UNIVERSITY OF CAPE COAST

INDIFFERENCE PRICING UNDER EPSTEIN-ZIN UTILITY: MAXIMUM PRINCIPLE

BY

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DECLARATION

Candidate's Declaration

I hereby declare that this thesis is the result of my own original research and that no part of it has been presented for another degree in this university or elsewhere.

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Supervisors' Declaration

We hereby declare that the preparation and presentation of the thesis were supervised in accordance with the guidelines on supervision of thesis laid down by the University of Cape Coast.

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ABSTRACT

The problem of dealing with claims on asset which is not traded has attracted a lot of interests recently. Naturally the approach consist of choosing a related traded asset or index to use for hedging purposes. In this thesis, we consider a model for which the non-traded asset is driven by a Ornstein-Uhlenbeck process. We introduce it into a consumption-investment problem with factor model under recursive utility of Epstein-Zin type. Due to the second Brownian motion, we are working in an incomplete market in which the objective of an agent is pricing and hedging this random payoff. Making use of the maximum principle method, we solve our forward-backward system and find the optimal consumption and investment strategies and a relation given the indifference price. Since a closed form formula for the indifference price is not obtained, a finite difference method is applied to estimate its value. For numerical purpose, we consider a one period model. We perform some numerical analysis on the optimal investment in presence of a claim and on the indifference price. In general, we observe that, for the parameters specification considered, the optimal investment becomes an increasing function with regard to initial wealth of the agent so as to be higher than its value in the no claim case. However, it is rather a decreasing function with respect to the correlation between non-traded and traded assets and is always net off the investment with zero claim. Regarding the indifference price, we observe that it increases when the traded asset becomes more and more correlated to the non-traded one. Then, analysing also the dependency of the indifference price to the risk aversion, we obtain that an agent is willing to pay less for the non-traded asset when he/she becomes less tolerant of risk. Finally, we notice from the indifference price versus the initial wealth that an agent is less willing to take on more risk.

KEY WORDS

Epstein-Zin Utility

Factor Model

Forward-Backward Stochastic Differential Equations

Indifference Pricing

Maximum Principle

Stochastic Optimal Control

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DEDICATION

To my family

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CHAPTER ONE

INTRODUCTION

Valuation of claims on assets that cannot be traded appears to be of great interest nowadays in option pricing theory. An investor expects to receive or pay out an unhedgeable claim on an asset, and must decide how to best manage this risk. The problem is to know how much should be the adjustment on the initial wealth of the investor in order to be indifferent before and after buying or selling the claim. This problem commonly known as utility indifference pricing can be seen as two stochastic optimal control problems. The first optimal investment problem considers that the agent has not taken any position on the claim whereas the second optimal investment problem assumes a buying or selling of the claim by the agent. A usual approach is to hedge the claim with the help of a traded asset correlated to the non-traded one. There is often a risk associated with the non-traded asset and that cannot be hedge perfectly, making the market incomplete. To the best of our knowledge, most of the work done are making use of the so called ”time separable utilities” especially exponential utility, power utility or logarithmic utility. In this thesis, we use the standard recursive utility of Epstein-Zin type to capture the fact that the utility of an agent at any time depends on his/her utility at all the previous time. To solve these problems the maximum principle method (Boltyanskii et al., 1960) is used.

Background to the Study

Think of:

(i) a representive of a corporation, who wishes to determine the correct price for a real option (a capital investment),

(ii) a gas-fired power-plant owner who wants to reduce the risk attached to the rising gas prices and declining power prices,

(iii) a representative of an insurance (reinsurance) company, who needs to determine the price of an insurance contract.

In each case, we want to determine the optimal price of a financial contract market consistently, by exploiting correlations between the pay off of our investment and the global stock market.

Statement of the Problem

A first approach to resolve utility indifference pricing problems has been to consider the exponential utility. But, the critical drawback of this approach is that the indifference price found does not depend on the initial wealth which makes it unrealistic (Malamud et al., 2013). Due to that limitation, several researches have been conducted now looking at the cases where the utility is either power or logarithmic. However, we observed that they were almost all focusing on time separable utilities even though it was shown that these types of utility generate a vast literature on asset pricing anomalies such as equity premium puzzle, excess volatility puzzle, credit spread puzzle and risk-free rate puzzle (Xing, 2017). Furthermore, as mentioned above, an utility indifference pricing problem is seen as a combination of two investment problems, herein we consider a model in which the dynamics of the traded asset depend on a correlated stochastic factor; such models appear to be an open problem in stochastic optimal investment models (Zariphopoulou, 2009). In this thesis, we seek to enrich the literature on option pricing by putting all together recursive utility, stochastic factor model and maximum principle. To our knowledge this is the first time indifference pricing under Epstein-Zin utility is solved for factor model and the method we use is the well known maximum principle method.

Research Objectives

The objectives of this thesis are as follow:

(i) to find the investment and consumption strategies of an agent with preferences described by a recursive utility of Epstein-Zin type,

(ii) to find the utility indifference price on a derivative written on the asset,

(iii) to numerically find the indifference price that an agent is willing to pay to hedge the claim.

Significance of the Study

Due to the fact that recursive utilities are more realistic to model preferences of an agent and also to the fact that indifference pricing has recently been applied to many incomplete market setting, the results of this thesis can be used:

(i) to help policymakers and investors (from various areas such as banks, hedge funds, insurance and reinsurance companies) to make better decisions,

(ii) by mathematicians to better understand the option pricing theory as it appears in various areas as portfolio optimisation, weather derivatives (temperature options, rainfall options) and energy contracts (commodity derivatives).

Delimitation

In this thesis, we make three general assumptions:

(i) the consumption stream is positive at each time,

(ii) we consider only self-financing portfolio-consumption pairs; that is, purchasing a new portfolio, as well as all consumption, must be financed uniquely by selling assets already in the portfolio,

(iii) the parameters specification of the recursive utility of Epstein-Zin type are both less than 1,

(iv) the factor model is observable,

(v) the price processes are all continuous.

Limitation

Due to the fact that it is only in very few cases that we can find an exact expression of the utility indifference price, we give here an approximation of its value using finite difference method making our results more difficult to interpret.

Definitions of Terms

We now introduce some notions and concepts that will be used in this thesis. For more details, the reader may consult the books by Øksendal (2003), Björk (2009) and Cohen & Elliott (2015).

Definition 1 (σ-algebra)

Let Ω be some set, and let 2^{Ω} represent its power set. Then a subset $\mathcal{F} \subseteq 2^{\Omega}$ is called a σ-algebra if it satisfies the following three properties:

(i) $\Omega \in \mathcal{F}$,

(ii) $\mathcal{F}$ is closed under complementation: If A is in $\mathcal{F}$, then so is its complement, $\Omega \setminus A \equiv A^c$,

(iii) $\mathcal{F}$ is closed under countable unions: If A_1, A_2, A_3,... are in $\mathcal{F}$, then so is $A = A_1 \cup A_2 \cup A_3 \cup ... := \bigcup_{i=1}^{\infty} A_i$.

The pair $(\Omega, \mathcal{F})$ is called a measurable space. Adding to it a probability measure $\mathbb{P}$, the triple $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space.

Definition 2 (Random variable)

Given $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space. A function $X : \Omega \to \mathbb{R}^n$ is said to be a random variable if it is $\mathcal{F}$-measurable. That is, if

$$X^{-1}(U) := \{\omega \in \Omega; X(\omega) \in U\} \in \mathcal{F}.$$

Definition 3 (Stochastic process)

A stochastic process is defined as a collection of random variables defined

on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where the random variables indexed by some set $\mathcal{T}$ take values in the same mathematical space S, which must be measurable with respect to some σ-algebra Σ.
In other words, for a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a measurable space (S, Σ), a stochastic process is a collection of S-valued random variables, which can be written as: $\{X_t : t \in \mathcal{T}\}$.
In mathematical finance, a stochastic process is used, for instance, to model the price of a stock measured on daily basis.

Definition 4 (Brownian motion)

A stochastic process B_t is called a Brownian motion if the following conditions hold.

(i) $B_0 = 0$,

(ii) The process B_t has independent increments, that is if $r < s \leq t < u$ then $B_u - B_t$ and $B_s - B_r$ are independent stochastic variables,

(iii) For $s < t$, the stochastic variable $B_t - B_s$ has the Gaussian distribution $\mathcal{N}(0, \sqrt{t-s})$, where $\sqrt{t-s}$ stands for the standard deviation,

(iv) B_t has continuous trajectories.

The Brownian motion appears to be the simplest stochastic process and used to describe the motion of a pollen grain in water.

Definition 5 (Filtration)

A filtration $\{\mathcal{F}_t\}_{t\geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a collection of sub-σ-algebras of $\mathcal{F}$ satisfying $\mathcal{F}_s \subseteq \mathcal{F}_t$ whenever $s \leq t$.
The idea is that $\mathcal{F}_t$ represents the set of events observable (or information known) at time t. The probability space taken together with the filtration $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ is called a *filtered probability space.*

Definition 6 (Adapted process)

A stochastic process X_t is said to be adapted to the filtration $\{\mathcal{F}_t\}_{t\in\mathcal{T}}$

if X_t is $\mathcal{F}_t$-measurable for each $t \in \mathcal{T}$.

For instance if X_t represents the stock price at time t, saying that X_t is adapted to the filtration $\{\mathcal{F}_t\}_{t\in\mathcal{T}}$ means we know all the values of the stock price up to time t.

Definition 7 (Stochastic differential equation)

A stochastic differential equation (SDE) is a differential equation in which one or more of the terms is a stochastic process. The general form of a k-dimensional SDE is

$$dX_t = f(t, X_t)dt + g(t, X_t)dB_t, \quad X_0 = x,$$

where $f : [0, T] \times \mathbb{R}^k \to \mathbb{R}^k$, $g : [0, T] \times \mathbb{R}^k \to \mathbb{R}^{k\times n}$ are known and represent the drift and diffusion coefficient and B_t is an n-dimensional Brownian motion.

Definition 8 (Backward stochastic differential equation)

A Backward stochastic differential equation (BSDE) is a new class of stochastic differential equations, whose value is prescribed at the terminal time T. The general form of a k-dimensional BSDE is

$$\begin{cases} dY_t = -f(t, Y_t, Z_t)dt + Z_t dB_t \\ Y_T = \xi, \end{cases} \tag{1.1}$$

where $f : [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k\times n} \to \mathbb{R}^k$ is called generator, $\{Y_t,\ t \in [0, T]\}$ is a continuous $\mathbb{R}^k$-valued adapted process, B_t is an n-dimensional Brownian motion , $\{Z_t,\ t \in [0, T]\}$ is an $\mathbb{R}^{k\times n}$-valued predictable process and $\xi \in \mathcal{L}^2(\mathbb{R}^k)$.

Definition 9 (Solution of a BSDE)

A solution to the backward stochastic differential equation (1.1) is a pair $(Y, Z) \in \mathbb{S}^2(0, T) \times \mathbb{H}^2(0, T)^d$ satisfying

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s \mathring{d}W_s, \quad 0 \leq t \leq T,$$

where $\mathbb{S}^2(0,T)$ is the set of real-valued progressively measurable processes Y such that

$$\mathbb{E}[\sup_{0\leq t\leq T}|Y_t|^2] < \infty,$$

and $\mathbb{H}^2(0,T)$ denotes the set of $\mathbb{R}^d$-valued progressively measurable processes Z such that

$$\mathbb{E}\Big[\int_0^T |Z_t|^2 \mathrm{d}t\Big] < \infty.$$

Theorem 1 (Uniqueness result). *Given a pair (ξ, f) satisfying conditions (i) and (ii), there exists a unique solution (Y,Z) to the backward stochastic differential equation (1.1).*
Where the conditions are given by

(i) $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}, \mathbb{R})$,

(ii) $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ *such that*

- $f(.,t,y,z)$ *written for simplicity* $f(t,y,z)$ *is progressively measurable for all* y, z,
- $f(t,0,0) \in \mathbb{H}^2(0,T)$,
- f *satisfies a uniform Lipschitz condition in* (y,z), *that is, there exists a constant* C_f *such that*

$$|f(t,y_1,z_1) - f(t,y_2,z_2)| \leq C_f(|y_1-y_2| + |z_1-z_2|),$$

$$\textit{for all } y_1,y_2,z_1,z_2, \quad \mathrm{d}t\otimes\mathrm{d}\mathbb{P} \textit{ almost everywhere.}$$

Definition 10 (Conditional expectation)

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a random variable X, the conditional expectation of X given $\mathcal{H} \subset \mathcal{F}$ a σ-algebra, denoted by $\mathbb{E}[X|\mathcal{H}]$, is the unique function from Ω to $\mathbb{R}^n$ satisfying:

(i) $\mathbb{E}[X|\mathcal{F}]$ is $\mathcal{F}$-measurable,

(ii) $\int_H \mathbb{E}[X|\mathcal{F}]d\mathring{\mathbb{P}} = \int_H X d\mathring{\mathbb{P}}$, for all $H \in \mathcal{H}$.

We have the following properties

(i) If Y_t and Z_t are stochastic variables and Z_t is $\mathcal{F}_t$-measurable, then

$$\mathbb{E}[Z_t \cdot Y_t|\mathcal{F}_t] = Z_t \cdot \mathbb{E}[Y_t|\mathcal{F}_t].$$

(ii) If Y_t is a stochastic variable and $s < t$, then

$$\mathbb{E}[\mathbb{E}[Y|\mathcal{F}_t]|\mathcal{F}_s] = \mathbb{E}[Y|\mathcal{F}_s].$$

Definition 11 (Martingale)

Working with a filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, a stochastic process X_t is called a $(\mathcal{F}_t, \mathbb{P})$-martingale if the following conditions hold

(i) X_t is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

(ii) For all t, $\mathbb{E}[|X_t|] < \infty$.

(iii) For all s and t with $s \leq t$ the following relation holds

$$X_s = \mathbb{E}[X_t|\mathcal{F}_s].$$

For all s and t with $s \leq t$, a process X_t satisfying (i), (ii) and the following inequality

$$X_s \leq \mathbb{E}[X_t|\mathcal{F}_s]$$

is called a submartingale, and a process satisfying (i), (ii) and the following inequality

$$X_s \geq \mathbb{E}[X_t|\mathcal{F}_s]$$

is called a supermartingale.

Theorem 2 (Martingale representation theorem). *Let $\{B_t, 0 \leq t \leq T\}$ be a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\{\mathcal{F}_t\}_{t\in[0,T]}$ be the filtration generated by this Brownian motion.*

Let $\{X_t, 0 \leq t \leq T\}$ be a martingale (under $\mathbb{P}$) relative to this filtration. Then there is an adapted process $\{Z_t, 0 \leq t \leq T\}$, such that

$$X_t = X_0 + \int_0^t Z_u dB_u, \quad 0 \leq t \leq T.$$

Theorem 3 (Girsanov's theorem). *Suppose that the filtration $\{\mathcal{F}_t\}_{t\in[0,\infty)}$ is the usual augmentation of the natural filtration generated by a Brownian motion $\{B_t\}_{t\in[0,\infty)}$.*

(i) Let $\mathbb{Q}$ equivalent to $\mathbb{P}$ be a probability measure on $\mathcal{F}$ and let $\{Z_t\}_{t\in[0,\infty)}$ be the corresponding density process, that is, $Z_t = \mathbb{E}\Big[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_t\Big]$. Then, there exists a predictable process $\{\theta_t\}_{t\in[0,\infty)}$ such that $Z = \varepsilon(\int_0^{\cdot} \theta_u dB_u)$ and

$$B_t - \int_0^t \theta_u du \text{ is a } \mathbb{Q}\text{-Brownian motion.}$$

(ii) Conversely, let the process $\{\theta_t\}_{t\in[0,\infty)}$ have the property that the process $Z = \varepsilon(\int_0^{\cdot} \theta_u dB_u)$ is a uniformly-integrable martingale with $Z_\infty > 0$ a.s. For any probability $\mathbb{Q} \sim \mathbb{P}$ such that $\mathbb{E}\Big[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_\infty\Big] = Z_\infty$,

$$B_t - \int_0^t \theta_u du, \; t \geq 0,$$

is a $\mathbb{Q}$-Brownian motion.

Theorem 4 (Itô's formula). *Assume that the dynamics of the process X_t is given by the following stochastic differential equation*

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t,$$

where μ_t and σ_t are adapted processes, and let f be a $C^{1,2}$-function. Define the process Z_t by $Z_t = f(t, X_t)$. Then Z_t has a stochastic differential given by

$$df(t, X_t) = \frac{\partial f}{\partial t}(t, X_t)dt + \frac{\partial f}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X_t)(dX_t)^2,$$

where we use the following relations

$$\begin{cases} (dt)^2 = 0, \\ dt \cdot dB_t = 0, \\ (dB_t)^2 = dt. \end{cases}$$

Definition 12 (Feynman-Kăc formula)

Let $f \in \mathcal{C}_0^2(\mathbb{R}^n)$ and $q \in \mathcal{C}(\mathbb{R}^n)$. Assume that q is lower bounded.

(i) Put

$$v(t, x) = \mathbb{E}^x\Big[\exp\Big(-\int_0^t q(X_S)\mathrm{d}s\Big)f(X_t)\Big]. \tag{1.2}$$

Then

$$\frac{\partial v}{\partial t} = Av - qv; \quad t > 0, \ x \in \mathbb{R}^n \tag{1.3}$$

$$v(0, x) = f(x); \quad x \in \mathbb{R}^n \tag{1.4}$$

(ii) Moreover, if $w(t, x) \in \mathcal{C}^{1,2}(\mathbb{R} \times \mathbb{R}^n)$ is bounded on $\mathbb{K} \times \mathbb{R}^n$ for each compact $\mathbb{K} \subset \mathbb{R}$ and w solves Equations (1.3), (1.4), then $w(t, x) = v(t, x)$, given by Equation (1.2).

Definition 13 (Lebesgue integrable space)

We have the following properties

(i) We say that the process X_t belongs to $\mathcal{L}^2[a, b]$ if the following conditions are satisfied

- $\int_a^b \mathbb{E}[X_s^2]ds < \infty$.

- The process X_t is adapted to the $\mathcal{F}_t$-filtration.

(ii) We say that the process X_t belongs to $\mathcal{L}^2$ if $X \in \mathcal{L}^2[0,t]$ for all $t > 0$.

Definition 14 (Lebesgue's dominated convergence theorem)

Let $\{f_n\}$ be a sequence of real-valued measurable functions on a measurable space (S, Σ, μ). Suppose that the sequence converges pointwise to a function f and it is dominated by some integrable function g in the sense that

$$|f_n(x)| \leq g(x)$$

for all numbers n in the index set of the sequence and all points $x \in S$. Then f is integrable and

$$\lim_{n\to\infty} \int_S |f_n - f| d\mu = 0$$

which also implies

$$\lim_{n\to\infty} \int_S f_n d\mu = \int_S f d\mu.$$

By g integrable we mean in the sense of Lebesgue, that is

$$\int_S |g| d\mu < \infty.$$

Definition 15 (Differentiability)

Let U be an open subset of a Banach space χ and let $G : U \to \mathbb{R}$.

(i) Saying that G has a "directional derivative" at $x \in U$ in the direction $y \in \chi$ means

$$D_y G(x) := \lim_{\epsilon\to 0} \frac{1}{\epsilon}(G(x+\epsilon y) - G(x)) \quad \text{exists.}$$

(ii) Saying that G is "Frechet differentiable" at $x \in U$ means that there exists a linear map

$$\mathcal{L} : \chi \to \mathbb{R}$$

so that we have

$$\lim_{\substack{\delta \in \chi \\ \delta \to 0}} \frac{1}{||\delta||} |G(x+\delta) - G(x) - \mathcal{L}(\delta)| = 0.$$

Then $\mathcal{L}$ is called the "Fréchet derivative" of G at x.

The notation for the "Fréchet derivative" of G at x is

$$\mathcal{L} := \nabla_x G.$$

(iii) All Fréchet differentiable map G has a directional derivative in all directions $y \in \chi$ and

$$D_y G(x) = \nabla_x G(y).$$

Definition 16 (Space of square integrable process)

Given a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$.

(i) We say that the process X_t belongs to the class $\mathcal{L}^2[a,b]$ if the following conditions are satisfied:

- $\int_0^T \mathbb{E}[X_s^2] \mathrm{d}s$
- The process X_t is adapted to the $(\mathcal{F}_t, \mathbb{P})$-filtration

(ii) We say that the process X_t belongs to the class $\mathcal{L}^2$ if X_t belongs to $\mathcal{L}^2[0,t]$ for all $t > 0$.

Definition 17 (Financial market)

A financial market describe a marketplace where buyers and sellers participate in the trade of financial instruments or assets such currencies, bonds, derivatives, etc.

A market in which all the claims are replicable is said to be complete otherwise it is incomplete.

Definition 18 (Option market)

In finance, an option is a contract which gives the owner of the option the right, but not the obligation, to buy or sell an underlying asset or instrument at a specified strike price on a specified date, depending on the form of the option.

Definition 19 (Strategy)

Let the N-dimensional price process $\{S_t;\ t \geq 0\}$ be given.

(i) A *portfolio strategy* is any $\mathcal{F}_t^S$-adapted N-dimensional process $\{h_t;\ t \geq 0\}$.

(ii) A *consumption process* is any $\mathcal{F}_t^S$-adapted one dimensional process $\{c_t;\ t \geq 0\}$.

A self-financing portfolio–consumption pair is a portfolio for which, apart of course from the consumption, there is no exogenous infusion or withdrawal of money.

Organisation of the Study

This thesis is structured as follows. This chapter presents the background of the research including its objectives and some basics definitions and theorems. Chapter Two presents a literature review on utility indifference pricing problem along with some key concepts around which the study is built.

In Chapter Three, we first state and prove a sufficient maximum principle for general forward-backward differential equations assuming that the Hamiltonian is concave. We then apply it to study two investment-consumption problems with and without claim. In latter case, it is worth mentioning that the claim is written on a non traded asset and the investor wishes to know the optimal investment consumption if he/she purchases a number λ of non-traded asset. Notice that in these cases, one needs to transform the generator of the recursive utility in order to obtain the concavity assumption.

Chapter Four talks about the indifference bid price for an agent who buys an option on the non-traded asset. Unlike in the case of classical utility it is not possible to derive an explicit solution for the indifference price. As such, we use the finite difference method to approximate its value at each time. Since it is numerically demanding we use a one period model, that is, we consider only two periods $t = 0$ and $t = T$. We end this chapter by examining the sensitivity of the price with respect to some of its parameters; that is, we examine the impact that a small change on the price parameter has on the price itself. We also, perform the same sensitivity analysis for the new hedging strategy when purchasing the claim.

In Chapter Five, summary of the thesis, conclusion and recommendations are given.

Chapter Summary

This chapter introduced the thesis, by first given the motivation for studying the problem contain in this thesis. We then moved on to state the related problem, announce our research objectives and the importance of our results both practically and mathematically. Also, in addition to the definition of some terms, we gave the delimitation and the limitation of this thesis. We concluded the chapter by describing the structure of this thesis.

CHAPTER TWO

LITERATURE REVIEW

Introduction

An incomplete financial market is by definition a market in which it is not possible to replicate all claims. In this case, the writing of a claim involves a real risk, and its pricing and hedging can only be done with regard to the agent's preferences towards such risk. A classical approach used by economists is to specify the agent's utility function which represents a mathematical concept that measures preferences of an agent over a set of goods and services. A first use of this approach in incomplete market was made by Hodges & Neuberger (1989) and Davis (1997). Hodges & Neuberger (1989) made use of it in the context of option pricing under transaction costs in the Black-Scholes model. In this methodology, the agent seeks to find the amount of money that his/her initial wealth should vary in order to get the same maximal expected utility before and after buying or selling claims. Other applications of utility-based hedging have been to stochastic volatility models (Sircar & Zariphopoulou, 2004) and to the pricing of volatility derivatives (Grasselli & Hurd, 2007). The general theory of utility-based pricing, with a particular emphasis on relations with the dual to the primal utility maximisation problem, has been studied by Delbaen et al. (2002), by Becherer (2004), and by Hugonnier et al. (2005).

