such that they possess certain properties. When these properties of V and V' are shown, the stability behaviour of the system is known. The direct method is via a special function called the Lyapunov function.

Definition 2 (Positive definite)

Let $Q \subset R^n$ be a domain containing the zero vector. A continuous function $V : Q \to [0, \infty)$ is called positive definite if $V(x) > 0$ for $x \neq 0$.

Definition 3 (Lyapunov Function)

Let $Q \subset R^n$ be a domain containing the zero vector. A function $V : Q \to [0, \infty)$ is called a Lyapunov function if;

1. $V(0) = 0$,
2. $V(x)$ is positive definite and
3. has continuous first-order partial derivatives.

The use of Lyapunov functionals allowed us to deduce inequalities that all solutions must satisfy and from which the exponential stability and instability is deduced.

However, the choice of a Lyapunov function plays an important role in the study of Lyapunov stability theory. Once a Lyapunov function has been found in some region around the origin, it becomes possible to test the stability, exponential stability or instability of the zero solution of a given system. In case of failure to find such a Lyapunov function, one cannot study the stability of a given system using the Lyapunov stability theory.

In general, no satisfactory technique is given which provides suitable Lyapunov function, particularly for nonlinear systems. However in the literature, some methods are there which are applicable to both linear and nonlinear systems to construct suitable Lyapunov functions.

Chapter Summary

This chapter presented the method that was used in conducting the research. It focused on the Lyapunov's direct method which was used to deduce inequalities in this thesis.

CHAPTER FOUR

RESULTS AND DISCUSSION

Introduction

This chapter covers the results of the study. In particular, the results of exponential stability and instability criteria of the nonlinear ordinary differential equation are presented and discussed. Results are presented based on the objectives of the study.

Preliminary Results

Consider the scalar nonlinear differential equation with multi delay as follows;

$$x'(t) = a(t)f(x(t)) + \sum_{i=1}^{n} b_i(t)f(x(t-h_i)), \tag{4.1}$$

where, a, b are continuous with $0 < h_i \leq h^*$ for $i = 1, ..., n$ for some positive constant h^* and $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ to be continuous.

Let

$$f_1(x) = \begin{cases} \frac{f(x)}{x}, & x \neq 0 \\ f'(0), & x = 0. \end{cases}$$

In Lemma 1, an equivalent form of Equation (4.1) is provided which will be used extensively in the rest of the thesis.

Lemma 1.

Equation (4.1) is equivalent to the equation

$$\begin{aligned} x'(t) &= \Big(a(t) + \sum_{i=1}^{n} b_i(t+h_i)\Big) f_1(x(t))x(t) \\ &\quad - \sum_{i=1}^{n} \frac{d}{dt} \int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds. \end{aligned} \tag{4.2}$$

Proof.

Differentiating the integral term in Equation (4.2), gives

$$
\begin{aligned}
x'(t) &= \Big(a(t) + \sum_{i=1}^{n} b_i(t+h_i)\Big) f_1(x(t))x(t) \\
&\quad - \sum_{i=1}^{n} \Big[b_i(t+h_i) f(x(t)) \\
&\quad - b_i(t-h_i+h_i) f(x(t-h_i)) \Big] \\
&= a(t) f_1(x(t))x(t) \\
&\quad + \sum_{i=1}^{n} b_i(t+h_i) f_1(x(t))x(t) \\
&\quad - \sum_{i=1}^{n} \Big[b_i(t+h_i) f(x(t)) \\
&\quad - b_i(t) f(x(t-h_i)) \Big] \\
&= a(t) f_1(x(t))x(t) \\
&\quad + \sum_{i=1}^{n} b_i(t+h_i) f_1(x(t))x(t) \\
&\quad - \sum_{i=1}^{n} b_i(t+h_i) f(x(t)) \\
&\quad + \sum_{i=1}^{n} b_i(t) f(x(t-h_i)). \qquad (4.3)
\end{aligned}
$$

But, $f(x(t)) = f_1(x(t))x(t)$. Therefore, Equation (4.3) becomes

$$\begin{aligned} x'(t) &= a(t)f(x(t)) \\ &\quad + \sum_{i=1}^{n} b_i(t+h_i)f(x(t)) \\ &\quad - \sum_{i=1}^{n} b_i(t+h_i)f(x(t)) \\ &\quad + \sum_{i=1}^{n} b_i(t)f(x(t-h_i)) \\ &= a(t)x(t) + \sum_{i=1}^{n} b_i(t)x(t-h_i). \end{aligned}$$

This completes the proof.

In lemma 2, the Lyapunov functionals that will be used to obtain results for the exponential stability and instability are proposed.

Lemma 2.

Let δ and H be constants such that $\delta > 0$ and $H > 0$. If $f(0) = 0$, then the functionals defined by

$$\begin{aligned} V(t,x) &= \left[x(t) + \sum_{i=1}^{n} \int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right]^2 \\ &\quad + \delta \sum_{i=1}^{n} \int_{-h_i}^{0} \int_{t+s}^{t} b_i^2(z+h_i)f^2(x(z))dzds \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} V(t,x) &= \left[x(t) + \sum_{i=1}^{n} \int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right]^2 \\ &\quad - H \sum_{i=1}^{n} \int_{t-h_i}^{t} b_i^2(s+h_i)f^2(x(s))ds, \end{aligned} \tag{4.5}$$

are Lyapunov functionals.

Proof.

To verify that Equation (4.4) is a Lyapunov functional, consider

$$V(t,0) = \left[0 + \sum_{i=1}^{n} \int_{t-h_i}^{t} b_i(s+h_i) f(0) ds\right]^2$$

$$+ \delta \sum_{i=1}^{n} \int_{-h_i}^{0} \int_{t+s}^{t} b_i^2(z+h_i) f^2(0) dz ds$$

$$= 0.$$

Now, it is clear from the definition of Equation (4.4) that $V(t,x) > 0$ for all x, except $x = 0$. Thus, $V(t,x)$ is positive definite.

Finally,

$$\frac{\partial V}{\partial x} = 2\left[x(t) + \sum_{i=1}^{n} \int_{t-h_i}^{t} b_i(s+h_i) f(x(s)) ds\right],$$

which is continuous.

Therefore, $V(t,x)$ defined by Equation (4.4) is a Lyapunov functional. Equation (4.5) can similarly be shown to be a Lyapunov functional.

This complete the proof.

Main Results

In this section, results concerning exponential stability and instability of Equation (4.1) is obtained. The following notation is given before the statement of the main results.

