UNIVERSITY OF CAPE COAST

EXPONENTIAL STABILITY AND INSTABILITY IN MULTIPLE FINITE DELAY VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

RICHARD OPPONG

2020

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BY

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Thesis submitted to the Department of Mathematics of the School of Physical Sciences, College of Agriculture and Natural Sciences, University of Cape Coast, in partial fulfilment of the requirements for the award of Masters of Philosophy degree in Mathematics

JULY 2020

DECLARATION

Candidate's Declaration

I hereby declare that this thesis is the result of my own original research and that no part of it has been presented for another degree in this university or elsewhere.

Candidate's Signature .... Date ....

Name: Richard Oppong

Supervisor's Declaration

We hereby declare that the preparation and presentation of the thesis were supervised in accordance with the guidelines on supervision of thesis laid down by the University of Cape Coast.

Principal Supervisor's Signature .... Date ....

Name: Prof. Ernest Yankson

ABSTRACT

In this thesis, sufficient conditions that guarantee the exponential stability and also instability of the zero solution of a certain class of Volterra integro-differential equations with multiple finite delays are obtained. Two Lyapunov functionals are constructed in the thesis. These Lyapunov functionals are used to obtain inequalities that the solutions of the Volterra integro-differential equations satisfy. These inequalities are used to deduce exponential stability results for the zero solution of the Volterra integro-differential equations, as well as a criteria for instability of the zero solution of the Volterra integro-differential equations.

KEY WORDS

Exponential Stability

Lyapunov functional

Integro-differential equation

Instability

Stability

Volterra integro-differential equation

ACKNOWLEDGEMENTS

First of all I would like to thank the Almighty God in my life for His protection, provision, favour, and blessings.

I would like to convey to my thesis supervisor Prof. Ernest Yankson, my greatest and most heartfelt appreciation for his encouragement, persistence, friendliness, passion and tremendous expertise that he used in directing me to write this thesis. Ayikoo Prof.

Special thanks also goes to Prof. Essel for all the pieces of advice he gave to encourage me.Am grateful to my wife: Juliet and children: Nana, Katakyie, Nhyira and Adom for their sacrifices, prayers, love and support they offered me.

I am also grateful to my family for their unwavering love for me throughout my life. Finally, I would like to thank my friends Adde, Akpah, Adwoa and my course mates Victor, Edem, Catherine and Bernard for the various ways they contributed to my success.

God bless you all.

DEDICATION

To my wife, Juliet Akosua Opokuaa Okyere

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CHAPTER ONE

INTRODUCTION

The theories of Volterra integral equations and Volterra integro-differential equations, and functional differential equations, have experienced extensive improvements over the last five decades. Volterra integral and integro-differential equations which started with a few fragmented papers on specific equations, and on specific problems, has developed into large-scale fields of applied science, with complex systems of their own. The large number of applications found in physics, engineering, biology, etc., has strongly promoted the growth. Our knowledge of those fairly basic, typically linear problems that were present from the start has greatly improved. Additionally, knowledge of more general and more complicated equations has been obtained. In the last five decades the theory of linear and nonlinear Volterra integro-differential equations has developed massively according to Wazwaz (2011).

Background to the Study

The key motivation to solve these differential equations is to try and know more about an actual physical phenomenon that is thought to be representing the equation. This is essential to the significance of differential, integral and integro-differential equations as the simplest equation leads to practical physical models, such as exponential growth and decay, spring mass processes, electrical circuits etc.

An integro-differential equation is an equation containing both an integral and an unknown function's derivatives. An integro-differential equation is essentially an equation that uses in the same equation, the integral and differential operators.

Vito Volterra, known for his contributions to mathematical biology and integral equations, was an Italian mathematician and physicist. Volterra's work on elasticity was the basis of his theory of integro-differential equations. He discovered that electrical or magnetic polarization at that moment depends not just on the magnetic field for those substances, but also on the analysis of the electromagnetic state of the matter at all earlier instants. Integro-differential equations model these physical facts.

Mathematical models are strong functional methods used in mathematical language and the principle is to explain real world problems. One of the notable mathematical models in ordinary differential equations (ODEs) is known as the Volterra integro-differential equations (VIDEs), which appeared in 1926 after Vito Volterra had established it. This type of model today has numerous applications in physics, chemistry, biology, engineering and several other fields as stated by Volterra (1959), Burton (1982) and Wazwaz (2011).

Hatamzadeh et al. (2008), Jaswon et al.(1977) and Kyselka (1977) clearly indicated that, in many real-world phenomena, the principle of linear and nonlinear Volterra integro-differential equations and their solutions play a significant role in science and engineering such as atomic energy, control theory, environment, manufacturing methods, fluid dynamics, biology, physics, medicine and many more.

According to Lakshmikantham & Rama Mohana Rao (1995), integro-differential equations with unbounded delay has evolved into a new branch of contemporary research in recent years because of its connection to numerous fields such as continuum mechanics, population dynamics, ecology, systems theory and nuclear reactor dynamics.

Significance of the study

This research is important because it will shed more light on the study of integro-differential equations, particularly Volterra integro-differential equations with multiple finite delays. We will present results for the exponential stability and instability of Volterra integro-differential equations with multiple finite delays in this thesis, which will contribute to our understanding of integro-differential equations.

Statement of the Problem

A lot of research has been conducted on the qualitative properties of integro-differential equations. A number of researchers have studied the qualitative behaviour of Volterra integro-differential equations with and without delays.

The results by Adivar & Raffoul (2012) and Funakubo et al. (2006) on stability of solutions of Volterra integro-differential equations (VIDEs) does not extend to linear Volterra integro-differential equations of the form

$$x'(t) = p(t)x(t) - \sum_{i=1}^{n} \int_{t-\tau_i}^{t} q_i(x,t)x(s)ds, \tag{1.1}$$

where $\tau_i > 0$ is a constant, $q_i : [0,\infty) \times [-\tau,\infty)$ and $p : [0,\infty) \to \mathbb{R}$ with $0 < \tau_i \leq \tau$ for $i = 1,...,n$; Equation (1.1) is an integro-differential equation with multiple finite delays.

For this reason, further investigation is needed to study the exponential stability of the zero solution of Volterra integro-differential equations with multiple finite delays.

Research Objectives

The objectives of the thesis are to:

1. Construct a Lyapunov functional which yields results regarding the exponential stability of the zero solution of the equation

$$x'(t) = p(t)x(t) - \sum_{i=1}^{n} \int_{t-\tau_i}^{t} q_i(x,t)x(s)ds.$$

2. Obtain sufficient conditions to ensure that the zero solution of the equation

$$x'(t) = p(t)x(t) - \sum_{i=1}^{n} \int_{t-\tau_i}^{t} q_i(x,t)x(s)ds,$$

is exponentially stable.

3. Give sufficient conditions that guarantee the instability of the zero solution

of the equation

$$x'(t) = p(t)x(t) - \sum_{i=1}^{n} \int_{t-\tau_i}^{t} q_i(x,t)x(s)ds.$$

Chapter Summary

The chapter began with an introduction to integro- differential equations. We gave the background of the study and some applications of integro-differential equations. We then proceeded to discuss the problem we intend to study and then, the importance of the research. We outlined the primary goals of the research. The full structure of the thesis was described in this chapter

CHAPTER TWO

LITERATURE REVIEW

Introduction

In this chapter, we shall look at differential and integral equations in the broad sense and review some of the earlier works on integro-differential equations.

Differential Equations

Differential equations dated all the way back to the mid-seventeenth century, when Newton and Leibniz pioneered the calculus independently. Modern mathematical physics began basically with "Newton's Principia" in which he not only introduced calculus, but also outlined his three basic laws of motion that made mathematical modelling of physical phenomena possible.

Driver (1962) and Miller (1970) indicated that, historically, breakthroughs in the theory of differential equations have emerged from knowledge acquired in treating and managing specific physical models. The topic has become a well-defined and credible field of mathematics, given this rather small incremental growth.

In Mathematics, a differential equation is an equation that applies to one or more functions and their derivatives. The function usually represent physical changes in applications, the derivatives represent their rate of change and the differential equation establishes the relationship between the two. Such relationships are widespread so differential equations play a significant role in many disciplines such as engineering, physics, economics, biology and so on.

There are two main types of differential equations; Ordinary Differential Equations (ODEs) and Partial Differential Equations (PDEs). An ordinary differential equation is an equation that contains an unknown function of a single independent variable and one or more of its derivatives. For example

$$\frac{dy}{dt} = ky(t),$$

is an ODE.

A partial differential equation on the other hand is a differential equation involving one or more partial derivatives of an unknown function of several variables. For example, the heat in a rod at position x and time t, obeys the heat equation

$$\frac{\partial h}{\partial t} = k\frac{\partial^2 h}{\partial x^2},$$

which is a PDE. Another example of a PDE is the Laplace's equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Linearity is another essential concept in the classification of differential equations. Generally, linear problems can be explicitly resolved. Most often, we are unable to find explicit solutions to non-linear problems. When we attempt to solve a differential equation, we can get different types of solutions, based on the equation. Ideally, we would have found a completely explicit solution, that is, the dependent variable is expressly given as a combination of the independent variable's elementary functions.

We can expect to find such a completely explicit solution only for limited number of equations. It turns out that, for non-linear Volterra integro-differential equations, closed form solutions cannot be found. However, the qualitative behaviour of the solution can be fully understood for certain equations as stated by Wazwaz (2011) and Tunc (2016). That is, some explanations about how the solutions are behaving can be offered, even if we cannot explicitly state the exact solution. Such an interpretation is often best explained in graphic terms using phase portraits for differential equation solutions.

Burton et al. (1985) and Wazwaz (2015) asserts that, all analytical tools may fail for some differential equations, and in this case we use the computer to approximate the solution, usually called a numerical solution. An approximation of x in terms of t is typically a numerical solution of a differential equation. An n^{th} order ordinary differential equation for $y(t)$ is said to be linear if it can be written in the form

$$a_n(t)\frac{d^ny}{dt^n} + a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + ... + a_1(t)\frac{dy}{dt} + a_0(t)y = f(t).$$

where; $a_n(t), a_{n-1}(t), ..., a_1(t), a_0(t)$ and $f(t)$ are continuous functions.