Real options is another area where claims on non-traded assets appear frequently, as can be seen in Dunbar (2000) and the book by Dixit et al. (1994). As examples of real options problems we have extraction rights to an oil reserve or the option to start up a research and development (R&D) venture.

He & Pages (1993) introduced the study of problems assuming that the volatility is stochastic. Since then in the literature, Cuoco (1997) and El Karoui & Jeanblanc-Picqué (1998) assume both incomes covered by assets but with liquidity constraint. Duffie & Zariphopoulou (1993), Duffie et al. (1997) and Koo (1998) worked on infinite horizon optimal consumption-investment with stochastic income and risky asset imperfectly correlated. Using a Markov chain approximation Munk (2000) gave numerical solutions. Some simple examples were considered by Duffie & Jackson (1990) and Svensson & Werner (1993) and explicit solutions under quadratic utility were found by Duffie & Richardson (1991).

Under constant relative risk aversion (CRRA), a related general problem was studied by Malamud et al. (2013) and Zariphopoulou (2001). In the latter, she obtained a non-linear partial differential equation and performed a transformation to reduce it to a linear one. She introduced unhedgeable risks by considering a dependency between the coefficients of the diffusion price process for a traded asset and a "stochastic factor" correlated with the asset price. Henderson (2002) and Henderson & Hobson (2002) differ from Zariphopoulou (2001) by directly pricing a claim on a non-traded asset by including it in the utility from wealth. Socgnia & Pamen (2018) studied a pricing and hedging problem of a commodity derivative at a given location for a not observable convenience yield. They considered an optimal control for a three-factor stochastic factor model assuming that one of the factors is not observed. With the use of the classical filtering technique (Bensoussan, 2004) they transformed the partial observation control problem for stochastic differential equation (SDE) to a full observation control problem for stochastic partial differential equation (SPDE). In this chapter, we discuss some of the studies related to this thesis. We explore some key concepts around which the study is built. We end up by a summary of the main points that have emerged from this literature review and their implications for the development of this thesis.

Stochastic Optimal Control

Stochastic optimal control problems regularly emerge in a variety of settings such as economics, ecology, engineering, finance, etc.(see Yong & Zhou

(1999) and Pham (2009)), where a criterion (defined as a functional) that measures the performance of the decisions is optimised (maximise or minimise) by choosing the inputs to a stochastic differential equation. In finance, a common problem concerns a utility maximization in consumption-investment models over a fixed time-horizon when the current utility depends also on the wealth process. These utilities, describing preferences depend in general on two parameters, risk aversion and elasticity of intertemporal substitution (EIS). Whereas EIS regulates an agent's willingness to substitute consumption over time, risk aversion measures an agent's attitude toward risk. Nevertheless, frequently used time-separable utilities assume EIS and risk aversion to be reciprocal, generating a vast literature on asset pricing anomalies such as equity premium puzzle, excess volatility puzzle, credit spread puzzle and risk-free rate puzzle. To tackle the latter anomalies, untying EIS and risk aversion is necessary and this can be done by using recursive utility of Epstein-Zin type and their continuous-time analogues.

By consumption-investment problem we mean a market in which an agent with a positive initial wealth can invest in risk-free and risky assets and at the same time decides to consume a part of the new wealth generated. The objective of an agent facing that problem is to maximize his/her overall utility or preference, during the time-horizon, from consumption and terminal wealth. In that case, optimal strategies take the corresponding values of strategies for overall maximum utility. Backward stochastic differential equations are used here since the agent's wealth varies stochastically with time and we assume that his/her objective is to reach to a certain terminal wealth.

Epstein-Zin Utility

The consumption-based capital asset pricing model (CCAPM) introduced by Lucas & Robert (1978) and Breeden (1979) considered as the conventional asset pricing model in financial economics, assumes that the preferences of an agent have a time-separable von Neumann-Morgenstern representation.

However, the model has been criticised for two reasons. First, it does not perform well empirically (Duffie & Epstein, 1992). Second, this specification confounds risk aversion and elasticity of intertemporal substitutability while it would be advantageous to be able to disentangle these two conceptually different aspects of preference.

The first researches to overcome these two drawbacks of the standard model were conducted by Duffie & Epstein (1989) and Weil (1990) who introduce recursive utilities in a discrete-time setting. These utility functions not only permit a degree of separation between risk aversion and substitution, but also imply relations between asset returns and rates of consumption that match data more closely. In addition, Duffie & Epstein (1992) move on to define a continuous-time form of recursive utility.

Factor Model

A stochastic factor model is a model in which the coefficients depend on a random external economic factor. It can be considered as the simplest and most direct extension of the celebrated Merton model in which stock dynamics are taken to be lognormal (see Merton (1969) and Merton (1971)).

Technically the factor model is driven by a Brownian motion correlated to the one that drives the underlying stock making the market incomplete. Even though little is known about the maximal expected utility as well as the form and properties of the optimal policies once the lognormality assumption is relaxed and correlation between the stock and the factor is introduced, stochastic factor models is widely used in incomplete markets situation in financial stochastic optimization (Zariphopoulou, 2009). Examples include modelling the time-varying predictability of stock returns, the volatility of stocks as well as the stochastic interest rates.

Classical Maximum Principle

One of the principal approaches in solving optimization problem consider deriving a set of necessary conditions that any optimal solution should satisfy.

For example, the use of zero-derivative condition (for the unconstrained case) or the Kuhn-Tucker condition (for the constrained case) allows us to obtain necessary conditions for an optimum of a finite-dimensional function. These necessary conditions become sufficient under some convexity conditions on the objective or constraint functions. An optimal control problem similar to an optimisation problem in infinite-dimensional spaces are difficult to solve. The maximum principle, formulated and derived by Boltyanskii et al. (1960), is truly a milestone of optimal control theory. It states that any optimal control along with optimal state trajectory must solve the so-called Hamiltonian system, which is a backward stochastic differential equation (in the classical case) or a forward-backward stochastic differential (for forward-backward systems), plus a maximum condition of a function called Hamiltonian. The significance of the maximum principle is due to two major facts: firstly, maximising the Hamiltonian is much easier than the infinite-dimensional original control problem and also there is no need to attached a Markovian property to our system.

Utility Indifference Pricing

Considering an agent going for a derivative or contingent claim offering payoff $h(X_T)$ at a future time $T > 0$. For a complete market, pricing and hedging has a unique solution. In this case we make use of the concept of replication; a portfolio in risk-free and risky asset recreates the terminal payoff of the option removing *the facto* all risk and uncertainty.

However, due to transactions cost, portfolio constraints and non-traded assets, most situations are incomplete in reality thus complete models are only approximation of it. In such situations, there is no longer a unique price neither nor prefect hedging. Nevertheless, an agent can still maximise his/her expected utility of wealth and may be able to reduce the risk due to the uncertainty of the payoff through dynamic trading. He/she would be willing to pay a certain amount today for the right to receive the claim such that he/she is no worse off in expected utility terms than he/she would have been without claim (Henderson (2002), Henderson & Hobson (2004), Carmona (2008)).

In addition to its economic justification and incorporarion of risk aversion, the utility indifference pricing carries some advantages:

(i) prices are non-linear in the number of units of claims, which contrast to prices in complete markets,

(ii) it is equivalent to the complete market price if the market is complete and the claim is replicable,

(iii) it incorporates wealth dependence; the price an agent is willing to pay could well depend on the current amount of her wealth,

(iv) it gives also an explicit identification of the hedge position; found naturally as part of the optimisation problem.

Chapter Summary

This chapter talked about some studies in relation to this thesis and explored some key concepts around which the study is built. Recently the concept of utility indifference pricing attracted lot of interests in option pricing theory. In this theory ones needs to consider, before any development, the type of the market (complete or incomplete). Thus, while in complete market pricing and hedging leads to a unique solution, in the incomplete case the price is not unique and the perfect hedging is no longer possible. Since the well known Merton problem, researchers have developed several models including factor models in order to consider the dependency of the stock on a certain external economic factor that is sometimes supposed to be observed. Technically a utility indifference pricing problem is a combination of two stochastic optimal control problems; the first in a situation of zero claim and the second assuming that an agent goes for a claim. The optimal consumption-investment problems that result rely strongly on the choice of the utility function. In contrast to the time-separable utility that was first used for such problems, a recent approach make use of recursive utility of Epstein-Zin type and their continuous-time analogues to capture the fact that an agent would like to separate his/her conception of risk aversion and intertemporal substitutability. Finally, in order to work in a more general framework (relaxing the Markovian property), the use of the maximum principle is becoming more and more frequent.

CHAPTER THREE

METHODOLOGY

Introduction

This chapter is divided into two parts. The first part presents a proof of the sufficient maximum principle for a forward-backward stochastic system. We restrict our proof in the case of a concave generator of the controlled process satisfying the backward stochastic differential equation (BSDE) of the system. In the second part, we apply the sufficient maximum principle to study two investment-consumption problems. More precisely, we wish to find an investment-consumption plan that maximises the recursive expected utility of an investor. We consider a recursive utility of Epstein-Zin type with parameters specification, risk aversion γ and elasticity of intertemporal substitution ψ, less than 1. We build for each problem the optimal consumption and investment strategies allowing us to get the optimal utility. In the first section, we study the problem assuming that the underlying traded asset S_t depends on an external factor X_t that we assume to be observed. Next, we rather consider this external factor as a non-traded asset and we introduce a claim on it. The new goal for our agent is to maximise his/her expected utility during the overall period. We consider that the agent in addition to funds generated by trading on the traded asset S_t, benefits of λ units of the claim $h(X_t)$. Throughout this chapter we consider a filtered probability space $(\Omega, (\mathcal{F}_{0\leq t\leq T}), \mathcal{F}, \mathbb{P})$, where $(\mathcal{F}_{0\leq t\leq T})$ is the augmented filtration generated by a two dimensional Brownian motion $(B, \tilde{B})$ for which each component satisfied the usual hypothesis of right-continuity and completeness. For $t < T$, c_t represents the consumption rate at time t.

Sufficient Stochastic Maximum Principle

Inspired by the work of Øksendal & Sulem (2009) and Pamen (2015), in this section we present a sufficient maximum principle for stochastic optimal control of a forward and backward SDE system.

Consider a state system $(\mathcal{W}_t, V_t)$ described by the following coupled system of forward-backward SDEs.

Forward system:

$$\begin{cases} \mathrm{d}\mathcal{W}_t = b(t, \mathcal{W}_t, u_t)\mathrm{d}t + \sigma(t, \mathcal{W}_t, u_t)\mathrm{d}B_t, \ t \in [0, T] \\ \mathcal{W}_0 = \omega. \end{cases} \tag{3.1}$$

Backward system:

$$\begin{cases} \mathrm{d}V_t = -g(t, \mathcal{W}_t, V_t, Z_t, u_t)\mathrm{d}t + Z_t\mathrm{d}B_t \\ V_T = c\mathcal{W}_T, \ \text{where } c \in \mathbb{R} - \{0\} \text{ is a given constant.} \end{cases} \tag{3.2}$$

Value function:

$$J(u) = \mathbb{E}\Big[\int_0^T f(t, \mathcal{W}_t, V_t, Z_t, u_t)\mathrm{d}t + h_1(V_0) + h_2(\mathcal{W}_T)\Big]; \ u \in \mathcal{A}, \tag{3.3}$$

where

$\mathcal{A}$ is a given family of controls, contained in the set of $\mathcal{F}_t-$predictable controls u_t such that the FBSDE system has a unique strong solution and

$$\mathbb{E}\Big[\int_0^T |f(t, \mathcal{W}_t, V_t, Z_t, u_t)|\mathrm{d}t + |h_1(V_0)| + |h_2(\mathcal{W}_T)|\Big] < \infty.$$

We consider the following problem, which is considered as a full observation optimal control of forward-backward stochastic differential equations,

Problem *(Full observation optimal control of FBSDEs).* Find the optimal value function $\phi \in \mathbb{R}$ and the optimal control u^* in the set of controls $\mathcal{A}$ such that

$$\phi = \sup_{u \in \mathcal{A}} J(u) = J(u^*). \tag{3.4}$$

Let us now define the Hamiltonian and the adjoint equations of the above optimal control problem.

The Hamiltonian is given by

$$\begin{aligned}&\mathcal{H}(t,\omega,v,z,u,\lambda,p,q)\\&=f(t,\omega,v,z,u)+g(t,\omega,v,z,u)\lambda+b(t,\omega,u)p+\sigma(t,\omega,u)q,\end{aligned}$$

with $\mathcal{H}$ Fréchet differentiable (C^1) with respect to ω, v, z (see Definition 15). Thus, the associated pair of adjoint equations are

$$\begin{cases}\mathrm{d}\lambda_t = \frac{\partial\mathcal{H}}{\partial v}(t,\mathcal{W}_t,V_t,Z_t,u_t,\lambda_t,p_t,q_t)\mathrm{d}t+\frac{\partial\mathcal{H}}{\partial z}(t,\mathcal{W}_t,V_t,Z_t,u_t,\lambda_t,p_t,q_t)\mathrm{d}B_t\\ \lambda_0 = h_1^{'}(V_0),\end{cases} \tag{3.5}$$

and

$$\begin{cases}\mathrm{d}p_t = -\frac{\partial\mathcal{H}}{\partial\omega}(t,\mathcal{W}_t,V_t,Z_t,u_t,\lambda_t,p_t,q_t)\mathrm{d}t+q_t\mathrm{d}B_t\\ p_T = c\lambda_T + h_2^{'}(\mathcal{W}_T).\end{cases} \tag{3.6}$$

We then obtain the following theorem.

Theorem 5 (Sufficient maximum principle)**.** *Let $\hat{\mathcal{W}}_t$, $\hat{V}_t$, $\hat{Z}_t$, $\hat{\lambda}_t$, $\hat{p}_t$, $\hat{q}_t$ be the corresponding solutions of a control $\hat{u}\in\mathcal{A}$ of Equations* (3.2), (3.5) *and* (3.6)*. Suppose that*

(Concavity property) The functions $x\to h_i(x)$; $i=1,2$ and

$(\omega,v,z,u)\to\mathcal{H}(t,\omega,v,z,u,\hat{\lambda}_t,\hat{p}_t,\hat{q}_t)$ are concave for all $t\in[0,T]$.

(The conditional maximum principle)

$$\max_{u\in U}\mathcal{H}(t,\hat{\mathcal{W}}_t,\hat{V}_t,\hat{Z}_t,u,\hat{\lambda}_t,\hat{p}_t,\hat{q}_t)=\mathcal{H}(t,\hat{\mathcal{W}}_t,\hat{V}_t,\hat{Z}_t,\hat{u}_t,\hat{\lambda}_t,\hat{p}_t,\hat{q}_t).$$

Also, suppose that for all $u\in\mathcal{A}$ the following integrability conditions are *satisfied:*

$$\mathbb{E}\Big[\int_0^T V_t^2\Big(\frac{\partial \hat{\mathcal{H}}}{\partial z}\Big)_t^2 \mathrm{d}t\Big] < \infty,\ \mathbb{E}\Big[\int_0^T Z_t^2\hat{\lambda}_t^2 \mathrm{d}t\Big] < \infty,$$
$$\mathbb{E}\Big[\int_0^T \mathcal{W}_t^2\hat{q}_t^2 \mathrm{d}t\Big] < \infty,\ \mathbb{E}\Big[\int_0^T \sigma_t^2\hat{p}_t^2 \mathrm{d}t\Big] < \infty.$$

Then $\hat{u}_t$ *is an optimal control for the problem. That means, it satisfies Equation* (3.4),

$$J(\hat{u}) = \sup_{u\in\mathcal{A}} J(u).$$

Proof. Choose $u \in \mathcal{A}$ with corresponding solutions $\mathcal{W}_t,\ V_t,\ Z_t,\ \lambda_t,\ p_t,\ q_t$. Then

$$\begin{aligned}
\hat{\mathcal{H}}(t) &= \mathcal{H}(t, \hat{\mathcal{W}}_t, \hat{V}_t, \hat{Z}_t, \hat{u}_t, \hat{\lambda}_t, \hat{p}_t, \hat{q}_t)\\
&= f(t, \hat{\mathcal{W}}_t, \hat{V}_t, \hat{Z}_t, \hat{u}_t) + g(t, \hat{\mathcal{W}}_t, \hat{V}_t, \hat{Z}_t, \hat{u}_t)\hat{\lambda}_t + b(t, \hat{\mathcal{W}}_t, \hat{u}_t)\hat{p}_t\\
&\quad + \sigma(t, \hat{\mathcal{W}}_t, \hat{u}_t)\hat{q}_t,\\
\mathcal{H}(t) &= \mathcal{H}(t, \mathcal{W}_t, V_t, Z_t, u_t, \hat{\lambda}_t, \hat{p}_t, \hat{q}_t)\\
&= f(t, \mathcal{W}_t, V_t, Z_t, u_t, \hat{\lambda}_t, \hat{p}_t, \hat{q}_t) + g(t, \mathcal{W}_t, V_t, Z_t, u_t)\hat{\lambda}_t + b(t, \mathcal{W}_t, u_t)\hat{p}_t\\
&\quad + \sigma(t, \mathcal{W}_t, u_t)\hat{q}_t,
\end{aligned}$$

also $\hat{f}(t) = f(t, \hat{\mathcal{W}}_t, \hat{V}_t, \hat{Z}_t, \hat{u}_t);\ f(t) = f(t, \mathcal{W}_t, V_t, Z_t, u_t)$, etc.

Then

$$J(\hat{u}) - J(u) = I_1 + I_2, \tag{3.7}$$

where

$$I_1 = \mathbb{E}\Big[\int_0^T \{\hat{f}(t) - f(t)\}\mathrm{d}t\Big] \tag{3.8}$$

and

$$I_2 = \mathbb{E}[h_1(\hat{\mathcal{W}}_0) - h_1(\mathcal{W}_0) + h_2(\hat{V}_T) - h_2(V_T)]. \tag{3.9}$$

The definition of $\mathcal{H}$ gives us

$$\begin{aligned} I_1 = \mathbb{E}\Big[\int_0^T \{&\hat{\mathcal{H}}(t) - \mathcal{H}(t) - (\hat{g}(t) - g(t))\hat{\lambda}_t - (\hat{b}(t) - b(t))\hat{p}_t \\ &- (\hat{\sigma}(t) - \sigma(t))\hat{q}_t\}\Big]. \end{aligned} \tag{3.10}$$

From the concavity of h_1 we get

$$h_1(\hat{\mathcal{W}}_0) - h_1(\mathcal{W}_0) \geq (\hat{\mathcal{W}}_0 - \mathcal{W}_0)h_1'(\hat{\mathcal{W}}_0) = (\hat{\mathcal{W}}_0 - \mathcal{W}_0)\lambda_0. \tag{3.11}$$

Similarly for h_2, we have

$$h_2(\hat{V}_T) - h_2(V_T) \geq (\hat{V}_T - V_T)h_2'(\hat{V}_T). \tag{3.12}$$

Applying the Ito's formula on $(\hat{\mathcal{W}}_t - \mathcal{W}_t)\lambda_t$ and using the two backward processes V_t and p_t we obtain

$$\begin{aligned} &\mathbb{E}[(\hat{\mathcal{W}}_0 - \mathcal{W}_0)\lambda_0] \\ &= \mathbb{E}[(\hat{\mathcal{W}}_T - \mathcal{W}_T)\lambda_T] - \mathbb{E}\Big[\int_0^T (\hat{\mathcal{W}}_t - \mathcal{W}_t)\mathrm{d}\lambda_t + \int_0^T \hat{\lambda}_t \mathrm{d}(\hat{\mathcal{W}}_t - \mathcal{W}_t) \\ &\quad + \int_0^T \Big(\frac{\partial \hat{\mathcal{H}}}{\partial z}\Big)_t (\hat{Z}_t - Z_t)\mathrm{d}t\Big] \\ &= \mathbb{E}[(\hat{\mathcal{W}}_t - \mathcal{W}_t)(\hat{p}_t - h_2'(\hat{\mathcal{W}}_T))] - \mathbb{E}\Big[\int_0^T \Big\{(\hat{V}_t - V_t)\Big(\frac{\partial \hat{\mathcal{H}}}{\partial v}\Big)_t \\ &\quad - \hat{\lambda}_t(\hat{g}(t) - g(t)) + \Big(\frac{\partial \hat{\mathcal{H}}}{\partial z}\Big)_t (\hat{Z}_t - Z_t)\Big\}\mathrm{d}t\Big] \\ &= \mathbb{E}\Big[\int_0^T (\hat{\mathcal{W}}_t - \mathcal{W}_t)\mathrm{d}\hat{p}_t + \int_0^T \hat{p}_t \mathrm{d}(\hat{\mathcal{W}}_t - \mathcal{W}_t) + \int_0^T (\hat{\sigma}_t - \sigma_t)\hat{q}_t \mathrm{d}t \\ &\quad - \int_0^T \Big\{(\hat{V}_t - V_t)\Big(\frac{\partial \hat{\mathcal{H}}}{\partial v}\Big)_t - \hat{\lambda}_t(\hat{g}(t) - g(t)) + (\hat{Z}_t - Z_t)\Big(\frac{\partial \hat{\mathcal{H}}}{\partial z}\Big)_t\Big\}\mathrm{d}t\Big] \end{aligned}$$

$$
\begin{aligned}
&- \mathbb{E}[(\hat{\mathcal{W}}_T - \mathcal{W}_T) h_2'(\hat{\mathcal{W}}_T)] \\
&= \mathbb{E}\Big[\int_0^T \Big\{(\hat{b}(t) - b(t))\hat{p}_t + (\hat{\sigma}_t - \sigma_t)\hat{q}_t - (\hat{\mathcal{W}}_t - \mathcal{W}_t)\Big(\frac{\partial \hat{\mathcal{H}}}{\partial \omega}\Big)_t \\
&- (\hat{V}_t - V_t)\Big(\frac{\partial \hat{\mathcal{H}}}{\partial v}\Big)_t + \hat{\lambda}_t(\hat{g}(t) - g(t)) - (\hat{Z}_t - Z_t)\Big(\frac{\partial \hat{\mathcal{H}}}{\partial z}\Big)_t\Big\}\mathrm{d}t\Big] \\
&- \mathbb{E}[(\hat{\mathcal{W}}_T - \mathcal{W}_T) h_2'(\hat{\mathcal{W}}_T)].
\end{aligned} \tag{3.13}
$$

Combining Equations (3.7)-(3.13) and using the definition of $\mathcal{H}$, we obtain

$$
\begin{aligned}
J(\hat{u}) - J(u) &= I_1 + I_2 \\
&\geq \mathbb{E}\Big[\int_0^T \Big\{\hat{\mathcal{H}}(t) - \mathcal{H}(t) - (\hat{\mathcal{W}}_t - \mathcal{W}_t)\Big(\frac{\partial \hat{\mathcal{H}}}{\partial \omega}\Big)_t \\
&\quad - (\hat{V}_t - V_t)\Big(\frac{\partial \hat{\mathcal{H}}}{\partial v}\Big)_t - (\hat{Z}_t - Z_t)\Big(\frac{\partial \hat{\mathcal{H}}}{\partial z}\Big)_t\Big\}\mathrm{d}t\Big] \\
&= \mathbb{E}\Big[\int_0^T \mathbb{E}\Big[\Big\{\hat{\mathcal{H}}(t) - \mathcal{H}(t) - (\hat{\mathcal{W}}_t - \mathcal{W}_t)\Big(\frac{\partial \hat{\mathcal{H}}}{\partial \omega}\Big)_t \\
&\quad - (\hat{V}_t - V_t)\Big(\frac{\partial \hat{\mathcal{H}}}{\partial v}\Big)_t - (\hat{Z}_t - Z_t)\Big(\frac{\partial \hat{\mathcal{H}}}{\partial z}\Big)_t\Big\}|\mathcal{F}_t\Big]\mathrm{d}t\Big].
\end{aligned} \tag{3.14}
$$

From the concavity of the function

$$
(\omega, v, z, u) \to \mathcal{H}(t, \omega, v, z, u, \hat{\lambda}_t, \hat{p}_t, \hat{q}_t),
$$

we get

$$
\begin{aligned}
\hat{\mathcal{H}}(t) - \mathcal{H}(t) \geq \Big(\frac{\partial \hat{\mathcal{H}}}{\partial \omega}\Big)_t(\hat{\mathcal{W}}_t - \mathcal{W}_t) + \Big(\frac{\partial \hat{\mathcal{H}}}{\partial v}\Big)_t(\hat{V}_t - V_t) \\
+ \Big(\frac{\partial \hat{\mathcal{H}}}{\partial z}\Big)_t(\hat{Z}_t - Z_t) + \Big(\frac{\partial \hat{\mathcal{H}}}{\partial u}\Big)_t(\hat{u}_t - u_t).
\end{aligned} \tag{3.15}
$$

Since $u = \hat{u}_t$ maximizes

$$
u \to \mathcal{H}(t, \hat{\mathcal{W}}_t, \hat{V}_t, \hat{Z}_t, u, \hat{\lambda}_t, \hat{p}_t, \hat{q}_t),
$$

we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}u}\mathcal{H}(t, \hat{\mathcal{W}}_t, \hat{V}_t, \hat{Z}_t, u, \hat{\lambda}_t, \hat{p}_t, \hat{q}_t)_{u=\hat{u}_t}(\hat{u}_t - u_t) \geq 0,$$

that is

$$\left(\frac{\partial \hat{\mathcal{H}}}{\partial u}\right)_t (\hat{u}_t - u_t) \geq 0. \tag{3.16}$$

From Equations (3.14), (3.15) and (3.16) we conclude that

$$J(\hat{u}) - J(u) \geq 0.$$

Since this holds for all $u \in \mathcal{A}$, then $\hat{u}$ is optimal. □

Consumption-Investment Problem in Incomplete Market

In this section, by the use of the sufficient maximum principle method we study a consumption-investment problem under a recursive utility of Epstein-Zin type in incomplete market. The incompleteness of the market comes from the fact that it is assumed there is a risk that cannot be hedged perfectly. Considering a recursive utility, written as a backward stochastic differential equation, the generator of the Epstein-Zin recursive utility is not concave and thus the above result cannot directly be applied. In order to use our sufficient maximum principle, we first transform the backward stochastic differential equation to an equivalent backward stochastic differential equation with a concave generator. Next, with the new generator we derived for our system the Hamiltonian $\mathcal{H}$ and its associated pair of adjoint processes p_t and λ_t. Then, from the terminal conditions of $\mathbb{Y}_t$ and p_t we assume a general form for each of them at optimality allowing us to build the optimal strategies. Finally, we end up getting the expression of the utility evaluated at these optimal strategies.