Let $\psi : [-h^*, 0] \to (-\infty, \infty)$ be a given bounded initial function with

$$||\psi|| = \max_{-h^* \le s \le 0} |\psi(s)|.$$

Also, denote the norm of a function $\varphi : [-h^*, \infty) \to (-\infty, \infty)$ with

$$||\varphi|| = \sup_{-h^* \leq s \leq \infty} |\varphi(s)|.$$

$x(t) \equiv x(t, t_0, \psi)$ is a solution of Equation (4.1), if $x(t)$ satisfies Equation (4.1) for $t \geq t_0$ and $x_{t_0} = x(t_0 + s) = \psi(s)$, $s \in [-h^*, 0]$.

Exponential Stability

In this section, inequalities regarding the exponential stability of Equation (4.1) is deduced.

To simplify notation, let

$$Q(t, x) = \Big(a(t) + \sum_{i=1}^{n} b_i(t + h_i)\Big) f_1(x(t)).$$

Lemma 3.

Let $V(t, x)$ be as defined in Equation (4.4). Assume for $\delta > 0$ that

$$\frac{\delta}{-(\delta + 1)h^*} \leq Q(t, x) \leq -\delta h^* \sum_{i=1}^{n} b_i^2(t + h_i) f_1^2(x(t)) \tag{4.6}$$

holds; Then along the solutions of Equation (4.1);

$$V'(t) \leq Q(t, x)V(t). \tag{4.7}$$

Proof.

Let $x(t) = x(t, t_0, \psi)$ be a solution of Equation (4.1) with $V(t, x)$ defined by Equation (4.4). It must be noted that $Q(t, x) < 0$ for all $t \geq 0$ in view of condition (4.6). Then along the solutions of Equation (4.2);

$$
\begin{aligned}
V'(t) &= 2\left[x(t)+\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right]\Big[x'(t)\\
&\quad + \sum_{i=1}^{n}\frac{d}{dt}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\Big]\\
&\quad + \delta\sum_{i=1}^{n}\int_{-h_i}^{0} b_i^2(t+h_i)f^2(x(t))ds\\
&\quad - \delta\sum_{i=1}^{n}\int_{-h_i}^{0} b_i^2(t+s+h_i)f^2(x(t+s))ds\\
&= 2\left[x(t)+\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right]Q(t,x)x(t)\\
&\quad + \delta\sum_{i=1}^{n}\int_{-h_i}^{0} b_i^2(t+h_i)f^2(x(t))ds\\
&\quad - \delta\sum_{i=1}^{n}\int_{-h_i}^{0} b_i^2(t+s+h_i)f^2(x(t+s))ds\\
&\leq 2\left[x(t)+\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right]Q(t,x)x(t)\\
&\quad + \delta h^*\sum_{i=1}^{n} b_i^2(t+h_i)f^2(x(t))\\
&\quad - \delta\sum_{i=1}^{n}\int_{-h_i}^{0} b_i^2(t+s+h_i)f^2(x(t+s))ds\\
&\leq 2Q(t,x)x(t)\left[x(t)+\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right]\\
&\quad + \delta h^*\sum_{i=1}^{n} b_i^2(t+h_i)f^2(x(t))\\
&\quad - \delta\sum_{i=1}^{n}\int_{-h_i}^{0} b_i^2(t+s+h_i)f^2(x(t+s))ds
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & 2Q(t,x)x^2(t) \\
& + 2Q(t,x)x(t)\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds \\
& + \delta h^*\sum_{i=1}^{n} b_i^2(t+h_i)f^2(x(t)) \\
& - \delta\sum_{i=1}^{n}\int_{-h_i}^{0} b_i^2(t+s+h_i)f^2(x(t+s))ds \\
\leq\ & Q(t,x)x^2(t) \\
& + 2Q(t,x)x(t)\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds \\
& + \delta h^*\sum_{i=1}^{n} b_i^2(t+h_i)f^2(x(t)) \\
& - \delta\sum_{i=1}^{n}\int_{-h_i}^{0} b_i^2(t+s+h_i)f^2(x(t+s))ds \\
& + Q(t,x)x^2(t) \\
\leq\ & Q(t,x)x^2(t) \\
& + 2Q(t,x)x(t)\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds \\
& + \delta h^*\sum_{i=1}^{n} b_i^2(t+h_i)f_1^2(x(t))x^2(t) \\
& - \delta\sum_{i=1}^{n}\int_{-h_i}^{0} b_i^2(t+s+h_i)f^2(x(t+s))ds \\
& + Q(t,x)x^2(t)
\end{aligned}
$$

$$
\begin{aligned}
&\leq Q(t,x)\left[x^2(t)+2x(t)\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right]\\
&+\delta h^*\sum_{i=1}^{n} b_i^2(t+h_i)f_1^2(x(t))x^2(t)\\
&-\delta\sum_{i=1}^{n}\int_{-h_i}^{0} b_i^2(t+s+h_i)f^2(x(t+s))ds\\
&+Q(t,x)x^2(t)\\
&\leq Q(t,x)\Big[x^2(t)\\
&+2x(t)\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\\
&+\left(\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right)^2\\
&-\left(\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right)^2\Big]\\
&+Q(t,x)\delta\sum_{i=1}^{n}\int_{-h_i}^{0}\int_{t+s}^{t} b_i^2(z+h_i)f^2(x(z))dzds\\
&-Q(t,x)\delta\sum_{i=1}^{n}\int_{-h_i}^{0}\int_{t+s}^{t} b_i^2(z+h_i)f^2(x(z))dzds\\
&-\delta\sum_{i=1}^{n}\int_{-h_i}^{0} b_i^2(t+s+h_i)f^2(x(t+s))ds\\
&+\delta h^*\sum_{i=1}^{n} b_i^2(t+h_i)f_1^2(x(t))x^2(t)\\
&+Q(t,x)x^2(t)
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & Q(t,x)\Big[x^2(t) + 2x(t)\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds \\
& + \left(\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right)^2 \\
& + \delta\sum_{i=1}^{n}\int_{-h_i}^{0}\int_{t+s}^{t} b_i^2(z+h_i)f^2(x(z))dzds\Big] \\
& - Q(t,x)\left(\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right)^2 \\
& - Q(t,x)\delta\sum_{i=1}^{n}\int_{-h_i}^{0}\int_{t+s}^{t} b_i^2(z+h_i)f^2(x(z))dzds \\
& - \delta\sum_{i=1}^{n}\int_{-h_i}^{0} b_i^2(t+s+h_i)f^2(x(t+s))ds \\
& + \delta h^*\sum_{i=1}^{n} b_i^2(t+h_i)f_1^2(x(t))x^2(t) \\
& + Q(t,x)x^2(t) \\
\leq\ & Q(t,x)\Big[\left(x(t) + \sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right)^2 \\
& + \delta\sum_{i=1}^{n}\int_{-h_i}^{0}\int_{t+s}^{t} b_i^2(z+h_i)f^2(x(z))dzds\Big] \\
& - Q(t,x)\delta\sum_{i=1}^{n}\int_{-h_i}^{0}\int_{t+s}^{t} b_i^2(z+h_i)f^2(x(z))dzds \\
& - Q(t,x)\left(\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right)^2 \\
& - \delta\sum_{i=1}^{n}\int_{-h_i}^{0} b_i^2(t+s+h_i)f^2(x(t+s))ds \\
& + \left(\delta h^*\sum_{i=1}^{n} b_i^2(t+h_i)f_1^2(x(t)) + Q(t,x)\right)x^2(t) \qquad (4.8)
\end{aligned}
$$