Such a linear equation is called homogeneous if $f(t) = 0$, and it is non-homogeneous if $f(t) \neq 0$. If the ODE is not linear then it is non-linear. An example of a non-linear differential equation is

$$\frac{d^2\theta}{dt^2} = -\omega_0^2 \sin(\theta) + f \cos(\omega t).$$

Integral Equations

Corduneanu (1973), Burton & Mahfoud (1982) and Burton & Mahfoud (1985) revealed that, there are so many scientific and engineering problems that are modelled with integral equations. A significant class of initial and boundary value problems can be converted into Volterra or Fredholm integral equations. Potential theory has contributed more than any field to give credence to an integral equation. Mathematical physics models, such as diffraction problems, quantum mechanics scattering, conformal mapping and water waves, also contributed to the emergence of an integral equation. An integral equation is an equation in which the unknown function appears under an integral sign. For instance, for an unknown function $u(t)$ an integral equation is given as

$$u(t) = f(t) + \lambda \int_{g(t)}^{h(t)} K(x,s)u(s)ds,$$

where $g(t)$ and $h(t)$ are the limits of the integration, λ is a constant parameter and $K(x, s)$ is a known function of two variables, x and s called the Kernel or the nucleus of the integral equation. The unknown function $u(t)$ appearing within the integral sign is yet to be determined. In several other instances, the unknown function $u(t)$ appears both under the integral and outside the integral sign. The limits of the integration $g(t)$ and $h(t)$, can both be variables, constants, or mixed. Integral equations are categorized into two main types depending on the limits of the integration.

When the limits of the integral equation are fixed, then the equation is called a Fredholm integral equation and this is of the form

$$u(t) = f(t) + \lambda \int_a^b K(x, s)u(s)ds,$$

where a and b are constants.

When at least one of the limits is variable, the equation is called a Volterra integral equation, which is of the form

$$u(t) = f(t) + \lambda \int_a^t K(x, s)u(s)ds,$$

where a and t are the limits of the integration.

Integro-Differential Equations

Integro-differential equations exist in many scientific applications , particularly when we transform both the initial value or boundary value problems into integral equations. The derivative of the unknown function may appear to any order. Integro-differential equations, as the name may suggest, are equations in which the integral and differential operators both appear. It is of the form

$$\frac{d^n u}{dt^n} = u^{(n)}(t) = f(t) + \int_{g(t)}^{h(t)} K(x, s)u(s)ds, u^k(0) = b_k, 0 \leq k \leq n-1, \quad (2.1)$$

where $g(t)$ and $h(t)$ are the limits of the integration.

According to Graef & Tunc (2015), Graef et al. (2016) and Wazwaz (2015), integro-differential equations may also be classified into two main types; Fredholm integro-differential equation and Volterra integro-differential equation.

Fredholm Integro-Differential Equation

In the case of Fredholm integro-differential equations the limits of integration are fixed just like Fredholm integral equations. An integro-differential equation is called a Fredholm integro-differential equation if the limits of the integral part are constants as opined by Volterra (1959). Fredholm integro-differential equation is of the form

$$\frac{d^n u}{dt^n} = u^{(n)}(t) = f(t) + \int_a^b K(x,s)u(s)ds, u^k(0) = b_k, 0 \leq k \leq n-1, \quad (2.2)$$

where a and b are constants and $u^{(n)}$ indicates the n^{th} derivative of $u(t)$. Fredholm integro-differential equation is best described by the existence of one or more derivatives outside the integral sign.

Volterra Integro-Differential Equation

Volterra analyzed population growth models and also studied the genetic factors. The work culminated in a particular subject where in the same equation both the differential and integral operators existed together. This particular kind of equation is called Volterra integro-differential equation.

An integro-differential equation is called Volterra integro-differential equation, when at least one of the limits of the integral part is a variable. Volterra integro-differential equation is of the form

$$\frac{d^n u}{dt^n} = u^{(n)}(t) = f(t) + \int_a^t K(x,s)u(s)ds, u^k(0) = b_k, 0 \leq k \leq n-1, \quad (2.3)$$

where a and t are the limits of the integration and $u^{(n)}$ indicates the n^{th} derivative of $u(t)$. Other less order derivatives may appear on the left side of the equation. Since the resulting equation combines both the integral operator and the differential operator, it is significant to establish the initial conditions for the determination of an exact solution of the Volterra integro-differential equation. Any Volterra integro-differential equation shall be defined by the presence of one or more derivatives outside the integral sign.

The integro-differential equations of Volterra may be obtained when we use Leibniz rule to transform an initial value problem to an integral equation as demonstrated in Volterra (1959). This then occurred in many physical applications such as glass forming process, heat transfer, diffusion process, neutron diffusion and biological species coexisting with increasing and decreasing rates of generation, nanohydrodynamics, and wind ripple in the desert.

Some Areas of Application of Integro-differentiation Equations

In this section we will indicate how integro-differential equations have been used in other fields of science and engineering. For example, according to Kheybari et al. (2017) a system of integro-differential equations model population trends and genetics in biological applications. Also an integro-differential equation's structure may explain the behaviour of interacting excitatory and inhibitory neurons in Biology. Destexhe &Sejnowski (2009), used the Wilson – Cowan model to assume a homogeneous population of excitatory and inhibitory subtypes of interacting neurons. He modelled the activity of an excitatory and inhibitory subtypes with a population by;

$$\begin{aligned} E(t+\tau) &= [1-\int_{t-\tau}^{t} E(t')dt']S_e\Big(\int_{-\infty}^{t}[\alpha(t-t')][c_1E(t')-c_2I(t')+P(t')]dt'\Big), \\ I(t+\tau) &= [1-\int_{t-\tau}^{t} I(t')dt']S_i\Big(\int_{-\infty}^{t}[\alpha(t-t')][c_3E(t')-c_4I(t')+Q(t')]dt'\Big), \end{aligned}$$

where $E(t)$ and $I(t)$ are respectively the proportion of excitatory and inhibitory cells firing at a time t, $\alpha(t)$ is a stimulus decay function, c_1, c_2, c_3 and c_4 are the connectivity coefficients, $P(t)$ is the external input to the excitatory population and $Q(t)$ is the internal input to the inhibitory population.

Integro-differential equations have again found applications in epidemiology, epidemic mathematical modelling, especially when the model contains "age structures" or spatial epidemics as stated by Brauer et al. (2012).

In Bellomo et al. (1999), a bifurcation analysis is developed for the initial value problem of a nonlinear integro-differential equation method, modelling interaction between tumour cells and the immune system. A vital bifurcation parameter is shown to distinguish conditions where the immune system regulates neoplastic development from those where tumour growth is not distinguished. This result refers to the hypothesis that cytokine signal activity will alter competition outputs. This same model is composed of two nonlinear integro-differential equations. That is

$$\begin{aligned}\frac{\delta N_1}{\delta t}(t,u) &= \alpha\int_{-1}^{1}\int_{-1}^{1}\Psi_{12}(u,m_{12}(v,w))N_1(t,v)N_2(t,w)dvdw\\ &\quad -\alpha N_1(t,u)\int_{-1}^{1}N_2(t,w)dw - N_1(t,u)\Big[\gamma_{12}\int_{-1}^{0}wN_2(t,w)dw\\ &\quad -\gamma_{13}\int_{0}^{1}wN_{30}(w)dw + \rho_{12}\int_{0}^{1}wN_2(t,w)dw\Big]H(u),\end{aligned}$$

and

$$\begin{aligned}\frac{\delta N_2}{\delta t}(t,u) &= \alpha\int_{-1}^{1}\int_{-1}^{1}\Psi_{12}(u,m_{21}(v,w))N_2(t,v)N_1(t,w)dvdw\\ &\quad -\alpha N_2(t,u)\Big[\gamma_{21}\int_{0}^{1}wN_1(t,w)dw - \gamma_{13}\int_{-1}^{1}N_1(t,w)dw\\ &\quad +\rho_{23}\int_{0}^{1}wN_{30}(w)dw\Big] + \gamma_2 H(u),\end{aligned}$$

where $H(u)$ is a Heaviside function and $H(u) = 1$ if $u \geq 0$ and $H(u) = 0$ if $u < 0$.

Furthermore, Verdugo (2018), studied the stability of the functional biological model with constructive feedback. He learned from the study that the architecture of the structures is based on biological gene networks and that the model takes the form of a delay integro-differential equation together with a partial differential equation given by

$$\begin{aligned}\frac{dm}{dt} &= -\mu_m m(t) + \int_0^1 e^{-|x-\bar{x}|} H(p(\bar{x}, t-T)) d\bar{x},\\ \frac{\partial p}{\partial t} &= m(t) - \mu_p p(t),\end{aligned}$$

where the time dependent variable are the mRNA concentration, $m(t)$ and its associated protein concentration $p(t)$ and the constants μ_m and μ_p are the decay rates of the mRNA and the protein molecules, respectively. The function $H(p(\bar{x}, t-T))$ represents the rate of delay production of mRNA.

In Physics, integro-differential equations model other conditions like the studies of circuit. For instance, by the voltage law of Kirchhoff, the net voltage drop across a closed loop equals the impressed voltage $E(t)$. For example, an RLC circuit is given by the equation;

$$E(t) = L\frac{d}{dt}I(t) + RI(t) + \frac{1}{C}\int_0^t I(\tau)d\tau,$$

where $I(t)$ is the electric current as function of time, R is the Resistor, L is the Inductor and C is the Capacitor as suggested by Hatamzadeh (2008).

In Bio-Chemistry, Dawson & Lugli (2015) tends to suggest that, the tumour budding cell is a typical example of an application of integro-differential equations. Tumour budding cells are individual cells or small groups of up to four or five elements isolated in the bulk of the main tumour from their counterparts. Tumour buds are considered to be the morphological equivalents of cancer cells with an epithelial-mesenchymal transition (EMT), an essential process for the growth of epithelial cancer, described as the presence of a tumour cell or of small clusters of up to 5 tumour cells in the peritumoral front (peritumor buds) or in the main tumore body (intratumoral buds).

Burton (2005) strongly indicates that, there are some areas of mathematical physics, such as diffraction problems, quantum mechanics scattering, conformal mapping and water waves, also contributed to the development of nonlinear integral equations. Since it is not always possible to find the exact answers to the problems raised in the physical sciences, a great deal of work is being undertaken to achieve methodological approximations that illustrate the solution structure.