Problem Formulation

Here, we study a consumption-investment optimization problem with Epstein-Zin utility under full observation. We assume that the objective of the policymaker is to maximise the Epstein-Zin utility, at time 0,

$$J(c, \pi) = V_0^{\pi, c} \tag{3.17}$$

driven by the backward stochastic differential equation (BSDE)

$$\begin{cases} -\mathrm{d}V_t = g(c_t, V_t)\mathrm{d}t - Z_t^c \mathrm{d}B_t - \tilde{Z}_t^c \mathrm{d}\tilde{B}_t, \\ V_T = U(\mathcal{W}_T) = \frac{\mathcal{W}_T^{1-\gamma}}{1-\gamma}, \end{cases} \tag{3.18}$$

subject to the forward stochastic differential equation (FSDE)

$$\begin{cases} \mathrm{d}\mathcal{W}_t = \mathcal{W}_t(r(X_t) + \pi_t \mu(X_t) - \tilde{c}_t)\mathrm{d}t + \mathcal{W}_t \pi_t \sigma(X_t)\mathrm{d}B_t, \\ \mathcal{W}_0 = \omega, \end{cases} \tag{3.19}$$

where

$$g(c, v) = \delta \frac{(1-\gamma)v}{1-\frac{1}{\psi}} \Big[\Big(\frac{c}{((1-\gamma)v)^{\frac{1}{1-\gamma}}} \Big)^{1-\frac{1}{\psi}} - 1 \Big]. \tag{3.20}$$

and

$$\begin{cases} \mathrm{d}X_t = b(X_t)\mathrm{d}t + \rho a(X_t)\mathrm{d}B_t + \tilde{\rho} a(X_t)\mathrm{d}\tilde{B}_t \\ X_0 = x \end{cases} \tag{3.21}$$

Thus, we state our full observation optimal control of forward-backward stochastic differential equations as follows:

Problem 1 *(Full observation optimal control of forward-backward SDEs).* Find the optimal value function $V^*(\omega, 0) \in \mathbb{R}$ and the optimal controls π^*, c^*

in the set of control $\mathcal{A}$ such that

$$V^*(\omega, 0) = \sup_{c,\pi \in \mathcal{A}} J(c, \pi) = J(c^*, \pi^*) \tag{3.22}$$

The solution of that problem is given by the following theorem.

Theorem 6. *The optimal investment and consumption strategies to the full observable utility maximisation problem (**problem** 1) are given by Equation* (3.41) *and the associated optimal value function takes the form as in Equation* (3.43).

The remaining of this section is devoted to the proof of Theorem 6.

Sufficient Maximum Principle

In this subsection, we apply the ideas shown in Theorem 5 to solve a full observation optimal control of forward-backward differential equations stated in ***problem*** 1. Since this method is based on works for a concave Hamiltonian (Øksendal & Sulem, 2009) which implies a concave generator, our first objective is to check for the concavity. We obtain the following

Lemma 1

The function g given by Equation (3.20) is not jointly concave in c and v.

Proof. The Hessian matrix of the function $(c, v) \to g(c, v)$ is defined by $H(c, v) = H(c, x) = \frac{\delta}{a} \begin{vmatrix} a(a-1)c^{a-2}x^d & dc^{a-1}x^{d-1} \\ dc^{a-1}x^{d-1} & d(d-1)c^a x^{d-2} \end{vmatrix}$ with trace $\mathrm{tr}(H)$ and determinant $\det(H) = \left(\frac{\delta}{a} c^{a-1} x^{d-1}\right)^2 \left(a(a-1)d(d-1) - d^2\right) < 0$ where $x = (1-\gamma)v > 0, c > 0, a = \frac{1}{1-\frac{1}{\psi}} (0 < a < 1)$ and $b = \frac{1}{1-\gamma} < 0$. The associated characteristic polynomial is $\lambda^2 - \mathrm{tr}(H)\lambda + \det(H) = 0$. The negative sign of the determinant means that $\lambda_1 \lambda_2 < 0$; so one of the eigenvalues is positive. That is, the Hessian matrix is not negative definite. Hence $(c, v) \to g(c, v)$ is not jointly concave. $\square$

Since our generator is not jointly concave with respect to c and v, we perform a transformation that allows us to get our desired property.

Lemma 2

Assuming that a triplet (V_t, Z_t, Z_t^c) is a solution of the backward differential equation (3.18). Consider the transformation

$$\mathbb{Y}_t = \frac{1}{1-\frac{1}{\psi}} e^{-\delta t}\left((1-\gamma)V_t\right)^{\frac{1}{\theta}}, \; (\mathbb{Z}_t, \tilde{\mathbb{Z}}_t) = e^{-\delta t}(1-\gamma)^{\frac{1}{\theta}-1} V_t^{\frac{1}{\theta}-1}(Z_t^c, \tilde{Z}_t^c).$$

Then $(\mathbb{Y}_t, \mathbb{Z}_t)$ satisfies

$$\mathbb{Y}_t = e^{-\delta T}\frac{c_T^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}} + \int_t^T \left(\delta e^{-\delta s}\frac{c_s^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}} + \frac{1}{2}(\theta-1)\frac{\mathbb{Z}_s^2}{\mathbb{Y}_s}\right) \mathrm{d}s - \int_t^T \mathbb{Z}_s \mathrm{d}B_s$$
$$- \int_t^T \tilde{\mathbb{Z}}_s \mathrm{d}\tilde{B}_s$$

and the generator in the the first integral is a concave function with respect to $(c, \mathbb{Y}, \mathbb{Z})$ when $\theta < 0$.

Proof. The function $c \to \frac{c^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}}$ is concave since the second derivative is negative, that is $-\frac{1}{\psi}c^{-1-\frac{1}{\psi}} < 0$ $(0 < \psi < 1)$. Moreover, the Hessian matrix of the function $(y, z) \mapsto \frac{z^2}{y}$ is defined by

$$H(y,z) = \begin{vmatrix} \frac{2z^2}{y^3} & -\frac{2z}{y^2} \\ -\frac{2z}{y^2} & \frac{2}{y} \end{vmatrix}$$

with trace $\mathrm{tr}(H) = \frac{2}{y}\left(\frac{z^2}{y^2}+1\right)$ and determinant $\det(H) = 0$. The associated characteristic polynomial is $\lambda^2 - \frac{2}{y}\left(\frac{z^2}{y^2}+1\right)\lambda = 0$, that gives $\lambda_1 = 0$ and $\lambda_2 = \frac{2}{y}\left(\frac{z^2}{y^2}+1\right) > 0$ as eigenvalues which means that the Hessian is positive semidefinite. So $(y,z) \mapsto \frac{z^2}{y}$ is a convex function since its Hessian matrix is positive semidefinite. Hence $(y,z) \mapsto \frac{1}{2}(\theta-1)\frac{z^2}{y}$ is concave since $\theta < 0$. Therefore, the generator is concave since it is a sum of concave functions.

Now, given that

$$e^{-\delta t}((1-\gamma)V_t)^{\frac{1}{\theta}} = \Big(1-\frac{1}{\psi}\Big)\mathbb{Y}_t,$$

and

$$e^{-\delta t}((1-\gamma)V_t)^{\frac{1}{\theta}-1}(Z_t^c, \tilde{Z}_t^c) = (\mathbb{Z}_t, \tilde{\mathbb{Z}}_t),$$

Itô's formula applied to $e^{-\delta t}((1-\gamma)V_t)^{\frac{1}{\theta}}$ gives

$$\begin{aligned}
&\mathrm{d}\Big(e^{-\delta t}((1-\gamma)V_t)^{\frac{1}{\theta}}\Big)\\
&= -\delta e^{-\delta t}((1-\gamma)V_t)^{\frac{1}{\theta}}\mathrm{d}t + \frac{1}{\theta}(1-\gamma)e^{-\delta t}((1-\gamma)V_t)^{\frac{1}{\theta}-1}\mathrm{d}V_t\\
&\quad + \frac{1}{2}\frac{1}{\theta}\Big(\frac{1}{\theta}-1\Big)(1-\gamma)^2 e^{-\delta t}((1-\gamma)V_t)^{\frac{1}{\theta}-2}(\mathrm{d}V_t)^2\\
&= \Big\{-\delta e^{-\delta t}((1-\gamma)V_t)^{\frac{1}{\theta}}\\
&\quad + \frac{1}{\theta}(1-\gamma)e^{-\delta t}((1-\gamma)V_t)^{\frac{1}{\theta}-1}\Big[-\frac{\delta}{1-\frac{1}{\psi}}c^{1-\frac{1}{\psi}}((1-\gamma)V_t)^{1-\frac{1}{\theta}}\\
&\quad + \frac{\delta}{1-\frac{1}{\psi}}(1-\gamma)V_t\Big] + \frac{1}{2}\frac{1}{\theta}\Big(\frac{1}{\theta}-1\Big)(1-\gamma)^2 e^{-\delta t}((1-\gamma)V_t)^{\frac{1}{\theta}-2}(Z_t^c)^2\\
&\quad + \frac{1}{2}\frac{1}{\theta}\Big(\frac{1}{\theta}-1\Big)(1-\gamma)^2 e^{-\delta t}((1-\gamma)V_t)^{\frac{1}{\theta}-2}(\tilde{Z}_t^c)^2\Big\}\mathrm{d}t\\
&\quad + \frac{1}{\theta}(1-\gamma)e^{-\delta t}((1-\gamma)V_t)^{\frac{1}{\theta}-1}Z_t^c\mathrm{d}B_t\\
&\quad + \frac{1}{\theta}(1-\gamma)e^{-\delta t}((1-\gamma)V_t)^{\frac{1}{\theta}-1}\tilde{Z}_t^c\mathrm{d}\tilde{B}_t\\
&= \Big\{-\delta e^{-\delta t}c^{1-\frac{1}{\psi}} + \frac{1}{2}\Big(1-\frac{1}{\psi}\Big)^2(1-\theta)e^{\delta t}((1-\gamma)V_t)^{-\frac{1}{\theta}}\mathbb{Z}_t^2\\
&\quad + \frac{1}{2}\Big(1-\frac{1}{\psi}\Big)^2(1-\theta)e^{\delta t}((1-\gamma)V_t)^{-\frac{1}{\theta}}\tilde{\mathbb{Z}}_t^2\Big\}\mathrm{d}t\\
&\quad + \Big(1-\frac{1}{\psi}\Big)\mathbb{Z}_t\mathrm{d}B_t + \Big(1-\frac{1}{\psi}\Big)\tilde{\mathbb{Z}}_t\mathrm{d}\tilde{B}_t\\
&= \Big\{-\delta e^{-\delta t}c^{1-\frac{1}{\psi}} + \frac{1}{2}\Big(1-\frac{1}{\psi}\Big)(1-\theta)\frac{\mathbb{Z}_t^2+\tilde{\mathbb{Z}}_t^2}{\mathbb{Y}_t}\Big\}\mathrm{d}t\\
&\quad + \Big(1-\frac{1}{\psi}\Big)\mathbb{Z}_t\mathrm{d}B_t + \Big(1-\frac{1}{\psi}\Big)\tilde{\mathbb{Z}}_t\mathrm{d}\tilde{B}_t\\
&= \Big(1-\frac{1}{\psi}\Big)\mathrm{d}\mathbb{Y}_t.
\end{aligned}$$

Hence

$$\mathrm{d}\mathbb{Y}_t = \Big(- \delta e^{-\delta t} \frac{c^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}} + \frac{1}{2}(1-\theta)\frac{\mathbb{Z}_t^2 + \tilde{\mathbb{Z}}_t^2}{\mathbb{Y}_t}\Big)\mathrm{d}t + \mathbb{Z}_t \mathrm{d}B_t + \tilde{\mathbb{Z}}_t \mathrm{d}\tilde{B}_t.$$

Integrating both sides from t to T, we have

$$\mathbb{Y}_t = e^{-\delta T}\frac{\mathcal{W}_T^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}} + \int_t^T \Big[\delta e^{-\delta s}\frac{c_s^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}} + \frac{1}{2}(\theta-1)\frac{\mathbb{Z}_s^2+\tilde{\mathbb{Z}}_s^2}{\mathbb{Y}_s}\Big]\mathrm{d}s - \int_t^T \mathbb{Z}_s \mathrm{d}B_s$$
$$- \int_t^T \tilde{\mathbb{Z}}_s \mathrm{d}\tilde{B}_s.$$

□

The generator g, for $c_t = \tilde{c}_t \mathcal{W}_t$, is then given by

$$g(\tilde{c}, \mathcal{W}, \mathbb{Y}, \mathbb{Z}) = \delta e^{-\delta t}\frac{\tilde{c}_t^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}}\mathcal{W}_t^{1-\frac{1}{\psi}} + \frac{1}{2}(\theta-1)\frac{\mathbb{Z}_t^2+\tilde{\mathbb{Z}}_t^2}{\mathbb{Y}_t}. \tag{3.23}$$

Then our equivalent value function is

$$J(c,\pi) = \mathbb{Y}_0^{\pi,c}. \tag{3.24}$$

Thus the Hamiltonian becomes

$$\mathcal{H} = \Big(\delta e^{-\delta t}\frac{\tilde{c}^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}}\mathcal{W}_t^{1-\frac{1}{\psi}} + \frac{1}{2}(\theta-1)\frac{\mathbb{Z}_t^2+\tilde{\mathbb{Z}}_t^2}{\mathbb{Y}_t}\Big)\lambda_t \tag{3.25}$$
$$+ \mathcal{W}_t(r_t + \pi_t\mu_t - \tilde{c}_t)p_t + \mathcal{W}_t\pi_t\sigma_t q_t.$$

Then follows the associated pair of adjoint processes

$$\begin{cases} \mathrm{d}p_t = -\Big[\delta e^{-\delta t}\tilde{c}_t^{1-\frac{1}{\psi}}\mathcal{W}_t^{-\frac{1}{\psi}}\lambda_t + (r_t+\pi_t\mu_t-\tilde{c}_t)p_t \\ \qquad +\pi_t\sigma_t q_t\Big]\mathrm{d}t + q_t\mathrm{d}B_t + \tilde{q}_t\mathrm{d}\tilde{B}_t, \\ p(T) = e^{-\delta T}\mathcal{W}_T^{-\frac{1}{\psi}}\lambda(T), \end{cases} \tag{3.26}$$

and

$$\begin{cases} \mathrm{d}\lambda_t = \lambda_t\Big(-\frac{1}{2}(\theta-1)\frac{\mathbb{Z}_t^2+\tilde{\mathbb{Z}}_t^2}{\mathbb{Y}_t^2}\mathrm{d}t + (\theta-1)\frac{\mathbb{Z}_t}{\mathbb{Y}_t}\mathrm{d}B_t + (\theta-1)\frac{\tilde{\mathbb{Z}}_t}{\mathbb{Y}_t}\mathrm{d}\tilde{B}_t\Big), \\ \lambda(0) = 1. \end{cases} \tag{3.27}$$

Taking the first derivative of $\mathcal{H}$ w.r.t $\tilde{c}$ and setting it to zero, we get

$$\begin{aligned} 0 &= \frac{\partial \mathcal{H}}{\partial \tilde{c}} \\ &= \delta e^{-\delta t}\tilde{c}^{-\frac{1}{\psi}}\mathcal{W}_t^{1-\frac{1}{\psi}}\lambda_t - \mathcal{W}_t p_t, \end{aligned}$$

then

$$\tilde{c}_t = \delta^{\psi} e^{-\delta\psi t}\mathcal{W}_t^{-1}\left(\frac{\lambda_t}{p_t}\right)^{\psi}. \tag{3.28}$$

Similarly, for the first derivative of $\mathcal{H}$ w.r.t π, we obtain

$$\begin{aligned} 0 &= \frac{\partial \mathcal{H}}{\partial \pi} \\ &= \mathcal{W}_t\mu_t p_t + \mathcal{W}_t\sigma_t q_t \\ &= \mu_t p_t + \sigma_t q_t, \end{aligned}$$

then

$$\mu_t p_t + \sigma_t q_t = 0. \tag{3.29}$$

We define

$$\phi(\mathcal{W}, \lambda, X, t) = e^{-\delta t}\mathcal{W}_t^{-\frac{1}{\psi}}\lambda_t e^{D(X,t)} = p_t,$$

with $D(X,t)$ the process which satisfies the backward stochastic differential equation

$$\begin{cases} \mathrm{d}D(X,t) = -F_1(X,D,G_1,t)\mathrm{d}t + \rho G_1(X,t)\mathrm{d}B_t + \tilde{\rho} G_1(X,t)\mathrm{d}\tilde{B}_t \\ D(X,T) = 0, \end{cases} \tag{3.30}$$

where $F_1(X,D,G,t)$ and $G_1(X,t)$ will be determined.

We first give the conditional form of the process p_t.

Integrating both sides of Equation (3.32) from t to T, we obtain

$$\begin{aligned} p(T) - p_t = \int_t^T \Big\{ - \Big[\delta e^{-\delta s} \tilde{c}_s^{1-\frac{1}{\psi}} \mathcal{W}_s^{-\frac{1}{\psi}} \lambda_s + (r_s + \pi_s \mu_s - \tilde{c}_s) p_s \\ + \pi_s \sigma_s q_s \Big] \mathrm{d}s + q_s \mathrm{d}B_s + \tilde{q}_s \mathrm{d}\tilde{B}_s \Big\}. \end{aligned}$$

Applying the conditional expectation on both sides yields

$$\begin{aligned} p_t = \mathbb{E}\Big[\int_t^T \Big\{ \delta e^{-\delta s} \tilde{c}_s^{1-\frac{1}{\psi}} \mathcal{W}_s^{-\frac{1}{\psi}} \lambda_s + (r_s + \pi_s \mu_s - \tilde{c}_s) p_s \\ + \pi_s \sigma_s q_s \Big\} \mathrm{d}s + p(T) | \mathcal{F}_t \Big], \end{aligned} \tag{3.31}$$

where,

$$\mathbb{E}\Big[\int_t^T q_s \mathrm{d}B_s | \mathcal{F}_t \Big] = 0 \quad \text{and} \quad \mathbb{E}\Big[\int_t^T \tilde{q}_s \mathrm{d}\tilde{B}_s | \mathcal{F}_t \Big] = 0,$$

since, by definition of a backward stochastic differential equation q_s and $\tilde{q}_s$ belong to $\mathcal{L}^2[t,T]$ for all $s \in [t,T]$.