$$
\begin{aligned}
\leq \quad & Q(t,x)V(t) \\
& - Q(t,x)\delta \sum_{i=1}^{n} \int_{-h_i}^{0} \int_{t+s}^{t} b_i^2(z+h_i) f^2(x(z)) dz ds \\
& - Q(t,x) \left(\sum_{i=1}^{n} \int_{t-h_i}^{t} b_i(s+h_i) f(x(s)) ds \right)^2 \\
& - \delta \sum_{i=1}^{n} \int_{-h_i}^{0} b_i^2(t+s+h_i) f^2(x(t+s)) ds \\
& + \left(\delta h^* \sum_{i=1}^{n} b_i^2(t+h_i) f_1^2(x(t)) + Q(t,x) \right) x^2(t). \qquad (4.9)
\end{aligned}
$$

If $u = t + s$, then

$$
\begin{aligned}
& \delta \sum_{i=1}^{n} \int_{-h_i}^{0} b_i^2(t+s+h_i) f^2(x(t+s)) ds \\
& = \delta \sum_{i=1}^{n} \int_{t-h_i}^{t} b_i^2(u+h_i) f^2(x(u)) ds. \qquad (4.10)
\end{aligned}
$$

By Holder's inequality;

$$
\begin{aligned}
& -Q(t,x) \left(\sum_{i=1}^{n} \int_{t-h_i}^{t} b_i(s+h_i) f(x(s)) ds \right)^2 \\
& \leq -Q(t,x) h^* \sum_{i=1}^{n} \int_{t-h_i}^{t} b_i^2(s+h_i) f^2(x(s)) ds. \qquad (4.11)
\end{aligned}
$$

It is observe that,

$$
\begin{aligned}
& \delta \sum_{i=1}^{n} \int_{-h_i}^{0} \int_{t+s}^{t} b_i^2(z+h_i) f^2(x(z)) dz ds \\
& \leq \delta h^* \int_{t-h_i}^{t} \sum_{i=1}^{n} b_i^2(s+h_i) f^2(x(s)) ds. \qquad (4.12)
\end{aligned}
$$

Substituting (4.9), (4.10), and (4.11) into (4.8) and making use of (4.6),

$$
\begin{aligned}
V'(t) &\leq Q(t,x)V(t) \\
&\quad - Q(t,x)\delta h^* \int_{t-h_i}^{t} \sum_{i=1}^{n} b_i^2(s+h_i) f^2(x(s))ds \\
&\quad - Q(t,x)h^* \sum_{i=1}^{n} \int_{t-h_i}^{t} b_i^2(s+h_i) f^2(x(s))ds \\
&\quad - \delta \sum_{i=1}^{n} \int_{t-h_i}^{t} b_i^2(u+h_i) f^2(x(u))ds \\
&\quad + \left(\delta h^* \sum_{i=1}^{n} b_i^2(t+h_i) f_1^2(x(t)) + Q(t,x) \right) x^2(t) \\
&\leq Q(t,x)V(t) \\
&\quad + \Big[- Q(t,x)\delta h^* - Q(t,x)h^* \\
&\quad - \delta \Big] \sum_{i=1}^{n} \int_{t-h_i}^{t} b_i^2(s+h_i) f^2(x(s))ds \\
&\quad + \left(\delta h^* \sum_{i=1}^{n} b_i^2(t+h_i) f_1^2(x(t)) + Q(t,x) \right) x^2(t) \\
&= Q(t,x)V(t) \\
&\quad + \Big[- Q(t,x)h^*(\delta+1) \\
&\quad - \delta \Big] \sum_{i=1}^{n} \int_{t-h_i}^{t} b_i^2(s+h_i) f^2(x(s))ds \\
&\quad + \left(\delta h^* \sum_{i=1}^{n} b_i^2(t+h_i) f_1^2(x(t)) + Q(t,x) \right) x^2(t)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \quad Q(t,x)V(t) + \Big[\frac{\delta}{(\delta+1)h^*}h^*(\delta+1) \\
&\quad - \delta\Big]\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i^2(s+h_i)f^2(x(s))ds \\
&\quad + \Big(\delta h^*\sum_{i=1}^{n} b_i^2(t+h_i)f_1^2(x(t)) \\
&\quad - \delta h^*\sum_{i=1}^{n} b_i^2(t+h_1)f_1^2(x(t))\Big)x^2(t)
\end{aligned}
$$

$$
V'(t) \quad \leq \quad Q(t,x)V(t).
$$

This completes the proof.

In the next theorem, two inequalities; one for $t \geq t_0 + h^*/2$ and the other for $t \in [t_0, t_0 + h^*/2]$ are proposed.

Theorem 4.

Suppose that condition (4.6) holds. Then any solution $x(t) = x(t, t_0, \psi)$ satisfies the exponential inequality

$$
|x(t)| \quad \leq \quad \sqrt{2\Big(\frac{2+\delta}{\delta}\Big)V(t_0)e^{-\frac{\delta h^*}{2}\int_{t_0}^{t-\frac{h_i}{2}}[\sum_{i=1}^{n} b_i^2(s+h_i)f_1^2(x(s))]ds}}, \quad (4.13)
$$

for $t \geq t_0 + h^*/2$ and

$$
|x(t)| \quad \leq \quad e^{\int_0^t a(s)f_1(x)ds}\Big[||\psi|| + \int_{t_0}^{t_0+h^*/2}\sum_{i=1}^{n}|b_i(u)||f(\psi(u-h_i))|e^{-\int_0^t a(s)f_1(x)ds}du\Big], \tag{4.14}
$$

for $t \in [t_0, t_0 + h^*/2]$.

Proof.