Review of Related Literature

According to Tunc & Mohammed (2017) and Tunc & Tunc (2018b), few types of integro-differential Volterra equations and systems can be explicitly solved. In the course of scientific investigations, researchers need to find analytical methods that allow them to study the qualitative behaviour of Volterra integro-differential equations. We therefore need to find analytical techniques for studying qualitative behaviours such as stability, boundedness, asymptotic stability, exponential stability, the existence of periodic solutions, integrability, square integrability, etc. of solutions of Volterra integro-differential equations without finding their exact solutions.

According to Adivar & Raffoul (2012), Tunc (2017a), Tunc (2017b) and Staffans (1988), a study of the qualitative behaviour of linear and non-linear Volterra integro-differential equations play an important role in engineering and science. Over the last five decades, researchers have stepped up their research and developed methods and techniques to achieve the required results. Some of these methods and techniques include the second Lyapunov method, the fixed point theory, the continuation method, the perturbation theory, the iterative techniques, and many more.

In Funakubo et al. (2006), the authors considered the equation

$$x'(t) = ax(t) - a\int_{t-\tau}^{t} x(s)ds.$$

This is a linear Volterra integro-differential equation with single delay τ and a constant kernel. They provided conditions that must be met for the uniform asymptotic stability of the zero solution of the linear Volterra integro-differential equation.

Also, Advar & Raffoul (2012) considered the Volterra integro-differential equation

$$x'(t) = p(t)x(t) - \int_{t-\tau}^{t} q(t,s)x(s)ds.$$

This is a linear Volterra integro-differential equation with single delay τ and kernel $q(t,s)$. They obtained conditions that guarantee the exponential stability of the zero solution of the finite delay linear Volterra integro-differential equation.

Some Methods of Solving Integro-differential Equations

According to Graef et al. (2016), Raffoul (2013) and Wazwaz (2003), there are different techniques for solving integro-differential equations. Some examples are Adomian decomposition method (ADM), Galerkin method, and rationalized Haar functions method.

Adomian Decomposition Method (ADM)

George Adomian invented and developed the Adomian decomposition process. It requires breaking down the unknown element into the sum of an infinite number of components identified by the sequence of decompositions. That is, ADM decomposes every solution as an infinite sum of components, where those components are simultaneously determined. This method would be used in its standard form, or merged with the phenomenon of noise terms as indicated by Wazwaz (2011).

Variational Iteration Method (VIM)

The variation iteration method can be applied to a wide range of linear, non-linear, homogeneous and non-homogeneous equations. This method, according to Wazwaz (2015) and Graef & Tunc (2015), provides rapidly convergent successive approximations of the exact solution if there is such a closed form solution, and not components as in ADM. Two essential steps are used to apply VIM. First, it is vital to ascertain the Lagrange multiplier, which can be optimally determined by integration by parts and by using a limited variation, to determine the successive approximation of the solutions.

The Successive Approximation Method

Burton (2005) and Wazwaz (2015), indicated that, the successive method of approximation also called the Picards iteration method offers a framework that can be used to solve integral equations. By initiation of an initial supposition, namely zeroth approximation, this procedure solves any problem. The zeroth approximation is any real-valued function used to define the other approximations in a repeated relationship.

Chapter Summary

We examined differential equations and integral equations in this chapter and proceeded to discuss relevant researches in the area. We discussed the various areas of applications of integro-differential equations. We also examined some of the methods of solving integro-differential equations after reviewing some relevant literature.

CHAPTER THREE

METHODOLOGY

Introduction

Corduneanu (2008) and Wazwaz (2011) indicated that, linear and non-linear Volterra integro-differential equations and systems theory offers significant mathematical models of science and engineering for many real-world phenomena. So, it is very critical to provide knowledge about the qualitative properties of linear and non-linear Volterra integro-differential equations and structures without solving them during studies that are related to science and engineering. Yet surprisingly, relatively few types of integro-differential equations can be directly solved. And hence, researchers need to find analytical techniques and tools during scientific investigations that allow them to analyze the qualitative behaviour of linear and non-linear Volterra integro-differential equations without solving them. The key ideas, techniques and approaches used by researchers to analyze the qualitative behaviour of Volterra integro-differential equations without discovering their empirical solutions include the second Lyapunov approach, continuity approaches, theory of coincidence degree, perturbation theory, fixed point approach or theory, iterative techniques, the variation of constants formula and so on.

Moreover, in many scientific fields, such as atomic energy, biology, bi-dimensional gravity chemistry, control theory, differential geometry, economy, engineering techniques, fluid mechanics, information theory, medicine, Jacobi fields, population dynamics, physics and many others investigations on stability, boundedness, asymptotic stability, uniform asymptotic stability, integrability and square integrability of solutions, the existence of periodic solutions, etc., of linear and non-linear Volterra integro-differential equations and models as indicated by Wang (2004), Becker (2009) and Tunc (2018a).

Lyapunov Second Method

A Lyapunov function is a scalar function defined on a continuous, positive-definite region with continuous first-order partial derivatives at all points. Lyapunov's second approach as stated by Lakshmikantham & Rama Mohana Rao (1995) is well-known as one of the fundamental techniques for studying the qualitative behaviour of linear and nonlinear differential equations. Lyapunov functionals or functions can be used to extend this method to either differential equations with delay or Volterra integro-differential equations. We find that when constructing Lyapunov functionals in applications, one must always use a combination of a Lyapunov function and a functional in order to estimate the corresponding derivatives properly without requiring previous knowledge of the solutions.

On the other hand, utilizing a Lyapunov function depends critically on choosing adequate minimal sets of suitable space of continuous functions along the derivative of the Lyapunov function which permits a convenient approximation. When we look at how Lyapunov functional are constructed in practice, we notice that they always use a combination of Lyapunov function and a functional to estimate the corresponding derivative adequately without requiring the minimal class of functions or previous knowledge of the solution. As a result of this, the Lyapunov functions method for product spacing was developed. We shall examine in this section, the stability properties of linear integro-differential equations by constructing Lyapunov functionals. We shall also use this set up to discuss stability properties using equivalent linear differential and linear integro-differential equations.

As per assertions by Burton (1982), Staffans (1988) and Raffoul & Unal (2014), the second or direct technique used by Lyapunov is one of the most common approaches used to analyze the stability of dynamic systems. This technique uses an auxiliary function known as a Lyapunov function to determine stability for a particular system without the need to develop system solutions.

In this thesis, the Lyapunov's direct method will be employed in obtaining the results concerning the exponential stability of the integro-differential equation considered in the thesis. With this method, we construct a suitable Lyapunov functional that guarantees the exponential stability of Equation (1.1). This Lyapunov functional must satisfy the following conditions;

- must be zero at $x = 0$, that is, $V(t, x) = 0$ at $x = 0$.
- must be positive definite, that is, $V(t, x) \geq 0$ except $x = 0$,
- must have continuous first-order partial derivatives, that is, $\frac{\partial V}{\partial x}$ must be continuous.

We then proceed to use the simple scalar integro-differential equation;

$$u' = \alpha u + \int_0^t a(t-s)u(s)ds \tag{3.1}$$

Where α is a constant and $a(t)$ is continuous for $0 \leq t < \infty$, with a strong sign condition. The following results for equation (3.1) is useful and interesting in subsequent discussions.

Theorem 3.1

Suppose that $\alpha < 0$ and $\alpha + \int_0^\infty a(t)dt \neq 0$

Then the following statements are equivalent:

a) all the solutions of equation (3.1) tends to zero.

b) $\alpha + \int_0^\infty a(t)dt < 0$

c) each solution of equation (3.1) is in $L'(R_+)$

d) the zero solution of equation (3.1) is uniformly asymptotically stable.

e) the zero solution of equation (3.1) is asymptomatically stable.

Suppose that (a) holds, but $\alpha + \int_0^\infty a(t)dt > 0$

We choose $t_0 > 0$ so that $\alpha + \int_0^\infty a(t)dt \geq 0$

Let $u(t, t_0, \phi)$ be a solution of equation (3.1) with the initial function $\phi(t) = 2$ on $[0, t_0]$.

We claim that $u(t) = u(t, t_0, \phi) > 1$ on $[t_0, \infty)$. If not, there exist a first $t_1 > t_0$ with $u(t_1) = 1$ and hence $u'(t_1) \leq 0$ It follows from equation (3.1) that;

$$
\begin{aligned}
u'(t_1) &= \alpha u(t_1) + \int_0^{t_1} a(t_1 - s)u(s)ds \\
&= \alpha + \int_0^{t_1} a(s)u(t_1 - s)ds
\end{aligned}
$$

$$
\begin{aligned}
u'(t) &\geq \alpha + \int_{t_1}^{0} a(s)\, ds \\
&\geq \alpha + \int_{t_1}^{0} a(s)\, ds \geq 0
\end{aligned}
$$

which is a contradiction. Hence (a) implies (b).// To prove that (b) implies (c), we choose a Lyapunov functional

$$
V\big(t, u(.)\big) = |u| + \int_0^t \int_t^\infty a(\tau - s)d\tau |u(s)|ds
$$

If $u(t)$ is a solution of equation (3.1) then, for $u(t) \neq 0$, it follows that,

$$
\begin{aligned}
V'\left(t, u(.)\right) &\leq \alpha|u| + \int_0^t a(t-s)|u(s)|ds \\
&\quad + \int_t^\infty a(\tau - t)d\tau|u| - \int_0^t a(t-s)|u(s)|ds \\
&= \left[\alpha + \int_t^\infty a(\tau - t)d\tau\right]|u| \\
&= \left[\alpha + \int_0^\infty a(\tau)d\tau\right]|u| = -\beta|u|
\end{aligned}
$$

for some constant $\beta > 0$.

This we have $0 \leq V(t, u(.)) \leq V(t_0, \phi(.)) - \int\limits_{t_0}^{t} |u(s)ds$

which implies that $\int\limits_{t_0}^{t} |u(s)ds < \infty$ and hence $u \in L'(R_+)$

This completes the proof

By slightly modifying the Lyapunov functional used in the proof of Theorem 3.1, we can relax the sign condition of $a(t)$, as the following result shows.

Theorem 3.2

Suppose that $\alpha + \int\limits_0^{\infty} a(t)dt < 0$, then the zero solution of equation (3.1) is uniformly asymptotically stable.

Proof.