On first hand, let us apply the Ito's formula on the process p_t. Calculation yields

$$\mathrm{d}\mathcal{W}_t^{-\frac{1}{\psi}} = \mathcal{W}_t^{-\frac{1}{\psi}} \Big[\Big(-\frac{1}{\psi}(r_t + \pi_t \mu_t - \tilde{c}_t) + \frac{1}{2}\frac{1}{\psi}\Big(1 + \frac{1}{\psi}\Big)\pi_t^2 \sigma_t^2 \Big) \mathrm{d}t - \frac{1}{\psi} \pi_t \sigma_t \mathrm{d}B_t \Big],$$

also

$$
\begin{aligned}
&\mathrm{d}e^{-\delta t+D(X,t)}\\
&=e^{-\delta t+D(X,t)}\Big[\Big(-\delta-F_1(X,D,G_1,t)+\frac{1}{2}G_1^2(X,t)\Big)\mathrm{d}t+\rho G_1(X,t)\mathrm{d}B_t\\
&\qquad+\tilde{\rho}G_1(X,t)\mathrm{d}\tilde{B}_t\Big],
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathrm{d}\Big(\mathcal{W}_t^{-\frac{1}{\psi}}\lambda_t\Big)\\
&=\mathcal{W}_t^{-\frac{1}{\psi}}\lambda_t\Big\{\Big[-\frac{1}{\psi}(r_t+\pi_t\mu_t-\tilde{c}_t)+\frac{1}{2}\frac{1}{\psi}\Big(1+\frac{1}{\psi}\Big)\pi_t^2\sigma_t^2-\frac{1}{2}(\theta-1)\frac{\mathbb{Z}_t^2+\tilde{\mathbb{Z}}_t^2}{\mathbb{Y}_t^2}\\
&\qquad-\frac{1}{\psi}\pi_t\sigma_t(\theta-1)\frac{\mathbb{Z}_t}{\mathbb{Y}_t}\Big]\mathrm{d}t\\
&\qquad+\Big(-\frac{1}{\psi}\pi_t\sigma_t+(\theta-1)\frac{\mathbb{Z}_t}{\mathbb{Y}_t}\Big)\mathrm{d}B_t+(\theta-1)\frac{\tilde{\mathbb{Z}}_t}{\mathbb{Y}_t}\mathrm{d}\tilde{B}_t\Big\}.
\end{aligned}
$$

Thus, p_t satisfies the backward differential equation

$$
\begin{aligned}
\mathrm{d}p_t=p_t\Big\{&\Big[-\frac{1}{\psi}(r_t+\pi_t\mu_t-\tilde{c}_t)+\frac{1}{2}\frac{1}{\psi}\Big(1+\frac{1}{\psi}\Big)\pi_t^2\sigma_t^2\\
&-\frac{1}{2}(\theta-1)\frac{\mathbb{Z}_t^2+\tilde{\mathbb{Z}}_t^2}{\mathbb{Y}_t^2}-\frac{1}{\psi}\pi_t\sigma_t(\theta-1)\frac{\mathbb{Z}_t}{\mathbb{Y}_t}-\delta-F_1(X,D,G_1,t)\\
&+\frac{1}{2}G_1^2(X,t)+\Big(-\frac{1}{\psi}\pi_t\sigma_t+(\theta-1)\frac{\mathbb{Z}_t}{\mathbb{Y}_t}\Big)\rho G_1(X,t)\\
&+(\theta-1)\frac{\tilde{\mathbb{Z}}_t}{\mathbb{Y}_t}\tilde{\rho}G_1(X,t)\Big]\mathrm{d}t\\
&+\Big[-\frac{1}{\psi}\pi_t\sigma_t+(\theta-1)\frac{\mathbb{Z}_t}{\mathbb{Y}_t}+\rho G_1(X,t)\Big]\mathrm{d}B_t\\
&+\Big[(\theta-1)\frac{\tilde{\mathbb{Z}}_t}{\mathbb{Y}_t}+\tilde{\rho}G_1(X,t)\Big]\mathrm{d}\tilde{B}_t\Big\}.
\end{aligned}
\tag{3.32}
$$

Comparing the diffusion terms in Equations (3.26) and (3.32), we get

$$
q_t=p_t\Big[-\frac{1}{\psi}\pi_t\sigma_t+(\theta-1)\frac{\mathbb{Z}_t}{\mathbb{Y}_t}+\rho G_1(X,t)\Big],\tag{3.33}
$$

and

$$\tilde{q}_t = p_t\Big[(\theta-1)\frac{\tilde{\mathbb{Z}}_t}{\mathbb{Y}_t} + \tilde{\rho}G_1(X,t)\Big]. \tag{3.34}$$

Secondly, by the non-linear Feynman-Kăc formula applying to Equation (3.31), we get that $\phi(\mathcal{W},\lambda,X,t)$ satisfies the following partial differential equation

$$\begin{aligned}&\phi_t + b\phi_x + \alpha_1\phi_{\mathcal{W}} + \alpha_2\phi_\lambda + \frac{1}{2}a^2\phi_{xx} + \frac{1}{2}\beta_1^2\phi_{\mathcal{W}\mathcal{W}} + \beta_1\rho a\phi_{x\mathcal{W}}\\ &+\frac{1}{2}(\beta_2^2+\tilde{\beta}_2^2)\phi_{\lambda\lambda} + (\beta_2\rho a + \tilde{\beta}_2\tilde{\rho}a)\phi_{x\lambda} + \beta_1\beta_2\phi_{\lambda\mathcal{W}} + f(X,t)\phi = 0,\end{aligned} \tag{3.35}$$

where we denote

$$\phi_t = \frac{\partial\phi}{\partial t},\ \phi_x = \frac{\partial\phi}{\partial x},\ \phi_{xx} = \frac{\partial^2\phi}{\partial x^2},$$

$$\begin{aligned}\alpha_1(\mathcal{W},t) &= \mathcal{W}_t(r(X_t) + \pi_t\mu(X_t) - \tilde{c}_t),\\ \beta_1(\mathcal{W},t) &= \mathcal{W}_t\pi_t\sigma(X_t),\\ \alpha_2(\lambda,t) &= -\frac{1}{2}(\theta-1)\lambda_t\frac{\mathbb{Z}_t^2+\tilde{\mathbb{Z}}_t^2}{\mathbb{Y}_t^2}\\ \beta_2(\lambda,t) &= (\theta-1)\lambda_t\frac{\mathbb{Z}_t}{\mathbb{Y}_t}\\ \tilde{\beta}_2(\lambda,t) &= (\theta-1)\lambda_t\frac{\tilde{\mathbb{Z}}_t}{\mathbb{Y}_t},\end{aligned}$$

and

$$\mathrm{d}X_t = b(X_t,t)\mathrm{d}t + \rho a(X_t,t)\mathrm{d}B_t + \tilde{\rho}a(X_t,t)\mathrm{d}\tilde{B}_t.$$

Moreover $f(X,t)p_t$ is the drift of p_t.

Also,

$$\begin{aligned}\phi_t = D_t\phi - \delta\phi,\ \phi_x = D_x\phi,\ \mathcal{W}_t\phi_{\mathcal{W}} = -\frac{1}{\psi}\phi,\ \lambda_t\phi_\lambda = \phi,\\ \phi_{xx} = D_{xx}\phi + (D_x\phi)^2,\ \mathcal{W}_t^2\phi_{\mathcal{W}\mathcal{W}} = \frac{1}{\psi}\Big(1+\frac{1}{\psi}\Big)\phi,\ \phi_{\lambda\lambda} = 0,\\ \mathcal{W}_t\phi_{x\mathcal{W}} = -\frac{1}{\psi}D_x\phi,\ \lambda_t\phi_{x\lambda} = D_x\phi,\ \mathcal{W}_t\lambda_t\phi_{\mathcal{W}\lambda} = -\frac{1}{\psi}\phi.\end{aligned}$$

Substituting the later expressions into Equation (3.35) gives

$$
\begin{aligned}
D_t - \delta + bD_x - \frac{1}{\psi}\frac{\alpha_1}{\mathcal{W}_t} + \frac{\alpha_2}{\lambda_t} + \frac{1}{2}a^2 D_{xx} + \frac{1}{2}a^2 (D_x)^2 \\
+\frac{1}{2}\frac{1}{\psi}\Big(1+\frac{1}{\psi}\Big)\frac{\beta_1^2}{\mathcal{W}_t^2} - \frac{1}{\psi}\frac{\beta_1}{\mathcal{W}_t}(\rho a D_x) - \frac{1}{\psi}\frac{\beta_1\beta_2}{\mathcal{W}_t\lambda_t} \\
+\frac{\beta_2\rho + \tilde{\beta}_2\tilde{\rho}}{\lambda_t}(aD_x) + f(X,t) = 0.
\end{aligned}
$$

Thus, from the non-linear Feynman-Kăc formula, we obtain

$$
\begin{aligned}
F_1(X,D,G_1,t) &= -\delta - \frac{1}{\psi}\frac{\alpha_1}{\mathcal{W}_t} + \frac{\alpha_2}{\lambda_t} + \frac{1}{2}\frac{1}{\psi}\Big(1+\frac{1}{\psi}\Big)\frac{\beta_1^2}{\mathcal{W}_t^2} \\
&\quad + \frac{1}{2}G_1^2(X,t) + \Big(-\frac{1}{\psi}\frac{\beta_1\rho}{\mathcal{W}_t} + \frac{\beta_2\rho + \tilde{\beta}_2\tilde{\rho}}{\lambda_t}\Big)G_1(X,t) \\
&\quad - \frac{1}{\psi}\frac{\beta_1\beta_2}{\mathcal{W}_t\lambda_t} + f(X,t), \qquad (3.36) \\
G_1(X,t) &= aD_x.
\end{aligned}
$$

Substituting $\alpha_1,\ \alpha_2,\ \beta_1,\ \tilde{\beta}_1,\ \beta_2,\ \tilde{\beta}_2$ by their respective expressions gives

$$
\begin{aligned}
F_1(X,D,G_1,t) &= -\delta - \frac{1}{\psi}(r_t + \pi_t\mu_t - \tilde{c}_t) + \frac{1}{2}\frac{1}{\psi}\Big(1+\frac{1}{\psi}\Big)\pi_t^2\sigma_t^2 \\
&\quad - \frac{1}{2}(\theta-1)\frac{\mathbb{Z}_t^2 + \tilde{\mathbb{Z}}_t^2}{\mathbb{Y}_t^2} + \frac{1}{2}G_1^2(X,t) \\
&\quad + \Big(-\frac{1}{\psi}\pi_t\sigma_t\rho_t + (\theta-1)\frac{\rho\mathbb{Z}_t + \tilde{\rho}\tilde{\mathbb{Z}}_t}{\mathbb{Y}_t}\Big)G_1(X,t) \\
&\quad - \frac{1}{\psi}\pi_t\sigma_t(\theta-1)\frac{\mathbb{Z}_t}{\mathbb{Y}_t} + f(X,t). \qquad (3.37)
\end{aligned}
$$

The computation of $f(X,t)$ is as follow

$$
\begin{aligned}
f(X,t) &= \frac{1}{p_t}\Big[\delta e^{-\delta t}\tilde{c}_t^{1-\frac{1}{\psi}}\mathcal{W}_t^{-\frac{1}{\psi}}\lambda_t + (r_t + \pi_t\mu_t - \tilde{c}_t)p_t + \pi_t\sigma_t q_t\Big] \\
&= \frac{1}{p_t}\Big[\tilde{c}_t p_t + (r_t + \pi_t\mu_t - \tilde{c}_t)p_t + \pi_t\sigma_t p_t\Big[-\frac{1}{\psi}\pi_t\sigma_t \\
&\qquad + (\theta-1)\frac{\mathbb{Z}_t}{\mathbb{Y}_t} + \rho G_1(X,t)\Big]\Big] \\
&= r_t + \pi_t\mu_t + \pi_t\sigma_t\Big[-\frac{1}{\psi}\pi_t\sigma_t + (\theta-1)\frac{\mathbb{Z}_t}{\mathbb{Y}_t} + \rho G_1(X,t)\Big].
\end{aligned}
$$

Then

$$
\begin{aligned}
&F_1(X, D, G_1, t)\\
&= -\delta - \frac{1}{\psi}(r_t + \pi_t\mu_t - \tilde{c}_t) + \frac{1}{2}\frac{1}{\psi}\Big(1 + \frac{1}{\psi}\Big)\pi_t^2\sigma_t^2 - \frac{1}{2}(\theta - 1)\frac{\mathbb{Z}_t^2 + \tilde{\mathbb{Z}}_t^2}{\mathbb{Y}_t^2}\\
&\quad + \frac{1}{2}G_1^2(X,t) + \Big(- \frac{1}{\psi}\pi_t\sigma_t\rho + (\theta - 1)\frac{\rho\mathbb{Z}_t + \tilde{\rho}\tilde{\mathbb{Z}}_t}{\mathbb{Y}_t}\Big)G_1(X,t)\\
&\quad - \frac{1}{\psi}\pi_t\sigma_t(\theta - 1)\frac{\mathbb{Z}_t}{\mathbb{Y}_t} + f(X,t)\\
&= -\delta + \Big(1 - \frac{1}{\psi}\Big)r_t + \Big(1 - \frac{1}{\psi}\Big)\pi_t\mu_t + \frac{1}{\psi}\tilde{c}_t - \frac{1}{2}\frac{1}{\psi}\Big(1 - \frac{1}{\psi}\Big)\pi_t^2\sigma_t^2\\
&\quad - \frac{1}{2}(\theta - 1)\frac{\mathbb{Z}_t^2 + \tilde{\mathbb{Z}}_t^2}{\mathbb{Y}_t^2} + \Big(\Big(1 - \frac{1}{\psi}\Big)\pi_t\sigma_t\rho + (\theta - 1)\frac{\rho\mathbb{Z}_t + \tilde{\rho}\tilde{\mathbb{Z}}_t}{\mathbb{Y}_t}\Big)G_1(X,t)\\
&\quad + \frac{1}{2}G_1^2(X,t) + \Big(1 - \frac{1}{\psi}\Big)\pi_t\sigma_t(\theta - 1)\frac{\mathbb{Z}_t}{\mathbb{Y}_t}.
\end{aligned}
$$

From the terminal condition of the process $\mathbb{Y}_t$, we assume at the optimal strategies that

$$\mathbb{Y}_t = \frac{1}{1 - \frac{1}{\psi}} e^{-\delta t}\mathcal{W}_t^{1-\frac{1}{\psi}} e^{E(X,t)},$$

where the process $E(X,t)$ satisfies the BSDE

$$
\begin{cases}
\mathrm{d}E(X,t) = -F_2(X, E, G_2, t)\mathrm{d}t + \rho G_2(X,t)\mathrm{d}B_t + \tilde{\rho}G_2(X,t)\mathrm{d}\tilde{B}_t,\\
E(X,T) = 0.
\end{cases}
$$

The diffusion terms of its associated BSDE are given by

$$\mathbb{Z}_t = \mathbb{Y}_t\Big[\Big(1 - \frac{1}{\psi}\Big)\pi_t\sigma_t + \rho G_2(X,t)\Big], \tag{3.38}$$

$$\tilde{\mathbb{Z}}_t = \mathbb{Y}_t\tilde{\rho}G_2(X,t), \tag{3.39}$$

and its drift is

$$
\begin{aligned}
&- \mathbb{Y}_t\Big[- \delta + \Big(1 - \frac{1}{\psi}\Big)(r_t + \pi_t\mu_t - \tilde{c}_t) - \frac{1}{2}\frac{1}{\psi}\Big(1 - \frac{1}{\psi}\Big)\pi_t^2\sigma_t^2\\
&\qquad - F_2(X, E, G_2, t) + \frac{1}{2}G_2^2(X,t) + \Big(1 - \frac{1}{\psi}\Big)\pi_t\sigma_t\rho G_2(X,t)\Big].
\end{aligned} \tag{3.40}
$$

Let us now define $F_2(X, E, G_2, t)$. In order to do that we equate expressions (3.23) and (3.40).

$$
\begin{aligned}
&\delta e^{-\delta t}\frac{\tilde{c}_t^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}}\mathcal{W}_t^{1-\frac{1}{\psi}} + \frac{1}{2}(\theta-1)\frac{\mathbb{Z}_t^2+\tilde{\mathbb{Z}}_t^2}{\mathbb{Y}_t}\\
&+\mathbb{Y}_t\Big[-\delta+\Big(1-\frac{1}{\psi}\Big)(r_t+\pi_t\mu_t-\tilde{c}_t)-\frac{1}{2}\frac{1}{\psi}\Big(1-\frac{1}{\psi}\Big)\pi_t^2\sigma_t^2\\
&\qquad -F_2(X,E,G_2,t)+\frac{1}{2}G_2^2(X,t)+\Big(1-\frac{1}{\psi}\Big)\pi_t\sigma_t\rho G_2(X,t)\Big]=0
\end{aligned}
$$

Factoring by $\mathbb{Y}_t$, we get

$$
\begin{aligned}
&\mathbb{Y}_t\Big[\tilde{c}_t e^{D(X,t)-E(X,t)}+\frac{1}{2}(\theta-1)\frac{\mathbb{Z}_t^2+\tilde{\mathbb{Z}}_t^2}{\mathbb{Y}_t^2}\\
&\quad -\delta+\Big(1-\frac{1}{\psi}\Big)(r_t+\pi_t\mu_t-\tilde{c}_t)-\frac{1}{2}\frac{1}{\psi}\Big(1-\frac{1}{\psi}\Big)\pi_t^2\sigma_t^2\\
&\quad -F_2(X,E,G_2,t)+\frac{1}{2}G_2^2(X,t)+\Big(1-\frac{1}{\psi}\Big)\pi_t\sigma_t\rho G_2(X,t)\Big]=0,
\end{aligned}
$$

then simplifying $\mathbb{Y}_t$ and substituting $\mathbb{Z}_t$ by its expression in Equation (3.38), it follows that

$$
\begin{aligned}
&F_2(X,E,G_2,t)\\
&=-\delta+\Big(1-\frac{1}{\psi}\Big)r_t+\Big(1-\frac{1}{\psi}\Big)\pi_t\mu_t+\frac{1}{\psi}\tilde{c}_t+\Big(e^{D(X,t)-E(X,t)}-1\Big)\tilde{c}_t\\
&\quad-\frac{1}{2}\frac{1}{\psi}\Big(1-\frac{1}{\psi}\Big)\pi_t^2\sigma_t^2+\frac{1}{2}(\theta-1)\frac{\mathbb{Z}_t^2+\tilde{\mathbb{Z}}_t^2}{\mathbb{Y}_t^2}+\frac{1}{2}G_2^2(X,t)\\
&\quad+\Big(1-\frac{1}{\psi}\Big)\pi_t\sigma_t\rho G_2(X,t).
\end{aligned}
$$

Now we show that $D(X,t)=E(X,t)=\frac{1}{\theta}Y_t$.

Assuming $G_1(X,t)=G(X,t)=G_2(X,t)$ and substituting $\mathbb{Y}_t$, $\mathbb{Z}_t$ by their respective expressions in Equation (3.37), we get

$$
\begin{aligned}
&F_1(X,D,G,t)\\
&=F_2(X,E,G,t)-(\theta-1)\frac{\mathbb{Z}_t^2+\tilde{\mathbb{Z}}_t^2}{\mathbb{Y}_t^2}+(\theta-1)\frac{\rho\mathbb{Z}_t+\tilde{\rho}\tilde{\mathbb{Z}}_t}{\mathbb{Y}_t}G(X,t)\\
&\quad+\Big(1-\frac{1}{\psi}\Big)\pi_t\sigma_t(\theta-1)\frac{\mathbb{Z}_t}{\mathbb{Y}_t}-\Big(e^{D(X,t)-E(X,t)}-1\Big)\tilde{c}_t\\
&=F_2(X,E,G,t).
\end{aligned}
$$

Thus, from Equations (3.28) and (3.29), the optimal strategies are given by

$$\tilde{c}_t^* = \delta^{\psi} e^{-\frac{\psi}{\theta} Y_t} \quad \text{and} \quad \pi_t^* = \frac{1}{\gamma}\frac{1}{\sigma^2}(\mu_t + \sigma_t \rho Z_t), \tag{3.41}$$

where (Y_t, Z_t) solution of the BSDE (Xing, 2017)

$$\begin{cases} -\mathrm{d}Y_t = F(X, Y, Z, t)\mathrm{d}t - \rho Z_t \mathrm{d}B_t - \tilde{\rho} Z_t \mathrm{d}\tilde{B}_t, \\ Y_T = 0, \end{cases}$$

with

$$\begin{aligned} &F(X, Y, Z, t) \\ &= -\theta\delta + (1-\gamma)r_t + \frac{1}{2}Z_t^2 + \frac{\theta}{\psi}\tilde{c}_t^* + (1-\gamma)\pi_t^*(\mu_t + \sigma_t \rho Z_t) \\ &\quad - \frac{1}{2}\gamma(1-\gamma)(\pi_t^*)^2\sigma_t^2 \\ &= -\theta\delta + (1-\gamma)r_t + \frac{1-\gamma}{2\gamma}\frac{\mu_t^2}{\sigma_t^2} + \theta\frac{\delta^{\psi}}{\psi}e^{-\frac{\psi}{\theta}Y_t} + \frac{1-\gamma}{\gamma}\frac{\mu_t \rho}{\sigma_t}Z_t \\ &\quad + \frac{1}{2}\left(1 + \frac{1-\gamma}{\gamma}\rho^2\right)Z_t^2. \end{aligned} \tag{3.42}$$

Let us now compute the optimal value function. That is,

$$V^*(\omega, 0) = \sup_{c,\pi \in \mathcal{A}} \mathbb{E}\Big[\int_0^T f(c_s, V_s)\mathrm{d}s + \frac{\mathcal{W}_T^{1-\gamma}}{1-\gamma}\Big].$$

Since

$$\pi_t^* = \frac{1}{\gamma}\frac{1}{\sigma^2}(\mu_t + \sigma_t \rho Z_t) \quad \text{and} \quad c_t^* = \mathcal{W}_t^* \tilde{c}_t = \delta^{\psi} \mathcal{W}_t^* e^{-\frac{\psi}{\theta}Y_t},$$

then

$$\begin{aligned} f(c_s^*, V_s^*) &= \delta\frac{1}{1-\frac{1}{\psi}}\left(\delta^{\psi}\mathcal{W}_s^* e^{-\frac{\psi}{\theta}Y_s}\right)^{1-\frac{1}{\psi}}((1-\gamma)V_s^*)^{1-\frac{1}{\theta}} - \delta\theta V_s^* \\ &= \delta\frac{1}{1-\frac{1}{\psi}}\delta^{\psi-1}(\mathcal{W}_s^*)^{1-\frac{1}{\psi}}e^{-\frac{\psi-1}{\theta}Y_s}\left((\mathcal{W}_s^*)^{1-\gamma}e^{Y_s}\right)^{1-\frac{1}{\theta}} \\ &\quad - \delta\theta\frac{1}{1-\gamma}(\mathcal{W}_s^*)^{1-\gamma}e^{Y_s} \\ &= \frac{\theta}{1-\gamma}\delta^{\psi}(\mathcal{W}_s^*)^{1-\gamma}e^{\left(1-\frac{\psi}{\theta}\right)Y_s} - \frac{\theta}{1-\gamma}\delta(\mathcal{W}_s^*)^{1-\gamma}e^{Y_s} \\ f(c_s^*, V_s^*) &= \frac{\theta}{1-\gamma}\delta(\mathcal{W}_s^*)^{1-\gamma}e^{Y_s}\left(\delta^{\psi-1}e^{-\frac{\psi}{\theta}Y_s} - 1\right). \end{aligned}$$

Thus

$$V^*(\omega,0)=\frac{1}{1-\gamma}\mathbb{E}\Big[\theta\delta\int_0^T(\mathcal{W}_s^*)^{1-\gamma}e^{Y_s}\Big(\delta^{\psi-1}e^{-\frac{\psi}{\theta}Y_s}-1\Big)ds+(\mathcal{W}_T^*)^{1-\gamma}\Big]. \tag{3.43}$$

Consumption-Investment Problem with Claim

In this section, we consider the external factor as a process that represents a non-traded asset. We assume here that even though the asset can not be traded directly, we can still observe its value. Our goal now is to solve a utility maximisation problem taking into account the introduction of a claim into our model. We still have a forward-backward stochastic differential equation system. Proceeding as in the previous sections, this section is divided into three parts. Firstly working with the transformed utility, we derived the new Hamiltonian $\mathcal{H}^h$ and its associated pair of adjoint processes p_t^h and λ_t^h driven respectively by a BSDE and a FSDE. Then, from the terminal conditions of $\mathbb{Y}_t$ and p_t^h we assume a general form for each of them at optimality allowing us to build the optimal strategies. Finally, we end up getting the expression of the utility evaluated at these optimal strategies.

Problem Formulation

Here, we consider the case of contingent claims hedged with the traded asset and the savings account. We assume that the claim pay off is of the form $\xi=h(X_T)$ for some bounded function h. We also assume that the objective of the policymaker is to maximize the Epstein-Zin utility, at time 0,

$$J^h(c,\pi)=(V_0^h)^{\pi,c} \tag{3.44}$$

driven by the backward stochastic differential equation (BSDE)

$$\begin{cases}-\mathrm{d}V_t^h=g(c_t,V_t^h)\mathrm{d}t-(Z_t^h)^c\mathrm{d}B_t-(\tilde{Z}_t^h)^c\mathrm{d}\tilde{B}_t,\\ V_T^h=U(\mathcal{W}_T^h)=\frac{(\mathcal{W}_T+\lambda h(X_T))^{1-\gamma}}{1-\gamma},\end{cases} \tag{3.45}$$

subject to the forward stochastic differential equation (FSDE)

$$\begin{cases} d\mathcal{W}_t^h = d\mathcal{W}_t + \lambda dh(X_t) \\ \mathcal{W}_0^h = \omega - v^b + \lambda h(x), \end{cases} \tag{3.46}$$

where

$$g(c,v) = \delta\frac{(1-\gamma)v}{1-\frac{1}{\psi}}\Big[\Big(\frac{c}{((1-\gamma)v)^{\frac{1}{1-\gamma}}}\Big)^{1-\frac{1}{\psi}} - 1\Big], \tag{3.47}$$

and $h(X_t)$ the claim at time t.