By changing the order of integration of the second term in $V(t,x)$ given by Equation (4.4) and using the fact that $t - \frac{h_i}{2} \leq z \leq t$ implies that $\frac{h_i}{2} \leq z - t + h_i \leq h_i$, yields;

$$
\begin{aligned}
& \delta\sum_{i=1}^{n}\int_{-h_i}^{0}\int_{t+s}^{t} b_i^2(z+h_i)f^2(x(z))dzds \\
= \;& \delta\sum_{i=1}^{n}\int_{t-h_i}^{t}\int_{-h_i}^{z-t} b_i^2(z+h_i)f^2(x(z)dzds \\
= \;& \delta\sum_{i=1}^{n}\int_{t-h_i}^{t}\int_{-h_i}^{z-t} b_i^2(z+h_i)f^2(x(z))dsdz \\
= \;& \delta\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i^2(z+h_i)f^2(x(z))(z-t+h_i)dz \\
= \;& \delta\sum_{i=1}^{n}\int_{t-h_i}^{t-\frac{h_i}{2}} b_i^2(z+h_i)f^2(x(z))(z-t+h_i)dz \\
& +\delta\sum_{i=1}^{n}\int_{t-\frac{h_i}{2}}^{t} b_i^2(z+h_i)f^2(x(z))(z-t+h_i)dz \\
\geq \;& \delta\sum_{i=1}^{n}\int_{t-\frac{h_i}{2}}^{t} b_i^2(z+h_i)f^2(x(z))(z-t+h_i)dz \\
\geq \;& \delta\sum_{i=1}^{n}\int_{t-\frac{h_i}{2}}^{t} b_i^2(z+h_i)f^2(x(z))\left(\frac{h_i}{2}\right)dz \\
\geq \;& \delta\sum_{i=1}^{n}\frac{h_i}{2}\int_{t-\frac{h_i}{2}}^{t} b_i^2(z+h_i)f^2(x(z))dz. \qquad (4.15)
\end{aligned}
$$

Thus, in view of Equation (4.14);

$$
\begin{aligned}
V(t) \;&\geq\; \delta\sum_{i=1}^{n}\int_{-h_i}^{0}\int_{t+s}^{t} b_i^2(z+h_i)f^2(x(z))dzds \\
&\geq\; \delta\sum_{i=1}^{n}\frac{h_i}{2}\int_{t-\frac{h_i}{2}}^{t} b_i^2(z+h_i)f^2(x(z))dz.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
V\left(t-\frac{h_i}{2}\right) &\geq \delta\sum_{i=1}^{n}\frac{h_i}{2}\int_{t-\frac{h_i}{2}-\frac{h_i}{2}}^{t-\frac{h_i}{2}} b_i^2(z+h_i)f^2(x(z))dz \\
&= \delta\sum_{i=1}^{n}\frac{h_i}{2}\int_{t-h_i}^{t-\frac{h_i}{2}} b_i^2(z+h_i)f^2(x(z))dz. \qquad (4.16)
\end{aligned}
$$

Due to the fact that $V'(t)\leq 0$, we have for $t\geq t_0+h^*/2$ such that

$$
0\leq V(t)+V\left(t-\frac{h_i}{2}\right)\leq 2V\left(t-\frac{h_i}{2}\right).
$$

Using (4.14) and (4.15), yields;

$$
\begin{aligned}
&V(t)+V\left(t-\frac{h_i}{2}\right) \\
&= \left[x(t)+\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right]^2 \\
&\quad + \delta\sum_{i=1}^{n}\int_{-h_i}^{0}\int_{t+s}^{t} b_i^2(z+h_i)f^2(x(z))dzds \\
&\quad +V\left(t-\frac{h_i}{2}\right) \\
&\geq \left[x(t)+\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right]^2 \\
&\quad + \delta\sum_{i=1}^{n}\frac{h_i}{2}\int_{t-\frac{h_i}{2}}^{t} b_i^2(s+h_i)f^2(x(s))ds \\
&\quad + \delta\sum_{i=1}^{n}\frac{h_i}{2}\int_{t-h_i}^{t-\frac{h_i}{2}} b_i^2(s+h_i)f^2(x(s))ds \qquad (4.17)
\end{aligned}
$$

$$\geq \left[x(t)+\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right]^2$$

$$+\,\delta\sum_{i=1}^{n}\frac{h_i}{2}\int_{t-h_i}^{t-\frac{h_i}{2}} b_i^2(s+h_i)f^2(x(s))ds$$

$$+\,\delta\sum_{i=1}^{n}\frac{h_i}{2}\int_{t-\frac{h_i}{2}}^{t} b_i^2(s+h_i)f^2(x(s))ds$$

$$\geq \left[x(t)+\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right]^2$$

$$+\,\delta\sum_{i=1}^{n}\frac{h_i}{2}\int_{t-h_i}^{t} b_i^2(s+h_i)f^2(x(s))ds. \tag{4.18}$$

By Schwartz inequality,

$$\frac{1}{2}\left(\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right)^2 \leq \sum_{i=1}^{n}\frac{h_i}{2}\int_{t-h_i}^{t} b_i^2(s+h_i)f^2(x(s))ds.$$

Thus, inequality (4.16) becomes

$$V(t)+V\left(t-\frac{h_i}{2}\right)$$

$$\geq \left[x(t)+\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right]^2$$

$$+\,\delta\left[\frac{1}{2}\left(\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right)^2\right]$$

$$\geq \left[x(t)+\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right]^2$$

$$+\,\delta\frac{1}{2}\left(\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right)^2$$

$$\geq \quad x(t)\left(x(t)+\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right)$$

$$+\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds$$

$$\left(x(t)+\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right)$$

$$+\,\delta\frac{1}{2}\left(\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right)^2$$

$$\geq \quad x^2(t)$$

$$+\,x(t)\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds$$

$$+\,x(t)\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds$$

$$+\left(\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right)^2$$

$$+\,\delta\frac{1}{2}\left(\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right)^2$$

$$\geq \quad x^2(t)$$

$$+\,2x(t)\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds$$

$$+\left[1+\frac{\delta}{2}\right]\left(\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right)^2$$

$$= \frac{2}{2+\delta}x^2(t)$$

$$+ \frac{\delta}{2+\delta}x^2(t)$$

$$+ 2x(t)\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds$$

$$+ \left[1+\frac{\delta}{2}\right]\left(\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right)^2$$

$$= \frac{\delta}{2+\delta}x^2(t)$$

$$+ \frac{1}{\sqrt{1+\frac{\delta}{2}}}x(t)$$

$$\left[\frac{1}{\sqrt{1+\frac{\delta}{2}}}x(t)+\sqrt{1+\frac{\delta}{2}}\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right]$$

$$+ \sqrt{1+\frac{\delta}{2}}\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\Big[\frac{1}{\sqrt{1+\frac{\delta}{2}}}x(t)$$

$$+ \sqrt{1+\frac{\delta}{2}}\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\Big]$$

$$= \frac{\delta}{2+\delta}x^2(t)$$

$$+ \left[\frac{1}{\sqrt{1+\frac{\delta}{2}}}x(t)+\sqrt{1+\frac{\delta}{2}}\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right]$$

$$\times \left[\frac{1}{\sqrt{1+\frac{\delta}{2}}}x(t)+\sqrt{1+\frac{\delta}{2}}\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right]$$

$$
\begin{aligned}
&= \frac{\delta}{2+\delta}x^2(t) \\
&\quad + \left[\frac{1}{\sqrt{1+\frac{\delta}{2}}}x(t) + \sqrt{1+\frac{\delta}{2}}\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right]^2 \\
&\geq \frac{\delta}{2+\delta}x^2(t).
\end{aligned}
$$