We choose the Lyapunov functional

$$V\big(t, u(.)\big) \;=\; |u| + \int_0^t \int_t^{\infty} a(\tau - s)d\tau |u(s)|ds$$

If $u(t) = u(t, t_0, \phi(.))$ is a solution of equation 3.1, then, for $u(t) \neq 0$, we obtain

$$\begin{aligned}
V'\big(t, u(.)\big) &\leq \alpha|u| + \int_0^t a(t-s)|u(s)|ds \\
&+ \int_t^{\infty} a(\tau - s)d\tau|u| - \int_0^t |a(t-s)||u(s)|ds \\
&\leq \Big[\alpha + \int_t^{\infty} a(\tau - t)d\tau\Big]|u| \\
&= \Big[\alpha + \int_0^{\infty} a(\tau)d\tau\Big]|u| = -\beta|u|
\end{aligned}$$

for some constant $\beta > 0$.

Hence $u \in L'(R_+)$. By theorem 3.1 this shows that the zero solution of equation (3.1) is uniformly asymptotically stable, and thus the proof of the theorem is complete.

We shall now consider a more general scalar integro-differential equation

$$u' = \alpha(t)u + \int_0^t a(t,s)u(s)ds \tag{3.2}$$

where $t \in R^+$ and given necessary and sufficient conditions for the zero solution, equation (3.2) is stable.

Theorem 3.3

Assume that:

i) α: $R_+ \to R$ is continuous and a is continuous for $0 \leq s \leq t < \infty$. ii) the integral $\int_t^\infty |a(\tau,t)d\tau$ is defined and finite for all $t \geq 0$ and suppose that there is a positive real number β such that ;

$$\int_0^t |a(t,s)|ds + \int_t^\infty |a(\tau,t)|d\tau - 2|\alpha(t)| \leq -\beta \tag{3.3}$$

Then the zero solution of equation 3.2 is stable iff and only if $\alpha(t) < 0$

Proof.

Suppose that$\alpha(t) < 0$, and consider a Lyapunov functional

$$V\big(t,u(.)\big) \quad = \quad u^2 + \int\limits_0^t \int\limits_t^\infty |a(\tau,s)|d\tau u^2(s)ds$$

Then the time derivative of $V\big(t,u(.)\big)$ along the solutions of equation (3.2) is given by

$$\begin{aligned} V'\big(t,u(.)\big) \quad &\leq \quad 2\alpha(t)u^2 + 2\int\limits_0^t |a(t,s)||u(s)||u|ds \\ &+ \int\limits_t^\infty |a(\tau,t)|d\tau u^2 - \int\limits_0^t |a(t,s)|u^2(s)ds \end{aligned}$$

Since we have that $2|u(s)||u| \leq u^2(s) + u^2$, it follows that;

$$\begin{aligned}
V'\big(t, u(.)\big) &\leq 2\alpha(t)u^2 + \int_0^t |a(t,s)|[u^2(s) + u^2]ds \\
&+ \int_t^\infty |a(\tau,t)|d\tau u^2 - \int_0^t |a(t,s)|u^2(s)ds \\
&= \Big[2\alpha(t) + \int_0^t |a(t,s)|ds + \int_t^\infty |a(\tau,t)|d\tau\Big]u^2
\end{aligned}$$

By invoking equation (3.3), we obtain

$$V'\big(t, u(.)\big) \leq -\beta u^2 \tag{3.4}$$

Since we have that V is positive definite and $V'\big(t, u(.)\big) \leq 0$. It follows that the zero solution of equation (3.2) is stable. Now suppose that $\alpha(t) \geq 0$ and consider the functional

$$W\big(t, u(.)\big) = u^2 + \int_0^t \int_t^\infty |a(\tau,s)|d\tau u^2(s)ds \tag{3.5}$$

Then we have

$$\begin{aligned}
W'\big(t, u(.)\big) &\geq 2\alpha(t)u^2 - 2\int_0^t |a(t,s)||u(s)||u|ds \\
&- \int_t^\infty |a(\tau,t)|d\tau u^2 - \int_0^t |a(t,s)|u^2(s)ds \\
&\geq 2\alpha(t)u^2 - \int_0^t |a(t,s)|[u^2(s) + u^2]ds \\
&- \int_t^\infty |a(\tau,t)|d\tau u^2 + \int_0^t |a(t,s)|u^2(s)ds \\
&= \Big[2\alpha(t) - \int_0^t |a(t,s)|ds - \int_t^\infty |a(\tau,t)|d\tau\Big]u^2
\end{aligned}$$

Therefore when we substitute equation (3.3), we obtain

$$W'\big(t, u(.)\big) \;\; \leq \;\; \beta u^2 \tag{3.6}$$

Now, given $t_0 \geq 0$ and $\delta > 0$, we can find a continuous function $\phi : [0, t_0] \to R$ with $|\phi(t)| < \delta$ and $W(t_0, \phi(.)) \geq 0$ such that if $u(t, t_0, \phi)$ is a solution of equation (3.3), then we have from equation (3.5) and equation (3.6) that

$$u^2 \geq W\big(t, u(.)\big) \geq W\big(t_0, \phi(.)\big) + \beta \int_{t_0}^{t} u^2(s) ds \tag{3.7}$$

$$\begin{aligned} &\geq \;\; W\big(t_0, \phi(.)\big) + \beta \int_{t_0}^{t} W\big(t_0, \phi(.)\big) ds \\ &= \;\; W\big(t_0, \phi(.)\big) + \beta W\big(t_0, \phi(.)\big)(t - t_0) \end{aligned}$$

Hence $|u(t)| \to \infty$, and the proof is complete.

Corollary 3.1

If equation (3.4) holds and $\alpha(t) < 0$ and bounded, then the zero solution of equation (3.3) is asymptomatically stable.

Proof.

From equation (3.6), we have ;

$$V'\big(t, u(.)\big) \;\; \leq \;\; -\beta u^2$$

This implies that $u^2 \in L'(R_1)$ and $u^2(t)$ is bounded on R_+. Therefore it follows from equation (3.3) and equation (3.4) that $u'(t)$ is bounded on R_+.

Hence $u(t) \to 0$ as $t \to \infty$.

Corollary 3.2

If equation (3.4) holds and $\alpha(t) > 0$ then the zero solution of equation (3.3) is completely unstable. Furthermore, for any $t_0 \geq 0$ and any $\delta > 0$, there is a continuous function $\phi : [0, t_0] \to R$ and a solution $u(t, t_0, \phi)$ of equation (3.3) with $|\phi(t) < \delta|$ and $|u(t, t_0, \phi| \geq [c_1 + c_2(t - t_0)]^{\frac{1}{2}}$ where c_1 and c_2 are positive constant depending on t_0 and ϕ.

Proof

This corollary is the immediate consequence of equation (3.7). We shall now obtain sufficient conditions for the asymptomatic stability of equation (3.3) in which $\alpha(t)$ is not necessarily negative and $a(t, s)$ need not be integrable on $R_+ \times R_+$.

It is clear that equation (3.3) is equivalent to

$$v'(t) = \mu(t)v(t) + \int_0^t b(t, s)v(s)ds + c(t) \tag{3.8}$$

where $t \in R_+$, $\mu(t) = \alpha(t) - q(t, t)$,

$$b(t, s) = a(t, s) + \frac{\partial}{\partial s}q(t, s) + q(t, s)\alpha(s) + \int_s^t q(t, \tau)a(\tau, s)d\tau$$

and $c(t) = q(t, 0)v(0)$, $q(t, s)$ being a continuously differentiable function for $[0 \leq s \leq t < \infty]$. We assume that there exist a continuously differentiable function $b(t, s)$ for $[0 \leq s \leq t < \infty]$ such that;

C_1: $|q(t, o)| \to 0$ as $t \to \infty|$ and $\int_0^\infty |q(t, 0)|dt < \infty$, and

$C_2 : \int_t^\infty |q(\tau, t)|d\tau$ is defined and finite for all $t \geq 0$.

Theorem 3.4

Suppose that C_1 and C_2 hold and there exist a constant $\beta > 0$ such that;

$$C_3 : \int_0^t |b(t, s)|ds + \int_t^\infty |b(\tau, t)|d\tau + |q(t, o)| - 2|\mu(t)| \leq -\beta$$

If $\mu(t) < 0$ and bounded, then the zero solution of equation (3.3) is asymptomatically stable.

Proof

Since we have equation (3.3) equivalent to equation (3.8), the proof of theorem

(3.4) is similar to corollary (3.1) with Lyapunov functional

$$V\big(t,u(.)\big) = u^2 + \int_0^t \int_t^\infty |b(\tau,s)|d\tau u^2(s)ds$$

Theorem 3.5

Suppose that C_1 and C_4 with $q(t,s) = q(t-s)$ and $b(t,s) = b(t-s)$ hold. If $\mu + \int_0^\infty |b(t)|dt < 0$, then the zero solution of equation (3.1) is uniformly asymptomatically stables.

Now we consider the scalar integro-differential equation

$$u' = -5u - \int_0^t 12e^{2(t-s)}u(s)ds \tag{3.9}$$

Here $\alpha = -5$ and $a(t) = -12e^{2t}$ in equation (3.1). Clearly $a(t)$ is not in $L'(R_+)$. We choose $q(t-s) = -3e^{-(t-s)} = q(t,s)$.

Then $\mu(t) = \alpha(t) - q(t,t) = -5 + 3 = -2 \leq 0$ and

$$\begin{aligned} b(t,s) &= a(t,s) + \frac{\partial}{\partial s}q(t,s) + q(t,s)\alpha(s) + \int_s^t q(t,\tau)a(\tau,s)d\tau \\ &= -12e^{2(t-s)} + 12e^{-(t-s)} + 36\int_s^t e^{(-t+2s+3\tau)}d\tau = 0 \end{aligned}$$

Thus all the conditions of Theorem (3.5) are satisfied, and therefore the zero solution of equation (3.9) is uniformly asymptotically stable.

We consider here again the scalar integro-differential equation

$$u' = \frac{u}{6+t} - \int_0^t \frac{6e^{-5(t-s)}(2+s)}{(2+t)^2}(3+s)u(s)ds \tag{3.10}$$

This implies that $\alpha(t) = \frac{1}{6+t}$, $a(t,s) = \frac{6e^{-5(t-s)}(2+s)}{(2+t)^2}(3+s)$ and $q(t,s) = \frac{6e^{-5(t-s)}(2+s)^2}{(2+t)^2}$.