Thus, we state our new full observation optimal control of forward-backward stochastic differential equations as follows:

Problem 2 *(Full observation optimal control of forward-backward SDEs).* Find the optimal value function $V^h(\omega - v^b, \lambda) \in \mathbb{R}$ and the optimal controls π^*, c^* in the set of controls $\mathcal{A}$ such that

$$V^h(\omega - v^b, \lambda) = \sup_{c,\pi\in\mathcal{A}} J^h(c,\pi) = J^h(c^*,\pi^*). \tag{3.48}$$

The solution of that problem is given by the following theorem.

Theorem 7. *The optimal consumption and investment strategies to the full observable utility maximisation problem with claim (**problem** 2) are respectively given by Equations* (3.62) *and* (3.63) *and the associated optimal value function takes the form as in Equation* (3.64).

The remaining of this section is devoted to the proof of Theorem 7.

Application to Optimal Consumption-Investment with Claim

In this subsection, we solve a full observation optimal control of forward-backward differential equations with claim stated in ***problem*** 2.

The value function is given by

$$V^h(\omega - v^b, \lambda) = \sup_{c,\pi\in\mathcal{A}} \mathbb{E}\Big[\int_0^T f(c_s, V_s^h)ds + \frac{(\mathcal{W}_T + \lambda h(X_T))^{1-\gamma}}{1-\gamma}\Big].$$

In order to get $V^h(\omega - v^b, \lambda)$ we proceed similarly as in the previous section. The new wealth is then given by

$$\begin{cases} \mathrm{d}\mathcal{W}_t^h = \mathrm{d}\mathcal{W}_t + \lambda \mathrm{d}h(X,t) \\ \qquad = (\mathcal{W}_t(r_t + \pi_t\mu_t - \tilde{c}_t) + \lambda B(X,t))\mathrm{d}t + (\mathcal{W}_t\pi_t\sigma_t + \lambda\rho A(X,t))\mathrm{d}B_t \\ \qquad\quad + \lambda\tilde{\rho}A(X,t)\mathrm{d}\tilde{B}_t \\ \mathcal{W}_0^h = \omega - v^b + \lambda h(X_0), \end{cases}$$

where

$$\mathrm{d}h(X,t) = B(X,t)\mathrm{d}t + \rho A(X,t)\mathrm{d}B_t + \tilde{\rho}A(X,t)\mathrm{d}\tilde{B}_t.$$

The Hamiltonian is now given by

$$\begin{aligned} \mathcal{H}^h =& \Big(\delta e^{-\delta t}\frac{\tilde{c}^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}}\mathcal{W}_t^{1-\frac{1}{\psi}} + \frac{1}{2}(\theta-1)\frac{\mathbb{Z}_t^2 + \tilde{\mathbb{Z}}_t^2}{\mathbb{Y}_t}\Big)\lambda_t^h \\ &+ (\mathcal{W}_t(r_t + \pi_t\mu_t - \tilde{c}_t) + \lambda B(X,t))p_t^h + (\mathcal{W}_t\pi_t\sigma_t + \lambda\rho A(X,t))q_t^h \\ &+ \lambda\tilde{\rho}A(X,t)\tilde{q}_t^h. \end{aligned} \tag{3.49}$$

Then follows the associated pair of adjoint processes

$$\begin{cases} \mathrm{d}p_t^h = q_t^h\mathrm{d}B_t + \tilde{q}_t^h\mathrm{d}\tilde{B}_t, \\ p^h(T) = e^{-\delta T}(\mathcal{W}_T + \lambda h(X_T))^{-\frac{1}{\psi}}\lambda^h(T), \end{cases} \tag{3.50}$$

and

$$\begin{cases} \mathrm{d}\lambda_t^h = \lambda_t^h\Big(-\frac{1}{2}(\theta-1)\frac{\mathbb{Z}_t^2+\tilde{\mathbb{Z}}_t^2}{\mathbb{Y}_t^2}\mathrm{d}t + (\theta-1)\frac{\mathbb{Z}_t}{\mathbb{Y}_t}\mathrm{d}B_t + (\theta-1)\frac{\tilde{\mathbb{Z}}_t}{\mathbb{Y}_t}\mathrm{d}\tilde{B}_t\Big), \\ \lambda^h(0) = 1. \end{cases} \tag{3.51}$$

Taking the first derivative of $\mathcal{H}^h$ w.r.t $\tilde{c}$ and setting it to zero, we get

$$\begin{aligned} 0 &= \frac{\partial\mathcal{H}^h}{\partial\tilde{c}} \\ &= \delta e^{-\delta t}\tilde{c}^{-\frac{1}{\psi}}\mathcal{W}_t^{1-\frac{1}{\psi}}\lambda_t^h - \mathcal{W}_t p_t^h, \end{aligned}$$

then

$$\tilde{c}_t = \delta^{\psi} e^{-\delta\psi t} \mathcal{W}_t^{-1} \Big(\frac{\lambda_t^h}{p_t^h}\Big)^{\psi}. \tag{3.52}$$

Similarly, for the first derivative of $\mathcal{H}^h$ w.r.t π, we obtain

$$\begin{aligned} 0 &= \frac{\partial \mathcal{H}^h}{\partial \pi} \\ &= \mathcal{W}_t \mu_t p_t^h + \mathcal{W}_t \sigma_t q_t^h \\ &= \mu_t p_t^h + \sigma_t q_t^h, \end{aligned}$$

then

$$\mu_t p_t^h + \sigma_t q_t^h = 0. \tag{3.53}$$

We define

$$\phi(\mathcal{W}, \lambda^h, X, t) = e^{-\delta t} (\mathcal{W}_t + \lambda h(X_t))^{-\frac{1}{\psi}} \lambda_t^h e^{D^h(X,t)} = p_t^h,$$

with $D^h(X,t)$ the process which satisfies the BSDE

$$\begin{cases} \mathrm{d}D^h(X,t) = -F_1(X, D^h, G_1^h, t)\mathrm{d}t + \rho G_1^h(X,t)\mathrm{d}B_t + \tilde{\rho} G_1^h(X,t)\mathrm{d}\tilde{B}_t \\ D^h(X,T) = 0, \end{cases}$$

where $F_1(X, D^h, G^h, t)$ and $G_1^h(t)$ will be determined.

We first give the conditional form of the process p_t^h.

Integrating both sides of Equation (3.50) from t to T, we obtain

$$p^h(T) - p_t^h = \int_t^T \Big\{ q_s^h \mathrm{d}B_s + \tilde{q}_s^h \mathrm{d}\tilde{B}_s \Big\}$$

Applying the conditional expectation on both sides yields

$$p_t^h = \mathbb{E}\Big[p^h(T) | \mathcal{F}_t\Big], \tag{3.54}$$

where

$$\mathbb{E}\Big[\int_t^T q_s^h \mathrm{d}B_s|\mathcal{F}_t\Big] = 0,$$

and

$$\mathbb{E}\Big[\int_t^T \tilde{q}_s^h \mathrm{d}\tilde{B}_s|\mathcal{F}_t\Big] = 0,$$

since, by definition of a backward stochastic differential equation q_s^h and $\tilde{q}_s^h$ belong to $\mathcal{L}^2[t,T]$ for all $s \in [t,T]$.

On first hand, let us apply the Ito's formula on the process p_t^h.

Calculation yields

$$\begin{aligned}
&\mathrm{d}e^{-\delta t+D^h(X,t)} \\
&= e^{-\delta t+D^h(X,t)}\Big[\Big(-\delta - F_1(X,D^h,G_1^h,t) + \frac{1}{2}(G_1^h)^2(X,t)\Big)\mathrm{d}t \\
&\qquad + \rho G_1^h(X,t)\mathrm{d}B_t + \tilde{\rho}G_1^h(X,t)\mathrm{d}\tilde{B}_t\Big],
\end{aligned}$$

$$\begin{aligned}
&\mathrm{d}(\mathcal{W}_t+\lambda h(X_t))^{-\frac{1}{\psi}} \\
&= \Big[-\frac{1}{\psi}(\mathcal{W}_t+\lambda h(X_t))^{-1-\frac{1}{\psi}}\mathcal{W}_t(r_t+\pi_t\mu_t-\tilde{c}_t) \\
&\quad + \frac{1}{2}\frac{1}{\psi}\Big(1+\frac{1}{\psi}\Big)(\mathcal{W}_t+\lambda h(X_t))^{-2-\frac{1}{\psi}}\mathcal{W}_t^2\pi_t^2\sigma_t^2 \\
&\quad - \frac{1}{\psi}\lambda h_x(\mathcal{W}_t+\lambda h(X_t))^{-1-\frac{1}{\psi}}b(X_t) \\
&\quad - \frac{1}{2}\frac{1}{\psi}\lambda h_{xx}(\mathcal{W}_t+\lambda h(X_t))^{-1-\frac{1}{\psi}}a^2(X_t) \\
&\quad + \frac{1}{2}\frac{1}{\psi}\Big(1+\frac{1}{\psi}\Big)\lambda^2 h_x^2(\mathcal{W}_t+\lambda h(X_t))^{-2-\frac{1}{\psi}}a^2(X_t) \\
&\quad + \frac{1}{\psi}\Big(1+\frac{1}{\psi}\Big)\lambda h_x(\mathcal{W}_t+\lambda h(X_t))^{-2-\frac{1}{\psi}}\rho a(X_t)\mathcal{W}_t\pi_t\sigma_t\Big]\mathrm{d}t \\
&\quad + \Big[-\frac{1}{\psi}(\mathcal{W}_t+\lambda h(X_t))^{-1-\frac{1}{\psi}}\mathcal{W}_t\pi_t\sigma_t \\
&\qquad - \frac{1}{\psi}\lambda h_x(\mathcal{W}_t+\lambda h(X_t))^{-1-\frac{1}{\psi}}\rho a(X_t)\Big]\mathrm{d}B_t \\
&\quad + \Big[-\frac{1}{\psi}\lambda h_x(\mathcal{W}_t+\lambda h(X_t))^{-1-\frac{1}{\psi}}\tilde{\rho}a(X_t)\Big]\mathrm{d}\tilde{B}_t,
\end{aligned}$$

$$
\begin{aligned}
&\mathrm{d}\Big((\mathcal{W}_t+\lambda h(X_t))^{-\frac{1}{\psi}}\lambda_t^h\Big)\\
=&(\mathcal{W}_t+\lambda h(X_t))^{-\frac{1}{\psi}}\lambda_t^h\Big[\Big\{-\frac{1}{\psi}(\mathcal{W}_t+\lambda h(X_t))^{-1}\mathcal{W}_t(r_t+\pi_t\mu_t-\tilde{c}_t)\\
&+\frac{1}{2}\frac{1}{\psi}\Big(1+\frac{1}{\psi}\Big)(\mathcal{W}_t+\lambda h(X_t))^{-2}\mathcal{W}_t^2\pi_t^2\sigma_t^2-\frac{1}{\psi}\lambda h_x(\mathcal{W}_t+\lambda h(X_t))^{-1}b(X_t)\\
&-\frac{1}{2}\frac{1}{\psi}\lambda h_{xx}(\mathcal{W}_t+\lambda h(X_t))^{-1}a^2(X_t)-\frac{\theta-1}{\psi}\frac{\tilde{\mathbb{Z}}_t}{\mathbb{Y}_t}\lambda h_x(\mathcal{W}_t+\lambda h(X_t))^{-1}\tilde{\rho}a(X_t)\\
&+\frac{1}{\psi}\Big(1+\frac{1}{\psi}\Big)\lambda h_x(\mathcal{W}_t+\lambda h(X_t))^{-2}\rho a(X_t)\mathcal{W}_t\pi_t\sigma_t\\
&-\frac{1}{2}(\theta-1)\frac{\mathbb{Z}_t^2+\tilde{\mathbb{Z}}_t^2}{\mathbb{Y}_t^2}-\frac{\theta-1}{\psi}\frac{\mathbb{Z}_t}{\mathbb{Y}_t}(\mathcal{W}_t+\lambda h(X_t))^{-1}\mathcal{W}_t\pi_t\sigma_t\\
&-\frac{\theta-1}{\psi}\frac{\mathbb{Z}_t}{\mathbb{Y}_t}\lambda h_x(\mathcal{W}_t+\lambda h(X_t))^{-1}\rho a(X_t)\\
&+\frac{1}{2}\frac{1}{\psi}\Big(1+\frac{1}{\psi}\Big)\lambda^2h_x^2(\mathcal{W}_t+\lambda h(X_t))^{-2}a^2(X_t)\Big\}\mathrm{d}t\\
&+\Big\{-\frac{1}{\psi}(\mathcal{W}_t+\lambda h(X_t))^{-1}\mathcal{W}_t\pi_t\sigma_t-\frac{1}{\psi}\lambda h_x(\mathcal{W}_t+\lambda h(X_t))^{-1}\rho a(X_t)\\
&+(\theta-1)\frac{\mathbb{Z}_t}{\mathbb{Y}_t}\Big\}\mathrm{d}B_t\\
&+\Big\{-\frac{1}{\psi}\lambda h_x(\mathcal{W}_t+\lambda h(X_t))^{-1}\tilde{\rho}a(X_t)+(\theta-1)\frac{\tilde{\mathbb{Z}}_t}{\mathbb{Y}_t}\Big\}\mathrm{d}\tilde{B}_t\Big].
\end{aligned}
$$

Thus, p_t^h satisfies the BSDE

$$
\begin{aligned}
&\mathrm{d}p_t^h\\
&=p_t^h\Big[-\delta-F_1(X,D^h,G_1^h,t)+\frac{1}{2}G_1^{h2}(X,t)\\
&\quad-\frac{1}{\psi}(\mathcal{W}_t+\lambda h(X_t))^{-1}\mathcal{W}_t(r_t+\pi_t\mu_t-\tilde{c}_t)\\
&\quad+\frac{1}{2}\frac{1}{\psi}\left(1+\frac{1}{\psi}\right)(\mathcal{W}_t+\lambda h(X_t))^{-2}\mathcal{W}_t^2\pi_t^2\sigma_t^2\\
&\quad-\frac{1}{\psi}\lambda h_x(\mathcal{W}_t+\lambda h(X_t))^{-1}b(X_t)-\frac{1}{2}\frac{1}{\psi}\lambda h_{xx}(\mathcal{W}_t+\lambda h(X_t))^{-1}a^2(X_t)\\
&\quad+\frac{1}{2}\frac{1}{\psi}\Big(1+\frac{1}{\psi}\Big)\lambda^2h_x^2(\mathcal{W}_t+\lambda h(X_t))^{-2}a^2(X_t)\\
&\quad+\frac{1}{\psi}\Big(1+\frac{1}{\psi}\Big)\lambda h_x(\mathcal{W}_t+\lambda h(X_t))^{-2}\rho a(X_t)\mathcal{W}_t\pi_t\sigma_t\\
&\quad-\frac{1}{2}(\theta-1)\frac{\mathbb{Z}_t^2+\tilde{\mathbb{Z}}_t^2}{\mathbb{Y}_t^2}-\frac{\theta-1}{\psi}\frac{\mathbb{Z}_t}{\mathbb{Y}_t}(\mathcal{W}_t+\lambda h(X_t))^{-1}\mathcal{W}_t\pi_t\sigma_t\\
&\quad-\frac{\theta-1}{\psi}\frac{\mathbb{Z}_t}{\mathbb{Y}_t}\lambda h_x(\mathcal{W}_t+\lambda h(X_t))^{-1}\rho a(X_t)-\frac{\theta-1}{\psi}\frac{\tilde{\mathbb{Z}}_t}{\mathbb{Y}_t}\lambda h_x(\mathcal{W}_t+\lambda h(X_t))^{-1}\\
&\quad-\frac{1}{\psi}(\mathcal{W}_t+\lambda h(X_t))^{-1}\mathcal{W}_t\pi_t\sigma_t\rho G_1^h(X,t)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{\psi}\lambda h_x(\mathcal{W}_t + \lambda h(X_t))^{-1}\rho^2 a(X_t)G_1^h(X,t) + (\theta-1)\frac{\mathbb{Z}_t}{\mathbb{Y}_t}\rho G_1^h(X,t) \\
&- \frac{1}{\psi}\lambda h_x(\mathcal{W}_t + \lambda h(X_t))^{-1}\tilde{\rho}^2 a(X_t)G_1^h(X,t) + (\theta-1)\frac{\tilde{\mathbb{Z}}_t}{\mathbb{Y}_t}\tilde{\rho} G_1^h(X,t)\Big]\mathrm{d}t \\
p_t^h\Big[&-\frac{1}{\psi}(\mathcal{W}_t + \lambda h(X_t))^{-1}\mathcal{W}_t\pi_t\sigma_t - \frac{1}{\psi}\lambda h_x(\mathcal{W}_t + \lambda h(X_t))^{-1}\rho a(X_t) \\
&+ (\theta-1)\frac{\mathbb{Z}_t}{\mathbb{Y}_t} + \rho G_1^h(X,t)\Big]\mathrm{d}B_t \\
p_t^h\Big[&-\frac{1}{\psi}\lambda h_x(\mathcal{W}_t + \lambda h(X_t))^{-1}\tilde{\rho} a(X_t) + (\theta-1)\frac{\tilde{\mathbb{Z}}_t}{\mathbb{Y}_t} + \tilde{\rho} G_1^h(X,t)\Big]\mathrm{d}\tilde{B}_t \quad (3.55)
\end{aligned}
$$

Comparing the diffusion terms in Equations (3.50) and (3.55), we get

$$
\begin{aligned}
q_t^h = p_t^h\Big[&-\frac{1}{\psi}(\mathcal{W}_t + \lambda h(X_t))^{-1}\mathcal{W}_t\pi_t\sigma_t - \frac{1}{\psi}\lambda h_x(\mathcal{W}_t + \lambda h(X_t))^{-1}\rho a(X_t) \\
&+ (\theta-1)\frac{\mathbb{Z}_t}{\mathbb{Y}_t} + \rho G_1^h(X,t)\Big],
\end{aligned}
$$

and

$$
\tilde{q}_t^h = p_t^h\Big[-\frac{1}{\psi}\lambda h_x(\mathcal{W}_t + \lambda h(X_t))^{-1}\tilde{\rho} a(X_t) + (\theta-1)\frac{\tilde{\mathbb{Z}}_t}{\mathbb{Y}_t} + \tilde{\rho} G_1^h(X,t)\Big].
$$

Secondly, by the non-linear Feynman-Kăc formula we get that $\phi(\mathcal{W}, \lambda^h, X, t)$ satisfies the following partial differential equation

$$
\begin{aligned}
&\phi_t + b\phi_x + \alpha_1\phi_{\mathcal{W}} + \alpha_2\phi_\lambda + \frac{1}{2}a^2\phi_{xx} + \frac{1}{2}\beta_1^2\phi_{\mathcal{W}\mathcal{W}} + \beta_1\rho a\phi_{x\mathcal{W}} \\
+&\frac{1}{2}(\beta_2^2 + \tilde{\beta}_2^2)\phi_{\lambda\lambda} + (\beta_2\rho a + \tilde{\beta}_2\tilde{\rho} a)\phi_{x\lambda} + \beta_1\beta_2\phi_{\lambda\mathcal{W}} + f^h(X,t)\phi = 0, \quad (3.56)
\end{aligned}
$$

where we denote

$$
\phi_t = \frac{\partial\phi}{\partial t}, \ \phi_x = \frac{\partial\phi}{\partial x}, \ \phi_{xx} = \frac{\partial^2\phi}{\partial x^2},
$$

also

$$
\begin{aligned}
&\alpha_1(\mathcal{W}, t) = \mathcal{W}_t(r(X_t) + \pi_t\mu(X_t) - \tilde{c}_t), \quad \beta_1(\mathcal{W}, t) = \mathcal{W}_t\pi_t\sigma(X_t), \\
&\alpha_2(\lambda, t) = -\frac{1}{2}(\theta-1)\lambda_t\frac{\mathbb{Z}_t^2 + \tilde{\mathbb{Z}}_t^2}{\mathbb{Y}_t^2}, \quad \beta_2(\lambda, t) = (\theta-1)\lambda_t^h\frac{\mathbb{Z}_t}{\mathbb{Y}_t}, \\
&\tilde{\beta}_2(\lambda, t) = (\theta-1)\lambda_t^h\frac{\tilde{\mathbb{Z}}_t}{\mathbb{Y}_t},
\end{aligned}
$$

and

$$\mathrm{d}X_t = b(X_t,t)\mathrm{d}t + \rho a(X_t,t)\mathrm{d}B_t + \tilde{\rho} a(X_t,t)\mathrm{d}\tilde{B}_t.$$

Moreover $f^h(X,t)p_t^h$ is the drift of p_t^h.