Consequently,

$$
\frac{\delta}{2+\delta}x^2(t) \leq V(t) + V\Big(t-\frac{h_i}{2}\Big) \leq 2V\Big(t-\frac{h_i}{2}\Big). \tag{4.19}
$$

An integration of (4.7), from t_0 to t yields

$$
\begin{aligned}
\ln V(s)|_{t_0}^{t} &\leq \int_{t_0}^{t} Q(s,x(s))ds \\
\Longrightarrow \ln V(t) - \ln V(t_0) &\leq \int_{t_0}^{t} Q(s,x(s))ds \\
\Longrightarrow \ln \frac{V(t)}{V(t_0)} &\leq \int_{t_0}^{t} Q(s,x(s))ds \\
\Longrightarrow \frac{V(t)}{V(t_0)} &\leq e^{\int_{t_0}^{t} Q(s,x(s))ds} \\
\Longrightarrow V(t) &\leq V(t_0)e^{\int_{t_0}^{t} Q(s,x(s))ds}.
\end{aligned}
$$

But

$$
Q(t,x) \quad = a(t)f_1(x(t)) + \textstyle\sum_{i=1}^{n} b_i(t+h_i)f_1(x(t)).
$$

Therefore

$$
V(t) \quad \leq V(t_0)e^{\int_{t_0}^{t}[a(s)f_1(x(s))+\sum_{i=1}^{n} b_i(s+h_i)f_1(x(s))]ds}.
$$

This implies that

$$V\Big(t-\frac{h_i}{2}\Big) \leq V(t_0)e^{\int_{t_0}^{t-\frac{h_i}{2}}[a(s)f_1(x(s))+\sum_{i=1}^{n}b_i(s+h_i)f_1(x(s))]ds}.$$

It follows from (4.17), that

$$\begin{aligned}
x^2(t) &\leq \Big(\frac{2+\delta}{\delta}\Big)2V\Big(t-\frac{h_i}{2}\Big)\\
&\leq 2\Big(\frac{2+\delta}{\delta}\Big)V(t_0)e^{\int_{t_0}^{t-\frac{h_i}{2}}[a(s)f_1(x(s))+\sum_{i=1}^{n}b_i(s+h_i)f_1(x(s))]ds}.
\end{aligned}$$

Thus,

$$\begin{aligned}
x(t) &\leq \sqrt{2\Big(\frac{2+\delta}{\delta}\Big)V(t_0)e^{\int_{t_0}^{t-\frac{h_i}{2}}[a(s)f_1(x(s))+\sum_{i=1}^{n}b_i(s+h_i)f_1(x(s))]ds}}\\
&= \sqrt{2\Big(\frac{2+\delta}{\delta}\Big)V(t_0)}\sqrt{e^{\int_{t_0}^{t-\frac{h_i}{2}}[a(s)f_1(x(s))+\sum_{i=1}^{n}b_i(s+h_i)f_1(x(s))]ds}}\\
&= \sqrt{2\Big(\frac{2+\delta}{\delta}\Big)V(t_0)}\left(e^{\int_{t_0}^{t-\frac{h_i}{2}}[a(s)f_1(x(s))+\sum_{i=1}^{n}b_i(s+h_i)f_1(x(s))]ds}\right)^{\frac{1}{2}}\\
&= \sqrt{2\Big(\frac{2+\delta}{\delta}\Big)V(t_0)}e^{\frac{1}{2}\int_{t_0}^{t-\frac{h_i}{2}}[a(s)f_1(x(s))+\sum_{i=1}^{n}b_i(s+h_i)f_1(x(s))]ds}.
\end{aligned}$$

Hence,

$$\begin{aligned}
|x(t)| &\leq \sqrt{2\Big(\frac{2+\delta}{\delta}\Big)V(t_0)}e^{\frac{1}{2}\int_{t_0}^{t-\frac{h_i}{2}}[a(s)f_1(x(s))+\sum_{i=1}^{n}b_i(s+h_i)f_1(x(s))]ds}\\
&\leq \sqrt{2\Big(\frac{2+\delta}{\delta}\Big)V(t_0)}e^{-\frac{\delta h^*}{2}\int_{t_0}^{t-\frac{h_i}{2}}[\sum_{i=1}^{n}b_i^2(s+h_i)f_1^2(x(s))]ds}.
\end{aligned}$$

Next, for $t \in [t_0, t_0 + h^*/2]$, Equation (4.1) can be written as

$$x'(t) = a(t)f(x(t)) + \sum_{i=1}^{n} b_i(t)f(\psi(t-h_i)).$$

Since ψ is the known initial function we solve for $x(t)$ using the variation of parameters formula. That is,

$$x(t) = e^{\int_0^t a(s)f_1(x)ds}\Big[\psi(t_0) + \int_{t_0}^{t}\sum_{i=1}^{n} b_i(u)f(\psi(u-h_i))e^{-\int_0^t a(s)f_1(x)ds}du\Big].$$

Thus, for $t \in [t_0, t_0 + h^*/2]$ the above expression implies

$$|x(t)| \leq e^{\int_0^t a(s)f_1(x)ds}\Big[||\psi|| + \int_{t_0}^{t_0+h^*/2}\sum_{i=1}^{n} |b_i(u)||f(\psi(u-h_i))|e^{-\int_0^t a(s)f_1(x)ds}du\Big].$$

This completes the proof.

In the next section, a corollary regarding the exponential stability of the zero solution of equation (4.1) is stated .

Corollary 5.

Suppose condition (4.6) hold and $f_1(x) \geq 1$. If

$$\sum_{i=1}^{n} b_i^2(s+h_i) \geq \gamma \tag{4.20}$$

for some positive constant γ and for all $t \geq t_0$ then the zero solution of Equation (4.1) is exponentially stable.

Proof.