Then $\mu(t) = \alpha(t) - q(t,t) = \frac{1}{6+t} - 2 < 0$ for all $t \geq 0$ Furthermore

$$\begin{aligned} b(t,s) &= a(t,s) + \frac{\partial}{\partial s}q(t,s) + q(t,s)\alpha(s) + \int_s^t q(t,\tau)a(\tau,s)d\tau \\ &= -\frac{6e^{-5(t-s)}(2+s)}{(2+t)^2}(3+s) + \frac{2e^{-2(t-s)}(2+s)^2}{(2+t)^2(6+t)} \end{aligned}$$

Therefore

$$\int_0^t |b(t,s)|ds + \int_t^\infty |b(\tau,t)|d\tau + |q(t,o)| - 2|\mu(t)|$$

$$\leq -\frac{118}{150} + \frac{23}{30} + 2\left|2 - \frac{1}{6+t}\right| \leq \frac{-17}{150}$$

and hence $\beta = \frac{17}{150}$ in condition C_3.

Thus all the conditions of Theorem (3.4) are satisfied, and therefore the zero solution of equation (3.10) is asymptotically stable.

However, Corollary (3.1) cannot be applied to equation (3.10) since $\alpha(t) = \frac{1}{6+t}$ is positive for all $t \geq 0$.

We are extending theorem (3.4) to the integro-differential system

$$x'(t) = A(t)x(t) + \int_0^t K(t,s)x(s)ds \qquad (3.11)$$

for $t \geq 0$ where $A(t)$ is an $n \times n$ continuous matrix for $0 \leq s \leq t < \infty$.

Equation (3.11) is equivalent to the system

$$y'(t) = B(t)y(t) + \int_0^t L(t,s)y(s)ds + F(t) \qquad (3.12)$$

where $t \in R_+$, $B(t) = A(t) - \phi(t,t)$,

$$L(t,s) \quad = \quad K(t,s) + \frac{\partial}{\partial s}\phi(t,s) + \phi(t,s)A(s) + \int_s^t \phi(t,u)K(u,s)du$$

and $F(t) = \phi(t,0)y_0$, $\phi(t,s)$ being an $n \times n$ matrix continuously differentiable for $0 \leq s \leq t < \infty$.

Let $H(t)$ be an $n \times n$ real symmetric matrix bounded and continuously differentiable for $0 \leq t < \infty$.

We shall also assume that there exists a constant $\delta > 0$ such that

$$y^T\Big[H'(t) + H(t)B(t) + B^T(t)H(t)\Big]y \leq -\delta|y|^2 \qquad (3.13)$$

for $y \in R^n$. Let $\int_t^\infty |\phi(u,t)|du$ be defined for all $t \geq 0$.

Theorem 3.6

Suppose that there exist an $n \times n$ matrix $\phi(t,s)$ continuously differentiable for $0 \leq s \leq t < \infty$ satisfying the following conditions:

i) $|\phi(t,0)| \to 0$ as $t \to \infty$ and $\int_0^\infty |\phi(t,0)|dt < \infty$.

ii) $\int_t^\infty |\phi(u,t)|du$ and $\int_t^\infty |L(u,t)|du$ are defined for $t \geq 0$ and

iii) there is a constant $\alpha_0 > 0$ such that

$$H_0\Big[\int_0^t |L(t,s)|ds + \int_t^\infty |L(u,t)|du + |\phi(t,0)|\Big] \leq \alpha_0 \tag{3.14}$$

where $H_0 = \sup_{t\geq 0} |H(t)|$.

If $\delta > \alpha_0$, equation (3.13) hold and $B(t)$ is bounded then the zero solution of equation (3.11) is asymptotically stable.

Proof

We consider the Lyapunov functional

$$V\big(t, y(.)\big) = y^T H(t)y + 2H_0 \int_0^t \int_t^\infty |L(u,s)|du|y^2(s)|ds$$

then along the solution $y(t)$ of equation (3.12), we have

$$\begin{aligned} V'\big(t, y(.)\big) &\leq y^T\Big[H'(t) + B^T(t)H(t) + H(t)B(t)\Big]y \\ &+ 2H_0|y||\phi(t.0)||y_0| + 2H_0 \int_0^t |L(t,s)|du|y(s)||y|ds \\ &+ H_0 \int_t^\infty |L(u,t)|du|y|^2ds + 2H_0 - H_0 \int_0^t |L(t,s)|du|y(s)|^2ds \end{aligned}$$

From equation (3.13) and the fact that $2|y||y_0| \leq |y|^2 + |y_0|^2$, it follows that;

$$\begin{aligned} V'(t, y(.)) &\leq -\delta|y|^2 + H_0|\phi(t,0)|(|y|^2 + |y_0|^2) \\ &+ H_0 \int_0^t |L(t,s)|(|y(s)|^2 + |y_0|^2)ds \\ &+ H_0 \int_t^\infty |L(u,t)|du|y|^2ds + -H_0 \int_0^t |L(t,s)||y(s)|^2ds \end{aligned}$$

Thus, equation (3.14) gives;

$$V'(t, y(.)) \leq \Big(-\delta + \alpha_0 \Big) |y|^2 + H_0 |\phi(t, 0)| |y_0|^2$$

Since $\delta > \alpha_0$, it follows from assumption (i) that $|y(t)|$ is in $L^2[0, \infty)$.

It can easily be seen from the assumptions of the theorem and equation (3.12) that $|y'(t)|$ is bounded.

Thus applying the Barbalat lemma yields $|y(t) \to 0|$ as $t \to \infty$, and this completes the proof.

Integro-differential equation with unbounded delays

In this section , we will look at related integro-differential equations of the form

$$x' = A(t)x(t) + \int_{-\infty}^{t} K(t, s)x(s)ds + f(t, x) \tag{3.15}$$

and demonstrate that the arguments similar to the previous section can be used to discuss the stability properties of equation (3.15). In this case A is an $n \times n$ matrix that is continuous for $-\infty < t < \infty$, K is an $n \times n$ continuous matrix for $-\infty < s \leq t < \infty$, $f \in C\Big[R \times S(p), R^n\Big]$, $S(p) = (x \in R^n : |x| < p)$ and $f(t, 0) \equiv 0$. Let $x(t) = x(t, 0, g)$ be a solution of the integro-differential equation (3.15) with initial function $g(t)$ on $(-\infty, 0]$, then equation (3.15) can be written as

$$x'(t) = A(t)x(t) + \int_{0}^{\infty} K(t, s)x(s)ds + f(t, x(t)) + F(t) \tag{3.16}$$

where $F(t) = \int_{-\infty}^{0} K(t, s)g(s)ds$

A modification of a theorem to suit the current situation can be found below.

Theorem 3.7

Let $\phi(t, s)$ be an $n \times n$ continuous matrix for $-\infty < s \leq t < \infty$. Then the equation (3.16) is equivalent to

$$y'(t) = B(t)y(t) + \int_0^t L(t,s)y(s)ds + G(t,y(t)) \tag{3.17}$$

and $y(0) = g(0) = x_0$ where $B(t) = A(t) - \phi(t,t)$,

$$L(t,s) = K(t,s) + \frac{\partial}{\partial s}\phi(t,s) + \phi(t,s)A(s) + \int_s^t \phi(t,u)K(u,s)du,$$

and

$$\begin{aligned} G(t,y(t)) &= f(t,y(t)) + F(t) + \phi(t,0)x_0 \\ &+ \int_0^t \phi(t,s)f(s,y(s))ds + \int_s^t \phi(t,s)F(s)ds, \end{aligned}$$

Then, $\phi(t,s)$ which is the differentiable resolvent corresponding to the kernel $K(t,s)$ satisfying the adjoint equation
$\frac{\partial}{\partial s}\phi(t,s) = -\phi(t,s)A(s) - \int_s^t \phi(t,u)K(u,s)du,\ \phi(t,t) = I$ then equation (3.17) gives the usual variation of parameters formula

$$y(t) = \phi(t,0)x_0 + \int_0^t \phi(t,s)f(s,y(s))ds + \int_0^t \phi(t,s)F(s)ds$$

We use the following assumptions to proceed further.

H_0 : There exist a continuous function $\lambda(t) > 0$ for $0 \leq t < \infty$ such that $|f(t,x)| \leq \lambda(t)$ for $(t,x) \in R^+ \times S(p)$ with $\lambda(t) \to 0$ as $t \to \infty$.

H_1 : There exist an $n \times n$ symmetric matrix,bounded and continuously

differentiable for $-\infty < s \leq t < \infty$ and a constant $\alpha > 0$ such that

$y^T\Big[H'(t) + H(t)B(t) + B^T(t)H(t)\Big]y \leq -\alpha|y|^2$ for $y \in R^n$.

H_2 : There exist an $n \times n$ matrix $\phi(t,s)$ that is differentiable for

$-\infty < s \leq t < \infty$ and satisfies the conditions $|\phi(t,0)| \to 0$ as $t \to \infty$ and $\int_0^\infty |\phi(t,0)|dt < \infty$.

H_3 : $\int_t^\infty |\phi(u,t)|du$ and $\int_t^\infty |L(u,t)|du$ are defined for all $t \in R$.

H_4 : There exist a continuously function $\eta(t)$ with the property;

$\eta(t) \to 0$ as $t \to \infty$ and $\eta(t) \in L'(R_+)$, where $\eta(t) = \int_{-\infty}^{0} |K(t,s)|ds + \int_{t}^{t} |\phi(t,s)| \int_{-\infty}^{0} |k(s,\tau)|d\tau ds$, and there is constant $\delta_0 > 0$ such that

$$H_0\Big[\int_0^t |L(t,s)|dt + \int_t^\infty |L(u,t)|du + |\phi(t,0)| + \eta(t)\Big]$$
$$+\lambda_0 H_0\Big[\int_0^t |\phi(t,s)|dt + \int_t^\infty |\phi(u,t)|du \le \alpha_0\Big]$$

where $\lambda_0 = \sup_{t\ge 0}|\lambda(t)|$ and $H_0 = \sup_{t\ge 0}|H(t)|$.

Theorem 3.8

Assume that $H_0 - H_4$ hold. If $\alpha > \delta_0 + 2\lambda_0 H_0$, $|(t)|$ is bounded and $|g(t)| < \delta$ on $(-\infty, 0]$, the the zero solution of equation (3.15) is asymptotically stable.