Also,

$$\begin{aligned}
\phi_t &= D_t^h\phi - \delta\phi,\\
\phi_x &= -\frac{1}{\psi}\lambda h_x(\mathcal{W}_t + \lambda h(X_t))^{-1}\phi + D_x^h\phi,\\
\phi_{\mathcal{W}} &= -\frac{1}{\psi}(\mathcal{W}_t + \lambda h(X_t))^{-1}\phi,\ \ \phi_{\mathcal{W}\lambda} = -\frac{1}{\psi}(\lambda_t^h)^{-1}(\mathcal{W}_t + \lambda h(X_t))^{-1}\phi,\\
\phi_{xx} &= -\frac{1}{\psi}\lambda h_{xx}(\mathcal{W}_t + \lambda h(X_t))^{-1}\phi + \frac{1}{\psi}\Big(1+\frac{1}{\psi}\Big)\lambda^2 h_x^2(\mathcal{W}_t + \lambda h(X_t))^{-2}\phi\\
&\quad -\frac{1}{\psi}\lambda h_x D_x^h(\mathcal{W}_t + \lambda h(X_t))^{-1}\phi + D_{xx}^h\phi + (D_x^h)^2\phi,\\
\phi_\lambda &= (\lambda_t^h)^{-1}\phi,\ \ \phi_{\lambda\lambda} = 0,\\
\phi_{\mathcal{W}\mathcal{W}} &= \frac{1}{\psi}\Big(1+\frac{1}{\psi}\Big)(\mathcal{W}_t + \lambda h(X_t))^{-2}\phi,\\
\phi_{x\mathcal{W}} &= \frac{1}{\psi}\Big(1+\frac{1}{\psi}\Big)\lambda h_x(\mathcal{W}_t + \lambda h(X_t))^{-2}\phi - \frac{1}{\psi}(\mathcal{W}_t + \lambda h(X_t))^{-1}D_x^h\phi,\\
\phi_{x\lambda} &= -\frac{1}{\psi}\lambda h_x(\lambda_t^h)^{-1}(\mathcal{W}_t + \lambda h(X_t))^{-1}\phi + (\lambda_t^h)^{-1}D_x^h\phi.
\end{aligned}$$

Substituting the later expressions into Equation (3.56) gives

$$\begin{aligned}
&D_t^h - \delta - \frac{1}{\psi}(\mathcal{W}_t + \lambda h(X_t))^{-1}b\lambda h_x + bD_x^h - \frac{1}{\psi}\alpha_1(\mathcal{W}_t + \lambda h(X_t))^{-1}\\
&+\alpha_2(\lambda_t^h)^{-1} + \frac{1}{2}a^2D_{xx}^h + \frac{1}{2}(aD_x^h)^2 - \frac{1}{2}\frac{1}{\psi}(\mathcal{W}_t + \lambda h(X_t))^{-1}a^2\lambda h_{xx}\\
&+\frac{1}{2}\frac{1}{\psi}\Big(1+\frac{1}{\psi}\Big)(\mathcal{W}_t + \lambda h(X_t))^{-2}\lambda^2a^2h_x^2 - \frac{1}{\psi}(\mathcal{W}_t + \lambda h(X_t))^{-1}a^2\lambda h_xD_x^h\\
&+\frac{1}{2}\frac{1}{\psi}\Big(1+\frac{1}{\psi}\Big)\beta_1^2(\mathcal{W}_t + \lambda h(X_t))^{-2} + \frac{1}{\psi}\Big(1+\frac{1}{\psi}\Big)(\mathcal{W}_t + \lambda h(X_t))^{-2}\beta_1\rho a\lambda h_x\\
&-\frac{1}{\psi}(\mathcal{W}_t + \lambda h(X_t))^{-1}(\beta_2\rho a + \tilde{\beta}_2\tilde{\rho}a)\lambda h_x(\lambda_t^h)^{-1} + (\beta_2\rho a + \tilde{\beta}_2\tilde{\rho}a)(\lambda_t^h)^{-1}D_x^h\\
&-\frac{1}{\psi}(\mathcal{W}_t + \lambda h(X_t))^{-1}\beta_1\beta_2(\lambda_t^h)^{-1} - \frac{1}{\psi}(\mathcal{W}_t + \lambda h(X_t))^{-1}\beta_1\rho aD_x^h\\
&+f^h(X,t) = 0.
\end{aligned}$$

Thus, from the non-linear Feynman-Kǎc formula, we obtain

$$
\begin{aligned}
&F_1(X, D^h, G_1^h, t) \\
&= -\delta - \frac{1}{\psi}(\mathcal{W}_t + \lambda h(X_t))^{-1} b\lambda h_x - \frac{1}{\psi}\alpha_1(\mathcal{W}_t + \lambda h(X_t))^{-1} + \alpha_2(\lambda_t^h)^{-1} \\
&\quad + \frac{1}{2}(G_1^h)^2 - \frac{1}{2}\frac{1}{\psi}(\mathcal{W}_t + \lambda h(X_t))^{-1} a^2 \lambda h_{xx} - \frac{1}{\psi}(\mathcal{W}_t + \lambda h(X_t))^{-1} a\lambda h_x G_1^h \\
&\quad + \frac{1}{2}\frac{1}{\psi}\Big(1 + \frac{1}{\psi}\Big)(\mathcal{W}_t + \lambda h(X_t))^{-2}\lambda^2 a^2 h_x^2 - \frac{1}{\psi}(\mathcal{W}_t + \lambda h(X_t))^{-1}\beta_1 \rho G_1^h \\
&\quad + \frac{1}{\psi}\Big(1 + \frac{1}{\psi}\Big)(\mathcal{W}_t + \lambda h(X_t))^{-2}\beta_1 \rho a \lambda h_x - \frac{1}{\psi}(\mathcal{W}_t + \lambda h(X_t))^{-1}\beta_1\beta_2(\lambda_t^h)^{-1} \\
&\quad - \frac{1}{\psi}(\mathcal{W}_t + \lambda h(X_t))^{-1}(\beta_2\rho + \tilde{\beta}_2\tilde{\rho}) a\lambda h_x (\lambda_t^h)^{-1} + (\beta_2\rho + \tilde{\beta}_2\tilde{\rho})(\lambda_t^h)^{-1} G_1^h \\
&\quad + \frac{1}{2}\frac{1}{\psi}\Big(1 + \frac{1}{\psi}\Big)\beta_1^2(\mathcal{W}_t + \lambda h(X_t))^{-2} + f^h(X, t)
\end{aligned} \tag{3.57}
$$

with $G_1^h = aD_x^h$.

Substituting α_1, α_2, β_1, $\tilde{\beta}_1$, β_2, $\tilde{\beta}_2$ by their respective expressions into Equation (3.57) gives

$$
\begin{aligned}
&F_1(X, D^h, G_1^h, t) \\
&= -\delta - \frac{1}{\psi}(\mathcal{W}_t + \lambda h(X_t))^{-1} b\lambda h_x - \frac{1}{\psi}(\mathcal{W}_t + \lambda h(X_t))^{-1}\mathcal{W}_t(r_t + \pi_t\mu_t - \tilde{c}_t) \\
&\quad - \frac{1}{2}(\theta - 1)\frac{\mathbb{Z}_t^2 + \tilde{\mathbb{Z}}_t^2}{\mathbb{Y}_t^2} + \frac{1}{2}(G_1^h)^2 - \frac{1}{2}\frac{1}{\psi}(\mathcal{W}_t + \lambda h(X_t))^{-1} a^2\lambda h_{xx} \\
&\quad + \frac{1}{2}\frac{1}{\psi}\Big(1 + \frac{1}{\psi}\Big)(\mathcal{W}_t + \lambda h(X_t))^{-2}\lambda^2 a^2 h_x^2 - \frac{1}{\psi}(\mathcal{W}_t + \lambda h(X_t))^{-1} a\lambda h_x G_1^h \\
&\quad + \frac{1}{2}\frac{1}{\psi}\Big(1 + \frac{1}{\psi}\Big)(\mathcal{W}_t + \lambda h(X_t))^{-2}\mathcal{W}_t^2\pi_t^2\sigma_t^2 + (\theta - 1)\frac{\rho\mathbb{Z}_t + \tilde{\rho}\tilde{\mathbb{Z}}_t}{\mathbb{Y}_t} G_1^h \\
&\quad + \frac{1}{\psi}\Big(1 + \frac{1}{\psi}\Big)(\mathcal{W}_t + \lambda h(X_t))^{-2}\mathcal{W}_t\pi_t\sigma_t\rho a\lambda h_x \\
&\quad - \frac{1}{\psi}(\mathcal{W}_t + \lambda h(X_t))^{-1}(\theta - 1)\frac{\rho\mathbb{Z}_t + \tilde{\rho}\tilde{\mathbb{Z}}_t}{\mathbb{Y}_t} a\lambda h_x \\
&\quad - \frac{1}{\psi}(\mathcal{W}_t + \lambda h(X_t))^{-1}\mathcal{W}_t\pi_t\sigma_t(\theta - 1)\frac{\mathbb{Z}_t}{\mathbb{Y}_t} - \frac{1}{\psi}(\mathcal{W}_t + \lambda h(X_t))^{-1}\mathcal{W}_t\pi_t\sigma_t\rho G_1^h.
\end{aligned} \tag{3.58}
$$

From the terminal condition of the process $\mathbb{Y}_t$, we assume at the optimal strategies that

$$\mathbb{Y}_t = \frac{1}{1-\frac{1}{\psi}} e^{-\delta t}(\mathcal{W}_t + \lambda h(X_t))^{1-\frac{1}{\psi}} e^{E^h(X,t)},$$

where the process $E^h(X,t)$ satisfies the BSDE

$$\begin{cases} \mathrm{d}E^h(X,t) = -F_2(X, E^h, G_2^h, t)\mathrm{d}t + \rho G_2^h(X,t)\mathrm{d}B_t + \tilde{\rho} G_2^h(X,t)\mathrm{d}\tilde{B}_t \\ E^h(X,T) = 0. \end{cases}$$

By the use of Itô's formula, we have the following

$$\begin{aligned}
&\mathrm{d}(\mathcal{W}_t + \lambda h(X_t))^{1-\frac{1}{\psi}} \\
&= \Big[\Big(1-\frac{1}{\psi}\Big)(\mathcal{W}_t + \lambda h(X_t))^{-\frac{1}{\psi}} \mathcal{W}_t(r_t + \pi_t \mu_t - \tilde{c}_t) \\
&\quad - \frac{1}{2}\frac{1}{\psi}\Big(1-\frac{1}{\psi}\Big)(\mathcal{W}_t + \lambda h(X_t))^{-1-\frac{1}{\psi}} \mathcal{W}_t^2 \pi_t^2 \sigma_t^2 \\
&\quad + \Big(1-\frac{1}{\psi}\Big)\lambda h_x (\mathcal{W}_t + \lambda h(X_t))^{-\frac{1}{\psi}} b(X_t) \\
&\quad + \frac{1}{2}\Big(1-\frac{1}{\psi}\Big)\lambda h_{xx} (\mathcal{W}_t + \lambda h(X_t))^{-\frac{1}{\psi}} a^2(X_t) \\
&\quad - \frac{1}{2}\frac{1}{\psi}\Big(1-\frac{1}{\psi}\Big)\lambda^2 h_x^2 (\mathcal{W}_t + \lambda h(X_t))^{-1-\frac{1}{\psi}} a^2(X_t) \\
&\quad - \frac{1}{\psi}\Big(1-\frac{1}{\psi}\Big)\lambda h_x (\mathcal{W}_t + \lambda h(X_t))^{-1-\frac{1}{\psi}} \rho a(X_t)\mathcal{W}_t \pi_t \sigma_t\Big]\mathrm{d}t \\
&\quad + \Big[\Big(1-\frac{1}{\psi}\Big)(\mathcal{W}_t + \lambda h(X_t))^{-\frac{1}{\psi}} \mathcal{W}_t \pi_t \sigma_t \\
&\qquad + \Big(1-\frac{1}{\psi}\Big)\lambda h_x (\mathcal{W}_t + \lambda h(X_t))^{-\frac{1}{\psi}} \rho a(X_t)\Big]\mathrm{d}B_t \\
&\quad + \Big[\Big(1-\frac{1}{\psi}\Big)\lambda h_x (\mathcal{W}_t + \lambda h(X_t))^{-\frac{1}{\psi}} \tilde{\rho} a(X_t)\Big]\mathrm{d}\tilde{B}_t
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{d}e^{-\delta t + E^h(X,t)} \\
&= e^{-\delta t + E^h(X,t)}\Big[\Big(-\delta - F_2(X, E^h, G_2^h, t) + \frac{1}{2}(G_2^h)^2(X,t)\Big)\mathrm{d}t \\
&\qquad + \rho G_2^h(X,t)\mathrm{d}B_t + \tilde{\rho} G_2^h(X,t)\mathrm{d}\tilde{B}_t\Big],
\end{aligned}$$

Then the diffusion terms of the associated BSDE of the process $\mathbb{Y}_t$ are given by

$$\mathbb{Z}_t = \mathbb{Y}_t\Big[\Big(1-\frac{1}{\psi}\Big)(\mathcal{W}_t+\lambda h(X_t))^{-1}\mathcal{W}_t\pi_t\sigma_t + \rho G_2^h(X,t)$$
$$+\Big(1-\frac{1}{\psi}\Big)\lambda h_x(\mathcal{W}_t+\lambda h(X_t))^{-1}\rho a(X_t)\Big] \tag{3.59}$$
$$\tilde{\mathbb{Z}}_t = \mathbb{Y}_t\Big[\Big(1-\frac{1}{\psi}\Big)\lambda h_x(\mathcal{W}_t+\lambda h(X_t))^{-1}\tilde{\rho} a(X_t) + \tilde{\rho} G_2^h(X,t)\Big], \tag{3.60}$$

and its drift is

$$-\mathbb{Y}_t\Big[-\delta - F_2(X,E^h,G_2^h,t) + \frac{1}{2}(G_1^h)^2(X,t)$$
$$+\Big(1-\frac{1}{\psi}\Big)(\mathcal{W}_t+\lambda h(X_t))^{-1}\mathcal{W}_t(r_t+\pi_t\mu_t-\tilde{c}_t)$$
$$-\frac{1}{2}\frac{1}{\psi}\Big(1-\frac{1}{\psi}\Big)(\mathcal{W}_t+\lambda h(X_t))^{-2}\mathcal{W}_t^2\pi_t^2\sigma_t^2$$
$$+\Big(1-\frac{1}{\psi}\Big)\lambda h_x(\mathcal{W}_t+\lambda h(X_t))^{-1}b(X_t)$$
$$+\frac{1}{2}\Big(1-\frac{1}{\psi}\Big)\lambda h_{xx}(\mathcal{W}_t+\lambda h(X_t))^{-1}a^2(X_t)$$
$$-\frac{1}{2}\frac{1}{\psi}\Big(1-\frac{1}{\psi}\Big)\lambda^2 h_x^2(\mathcal{W}_t+\lambda h(X_t))^{-2}a^2(X_t)$$
$$-\frac{1}{\psi}\Big(1-\frac{1}{\psi}\Big)\lambda h_x(\mathcal{W}_t+\lambda h(X_t))^{-2}\rho a(X_t)\mathcal{W}_t\pi_t\sigma_t$$
$$+\Big(1-\frac{1}{\psi}\Big)(\mathcal{W}_t+\lambda h(X_t))^{-1}\mathcal{W}_t\pi_t\sigma_t\rho G_2^h(X,t)$$
$$+\Big(1-\frac{1}{\psi}\Big)\lambda h_x(\mathcal{W}_t+\lambda h(X_t))^{-1}a(X_t)G_2^h(X,t)\Big] \tag{3.61}$$

In order to find $F_2(X,E^h,G_2^h,t)$ we equate expressions (3.47) and (3.61).

$$\delta e^{-\delta t}\frac{\tilde{c}_t^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}}\mathcal{W}_t^{1-\frac{1}{\psi}} + \frac{1}{2}(\theta-1)\frac{\mathbb{Z}_t^2+\tilde{\mathbb{Z}}_t^2}{\mathbb{Y}_t}$$
$$+\mathbb{Y}_t\Big\{-\delta - F_2(X,E^h,G_2^h,t) + \frac{1}{2}(G_1^h)^2(X,t)$$
$$+\Big(1-\frac{1}{\psi}\Big)(\mathcal{W}_t+\lambda h(X_t))^{-1}\mathcal{W}_t(r_t+\pi_t\mu_t-\tilde{c}_t)$$
$$-\frac{1}{2}\frac{1}{\psi}\Big(1-\frac{1}{\psi}\Big)(\mathcal{W}_t+\lambda h(X_t))^{-2}\mathcal{W}_t^2\pi_t^2\sigma_t^2$$
$$+\Big(1-\frac{1}{\psi}\Big)\lambda h_x(\mathcal{W}_t+\lambda h(X_t))^{-1}b(X_t)$$
$$+\frac{1}{2}\Big(1-\frac{1}{\psi}\Big)\lambda h_{xx}(\mathcal{W}_t+\lambda h(X_t))^{-1}a^2(X_t)$$
$$-\frac{1}{2}\frac{1}{\psi}\Big(1-\frac{1}{\psi}\Big)\lambda^2 h_x^2(\mathcal{W}_t+\lambda h(X_t))^{-2}a^2(X_t)$$

$$
\begin{aligned}
&- \frac{1}{\psi}\Big(1-\frac{1}{\psi}\Big)\lambda h_x(\mathcal{W}_t+\lambda h(X_t))^{-2}\rho a(X_t)\mathcal{W}_t\pi_t\sigma_t \\
&+ \Big(1-\frac{1}{\psi}\Big)(\mathcal{W}_t+\lambda h(X_t))^{-1}\mathcal{W}_t\pi_t\sigma_t\rho G_2^h(X,t) \\
&+ \Big(1-\frac{1}{\psi}\Big)\lambda h_x(\mathcal{W}_t+\lambda h(X_t))^{-1}a(X_t)G_2^h(X,t)\Big\} = 0
\end{aligned}
$$

Factoring by $\mathbb{Y}_t$, we get

$$
\begin{aligned}
&\mathbb{Y}_t\Big[\delta\tilde{c}_t^{1-\frac{1}{\psi}}(\mathcal{W}_t+\lambda h(X_t))^{-1+\frac{1}{\psi}}\mathcal{W}_t^{1-\frac{1}{\psi}}e^{-E^h(X,t)} + \frac{1}{2}(\theta-1)\frac{\mathbb{Z}_t^2+\tilde{\mathbb{Z}}_t^2}{\mathbb{Y}_t^2} \\
&- \delta - F_2(X,E^h,G_2^h,t) + \Big(1-\frac{1}{\psi}\Big)(\mathcal{W}_t+\lambda h(X_t))^{-1}\mathcal{W}_t(r_t+\pi_t\mu_t-\tilde{c}_t) \\
&- \frac{1}{2}\frac{1}{\psi}\Big(1-\frac{1}{\psi}\Big)(\mathcal{W}_t+\lambda h(X_t))^{-2}\mathcal{W}_t^2\pi_t^2\sigma_t^2 + \frac{1}{2}(G_1^h)^2(X,t) \\
&+ \Big(1-\frac{1}{\psi}\Big)\lambda h_x(\mathcal{W}_t+\lambda h(X_t))^{-1}b(X_t) \\
&+ \frac{1}{2}\Big(1-\frac{1}{\psi}\Big)\lambda h_{xx}(\mathcal{W}_t+\lambda h(X_t))^{-1}a^2(X_t) \\
&- \frac{1}{2}\frac{1}{\psi}\Big(1-\frac{1}{\psi}\Big)\lambda^2 h_x^2(\mathcal{W}_t+\lambda h(X_t))^{-2}a^2(X_t) \\
&- \frac{1}{\psi}\Big(1-\frac{1}{\psi}\Big)\lambda h_x(\mathcal{W}_t+\lambda h(X_t))^{-2}\rho a(X_t)\mathcal{W}_t\pi_t\sigma_t \\
&+ \Big(1-\frac{1}{\psi}\Big)(\mathcal{W}_t+\lambda h(X_t))^{-1}\mathcal{W}_t\pi_t\sigma_t\rho G_2^h(X,t) \\
&+ \Big(1-\frac{1}{\psi}\Big)\lambda h_x(\mathcal{W}_t+\lambda h(X_t))^{-1}a(X_t)G_2^h(X,t)\Big] = 0,
\end{aligned}
$$

then substituting $\mathbb{Z}_t$ and $\tilde{\mathbb{Z}}_t$ by their respective expressions in Equations (3.59) and (3.60) yields

$$
\begin{aligned}
&F_2(X,E^h,G_2^h,t) \\
&= \delta\tilde{c}_t^{1-\frac{1}{\psi}}(\mathcal{W}_t+\lambda h(X_t))^{-1+\frac{1}{\psi}}\mathcal{W}_t^{1-\frac{1}{\psi}}e^{-E^h(X,t)} - \delta + \frac{1}{2}\theta(G_1^h)^2(X,t) \\
&\quad + \Big(1-\frac{1}{\psi}\Big)(\mathcal{W}_t+\lambda h(X_t))^{-1}\mathcal{W}_t(r_t+\pi_t\mu_t-\tilde{c}_t) \\
&\quad - \frac{1}{2}\Big(1-\frac{1}{\psi}\Big)\gamma(\mathcal{W}_t+\lambda h(X_t))^{-2}\mathcal{W}_t^2\pi_t^2\sigma_t^2 \\
&\quad + \Big(1-\frac{1}{\psi}\Big)\lambda h_x(\mathcal{W}_t+\lambda h(X_t))^{-1}b(X_t) \\
&\quad + \frac{1}{2}\Big(1-\frac{1}{\psi}\Big)\lambda h_{xx}(\mathcal{W}_t+\lambda h(X_t))^{-1}a^2(X_t)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}\Big(1-\frac{1}{\psi}\Big)\gamma\lambda^2 h_x^2(\mathcal{W}_t+\lambda h(X_t))^{-2}a^2(X_t)\\
&-\Big(1-\frac{1}{\psi}\Big)\gamma\lambda h_x(\mathcal{W}_t+\lambda h(X_t))^{-2}\rho a(X_t)\mathcal{W}_t\pi_t\sigma_t\\
&+\theta\Big(1-\frac{1}{\psi}\Big)(\mathcal{W}_t+\lambda h(X_t))^{-1}\mathcal{W}_t\pi_t\sigma_t\rho G_2^h(X,t)\\
&+\theta\Big(1-\frac{1}{\psi}\Big)\lambda h_x(\mathcal{W}_t+\lambda h(X_t))^{-1}a(X_t)G_2^h(X,t).
\end{aligned}
$$

Similarly if $G_1^h(X,t)=G_2^h(X,t)=G^h(X,t)$ then $D^h(X,t)=E^h(X,t)$. Assuming $G_1^h(X,t)=G_2^h(X,t)$ and substituting $\mathbb{Z}_t$, $\tilde{\mathbb{Z}}_t$ by their respective expressions in Equations (3.59) and (3.60), we get

$$
\begin{aligned}
&F_1(X,D^h,G_1^h,t)\\
&=-\delta-\frac{1}{\psi}(\mathcal{W}_t+\lambda h(X_t))^{-1}b\lambda h_x-\frac{1}{\psi}(\mathcal{W}_t+\lambda h(X_t))^{-1}\mathcal{W}_t(r_t+\pi_t\mu_t-\tilde{c}_t)\\
&\quad-\frac{1}{2}(\theta-1)\Big(1-\frac{1}{\psi}\Big)^2(\mathcal{W}_t+\lambda h(X_t))^{-2}\mathcal{W}_t^2\pi_t^2\sigma_t^2\\
&\quad-\frac{1}{2}(\theta-1)\Big(1-\frac{1}{\psi}\Big)^2\lambda^2 h_x^2(\mathcal{W}_t+\lambda h(X_t))^{-2}a^2(X_t)+\frac{1}{2}(\theta-1)(G_2^h)^2(X,t)\\
&\quad-(\theta-1)\Big(1-\frac{1}{\psi}\Big)^2\lambda h_x(\mathcal{W}_t+\lambda h(X_t))^{-2}\mathcal{W}_t\pi_t\sigma_t\rho a(X_t)\\
&\quad-(\theta-1)\Big(1-\frac{1}{\psi}\Big)(\mathcal{W}_t+\lambda h(X_t))^{-1}\mathcal{W}_t\pi_t\sigma_t\rho G_2^h(X,t)\\
&\quad-(\theta-1)\Big(1-\frac{1}{\psi}\Big)\lambda h_x(\mathcal{W}_t+\lambda h(X_t))^{-1}a(X_t)G_2^h(X,t)\\
&\quad+\frac{1}{2}(G_1^h)^2-\frac{1}{2}\frac{1}{\psi}(\mathcal{W}_t+\lambda h(X_t))^{-1}a^2\lambda h_{xx}\\
&\quad+\frac{1}{2}\frac{1}{\psi}\Big(1+\frac{1}{\psi}\Big)(\mathcal{W}_t+\lambda h(X_t))^{-2}\lambda^2a^2h_x^2-\frac{1}{\psi}(\mathcal{W}_t+\lambda h(X_t))^{-1}a\lambda h_xG_1^h\\
&\quad+\frac{1}{2}\frac{1}{\psi}\Big(1+\frac{1}{\psi}\Big)(\mathcal{W}_t+\lambda h(X_t))^{-2}\mathcal{W}_t^2\pi_t^2\sigma_t^2\\
&\quad+\frac{1}{\psi}\Big(1+\frac{1}{\psi}\Big)(\mathcal{W}_t+\lambda h(X_t))^{-2}\mathcal{W}_t\pi_t\sigma_t\rho a\lambda h_x\\
&\quad-(\theta-1)\frac{1}{\psi}\Big(1-\frac{1}{\psi}\Big)\lambda h_x(\mathcal{W}_t+\lambda h(X_t))^{-2}\mathcal{W}_t\pi_t\sigma_t\rho a(X_t)\\
&\quad-(\theta-1)\frac{1}{\psi}\Big(1-\frac{1}{\psi}\Big)\lambda^2h_x^2(\mathcal{W}_t+\lambda h(X_t))^{-2}a^2(X_t)\\
&\quad-(\theta-1)\frac{1}{\psi}(\mathcal{W}_t+\lambda h(X_t))^{-1}a\lambda h_xG_2^h(X,t)\\
&\quad+(\theta-1)\Big(1-\frac{1}{\psi}\Big)(\mathcal{W}_t+\lambda h(X_t))^{-1}\mathcal{W}_t\pi_t\sigma_t\rho G_1^h
\end{aligned}
$$

$$
\begin{aligned}
&+(\theta-1)\Big(1-\frac{1}{\psi}\Big)\lambda h_x(\mathcal{W}_t+\lambda h(X_t))^{-1}a(X_t)G_1^h+(\theta-1)G_2^hG_1^h\\
&-(\theta-1)\frac{1}{\psi}\Big(1-\frac{1}{\psi}\Big)(\mathcal{W}_t+\lambda h(X_t))^{-2}\mathcal{W}_t^2\pi_t^2\sigma_t^2\\
&-(\theta-1)\frac{1}{\psi}\Big(1-\frac{1}{\psi}\Big)\lambda h_x(\mathcal{W}_t+\lambda h(X_t))^{-2}\mathcal{W}_t\pi_t\sigma_t\rho a(X_t)\\
&-(\theta-1)\frac{1}{\psi}(\mathcal{W}_t+\lambda h(X_t))^{-1}\mathcal{W}_t\pi_t\sigma_t\rho G_2^h(X,t)\\
&-\frac{1}{\psi}(\mathcal{W}_t+\lambda h(X_t))^{-1}\mathcal{W}_t\pi_t\sigma_t\rho G_1^h\\
=\;& F_2(X,E^h,G_2^h,t).
\end{aligned}
$$