From inequality (4.12) in Theorem 4;

$$\begin{aligned}
|x(t)| &\leq \sqrt{2\Big(\frac{2+\delta}{\delta}\Big)V(t_0)e^{-\frac{\delta h^*}{2}\int_{t_0}^{t-\frac{h_i}{2}}[\sum_{i=1}^{n} b_i^2(s+h_i)f_1^2(x(s))]ds}} \\
&\leq \sqrt{2\Big(\frac{2+\delta}{\delta}\Big)V(t_0)e^{-\frac{\delta h^*}{2}\int_{t_0}^{t-\frac{h_i}{2}}\gamma ds}} \\
&\leq \sqrt{2\Big(\frac{2+\delta}{\delta}\Big)V(t_0)e^{-\frac{\delta h^*}{2}\int_{t_0}^{t}\gamma ds}} \\
|x(t)| &\leq \sqrt{2\Big(\frac{2+\delta}{\delta}\Big)V(t_0)e^{-\frac{\delta h^*\gamma}{2}(t-t_0)}}.
\end{aligned}$$

Thus, showing that the zero solution of Equation (4.1) is exponentially stable. This completes the proof.

Instability Criteria

In this section, a non-negative definite Lyapunov functional is used to obtained a criterion that can easily be applied to test for instability of the zero solution of Equation (4.1).

Lemma 6.

Suppose there exists a positive constant $H > h^*$ such that

$$Q(t,x) - H\sum_{i=1}^{n} b_i^2(t+h_i)f_1^2(x(t)) \geq 0. \tag{4.21}$$

If $V(t,x)$ is as defined by (4.5), then along the solutions of Equation (4.1);

$$V'(t) \geq Q(t,x)V(t). \tag{4.22}$$

Proof.

In view of condition (4.19) it is clear that $Q(t,x) > 0$ for all $t \geq 0$. Let $x(t) = x(t,t_0,\psi)$ be a solution of Equation (4.1) with $V(t,x)$ defined by (4.5). Taking the time derivative of the functional $V(t,x)$ along the solution of Equation (4.1) yields;

$$\begin{aligned} V'(t) &= 2\left[x(t) + \sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right] \\ &\quad \times\left[x'(t) + \sum_{i=1}^{n}\frac{d}{dt}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right] \\ &\quad - H\Big[\sum_{i=1}^{n} b_i^2(t+h_i)f^2(x(t)) \\ &\quad - \sum_{i=1}^{n} b_i^2(t-h_i+h_i)f^2(x(t-h_i))\Big]. \end{aligned} \tag{4.23}$$

But

$$x'(t) = Q(t,x)x(t) - \sum_{i=1}^{n} \frac{d}{dt} \int_{t-h_i}^{t} b_i(s+h_i) f(x(s))ds.$$

Hence,

$$Q(t,x)x(t) = x'(t) + \sum_{i=1}^{n} \frac{d}{dt} \int_{t-h_i}^{t} b_i(s+h_i) f(x(s))ds.$$

Thus, Equation (4.21) becomes

$$\begin{aligned}
V'(t) &= 2\left[x(t) + \sum_{i=1}^{n} \int_{t-h_i}^{t} b_i(s+h_i) f(x(s))ds\right] Q(t,x)x(t) \\
&\quad - H\sum_{i=1}^{n} b_i^2(t+h_i) f^2(x(t)) \\
&\quad + H\sum_{i=1}^{n} b_i^2(t) f^2(x(t-h_i)) \\
&= 2\left[x(t) + \sum_{i=1}^{n} \int_{t-h_i}^{t} b_i(s+h_i) f(x(s))ds\right] Q(t,x)x(t) \\
&\quad - H\sum_{i=1}^{n} b_i^2(t+h_i) f^2(x(t)) \\
&\quad + H\sum_{i=1}^{n} b_i^2(t) f^2(x(t-h_i)) \\
&= 2Q(t,x)x(t)\left[x(t) + \sum_{i=1}^{n} \int_{t-h_i}^{t} b_i(s+h_i) f(x(s))ds\right] \\
&\quad - H\sum_{i=1}^{n} b_i^2(t+h_i) f^2(x(t)) \\
&\quad + H\sum_{i=1}^{n} b_i^2(t) f^2(x(t-h_i))
\end{aligned}$$

$$
\begin{aligned}
&= 2Q(t,x)x^2(t)) \\
&\quad + 2Q(t,x)x(t)\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds \\
&\quad - H\sum_{i=1}^{n} b_i^2(t+h_i)f^2(x(t)) \\
&\quad + H\sum_{i=1}^{n} b_i^2(t)f^2(x(t-h_i)) \\
&= Q(t,x)x^2(t) \\
&\quad + 2Q(t,x)x(t)\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds \\
&\quad - H\sum_{i=1}^{n} b_i^2(t+h_i)f^2(x(t)) \\
&\quad + H\sum_{i=1}^{n} b_i^2(t)f^2(x(t-h_i)) \\
&\quad + Q(t,x)x(t) \\
&= Q(t,x)x^2(t) \\
&\quad + 2Q(t,x)x(t)\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds \\
&\quad - H\sum_{i=1}^{n} b_i^2(t+h_i)f_1^2(x(t))x^2(t) \\
&\quad + H\sum_{i=1}^{n} b_i^2(t)f_1^2(x(t-h_i))x^2(t-h_i) \\
&\quad + Q(t,x)x^2(t)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & Q(t,x)\left[x^2(t) + 2x(t)\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right] \\
& - H\sum_{i=1}^{n} b_i^2(t+h_i)f_1^2(x(t))x^2(t) \\
& + H\sum_{i=1}^{n} b_i^2(t)f_1^2(x(t-h_i))x^2(t-h_i) \\
& + Q(t,x)x^2(t) \\
= \quad & Q(t,x)\Big[x^2(t) + 2x(t)\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds \\
& + \left(\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right)^2 \\
& - \left(\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right)^2\Big] \\
& + Q(t,x)H\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i^2(s+h_i)f^2(x(s))ds \\
& - Q(t,x)H\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i^2(s+h_i)f^2(x(s))ds \\
& - H\sum_{i=1}^{n} b_i^2(t+h_i)f_i^2(x(t))x^2(t) \\
& + H\sum_{i=1}^{n} b_i^2(t)f_i^2(x(t-h_i))x^2(t-h_i) \\
& + Q(t,x)x^2(t)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & Q(t,x)\Big[x^2(t) + 2x(t)\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds \\
& + \Big(\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\Big)^2\Big] \\
& + Q(t,x)H\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i^2(s+h_i)f^2(x(s))ds \\
& - Q(t,x)H\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i^2(s+h_i)f^2(x(s))ds \\
& - Q(t,x)\left(\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right)^2 \\
& + H\sum_{i=1}^{n} b_i^2(t)f_i^2(x(t-h_i))x^2(t-h_i) \\
& - H\sum_{i=1}^{n} b_i^2(t+h_i)f_i^2(x(t))x^2(t) \\
& + Q(t,x)x^2(t) \\
= \quad & Q(t,x)\Bigg[\left(x(t) + \sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right)^2 \\
& - H\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i^2(s+h_i)f^2(x(s))ds\Bigg] \\
& + Q(t,x)H\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i^2(s+h_i)f^2(x(s))ds \\
& - Q(t,x)\Big(\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\Big)^2 \\
& + H\sum_{i=1}^{n} b_i^2(t)f_i^2(x(t-h_i))x^2(t-h_i) \\
& + Q(t,x)x^2(t) \\
& - H\sum_{i=1}^{n} b_i^2(t+h_i)f_i^2(x(t))x^2(t)
\end{aligned}
$$