Proof

We consider the Lyapunov functional

$$\begin{aligned} V\big(t, y(.)\big) &= y^T H(t) y + H_0 \int_0^t \int_t^\infty |L(u,s)|du|y^2(s)|ds \\ &+ \lambda_0 H_0 \int_0^t \int_t^\infty |\phi(u,s)|du|y^2(s)|ds \end{aligned}$$

then the time derivative of V along the solution of equation (3.17), we have

$$\begin{aligned} V'\big(t, y(.)\big) &\le y^T\Big[H'(t) + B^T(t)H(t) + H(t)B(t)\Big]y \\ &+ 2H_0|y||G(t,y)||y_0| + 2H_0 \int_0^t |L(t,s)||y(s)||y|ds \\ &+ H_0 \int_t^\infty |L(u,t)|du|y|^2 - H_0 \int_0^t |L(t,s)||y(s)|^2 ds \\ &+ \lambda_0 H_0 \int_t^\infty |\phi(u,t)|du|y|^2 - \lambda_0 H_0 \int_0^t |\phi(t,s)||y(s)|^2 ds \end{aligned}$$

By the definition of $G(t,y)$ ad invoking hypothesis H_0, H_1 and H_4 and the fact that $2|y(s)||y| \leq |y(s)|^2 + |y|^2$, we obtain;

$$\begin{aligned}
V'(t, y(.)) &\leq \Big(-\alpha + 2\lambda_0 H_0 \Big)|y|^2 \\
&+ H_0\Big[|\phi(t,0)| + \int_t^{\infty} |L(y,t)|du + \int_0^t |L(t,s)|ds\Big]|y|^2 \\
&+ \lambda_0 H_0\Big[\int_0^t |\phi(t,s)|ds + \int_t^{\infty} |\phi(u,t)|du\Big]|y|^2 + H_0|\phi(t,0)||x_0|^2 \\
&+ H_0\Big[\int_{-\infty}^0 |K(t,s)|ds + \int_0^t |\phi(t,s)| \int_{-\infty}^0 |K(s,\tau)|d\tau ds\Big]|y|^2 \\
&+ H_0\Big[\int_{-\infty}^0 |K(t,s)||g(s)|^2 ds \\
&+ \int_0^t |\phi(t,s)| \int_{-\infty}^0 |K(s,\tau)||g(\tau)|^2 d\tau ds\Big]
\end{aligned}$$

This implies that;

$$V'(t, y(.)) \leq \Big(-\alpha + 2\lambda_0 H_0 + \delta_0 \Big)|y|^2 + H_0\lambda\eta(t) + H_0|\phi(t,0)||x_0|^2$$

Since $\alpha > \delta + 2\lambda_0 H_0$,

it follows from H_2 and H_4 that $|y(t)|$ is in $L^2[0, \infty)$.

It can also be shown from equation (3.17) that $|y'(t)|$ is bounded. Hence by applying the Barbalet lemma yields $|y(t)| \to 0$ as $t \to \infty$.

Thus the proof is complete.

Chapter Summary

In this chapter, we discussed the methodology for the research. We examined extensively the Lyapunov second method of solving integro-differential equations. We also discussed the conditions under which a functional can be classified as being a Lyapunov functional. We proceeded to prove some theorems concerning the Lyapunov second method of solving integro-differential equations.

CHAPTER FOUR

RESULTS AND DISCUSSION

Introduction

In this chapter, we obtain the main results of the thesis. The Lyapunov's direct method is used to obtain inequalities that guarantee exponential stability and instability of the zero solution of the Volterra integro-differential equations with multiple finite delays.

Exponential Stability

In this section, our goal is to achieve exponential stability of the zero solution of the Volterra integro-differential equations with multiple finite delays. To enable us establish exponential stability, we use Lyapunov functionals to obtain inequalities that can be used to deduce exponential stability for the Volterra integro-differential equation

$$x'(t) = p(t)x(t) - \sum_{i=1}^{n} \int_{t-\tau_i}^{t} q_i(t,s)x(s)ds, \tag{4.1}$$

where $\tau_i > 0$ is a constant, $q_i : [0,\infty) \times [-\tau,\infty)$ and $p : [0,\infty) \to \mathbb{R}$ with $0 < \tau_i \leq \tau$ for $i = 1, ..., n$.

Specifically, a Lyapunov functional denoted by $V(t,x) = V(t)$ is constructed and used to show that along the solutions of Equation (4.1),

$V'(t) \leq Q(t)V(t)$;

for some non-positive function $Q(t)$ under suitable conditions.

Define

$$\sum_{i=1}^{n} A_i(t,s) = \sum_{i=1}^{n} \int_{t-s}^{\tau_i} q_i(u+s,s)\,du \tag{4.2}$$

for $t \in [0.\infty)$ and $s \in [-\tau.\infty)$ then

$$\begin{aligned}
\sum_{i=1}^{n} A_i(t, t-\tau_i) \quad b &= \sum_{i=1}^{n} \int_{t-(t-\tau_i)}^{\tau_i} q_i(u+s, s)\, du \\
&= \sum_{i=1}^{n} \int_{-\tau_i}^{\tau_i} q_i(u+s, s)\, du \\
&= 0.
\end{aligned}$$

Thus,

$$\sum_{i=1}^{n} A_i(t, t-\tau_i) \equiv 0,$$

for all $t \geq 0$.

We assume that

$$\sum_{i=1}^{n} A_i(t, s) q_i(t, s) \geq 0 \text{ for all } t \in [0, \infty) \text{ and all } s \in [t-\tau, t) \tag{4.3}$$

and for $1 < \alpha \leq 2$

$$\sum_{i=1}^{n} A_i^2\Big(t - \frac{(\alpha-1)}{\alpha}\tau, z\Big) \geq \sum_{i=1}^{n} A_i^2(t, z) \tag{4.4}$$

for $t \in [0, \infty)$ and for $s \in \Big[t - \frac{(\alpha-1)}{\alpha}\tau, t - \frac{\tau}{\alpha}\Big]$.

It follows from (4.3) that, for all $t \in [0, \infty]$ and all $s \in [t-\tau, t]$ we have

$$\sum_{i=1}^{n} \int_{-\tau_i}^{0} \int_{t+s}^{t} A_i(t, z) \frac{\partial}{\partial t} A(t, z) x^2(z)\, dz ds \tag{4.5}$$

$$\begin{aligned}
&= -\sum_{i=1}^{n} \int_{t-\tau_i}^{t} \int_{-\tau_i}^{z-t} A_i(t, z) q_i(t, z) x^2(z)\, ds dz \\
&= -\sum_{i=1}^{n} \int_{t-\tau_i}^{t} (z - t + \tau_i) A_i(t, z) q_i(t, z) x^2(z)\, ds dz \\
&\leq 0.
\end{aligned} \tag{4.6}$$

Let $\psi : [-\tau, 0] \to (-\infty, \infty)$ be a given bounded initial function with

$$||\psi|| = \max_{-\tau \le s \le 0} |\psi(s)|.$$

We also denote the norm of a function $\varphi : [-\tau, \infty) \to (-\infty, \infty)$ by

$$||\varphi|| = \sup_{-\tau \le s < \infty} |\varphi(s)|.$$

We next consider the following Lemma which gives us an equivalent form of equation (4.1) which will be used extensively in the rest of the thesis.
In this lemma we will show that equation (4.1) can be re-written in an equivalent form by the use of the Leibniz formula.

Lemma 4.1 *Equation* (4.1) *is equivalent to the equation*

$$x'(t) = Q(t)x(t) + \frac{d}{dt}\sum_{i=1}^{n}\int\limits_{t-\tau_i}^{t} A_i(t,s)x(s)ds \tag{4.7}$$

where

$$Q(t) = p(t) - \sum_{i=1}^{n} A_i(t,t). \tag{4.8}$$

Proof.

Using the Leibniz formula which is given by

$$\begin{aligned}\frac{d}{dt}\Big(\int_{\alpha(t)}^{\beta(t)} f(x,s)\,ds\Big) &= f(x,\beta(t))\frac{d\beta(t)}{dt} - f(x,\alpha(t))\frac{d\alpha(t)}{dt} \\ &+ \int_{\alpha(t)}^{\beta(t)} \frac{\partial f(x,s)}{\partial t}\,ds\end{aligned}$$

we obtain

$$\begin{aligned}x'(t) &= Q(t)x(t) + \sum_{i=1}^{n} A_i(t,t)x(t) - \sum_{i=1}^{n} A_i(t,t-\tau_i)x(t-\tau_i) \\ &+ \sum_{i=1}^{n}\int\limits_{t-\tau_i}^{t} \frac{\partial}{\partial t}A_i(t,s)x(s)ds\end{aligned}$$

$$= \big(Q(t) + \sum_{i=1}^{n} A_i(t,t)\big)x(t) - \sum_{i=1}^{n} A_i(t, t-\tau_i)x(t-\tau_i)$$
$$- \sum_{i=1}^{n} \int_{t-\tau_i}^{t} q_i(t,s)x(s)ds$$

But $\sum_{i=1}^{n} A_i(t, t-\tau_i) \equiv 0$.
Thus

$$\begin{aligned} x'(t) &= \big(Q(t) + \sum_{i=1}^{n} A_i(t,t)\big)x(t) - \sum_{i=1}^{n} \int_{t-\tau_i}^{t} q_i(t,s)x(s)ds \\ &= \left(Q(t) + \sum_{i=1}^{n} A_i(t,t)\right) x(t) - \sum_{i=1}^{n} \int_{t-\tau_i}^{t} q_i(t,s)x(s)ds \\ &= p(t)x(t) - \sum_{i=1}^{n} \int_{t-\tau_i}^{t} q_i(t,s)x(s)ds. \end{aligned}$$

This completes the proof.

In the next lemma we propose the Lyapunov functionals that will be used in this thesis. These Lyapunov functionals satisfies the conditions stated earlier in Chapter three.