Thus, from Equations (3.52) and (3.53), the optimal strategies are given by

$$\tilde{c}_t^*=\delta^{\psi}\Big(1+\frac{\lambda h(X_t)}{\mathcal{W}_t}\Big)e^{-\frac{\psi}{\theta}Y_t^h} \tag{3.62}$$

and

$$\pi_t^*=\frac{1}{\gamma}\frac{1}{\sigma_t^2}\Big(1+\frac{\lambda h(X_t)}{\mathcal{W}_t}\Big)(\mu_t+\sigma_t\rho_tZ_t^h)-\frac{\lambda h_x}{\sigma_t\mathcal{W}_t}\rho a(X_t), \tag{3.63}$$

where (Y_t^h,Z_t^h) solution of the backward differential equation (provided that it exists)

$$\begin{cases}-\mathrm{d}Y_t^h=F^h(X,Y^h,Z^h,t)\mathrm{d}t-\rho Z_t^h\mathrm{d}B_t-\tilde{\rho}Z_t^h\mathrm{d}\tilde{B}_t\\ Y_T^h=0\end{cases}$$

with

$$
\begin{aligned}
&F^h(X,Y^h,Z^h,t)\\
&=-\theta\delta+\theta\frac{\delta^{\psi}}{\psi}e^{-\frac{\psi}{\theta}Y_t^h}+\frac{1}{2}(Z_t^h)^2\\
&\quad+\theta\Big(1-\frac{1}{\psi}\Big)(\mathcal{W}_t+\lambda h(X_t))^{-1}\mathcal{W}_t(r_t+\pi_t^*\mu_t-\tilde{c}_t^*)\\
&\quad-\frac{1}{2}\theta\Big(1-\frac{1}{\psi}\Big)\gamma(\mathcal{W}_t+\lambda h(X_t))^{-2}\mathcal{W}_t^2(\pi_t^*)^2\sigma_t^2\\
&\quad+\theta\Big(1-\frac{1}{\psi}\Big)\lambda h_x(\mathcal{W}_t+\lambda h(X_t))^{-1}b(X_t)\\
&\quad+\frac{1}{2}\theta\Big(1-\frac{1}{\psi}\Big)\lambda h_{xx}(\mathcal{W}_t+\lambda h(X_t))^{-1}a^2(X_t)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}\theta\Big(1-\frac{1}{\psi}\Big)\gamma\lambda^2 h_x^2(\mathcal{W}_t+\lambda h(X_t))^{-2}a^2(X_t)\\
&-\theta\Big(1-\frac{1}{\psi}\Big)\gamma\lambda h_x(\mathcal{W}_t+\lambda h(X_t))^{-2}\mathcal{W}_t\pi_t^*\sigma_t\rho a(X_t)\\
&+\theta\Big(1-\frac{1}{\psi}\Big)(\mathcal{W}_t+\lambda h(X_t))^{-1}\mathcal{W}_t\pi_t^*\sigma_t\rho Z_t^h\\
&+\theta\Big(1-\frac{1}{\psi}\Big)\lambda h_x(\mathcal{W}_t+\lambda h(X_t))^{-1}a(X_t)Z_t^h.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&(V^h)^*(\omega-v^b,\lambda)\\
&=\frac{1}{1-\gamma}\mathbb{E}\Big[\theta\delta\int_0^T(\mathcal{W}_s+\lambda h(X_s))^{1-\gamma}e^{Y_s^h}\Big(\delta^{\psi-1}e^{-\frac{\psi}{\theta}Y_s^h}-1\Big)ds\\
&\qquad\qquad+(\mathcal{W}_T+\lambda h(X_T))^{1-\gamma}\Big]. \qquad (3.64)
\end{aligned}
$$

Chapter Summary

This chapter was firstly concerned about proving the sufficient maximum principle theorem when we assume a concavity property on the Hamiltonian. With the proof of a sufficient maximum principle assuming we moved on to solve a consumption-investment problem in two different cases that appeared in the form of forward-backward stochastic differential equations systems. The first one considered a traded asset that depends on an observable external factor while the second one looked at the case where the external factor plays the role of a non-traded asset. As it is generally convenient, we assumed that the Brownian motion associated with the non-traded asset is correlated to the one of the traded asset. We were able to solve each problem using utility maximisation; finding the exact optimal consumption and investment strategies followed by the optimal utilities.

CHAPTER FOUR

RESULTS AND DISCUSSION

Introduction

In this chapter, we aim at finding the indifference bid price v^b. We recall that computing that price means, on first hand, solving two stochastic control problems; the first one assumes that the agent has not taken any position in the claim whereas the second assumes a buying or selling of the claim by the investor. We solved in the previous chapter the related two control problems. On second hand, the indifference bid price is the maximum amount of money that the agent is willing to pay from his/her initial wealth to be indifferent, in the sense of expected utility, from buying the claim or not doing so. That is, it consists of finding the indifference bid price from the equation $(V^h)^*(\omega - v^b, \lambda) = V^*(\omega, 0)$. Since a closed formula has not been obtained, we tackle our problem by a numerical approach making use of the finite difference method. That is, solving numerically

$$\begin{aligned}
&\frac{1}{1-\gamma}\mathbb{E}\Big[\theta\delta\int_0^T (\mathcal{W}^*)_s^{1-\gamma}e^{Y_s}\Big(\delta^{\psi-1}e^{-\frac{\psi}{\theta}Y_s}-1\Big)ds + (\mathcal{W}^*)_T^{1-\gamma}\Big]\\
=&\frac{1}{1-\gamma}\mathbb{E}\Big[\theta\delta\int_0^T (\mathcal{W}_s^* + \lambda h(X_s))^{1-\gamma}e^{Y_s^h}\left(\delta^{\psi-1}e^{-\frac{\psi}{\theta}Y_s^h}-1\right)ds\\
&\qquad + (\mathcal{W}_T^* + \lambda h(X_T))^{1-\gamma}\Big].
\end{aligned} \tag{4.1}$$

The hedging strategy of the agent is also crucial. But due to the incompleteness of the market, this hedge can not be perfect. This investment strategy comes from *problem* 2. We will end up this chapter by the sensitivity analysis of price with respect to its parameters.

We aim at proving the following

Theorem 8. *The indifference bid price v^b is the solution of Equation* (4.15).

Discretisation

In order to apply a finite difference method, let $0 = t_0 < ... < t_N = T$ be a partition of the interval $[0, T]$ with time step $\Delta t = t_n - t_{n-1}$ and corresponding increment of the Brownian motion $\Delta W_n = W_n - W_{n-1}$.

Considering the method in Bouchard & Touzi (2004), from the forward-backward differential equation

$$\begin{cases} \mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \quad X_0 = x \\ \mathrm{d}Y_t = H(t, X_t, Y_t, Z_t)\mathrm{d}t - Z_t\mathrm{d}W_t, \quad Y_T = 0 \end{cases}$$

we obtain the following algorithm

$$\begin{aligned} \hat{Y}_N &= Y_T = 0, \\ \hat{Y}_n &= \mathbb{E}_n[\hat{Y}_{n+1}] + \Delta t \mathbb{E}_n[H(t_n, \hat{X}_n, \hat{Y}_n, \hat{Z}_n)], \\ \hat{Z}_N &= Z_T = 0, \\ \hat{Z}_n &= \frac{1}{\Delta t}\mathbb{E}_n[\hat{Y}_{n+1}\Delta \hat{W}_{n+1}], \end{aligned}$$

with $(\hat{X}_n, \hat{Y}_n, \hat{Z}_n, \hat{W}_n)$the finite difference approximations of (X_t, Y_t, Z_t, W_t) at time $t = t_n$.

Due to the fact that it is numerically demanding, we restrict our numerical computations to a one period model. We assume a one period model for a problem with time horizon T. That is, we consider only two times $t_0 = t_{N-1} = 0,\ t_N = T$ and $\Delta t = T$.

Discrete Traded Asset

Let us now give a numerical approximation of $V(\omega, 0)$ given by Equation (3.43).

From Equation (3.42) the drift $F(X, Y, Z, t)$ of the process Y_t is given by

$$\begin{aligned} F(X, Y, Z, t) = \frac{1}{2}\Big(1 + \frac{(1-\gamma)\rho^2}{\gamma}\Big)Z_t^2 + \frac{1-\gamma}{\gamma}\frac{\mu(X_t)}{\sigma(X_t)}\rho Z_t + \theta\frac{\delta^\psi}{\psi}e^{-\frac{\psi}{\theta}Y_t} \\ + \frac{1-\gamma}{2\gamma}\frac{\mu^2(X_t)}{\sigma^2(X_t)} + (1-\gamma)r(X_t) - \theta\delta. \end{aligned} \quad (4.2)$$

From the algorithm, we have

$$
\begin{aligned}
\hat{Z}_0 &= \hat{Z}_{N-1} \\
&= \frac{1}{\Delta t}\mathbb{E}\Big[\hat{Y}_N \Delta \hat{W}_N\Big] \\
&= \frac{1}{T}\mathbb{E}\Big[\hat{Y}_N \Delta \hat{W}_N\Big] \\
&= 0, \text{ since } \hat{Y}_N = 0
\end{aligned}
$$

which means $\hat{Z}_0 = 0$ and $\hat{Z}_N = 0$.

Then

$$
\begin{aligned}
\hat{Y}_0 &= \mathbb{E}[\hat{Y}_N] + \Delta t \mathbb{E}[F(\hat{X}, \hat{Y}, \hat{Z}, 0)] \\
&= TF(\hat{X}, 0, 0, 0) \\
&= T\Big(\theta \frac{\delta^{\psi}}{\psi} e^{-\frac{\psi}{\theta}\hat{Y}_0} + \frac{1-\gamma}{2\gamma}\frac{\mu^2(x)}{\sigma^2(x)} + (1-\gamma)r(x) - \theta\delta\Big). \qquad (4.3)
\end{aligned}
$$

So, the optimal portfolio, at the initial time, is given by

$$
\begin{aligned}
\pi_0^* &= \frac{1}{\gamma}\frac{1}{\sigma^2(x)}(\mu(x) + \sigma(x)\rho\hat{Z}_0) \\
&= \frac{\mu(x)}{\gamma\sigma^2(x)}, \text{ since } Z_0 = 0. \qquad (4.4)
\end{aligned}
$$

The optimal consumption wealth ratio becomes

$$
\begin{aligned}
\widetilde{c}_0^* &= \delta^{\psi} e^{-\frac{\psi}{\theta}\hat{Y}_0} \\
&= \delta^{\psi} \exp\Big(-\frac{\psi}{\theta}T\Big(\theta\frac{\delta^{\psi}}{\psi}e^{-\frac{\psi}{\theta}\hat{Y}_0} + \frac{1-\gamma}{2\gamma}\frac{\mu^2(x)}{\sigma^2(x)} + (1-\gamma)r(x) - \theta\delta\Big)\Big). \qquad (4.5)
\end{aligned}
$$

Hence

$$
\begin{aligned}
(1-\gamma)V^*(\omega, 0) &= \mathbb{E}\Big[(\mathcal{W}_T^*)^{1-\gamma}\Big] + \theta\delta\mathbb{E}\Big[\Delta t(\mathcal{W}_0^*)^{1-\gamma}e^{Y_T}\Big(\delta^{\psi-1}e^{-\frac{\psi}{\theta}Y_T} - 1\Big)\Big] \\
&= \mathbb{E}\Big[(\mathcal{W}_T^*)^{1-\gamma}\Big] + \theta\delta T\omega^{1-\gamma}\Big(\delta^{\psi-1} - 1\Big).
\end{aligned}
$$

Applying the Ito's formula on the process $(\mathcal{W}_t^*)^{1-\gamma}$ and considering the initial condition $\omega^{1-\gamma}$, we have the system

$$\begin{aligned}
\mathrm{d}(\mathcal{W}_t^*)^{1-\gamma} &= (\mathcal{W}_t^*)^{1-\gamma}\Big[\Big((1-\gamma)(r_t+\pi_t^*\mu_t-\tilde{c}_t^*)-\frac{1}{2}\gamma(1-\gamma)(\pi_t^*)^2\sigma_t^2\Big)\mathrm{d}t \\
&\quad + (1-\gamma)\pi_t^*\sigma_t\rho\mathrm{d}B_t\Big] \\
(\mathcal{W}_0^*)^{1-\gamma} &= \omega^{1-\gamma}.
\end{aligned}$$

Integrating both sides from 0 to T and applying the expectation, we obtain the following equation

$$\begin{aligned}
\mathbb{E}\Big[\mathcal{W}_t^{1-\gamma}\Big] &= \omega^{1-\gamma}+\mathbb{E}\Big[\int_0^T(\mathcal{W}_s^*)^{1-\gamma}\Big((1-\gamma)(r(X_s)+\pi_s^*\mu(X_s)-\tilde{c}_s^*) \\
&\quad -\frac{1}{2}\gamma(1-\gamma)(\pi_s^*)^2\sigma^2(X_s)\Big)\Big] \\
&= \omega^{1-\gamma}+\mathbb{E}\Big[\Delta t\omega^{1-\gamma}\Big((1-\gamma)(r(X_0)+\pi_0^*\mu(X_0)-\tilde{c}_0^*) \\
&\quad -\frac{1}{2}\gamma(1-\gamma)(\pi_0^*)^2\sigma^2(X_0)\Big)\Big] \\
&= \omega^{1-\gamma}\Big[1+T\Big((1-\gamma)(r(x)+\pi_0^*\mu(x)-\tilde{c}_0^*) \\
&\quad -\frac{1}{2}\gamma(1-\gamma)(\pi_0^*)^2\sigma^2(x)\Big)\Big].
\end{aligned}$$

Hence,

$$\begin{aligned}
&(1-\gamma)V^*(\omega,0) \\
&= \omega^{1-\gamma}\Big[1+T\Big((1-\gamma)(r(x)+\pi_0^*\mu(x)-\tilde{c}_0^*)-\frac{1}{2}\gamma(1-\gamma)(\pi_0^*)^2\sigma^2(x)\Big) \\
&\quad +\theta\delta T\Big(\delta^{\psi-1}-1\Big)\Big]. \qquad (4.6)
\end{aligned}$$

Discrete Non-Traded Asset

Now, a numerical approximation of $(V^h)^*(\omega-v^b,\lambda)$ can be given. The drift $F^h(X,Y^h,Z^h,t)$ of the process Y_t^h is given by

$$\begin{aligned}
&F^h(X,Y^h,Z^h,t) \\
&= -\theta\delta+\theta\frac{\delta^\psi}{\psi}e^{-\frac{\psi}{\theta}Y_t^h}+\frac{1}{2}(Z_t^h)^2 \\
&\quad +\theta\Big(1-\frac{1}{\psi}\Big)(\mathcal{W}_t+\lambda h(X_t))^{-1}\mathcal{W}_t(r_t+\pi_t^*\mu_t-\tilde{c}_t^*)
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\theta\Big(1-\frac{1}{\psi}\Big)\gamma(\mathcal{W}_t+\lambda h(X_t))^{-2}\mathcal{W}_t^2(\pi_t^*)^2\sigma_t^2\\
&+\theta\Big(1-\frac{1}{\psi}\Big)\lambda h_x(\mathcal{W}_t+\lambda h(X_t))^{-1}b(X_t)\\
&+\frac{1}{2}\theta\Big(1-\frac{1}{\psi}\Big)\lambda h_{xx}(\mathcal{W}_t+\lambda h(X_t))^{-1}a^2(X_t)\\
&-\frac{1}{2}\theta\Big(1-\frac{1}{\psi}\Big)\gamma\lambda^2h_x^2(\mathcal{W}_t+\lambda h(X_t))^{-2}a^2(X_t)\\
&-\theta\Big(1-\frac{1}{\psi}\Big)\gamma\lambda h_x(\mathcal{W}_t+\lambda h(X_t))^{-2}\mathcal{W}_t\pi_t^*\sigma_t\rho a(X_t)\\
&+\theta\Big(1-\frac{1}{\psi}\Big)(\mathcal{W}_t+\lambda h(X_t))^{-1}\mathcal{W}_t\pi_t^*\sigma_t\rho Z_t^h\\
&+\theta\Big(1-\frac{1}{\psi}\Big)\lambda h_x(\mathcal{W}_t+\lambda h(X_t))^{-1}a(X_t)Z_t^h.
\end{aligned}
$$

From the numerical algorithm, we obtain

$$
\begin{aligned}
\hat{Z}_0^h &= \hat{Z}_{N-1}^h\\
&= \frac{1}{\Delta t}\mathbb{E}\Big[\hat{Y}_N^h\Delta\hat{W}_N\Big]\\
&= \frac{1}{T}\mathbb{E}\Big[\hat{Y}_N^h\Delta\hat{W}_N\Big]\\
&= 0,\ \text{since } \hat{Y}_N^h = 0
\end{aligned}
$$

which means $\hat{Z}_0^h = 0$ and $\hat{Z}_N^h = 0$.

Then

$$
\begin{aligned}
\hat{Y}_0^h &= \mathbb{E}[\hat{Y}_T^h] + \Delta t\mathbb{E}[F^h(\hat{X},\hat{Y}^h,\hat{Z}^h,T)]\\
&= TF^h(X,0,0,0).
\end{aligned}
\tag{4.7}
$$

So, the optimal portfolio, at the initial time, is given by

$$
\begin{aligned}
\pi_0^* &= \frac{1}{\gamma}\frac{1}{\sigma^2(X_0)}\Big(1+\frac{\lambda h(X_0)}{\mathcal{W}_0}\Big)(\mu(X_0)+\sigma(X_0)\rho\hat{Z}_0^h)\\
&\quad -\frac{\lambda h_x}{\sigma(X_0)\mathcal{W}_0}\rho a(X_0)\\
&= \frac{1}{\gamma}\frac{\mu(x)}{\sigma^2(x)}\Big(1+\frac{\lambda h(x)}{\omega-v^b}\Big)-\frac{\lambda h_x}{\sigma(x)(\omega-v^b)}\rho a(x),\ \text{since } \hat{Z}_0^h = 0.
\end{aligned}
\tag{4.8}
$$

The optimal consumption wealth ratio becomes

$$\begin{aligned}\widetilde{c}_0^* &= \delta^{\psi}\Big(1+\frac{\lambda h(X_0)}{\mathcal{W}_0}\Big)e^{-\frac{\psi}{\theta}\hat{Y}_0^h}\\ &= \delta^{\psi}\Big(1+\frac{\lambda h(x)}{\omega-v^b}\Big)e^{-\frac{\psi}{\theta}TF^h(x,0,0,0)},\end{aligned} \tag{4.9}$$

where

$$\begin{aligned}&F^h(\hat{X},0,0,0)\\ &= -\theta\delta+\theta\frac{\delta^{\psi}}{\psi}e^{-\frac{\psi}{\theta}Y_0^h}\\ &\quad -\frac{1}{2}\theta\Big(1-\frac{1}{\psi}\Big)\gamma(\mathcal{W}_0^*+\lambda h(X_0))^{-2}(\mathcal{W}_0^*)^2(\pi_0^*)^2\sigma_0^2\\ &\quad +\theta\Big(1-\frac{1}{\psi}\Big)(\mathcal{W}_0^*+\lambda h(X_0))^{-1}\mathcal{W}_0^*(r_0+\pi_0^*\mu_0-\tilde{c}_0^*)\\ &\quad +\theta\Big(1-\frac{1}{\psi}\Big)\lambda h_x(\mathcal{W}_0^*+\lambda h(X_0))^{-1}b(X_0)\\ &\quad +\frac{1}{2}\theta\Big(1-\frac{1}{\psi}\Big)\lambda h_{xx}(\mathcal{W}_0^*+\lambda h(X_0))^{-1}a^2(X_0)\\ &\quad -\frac{1}{2}\theta\Big(1-\frac{1}{\psi}\Big)\gamma\lambda^2h_x^2(\mathcal{W}_0^*+\lambda h(X_0))^{-2}a^2(X_0)\\ &\quad -\theta\Big(1-\frac{1}{\psi}\Big)\gamma\lambda h_x(\mathcal{W}_0^*+\lambda h(X_0))^{-2}\mathcal{W}_0^*\pi_0^*\sigma_0\rho a(X_0).\end{aligned} \tag{4.10}$$

We can now move on to give the approximation of $(1-\gamma)(V^h)^*(\omega-v^b,\lambda)$. We then obtain

$$\begin{aligned}&(1-\gamma)(V^h)^*(\omega-v^b,\lambda)\\ &=\mathbb{E}\Big[(\mathcal{W}_T^*+\lambda h(X_T))^{1-\gamma}\Big]\\ &\quad +\theta\delta\mathbb{E}\Big[\Delta t(\mathcal{W}_0^*+\lambda h(X_0))^{1-\gamma}e^{Y_T^h}\Big(\delta^{\psi-1}e^{-\frac{\psi}{\theta}Y_T^h}-1\Big)\Big]\\ &=\mathbb{E}\Big[(\mathcal{W}_T^*+\lambda h(X_T))^{1-\gamma}\Big]\\ &\quad +\theta\delta\mathbb{E}\Big[T(\omega-v^b+\lambda h(X_0))^{1-\gamma}\Big(\delta^{\psi-1}-1\Big)\Big]\\ &=\mathbb{E}\Big[(\mathcal{W}_T^*+\lambda h(X_T))^{1-\gamma}\Big]+\theta\delta T(\omega-v^b+\lambda h(x))^{1-\gamma}\Big(\delta^{\psi-1}-1\Big).\end{aligned} \tag{4.11}$$