$$
\begin{aligned}
&= Q(t,x)V(t) \\
&\quad + Q(t,x)H\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i^2(s+h_i)f^2(x(s))ds \\
&\quad - Q(t,x)\Big(\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\Big)^2 \\
&\quad + H\sum_{i=1}^{n} b_i^2(t)f_i^2(x(t-h_i))x^2(t-h_i) \\
&\quad + Q(t,x)x^2(t) \\
&\quad - H\sum_{i=1}^{n} b_i^2(t+h_i)f_i^2(x(t))x^2(t) \\
&= Q(t,x)V(t) \\
&\quad + Q(t,x)\Big[H\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i^2(s+h_i)f^2(x(s))ds \\
&\quad - \Big(\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\Big)^2\Big] \\
&\quad + \Big[Q(t,x) - H\sum_{i=1}^{n} b_i^2(t+h_i)f_i^2(x(t))\Big]x^2(t) \\
&\quad + H\sum_{i=1}^{n} b_i^2(t)f_i^2(x(t-h_i))x^2(t-h_i) \\
&\geq Q(t,x)V(t). \qquad (4.24)
\end{aligned}
$$

where condition (4.19) is used and the fact that by Holder's inequality,

$$h^* \sum_{i=1}^{n} \int_{t-h_i}^{t} b^2(s+h_i) f^2(x(s)) ds - \left(\sum_{i=1}^{n} \int_{t-h_i}^{t} b(s+h_i) f^2(x(s)) ds \right)^2 \geq 0.$$

This completes the proof.

In Theorem 7, conditions for instability for the zero solution of Equation (4.1) is provided.

Theorem 7.

Suppose that condition (4.19) hold and $f_1(x) \geq 1$. Then the zero solution of Equation (4.1) is unstable, provided that

$$\sum_{i=1}^{n} \int_{t_0}^{\infty} b_i^2(s+h_i) ds = \infty.$$

Proof.

Integrating inequality (4.23) from t_0 to t, yields;

$$\begin{aligned} \ln V(s)|_{t_0}^{t} &\geq \int_{t_0}^{t} Q(s, x(s)) ds \\ \implies \ln V(t) - \ln V(t_0) &\geq \int_{t_0}^{t} Q(s, x(s)) ds \\ \implies \ln \frac{V(t)}{V(t_0)} &\geq \int_{t_0}^{t} Q(s, x(s)) ds \\ \implies \frac{V(t)}{V(t_0)} &\geq e^{\int_{t_0}^{t} Q(s, x(s)) ds} \\ \implies V(t) &\geq V(t_0) e^{\int_{t_0}^{t} Q(s, x(s)) ds}. \end{aligned}$$

Thus

$$V(t) \geq V(t_0)e^{\int_{t_0}^{t}[a(s)f_1(x(s))+\sum_{i=1}^{n}b_i(s+h_i)f_1(x(s))]ds}. \quad (4.25)$$

With $V(t)$ given by (4.5), yields

$$\begin{aligned} V(t) &= x^2(t) \\ &+ x(t)\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds \\ &+ x(t)\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s)) \\ &+ \left(\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right)^2 \\ &- H\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i^2(s+h_i)f^2(x(s))ds \\ &= x^2(t) \\ &+ 2x(t)\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds \\ &+ \left(\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right)^2 \\ &- H\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i^2(s+h_i)f^2(x(s))ds. \quad (4.26) \end{aligned}$$

Let $\beta = H - h^*$. Then

$$\left(\frac{\sqrt{h}}{\sqrt{\beta}}a - \frac{\sqrt{\beta}}{\sqrt{h}}b\right)^2 \geq 0$$

$$\Longrightarrow \frac{h}{\beta}a^2 - ab - ab + \frac{\beta}{h}b^2 \geq 0$$

$$\Longrightarrow \frac{h}{\beta}a^2 + \frac{\beta}{h}b^2 - 2ab \geq 0$$

$$\Longrightarrow \frac{h}{\beta}a^2 + \frac{\beta}{h}b^2 \geq 2ab$$

$$\Longrightarrow 2ab \leq \frac{h}{\beta}a^2 + \frac{\beta}{h}b^2. \tag{4.27}$$

Using inequality (4.25), yields;

$$2x(t)\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds$$

$$\leq 2|x(t)||\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds|$$

$$\leq \frac{h_i}{\beta}x^2(t)$$

$$+\frac{\beta}{h_i}\left(\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right)^2. \tag{4.28}$$

But

$$\left(\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right)^2$$

$$\leq h_i\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i^2(s+h_i)f^2(x(s))ds.$$

Therefore, inequality (4.26), becomes

$$2x(t)\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds$$

$$\leq \frac{h^*}{\beta}x^2(t)$$

$$+\frac{\beta}{h_i}h_i\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i^2(s+h_i)f^2(x(s))ds$$

$$\leq \frac{h^*}{\beta}x^2(t)$$

$$+\beta\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i^2(s+h_i)f^2(x(s))ds. \qquad (4.29)$$

Substituting (4.27) into (4.24), yields;

$$V(t) \leq x^2(t)$$

$$+\frac{h^*}{\beta}x^2(t)$$

$$+\beta\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i^2(s+h_i)f^2(x(s))ds$$

$$+\left(\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i(s+h_i)f(x(s))ds\right)^2$$

$$-H\sum_{i=1}^{n}\int_{t-h_i}^{t} b_i^2(s+h_i)f^2(x(s))ds \qquad (4.30)$$

$$
\begin{aligned}
&\leq x^2(t) \\
&\quad + \frac{h^*}{\beta} x^2(t) \\
&\quad + \beta \sum_{i=1}^{n} \int_{t-h_i}^{t} b_i^2(s+h_i) f^2(x(s)) ds \\
&\quad + h^* \sum_{i=1}^{n} \int_{t-h_i}^{t} b_i^2(s+h_i) f^2(x(s)) ds \\
&\quad - H \sum_{i=1}^{n} \int_{t-h_i}^{t} b_i^2(s+h_i) f^2(x(s)) ds \\
&\leq x^2(t) \\
&\quad + \frac{h^*}{\beta} x^2(t) \\
&\quad + (\beta + h^* - H) \sum_{i=1}^{n} \int_{t-h_i}^{t} b_i^2(s+h_i) f^2(x(s)) ds \\
&\leq x^2(t) \\
&\quad + \frac{h^*}{\beta} x^2(t) \\
&\quad + (H - h^* + h^* - H) \sum_{i=1}^{n} \int_{t-h_i}^{t} b_i^2(s+h_i) f^2(x(s)) ds \\
&= x^2(t) + \frac{h^*}{\beta} x^2(t) \\
&= \frac{\beta + h^*}{\beta} x^2(t) \\
&= \frac{H}{H - h^*} x^2(t). \qquad (4.31)
\end{aligned}
$$