Lemma 4.2 *Let δ and λ be constants such that $\delta > 0$ and $\lambda > 0$. Then the functionals defined by*

$$\begin{aligned} V(t,x) &= \left[x(t) - \sum_{i=1}^{n} \int_{t-\tau_i}^{t} A_i(t,s)x(s)\,ds\right]^2 \\ &+ \sum_{i=1}^{n} \int_{-\tau_i}^{0} \int_{t+s}^{t} A_i^2(t,z)x^2(z)\,dzds \end{aligned} \tag{4.9}$$

and

$$V(t,x) = \left(x(t) - \sum_{i=1}^{n} \int_{t-\tau_i}^{t} A_i(t,s)x(s)\,ds \right)^2 - \lambda \sum_{i=1}^{n} \int_{t-\tau_i}^{t} A_i^2(t,z)x^2(z)\,dz \tag{4.10}$$

are Lyapunov functionals.

Proof.

We show that the functions defined in Equation (4.9) and (4.10) are Lyapunov functionals.

To this end, it must be noted that for $x = 0$ in Equation (4.9) that

$$\begin{aligned} V(t,0) &= \left[0 - \sum_{i=1}^{n} \int_{t-\tau_i}^{t} A_i(t,s)0\,ds \right]^2 \\ &\quad + \sum_{i=1}^{n} \int_{-\tau_i}^{0} \int_{t+s}^{t} A_i^2(t,z)0^2(z)\,dzds \\ &= 0 \end{aligned}$$

Thus $V(t,0) = 0$ for $x = 0$.

Now, $V(t,x) > 0$ except $x = 0$ for Equation (4.9). That is $V(t,x) > 0$ defined by Equation (4.9) is positive definite.

Finally,

$$\begin{aligned} \frac{\partial}{\partial x} V(t,x) &= 2 \left[x(t) - \sum_{i=1}^{n} \int_{t-\tau_i}^{t} A_i(t,s)x(s)\,ds \right] \\ &\quad \times \frac{\partial}{\partial x} \left[x(t) - \sum_{i=1}^{n} \int_{t-\tau_i}^{t} A_i(t,s)x(s)\,ds \right] \end{aligned}$$

which is continuous. Therefore, showing that the functional defined by Equation (4.9) is a Lyapunov functional.

It can be shown in a similar manner that the functional defined by Equation

(4.10) is also a Lyapunov functional.

This completes the proof.

We will use the Lyapunov functional defined in Equation (4.9) in the next lemma to obtain conditions that will enable us obtain appropriate inequalities that will be used in deducing our main results.

Lemma 4.3 *Let $Q(t)$ and $V(t)$ be defined as in Equations* (4.8) *and* (4.9) *respectively. Suppose* (4.3), (4.4) *and*

$$\frac{-1}{2\tau} \leq Q(t) \leq -\tau \sum_{i=1}^{n} A_i(t,t) \tag{4.11}$$

hold, then along the solution of Equation (4.1)*, we have*

$$V'(t) \leq Q(t)V(t). \tag{4.12}$$

Proof.

Let $x(t) = x(t, t_0, \psi)$ be a solution of Equation (4.1) with $V(t)$ defined by Equation (4.9). It must be noted that $Q(t) < 0$ for all $t \geq 0$ in view of condition inequality (4.11).

Differentiating Equation (4.9) with respect to t, then along the solutions of Equation (4.1) we have

$$\begin{aligned} V'(t) &= \frac{d}{dt}\left[x(t) - \sum_{i=1}^{n}\int_{t-\tau_i}^{t} A_i(t,s)x(s)\,ds\right]^2 \\ &\quad + \frac{d}{dt}\left(\sum_{i=1}^{n}\int_{-\tau_i}^{0}\int_{t+s}^{t} A_i(t,z)x^2(z)\,dzds\right) \\ &= 2\left[x(t) - \sum_{i=1}^{n}\int_{t-\tau_i}^{t} A_i(t,s)x(s)\,ds\right] \\ &\quad \times \left[x'(t) - \frac{d}{dt}\sum_{i=1}^{n}\int_{t-\tau_i}^{t} A_i(t,s)x(s)\,ds\right] \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{i=1}^{n} \tau_i A_i^2(t,t)x^2(t) - \sum_{i=1}^{n} \int_{-\tau_i}^{0} A_i(t,t+s)x^2(t+s)\,ds \\
&+ \sum_{i=1}^{n} \int_{-\tau_i}^{0} \int_{t+s}^{t} 2A_i(t,z)\frac{\partial}{\partial t}A_i(t,z)x^2(z)\,dzds \qquad (4.13)\\
= \;& 2[x(t) - \sum_{i=1}^{n} \int_{t-\tau_i}^{t} A_i(t,s)x(s)\,ds] \\
&\times[x'(t) - \frac{d}{dt}\sum_{i=1}^{n} \int_{t-\tau_i}^{t} A_i(t,s)x(s)\,ds] \\
&+ \sum_{i=1}^{n} \tau_i A_i^2(t,t)x^2(t) - \sum_{i=1}^{n} \int_{-\tau_i}^{0} A_i(t,t+s)x^2(t+s)\,ds \\
&+ \sum_{i=1}^{n} \int_{-\tau_i}^{0} \int_{t+s}^{t} 2A_i(t,z)\frac{\partial}{\partial t}A_i(t,z)x^2(z)\,dzds \\
= \;& 2[x(t) - \sum_{i=1}^{n} \int_{t-\tau_i}^{t} A_i(t,s)x(s)\,ds][Q(t)x(t) \\
&+\frac{d}{dt}\sum_{i=1}^{n} \int_{t-\tau_i}^{t} A_i(t,s)x(s)\,ds \\
&- \frac{d}{dt}\sum_{i=1}^{n} \int_{t-\tau_i}^{t} A_i(t,s)x(s)\,ds] \\
&+ \sum_{i=1}^{n} \tau_i A_i^2(t,t)x^2(t) - \sum_{i=1}^{n} \int_{-\tau_i}^{0} A_i(t,t+s)x^2(t+s)\,ds \\
&+ \sum_{i=1}^{n} \int_{-\tau_i}^{0} \int_{t+s}^{t} 2A_i(t,z)\frac{\partial}{\partial t}A_i(t,z)x^2(z)\,dzds
\end{aligned}
$$

(4.14)

$$
\begin{aligned}
&\leq 2[x(t) - \sum_{i=1}^{n}\int\limits_{t-\tau_i}^{t} A_i(t,s)x(s)\,ds]Q(t)x(t) \\
&\quad + \sum_{i=1}^{n}\tau_i A_i^2(t,t)x^2(t) - \sum_{i=1}^{n}\int\limits_{-\tau_i}^{0} A_i(t,t+s)x^2(t+s)\,ds \\
&\leq Q(t)\Big[x^2(t) - 2x(t)\sum_{i=1}^{n}\int\limits_{t-\tau_i}^{t} A_i(t,s)x(s)\,ds + \Big(\sum_{i=1}^{n}\int\limits_{t-\tau_i}^{t} A_i(t,s)x(s)\,ds)\Big)^2\Big] \\
&\quad + Q(t)\sum_{i=1}^{n}\int\limits_{-\tau_i}^{0}\int\limits_{t+s}^{t} A_i^2(t,z)x^2(z)\,dzds \\
&\quad - Q(t)\Big(\sum_{i=1}^{n}\int\limits_{t-\tau_i}^{t} A_i(t,s)x(s)\,ds\Big)^2 + \Big(Q(t) + \sum_{i=1}^{n}\tau A_i^2(t,t)\Big)x^2(t) \\
&\quad - \sum_{i=1}^{n}\int\limits_{-\tau_i}^{0} A_i(t,t+s)x^2(t+s)\,ds \\
&\quad - Q(t)\sum_{i=1}^{n}\int\limits_{-\tau_i}^{0}\int\limits_{t+s}^{t} A_i^2(t,z)x^2(z)\,dzds \\
&\leq Q(t)V(t) - Q(t)\Big(\sum_{i=1}^{n}\int\limits_{t-\tau_i}^{t} A_i(t,s)x(s)\,ds\Big)^2 \\
&\quad + \Big(Q(t) + \sum_{i=1}^{n}\tau A_i^2(t,t)\Big)x^2(t) - \sum_{i=1}^{n}\int\limits_{-\tau_i}^{0} A_i(t,t+s)x^2(t+s)\,ds \\
&\quad - Q(t)\sum_{i=1}^{n}\int\limits_{-\tau_i}^{0}\int\limits_{t+s}^{t} A_i^2(t,z)x^2(z)\,dzds \qquad (4.15)
\end{aligned}
$$

By the Schwartz inequality it follows that

$$
\begin{aligned}
\sum_{i=1}^{n}\Big(\int\limits_{t-\tau_i}^{t} A_i(t,s)x(s)\,ds\Big)^2 &\leq \sum_{i=1}^{n}\tau_i\int\limits_{t-\tau_i}^{t} A_i^2(t,z)x^2(s)\,ds \\
&\leq \sum_{i=1}^{n}\tau\int\limits_{t-\tau_i}^{t} A_i^2(t,z)x^2(s)\,ds \qquad (4.16)
\end{aligned}
$$

Also

$$\begin{aligned}\sum_{i=1}^{n}\int_{-\tau_i}^{0}\int_{t+s}^{t} A_i^2(t,z)x^2(z)\,dzds &\leq \sum_{i=1}^{n}\tau_i\int_{t-\tau_i}^{t} A_i^2(t,s)x^2(s)\,ds\\ &\leq \sum_{i=1}^{n}\tau\int_{t-\tau_i}^{t} A_i^2(t,s)x^2(s)\,ds, \qquad (4.17)\end{aligned}$$

and by change of variable, $t+s \to s$ we have

$$\sum_{i=1}^{n}\int_{-\tau_i}^{0} A_i^2(t,t+s)x^2(t+s)\,ds = \sum_{i=1}^{n}\int_{t-\tau_i}^{t} A_i^2(t,s)x^2(s)\,ds. \qquad (4.18)$$

Substituting (4.16), (4.17) and (4.18) into (4.19) we obtain

$$\begin{aligned}V'(t) &\leq Q(t)V(t) - Q(t)\sum_{i=1}^{n}\tau\int_{t-\tau_i}^{t} A_i^2(t,z)x^2(s)\\ &\quad + \Big(Q(t)+\sum_{i=1}^{n}\tau A_i^2(t,t)\Big)x^2(t) - \sum_{i=1}^{n}\int_{t-\tau_i}^{t} A_i^2(t,s)x^2(s)\,ds\\ &\quad - Q(t)\sum_{i=1}^{n}\tau\int_{t-\tau_i}^{t} A_i^2(t,s)x^2(s)\,ds\\ &\leq Q(t)V(t) + \left(Q(t)+\sum_{i=1}^{n}\tau A_i^2(t,t)\right)x^2(t)\\ &\quad + (-2\tau Q(t)-1)\sum_{i=1}^{n}\int_{t-\tau_i}^{t} A_i^2(t,s)x^2(s)\,ds.\end{aligned}$$

Therefore, invoking (4.11) we obtain

$$V'(t) \leq Q(t)V(t).$$

In theorem 4.4, we obtain an inequality that every solution of Equation (4.1) satisfy.