From Itô's formula, we obtain

$$
\begin{aligned}
&\mathrm{d}(\mathcal{W}_t + \lambda h(X_t))^{1-\gamma} \\
&= \Big[\Big(1-\gamma\Big)(\mathcal{W}_t + \lambda h(X_t))^{-\gamma}\mathcal{W}_t(r_t + \pi_t\mu_t - \tilde{c}_t) \\
&\quad - \frac{1}{2}\gamma\Big(1-\gamma\Big)(\mathcal{W}_t + \lambda h(X_t))^{-1-\gamma}\mathcal{W}_t^2\pi_t^2\sigma_t^2 \\
&\quad + \Big(1-\gamma\Big)\lambda h_x(\mathcal{W}_t + \lambda h(X_t))^{-\gamma} b(X_t) \\
&\quad + \frac{1}{2}\Big(1-\gamma\Big)\lambda h_{xx}(\mathcal{W}_t + \lambda h(X_t))^{-\gamma} a^2(X_t) \\
&\quad - \frac{1}{2}\gamma\Big(1-\gamma\Big)\lambda^2 h_x^2(\mathcal{W}_t + \lambda h(X_t))^{-1-\gamma} a^2(X_t) \\
&\quad - \gamma\Big(1-\gamma\Big)\lambda h_x(\mathcal{W}_t + \lambda h(X_t))^{-1-\gamma}\rho a(X_t)\mathcal{W}_t\pi_t\sigma_t\Big]\mathrm{d}t \\
&\quad + \Big[\Big(1-\gamma\Big)(\mathcal{W}_t + \lambda h(X_t))^{-\gamma}\mathcal{W}_t\pi_t\sigma_t \\
&\qquad + \Big(1-\gamma\Big)\lambda h_x(\mathcal{W}_t + \lambda h(X_t))^{-\gamma}\rho a(X_t)\Big]\mathrm{d}B_t \\
&\quad + \Big[\Big(1-\gamma\Big)\lambda h_x(\mathcal{W}_t + \lambda h(X_t))^{-\gamma}\tilde{\rho} a(X_t)\Big]\mathrm{d}\tilde{B}_t.
\end{aligned}
$$

Integrating both sides from 0 to T, and applying the expectation we end up getting

$$
\begin{aligned}
&\mathbb{E}\Big[((\mathcal{W}_T^h)^* + \lambda h(X_T))^{1-\gamma}\Big] \\
&= (\omega - v^b + \lambda h(X_0))^{1-\gamma} \\
&\quad T\Big[(1-\gamma)\lambda h_x(\omega - v^b + \lambda h(X_0))^{-\gamma} b(X_0) \\
&\quad - \frac{1}{2}\gamma(1-\gamma)(\omega - v^b + \lambda h(X_0))^{-1-\gamma}(\omega - v^b)^2(\pi_0^*)^2\sigma^2(X_0) \\
&\quad + (1-\gamma)(\omega - v^b + \lambda h(X_0))^{1-\gamma}(\omega - v^b)(r(X_0) + \pi_0^*\mu(X_0) - \tilde{c}_0^*) \\
&\quad + \frac{1}{2}(1-\gamma)\lambda h_{xx}(X_0)(\omega - v^b + \lambda h(X_0))^{-\gamma} a^2(X_0) \\
&\quad - \frac{1}{2}\gamma(1-\gamma)\lambda^2 h_x^2(\omega - v^b + \lambda h(X_0))^{-1-\gamma} a^2(X_0) \\
&\quad - \gamma(1-\gamma)\lambda h_x(\omega - v^b + \lambda h(X_0))^{-1-\gamma}(\omega - v^b)\rho a(X_0)\pi_0^*\sigma(X_0)\Big].
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&(1-\gamma)(V^h)^*(\omega - v^b, \lambda) \\
&= (\omega - v^b + \lambda h(X_0))^{1-\gamma} \\
&\quad T\Big[(1-\gamma)(\omega - v^b + \lambda h(X_0))^{1-\gamma}(\omega - v^b)(r(X_0) + \pi_0^*\mu(X_0) - \tilde{c}_0^*) \\
&\quad - \frac{1}{2}\gamma(1-\gamma)(\omega - v^b + \lambda h(X_0))^{-1-\gamma}(\omega - v^b)^2(\pi_0^*)^2\sigma^2(X_0) \\
&\quad + (1-\gamma)\lambda h_x(\omega - v^b + \lambda h(X_0))^{-\gamma} b(X_0) \\
&\quad + \frac{1}{2}(1-\gamma)\lambda h_{xx}(X_0)(\omega - v^b + \lambda h(X_0))^{-\gamma} a^2(X_0) \\
&\quad - \frac{1}{2}\gamma(1-\gamma)\lambda^2 h_x^2(\omega - v^b + \lambda h(X_0))^{-1-\gamma} a^2(X_0) \\
&\quad - \gamma(1-\gamma)\lambda h_x(\omega - v^b + \lambda h(X_0))^{-1-\gamma}(\omega - v^b)\rho a(X_0)\pi_0^*\sigma(X_0)\Big] \\
&\quad + \theta\delta T(\omega - v^b + \lambda h(X_0))^{1-\gamma}\Big(\delta^{\psi-1} - 1\Big).
\end{aligned}
\tag{4.12}
$$

Factoring by $(\omega - v^b + \lambda h(X_0))^{1-\gamma}$, we obtain

$$
\begin{aligned}
&(1-\gamma)(V^h)^*(\omega - v^b, \lambda) \\
&= (\omega - v^b + \lambda h(X_0))^{1-\gamma}\Big[1 + \\
&\quad T\Big((1-\gamma)(\omega - v^b)(r(X_0) + \pi_0^*\mu(X_0) - \tilde{c}_0^*) \\
&\quad - \frac{1}{2}\gamma(1-\gamma)(\omega - v^b + \lambda h(X_0))^{-2}(\omega - v^b)^2(\pi_0^*)^2\sigma^2(X_0) \\
&\quad + (1-\gamma)\lambda h_x(\omega - v^b + \lambda h(X_0))^{-1} b(X_0) \\
&\quad + \frac{1}{2}(1-\gamma)\lambda h_{xx}(X_0)(\omega - v^b + \lambda h(X_0))^{-1} a^2(X_0) \\
&\quad - \frac{1}{2}\gamma(1-\gamma)\lambda^2 h_x^2(\omega - v^b + \lambda h(X_0))^{-2} a^2(X_0) \\
&\quad - \gamma(1-\gamma)\lambda h_x(\omega - v^b + \lambda h(X_0))^{-2}(\omega - v^b)\rho a(X_0)\pi_0^*\sigma(X_0) \\
&\quad + \theta\delta\Big(\delta^{\psi-1} - 1\Big)\Big)\Big].
\end{aligned}
\tag{4.13}
$$

Making the change of variables $q = \omega - v^b + \lambda h(X_0)$ and $m = \theta\delta(\delta^{\psi-1} - 1)$, we obtain from $(V^h)^*(\omega - v^b, \lambda) = V^*(\omega, 0)$ the following equation

$$
\begin{aligned}
& q^{1-\gamma}\Big[1+T\Big\{(1-\gamma)(q-\lambda h(x))(r(x)+\pi_0^*\mu(x)-\tilde{c}_0^*) \\
& -\frac{1}{2}\gamma(1-\gamma)q^{-2}(q-\lambda h(x))^2(\pi_0^*)^2\sigma^2(x)+(1-\gamma)\lambda h_x q^{-1}b(x) \\
& +\frac{1}{2}(1-\gamma)\lambda h_{xx}q^{-1}a^2(x)-\frac{1}{2}\gamma(1-\gamma)\lambda^2 h_x^2 q^{-2}a^2(x) \\
& -\gamma(1-\gamma)\lambda h_x q^{-2}(q-\lambda h(x))\rho a(x)\pi_0^*\sigma(x)+\theta\delta\Big(\delta^{\psi-1}-1\Big)\Big\}\Big] \\
& -\omega^{1-\gamma}\Big[1+T\Big((1-\gamma)(r(x)+\pi_0^*\mu(x)-\tilde{c}_0^*) \\
& -\frac{1}{2}\gamma(1-\gamma)(\pi_0^*)^2\sigma^2(x)+m\Big)\Big]=0. \qquad (4.14)
\end{aligned}
$$

Substituting $\tilde{c}_0^*$ by its numerical approximation in Equation (4.9) into the previous equation, we get

$$
\begin{aligned}
& q^{1-\gamma}\Big[1+T\Big\{(1-\gamma)(q-\lambda h(x))\Big(r(x)+\pi_0^*\mu(x) \\
& -\delta^{\psi}\Big(1+\frac{\lambda h(x)}{q-\lambda h(x)}\Big)e^{-\frac{\psi}{\theta}TF^h(x,0,0,0)}\Big) \\
& -\frac{1}{2}\gamma(1-\gamma)q^{-2}(q-\lambda h(x))^2(\pi_0^*)^2\sigma^2(x)+(1-\gamma)\lambda h_x q^{-1}b(x) \\
& +\frac{1}{2}(1-\gamma)\lambda h_{xx}q^{-1}a^2(x)-\frac{1}{2}\gamma(1-\gamma)\lambda^2 h_x^2 q^{-2}a^2(x) \\
& -\gamma(1-\gamma)\lambda h_x q^{-2}(q-\lambda h(x))\rho a(x)\pi_0^*\sigma(x)+\theta\delta\Big(\delta^{\psi-1}-1\Big)\Big\}\Big] \\
& -\omega^{1-\gamma}\Big[1+T\Big\{(1-\gamma)(r(x)+\pi_0^*\mu(x)-\delta^{\psi}e^{-\frac{\psi}{\theta}TF(x,0,0,0)}) \\
& -\frac{1}{2}\gamma(1-\gamma)(\pi_0^*)^2\sigma^2(x)+m\Big\}\Big]=0, \qquad (4.15)
\end{aligned}
$$

where

$$
F(x,0,0,0)=\theta\frac{\delta^{\psi}}{\psi}e^{-\frac{\psi}{\theta}Y_0}+\frac{1-\gamma}{2\gamma}\frac{\mu^2(x)}{\sigma^2(x)}+(1-\gamma)r(x)-\theta\delta, \qquad (4.16)
$$

$$
\begin{aligned}
& F^h(x,0,0,0) \\
& =-\theta\delta+\theta\frac{\delta^{\psi}}{\psi}e^{-\frac{\psi}{\theta}Y_0^h}-\frac{1}{2}\theta\Big(1-\frac{1}{\psi}\Big)\gamma q^{-2}(q-\lambda h(x))^2(\pi_0^*)^2\sigma_0^2 \\
& +\theta\Big(1-\frac{1}{\psi}\Big)q^{-1}(q-\lambda h(x))\Big(r_0+\pi_0^*\mu_0-\delta^{\psi}\Big(1+\frac{\lambda h(x)}{\omega}\Big)e^{-\frac{\psi}{\theta}Y_0^h}\Big) \\
& +\theta\Big(1-\frac{1}{\psi}\Big)\lambda h_x q^{-1}b(x)+\frac{1}{2}\theta\Big(1-\frac{1}{\psi}\Big)\lambda h_{xx}q^{-1}a^2(x) \\
& -\theta\Big(1-\frac{1}{\psi}\Big)\gamma\lambda h_x q^{-2}(q-\lambda h(x))\pi_0^*\sigma_0\rho a(x)-\frac{1}{2}\theta\Big(1-\frac{1}{\psi}\Big)\gamma\lambda^2 h_x^2 q^{-2}a^2(x).
\end{aligned}
\qquad (4.17)
$$

Discussion

In this section, we perform some sensitivity analysis of the optimal investment and the indifference bid price with respect to the parameters. We will like to know the effect of the change of a parameters on those values.

Sensitivity of the Investment to the Correlation Coefficient

Here we assume that X_t is a geometric Brownian motion given by

$$\mathrm{d}X_t = bX_t\mathrm{d}t + \rho a X_t \mathrm{d}B_t + \tilde{\rho} a X_t \mathrm{d}\tilde{B}_t.$$

We consider the following parameter values: $b = 1.0$, $a = 0.3$, $r = 0.0014$, $\sigma = 0.24$, $\lambda = 0.5$, $x = 1.0$, $\gamma = 0.5$, $T = 1$, $\omega = 5.0$, $\psi = 0.3$, $\delta = 0.0052$. Figure 1 shows the investment strategy as a function of ρ. The negative value of the optimal investment suggests short selling. We observe that the investment $(\pi^h)^*$ is always less than the one with zero claim ($\pi^* = -34.72$). This choice of parameters tells that the agent holds less of the asset than the zero claim investment, and this decreases with correlation.

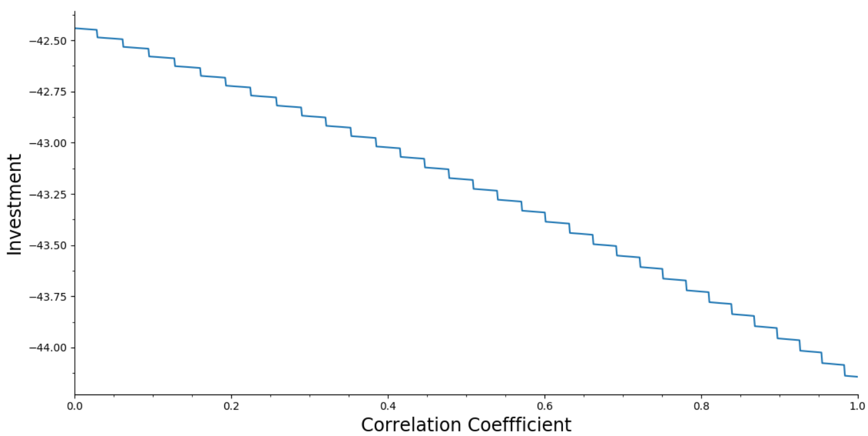

Figure 1: Change of the optimal investment with respect to the correlation coefficient ρ: $0.0 \leq \rho \leq 1.0$.

Sensitivity of the Investment to the Initial Wealth

We consider the following parameter values: $b = 1.0$, $a = 0.3$, $r = 0.0014$, $\sigma = 0.24$, $\lambda = 0.5$, $x = 1.0$, $\gamma = 0.5$, $T = 1$, $\rho = 0.5$, $\psi = 0.3$, $\delta = 0.0052$.

Figure 2 depicts the change of the optimal investment with respect to the initial value of the wealth. The negative value of the optimal investment suggests short selling and as initial wealth increases the optimal number of assets also increases.

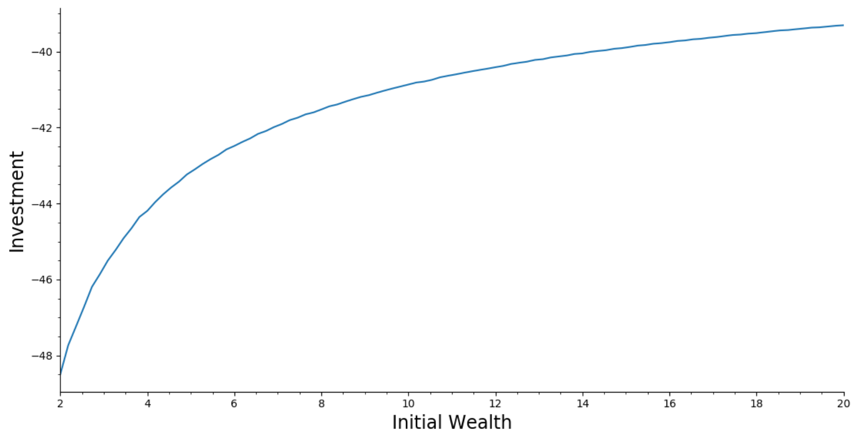

Figure 2: Change of the optimal investment$(\pi^h)^*$ with respect to the initial wealth ω: $2.0 \leq \omega \leq 20.0$.

Sensitivity of the Price to the Correlation Coefficient

We consider the following parameter values: $b = 1.0$, $a = 0.3$, $r = 0.0014$, $\sigma = 0.24$, $\lambda = 0.5$, $x = 1.0$, $\gamma = 0.5$, $T = 1$, $\omega = 5.0$, $\psi = 0.3$, $\delta = 0.0052$.

Figure 3 shows that the indifference bid price v^b increases with the correlation.

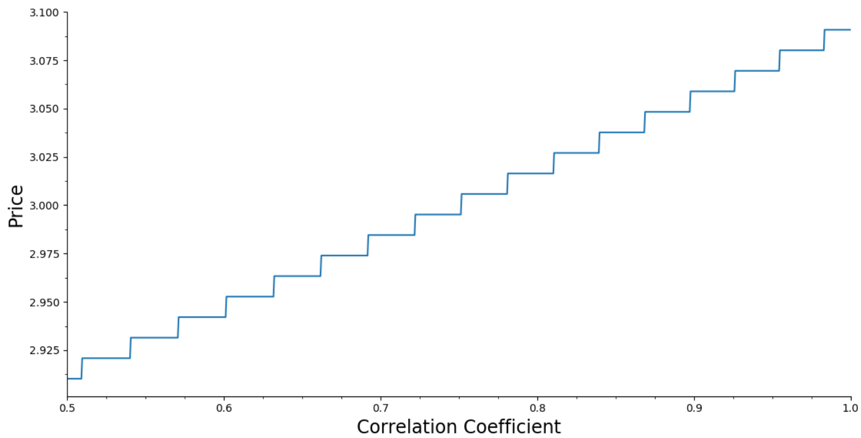

Figure 3: Change of the price v^b with respect to the correlation coefficient ρ: $0.5 \leq \rho \leq 1.0$.

Sensitivity of the Price to the Risk Aversion

We consider the following parameter values: $b = 1.0$, $a = 0.3$, $r = 0.0014$, $\sigma = 0.24$, $\lambda = 0.5$, $x = 1.0$, $T = 1$, $\omega = 5.0$, $\psi = 0.3$, $\delta = 0.0052$. Figure 4 shows that the bid price v^b decreases as the risk aversion γ increases.

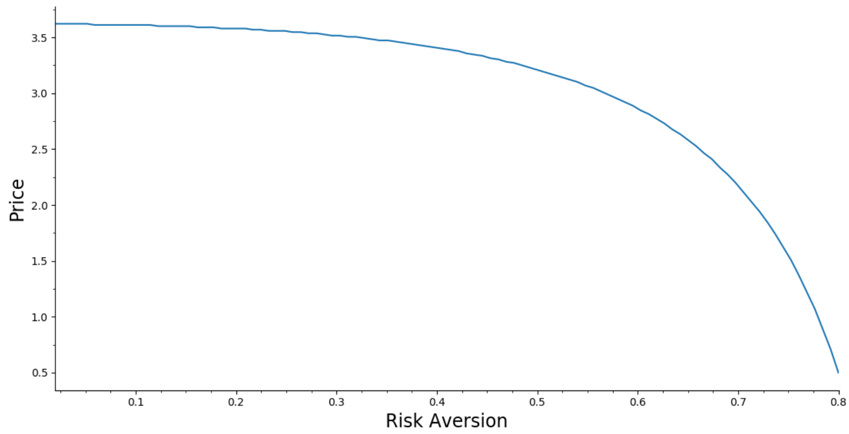

Figure 4: Change of the price v^b with respect to the risk aversion γ: $0.02 \leq \gamma \leq 0.75$.

Sensitivity of the Price to the Initial Wealth

We consider the following parameter values: $b = 1.0$, $a = 0.3$, $r = 0.0014$, $\sigma = 0.24$, $\lambda = 0.5$, $x = 1.0$, $T = 1$, $\omega = 5.0$, $\psi = 0.3$, $\delta = 0.0052$. Figure 5 shows the indifference bid price v^b for different values the risk aversion γ equal to 0.05, 0.25 and 0.5. The graph shows that agents with huge amounts of initial wealth and different perception on the risk market are likely to pay the same price to purchase the claim. That is, all the bid price for different value of the risk aversion tend to follow the same path with the increase of the initial wealth.

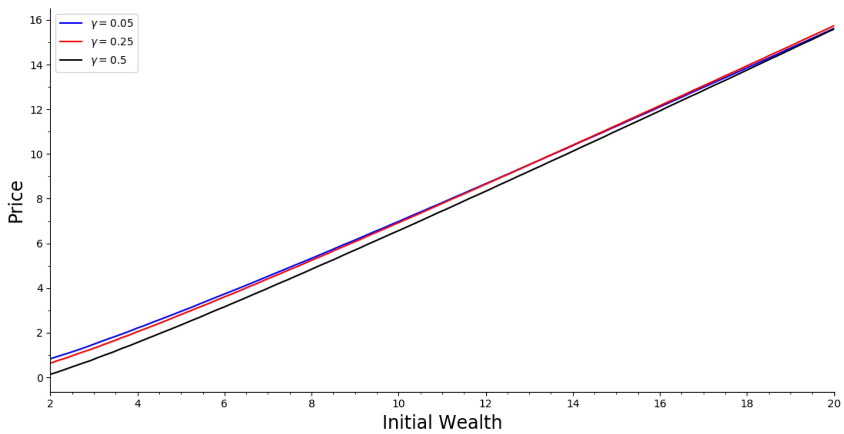

Figure 5: Change of the price v^b with respect to the initial wealth for different values of the risk aversion γ.

Chapter Summary

In this chapter, we used the finite difference method to first estimate the investment when an agent goes for a claim and then estimate the indifference price. In each case we examined their dependency to some of the parameters. In the first case, we observed that when fixing all the other parameters and making the correlation between traded and non-traded assets vary, the investment in presence of claim is always less than the no claim hedge. However, when the initial wealth increases this new investment increases so as to start being bigger than the no claim hedge.

For the dependency of the indifference price to the correlation between the two assets, we observed that the more the traded asset is correlated to the non-traded the more the agent is willing to pay for the risk. We also observe that as risk aversion increases so the indifference price falls. For the parameters specification that we chose our model does not fit the case where the risk aversion is between 0.75 and 1. We then move on to study the indifference price as function of the initial wealth of the agent for different values of the risk aversion 0.05, 0.1, 0.25 and 0.5. We obtained for a huge amount of initial wealth the price that an agent is willing to pay is less influenced by the risk aversion.

CHAPTER FIVE

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

Overview

Utility indifference pricing has attracted a lot of interests in mathematical finance and actuarial sciences as a concept of valuation of claims in incomplete market situation and has a number of applications as in option pricing with transaction costs and in portfolio constraints' context. Now consider the problem of an agent who faces receiving a claim on a risky asset on which trading is not possible. The question is: *"How best to price and hedge this claim in this incomplete market?"* Using maximum principle method, we solved this problem by the use of the utility indifference pricing concept. We incorporated into our model an observable external factor (factor model) and considered Epstein-Zin utility. To the best of our knowledge this is the first time such a problem is studied. In this chapter we summarise and conclude the work done in this thesis.

Summary

To compute utility indifference price of a claim, two stochastic control problems must be solved. In this thesis these problems were given in the form of forward-backward stochastic systems. We solved it with the use of the sufficient maximum principle for forward-backward system. That allowed us to find the optimal consumption and investment strategies and then a relation given the indifference price. As it is common in indifference pricing problem, a closed form formula for the indifference price was not found. So a numerical approach (one period finite difference method) was performed.

The first question we asked ourselves was: *"Is there any influence of the claim on the hedging strategy?"* The answer is that an agent holds less of the asset than in the zero claim strategy without any regard of the correlation between traded and non-traded assets (related to the parameters specifications we took).

Then, we wanted to compare again these two investment strategies for different value of the initial wealth. This study showed that the investment strategy is an increasing function of the initial wealth of the agent so as to be higher than its value in the zero claim case.

Our next question was: *"What effect has the changes in parameter values have on the indifference price?"* Our first study brought us to the conclusion that the higher the correlation coefficient, the higher the indifference price. Secondly, we obtained that an agent is willing to pay less for the non-traded asset as he/she becomes less tolerant of risk (for increasing risk aversion). Finally, we noticed that for large investors there is an indifference on the risk for them to decide which amount they are willing to pay.

Conclusion

A utility indifference pricing problem was solved using maximum principle for a model considering observable factor model under Epstein-Zin utility. Surprisingly, we found that an agent is going to spend a big proportion (between 46% and 87%) of his/her initial wealth in order to hedge the claim. However, our model confirmed the common sense that "an agent is willing to pay less for the non-traded as he/she becomes less tolerant of risk." These findings can be explained by the fact that we are considering a more realistic model.

Recommendations

In this thesis we considered a linear payoff λX_T. This assumption has greatly simplified some of the analysis. A natural consideration would be option payoffs such as call/put options defined by $\lambda(K - X_T)^+$, where K is a price of the stock fixed at initial time (strike price). Also, we assumed that the factor model is observed even though in reality it is not always the case. These are left for further research.

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