Using inequalities, (4.18), (4.23) and (4.28), yields;

$$
\begin{aligned}
\frac{H}{H-h^*}x^2(t) &\geq V(t_0)e^{\int_{t_0}^{t}[a(s)f_1(x(s))+\sum_{i=1}^{n}b_i(s+h_i)f_1(x(s))]ds} \\
|x(t)|^2 &= \frac{H-h^*}{H}V(t_0)e^{\int_{t_0}^{t}[a(s)f_1(x(s))+\sum_{i=1}^{n}b_i(s+h_i)f_1(x(s))]ds} \\
&\geq \frac{H-h^*}{H}V(t_0)e^{H\int_{t_0}^{t}\sum_{i=1}^{n}b_i^2(s+h_i)f_1^2(x(s))ds} \\
|x(t)| &\geq \sqrt{\frac{H-h^*}{H}V(t_0)e^{H\int_{t_0}^{t}\sum_{i=1}^{n}b_i^2(s+h_i)f_1^2(x(s))ds}} \\
&\geq \sqrt{\frac{H-h^*}{H}}V^{1/2}(t_0)e^{\frac{H}{2}\int_{t_0}^{t}\sum_{i=1}^{n}b_i^2(s+h_i)f_1^2(x(s))ds} \\
&\geq \sqrt{\frac{H-h^*}{H}}V^{1/2}(t_0)e^{\frac{H}{2}\int_{t_0}^{t}\sum_{i=1}^{n}b_i^2(s+h_i)ds} \\
&\geq \sqrt{\frac{H-h^*}{H}}V^{1/2}(t_0)e^{\frac{H}{2}\int_{t_0}^{t}\sum_{i=1}^{n}b_i^2(s+h_i)ds} \to \infty \quad as \quad t \to \infty.
\end{aligned}
$$

This completes the proof.

Chapter Summary

In this chapter, sufficient conditions for exponential decay of solution to zero of the nonlinear delay ordinary differential equation was obtained by means of Lyapunov functionals. The Lyapunov functionals constructed were used to deduce inequalities regarding the solutions of the nonlinear delay ordinary differential equation from which the exponential stability and instability of the nonlinear delay ordinary differential equation were obtained.

CHAPTER FIVE

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

Overview

This chapter provides the summary, conclusions as well as recommendation of the study. The summary briefly presents an overview of the research problem, objectives, method and results of the study. The conclusions encompasses the overall results of the study with respect to the research objectives of the study. Some recommendation based on the work done is also presented.

Summary

This study generally determined the sufficient conditions for exponential stability and instability of solutions of nonlinear delay ordinary differential equations. Specifically, the study was carried out to; construct a suitable Lyapunov functional that yields results concerning the exponential stability of the zero solution of nonlinear delay ordinary differential equations, obtain sufficient conditions for exponential stability of the zero solution of nonlinear delay ordinary differential equations and to obtain an instability criteria of the zero solution of nonlinear delay ordinary differential equations.

The Lyapunov's direct method was employed in the study. The direct method was via a special function called Lyapunov function. The Lyapunov functionals constructed were used to obtain inequalities regarding the solutions of the nonlinear ODEs from which the exponential stability of the zero solution was deduced. Also, instability criteria of the zero solution of nonlinear ODEs by means of Lyapunov functional was provided.

Conclusions

Suitable Lyapunov functionals that can be used to deduce exponential stability and instability for nonlinear ODEs with delay have been obtained.

Sufficient conditions for the zero solution of nonlinear ODEs with delay to be exponentially stable have been obtained.

A criteria for instability for nonlinear ODEs with delay has been established.

Recommendations

This problem can fruitfully be studied again by making $n = 1$, to reduce the ordinary differential equation.

REFERENCES

Ahmad, S., & Rao, M. R. M. (1999). Theory of ordinary differential equations. *With applications in biology and engineering. Affiliated East-West Press Pvt. Ltd. , New Delhi.*

Burton, T. (2003). Stability by fixed point theory or liapunov's theory: A comparison. *Fixed point theory*, *4*(1), 15–32.

Busenberg, S. N., & Cooke, K. L. (1984). Stability conditions for nonautonomous delay differential equations. *Quarterly of applied mathematics*, *42*(3), 295–306.

Cable, M., & Raffoul, Y. (2011). Lyapunov functionals and exponential stability and instability in multi delay differential equations. *International Journal of Mathematical Sciences and Applications*, *1*(2), 961–970.

Hale, J. (1977). *Theory of functional differential equations.* Springer-Verlag, New York.

Hatvani, L. (1997). Annulus arqument in the stability theory for functional differential equations. *Differential and Integral Equations*, *10*(5), 975–1002.

Ince, E. L. (1956). *Ordinary differential equations dover.* New York.

Knyazhishche, L. B., & Shcheglov, V. A. (1998). On the sign definiteness of liapunov functionals and stability of a linear delay equation. *Electronic Journal of Qualitative Theory of Differential Equations*, *1998*(8), 1–13.

La Salle, J., & Lefschetz, S. (1961). *Stability analysis by lyapunov's direct methods.* Academic, New York.

Lyapunov, A. M. (1892). The general problem of the stability of motion. *Annals of Mathematics Studies*, *17*.

Nagy, G. (2019). *Ordinary differential equations.* Michigan State University.

Parks, P. C. (1992). Am lyapunov's stability theory-100 years on. *IMA journal of Mathematical Control and Information*, *9*(4), 275–303.

Poincaré, H., & Magini, R. (1899). The new methods of celestial method é. *Il Nuovo Cimento(1895-1900)*, *10*(1), 128–130.

Sasser, J. E. (2005). *History of ordinary differential equations: The first hundred years* (Vol. 1). Citeseer.

Wang, T. (2004). Inequalities and stability for a linear scalar functional differential equation. *Journal of Mathematical Analysis and Applications*, *298*(1), 33–44.