Theorem 4.4 *Assume the hypothesis of lemma 2.1 hold and let* $1 < \alpha \leq 2$. *Then any solution of* $x(t) = x(t, t_0, \varphi)$ *of Equation* (4.1) *satisfies the inequality*

$$|x(t)|^2 \leq 2\left(\frac{2\alpha-1}{\alpha-1}\right) V(t_0) \exp\left(\int_{t_0}^{t-\frac{(\alpha-1)}{\alpha}\tau} [p(s) - \sum_{i=1}^{n} A_i(s,s)]\, ds\right) \tag{4.19}$$

for $t \geq t_0 + \frac{\alpha-1}{\alpha}\tau$.

Proof.

By changing the order of integration of the second term in the definition of $V(t)$ we obtain

$$\begin{aligned} \sum_{i=1}^{n} \int_{-\tau_i}^{0} \int_{t+s}^{t} A_i^2(t,z)x^2(z)\, dzds &= \sum_{i=1}^{n} \int_{t-\tau_i}^{t} \int_{-\tau_i}^{z-t} A_i^2(t,z)x^2(z)\, dsdz \\ &= \sum_{i=1}^{n} \int_{t-\tau_i}^{t} (z-t+\tau_i)A_i^2(t,z)x^2(z)\, dz \end{aligned} \tag{4.20}$$

For $1 < \alpha \leq 2$ and if $t - \frac{\tau}{\alpha} \leq z \leq t$, then $\frac{\alpha-1}{\alpha} \leq z - t + \tau \leq \tau$.

Then the expression (4.20) yields

$$\begin{aligned} \sum_{i=1}^{n} \int_{-\tau_i}^{0} \int_{t+s}^{t} A_i^2(t,z)x^2(z)\, dzds &= \sum_{i=1}^{n} \int_{t-\tau_i}^{t} (z-t+\tau)A_i^2(t,z)x^2(z)\, dz \\ &= \sum_{i=1}^{n} \int_{t-\tau_i}^{t-\frac{\tau_i}{\alpha}} (z-t+\tau)A_i^2(t,z)x^2(z)\, dz \\ &\quad + \sum_{i=1}^{n} \int_{t-\frac{\tau_i}{\alpha}}^{t} (z-t+\tau)A_i^2(t,z)x^2(z)\, dz \\ &\geq \sum_{i=1}^{n} \int_{t-\frac{\tau_i}{\alpha}}^{t} (z-t+\tau)A_i^2(t,z)x^2(z)\, dz \\ &\geq \frac{(\alpha-1)}{\alpha}\tau \sum_{i=1}^{n} \int_{t-\frac{\tau_i}{\alpha}}^{t} A_i^2(t,z)x^2(z)\, dz \end{aligned} \tag{4.21}$$

Let $V(t)$ be given by (4.9) then

$$\begin{aligned} V(t) &\geq \sum_{i=1}^{n} \int_{-\tau_i}^{0} \int_{t+s}^{t} A_i^2(t,z)x^2(z)\,dzds \\ &\geq \frac{(\alpha-1)}{\alpha}\tau \sum_{i=1}^{n} \int_{t-\frac{t-(\alpha-1)}{\alpha}\tau_i}^{t} A_i^2(t,z)x^2(z)\,dz \end{aligned}$$

This implies that for $1 < \alpha \leq 2$

$$\begin{aligned} V(t-\frac{(\alpha-1)}{\alpha}\tau) &\geq \frac{(\alpha-1)}{\alpha}\tau \sum_{i=1}^{n} \int_{t-\tau_i}^{t-\tau_i+\frac{\tau_i}{\alpha}} A_i^2(t,z)x^2(z)\,dz \\ &\geq \frac{(\alpha-1)}{\alpha}\tau \sum_{i=1}^{n} \int_{(t-\tau_i)}^{t-\frac{\tau_i}{\alpha}} A_i^2(t,z)x^2(z)\,dz \end{aligned} \tag{4.22}$$

Now $V'(t) \leq 0$, then we have for $t \geq t_0 + \frac{(\alpha-1)}{\alpha}\tau$ and that

$$0 \leq V(t) + V\left(t - \frac{(\alpha-1)}{\alpha}\tau\right) \leq 2V\left(t - \frac{(\alpha-1)}{\alpha}\tau\right)$$

$$\begin{aligned} V(t) + V\left(t - \frac{(\alpha-1)}{\alpha}\tau\right) & \\ &\geq \left(x(t) - \sum_{i=1}^{n} \int_{(t-\tau_i)}^{t} A_i(t,s)x^2(s)\,ds\right)^2 \\ &+ \sum_{i=1}^{n} \int_{-\tau_i}^{0} \int_{t+s}^{t} A_i^2(t,s)x^2(z)\,dzds \\ &+ \sum_{i=1}^{n} \frac{(\alpha-1)}{\alpha}\tau_i \int_{(t-\tau_i)}^{t-\frac{\tau_i}{\alpha}} A_i^2(t,z)x^2(z)\,dz \end{aligned}$$

It follows from (4.19), (4.21) and (4.22) that

$$
\begin{aligned}
&\geq \left(x(t) - \sum_{i=1}^{n} \int\limits_{(t-\tau_i)}^{t} A_i(t,s)x^2(s)\,ds \right)^2 \\
&\quad + \sum_{i=1}^{n} \frac{(\alpha-1)}{\alpha}\tau_i \int\limits_{(t-\frac{\tau_i}{\alpha})}^{t} A_i^2(t,z)x^2(z)\,dz \\
&\quad + \sum_{i=1}^{n} \frac{(\alpha-1)}{\alpha}\tau_i \int\limits_{(t-\tau_i)}^{t-\frac{\tau_i}{\alpha}} A_i^2(t,z)x^2(z)\,dz
\end{aligned}
$$

$$
\begin{aligned}
&= \left(x(t) - \sum_{i=1}^{n} \int\limits_{(t-\tau_i)}^{t} A_i(t,s)x^2(s)\,ds \right)^2 \\
&\quad + \sum_{i=1}^{n} \frac{(\alpha-1)}{\alpha}\tau_i \int\limits_{(t-\tau_i)}^{t} A_i^2(t,z)x^2(z)\,dz
\end{aligned}
$$

By the Schwartz inequality, we obtain

$$
\begin{aligned}
V(t) + V\left(t - \frac{(\alpha-1)}{\alpha}\tau\right) \\
&\geq \left(x(t) - \sum_{i=1}^{n} \int\limits_{(t-\tau_i)}^{t} A_i(t,s)x^2(s)\,ds \right)^2 \\
&\quad + \sum_{i=1}^{n} \frac{(\alpha-1)}{\alpha} \left(\int\limits_{(t-\tau_i)}^{t} A_i(t,s)x(s)\,ds \right)^2 \\
&= \frac{\alpha-1}{2\alpha-1}x^2(t) \\
&\quad + \left(\frac{1}{\sqrt{1+\frac{\alpha-1}{\alpha}}}x(t) \right. \\
&\quad \left. - \sqrt{1+\frac{\alpha-1}{\alpha}} \sum_{i=1}^{n} \int\limits_{t-\tau_i}^{t} A_i(t,s)x(s)\,ds \right)^2 \\
&\geq \frac{\alpha-1}{2\alpha-1}x^2(t) \qquad (4.23)
\end{aligned}
$$

Thus (4.23) show that

$$\frac{\alpha-1}{2\alpha-1}x^2(t) \leq V(t) + V\left(t - \frac{(\alpha-1)}{\alpha}\tau\right)$$
$$\leq \quad 2V\left(t - \frac{(\alpha-1)}{\alpha}\tau\right)$$

Integrating the inequality (4.12) from t_0 to t we obtain

$$\begin{aligned}
\ln V(s)|_{t_0}^{t} &\leq \int_{t_0}^{t} Q(s)\, ds \\
\implies V(t) &\leq V(t_0) \exp \int_{t_0}^{t} Q(s)\, ds \\
\implies V(t) &\leq V(t_0) \exp \int_{t_0}^{t} \left([p(s) - \sum_{i=1}^{n} A_i(s,s)]\, ds \right)
\end{aligned}$$

As a result

$$V\left(t - \frac{(\alpha-1}{\alpha}\tau\right) \leq V(t_0) \exp \int_{t_0}^{t-\frac{(\alpha-1)}{\alpha}\tau} \left([p(s) - \sum_{i=1}^{n} A_i(s,s)]\, ds \right)$$

and also

$$|x(t)|^2 \leq 2\Big(\frac{2\alpha-1}{\alpha-1}\Big) V(t_0) \exp \int_{t_0}^{t-\frac{(\alpha-1)}{\alpha}\tau} \left([p(s) - \sum_{i=1}^{n} A_i(s,s)]\, ds \right)$$

for $t \geq t_0 + \frac{\alpha-1}{\alpha}\tau$. This completes the proof.

The next corollary is as a direct consequence of Theorem 4.4. It contains sufficient conditions of Equation (4.11) to be exponential stable.

Corollary 4.5 *Suppose condition* (4.11) *hold. If*

$$\sum_{i=1}^{n} A_i(t,t) \geq \gamma \tag{4.24}$$

for some positive constant γ for all $t \geq t_0$ then the zero solution of (4.1) *is exponentially stable.*

Proof.

From Theorem 4.4 we have that

$$\begin{aligned}
|x(t)|^2 &\leq 2\Big(\frac{2\alpha-1}{\alpha-1}\Big)V(t_0)\exp \int_{t_0}^{t-\frac{(\alpha-1)}{\alpha}\tau} \Big([p(s) - \sum_{i=1}^{n} A_i(s,s)]\, ds\Big) \\
&\leq 2\Big(\frac{2\alpha-1}{\alpha-1}\Big)V(t_0)\exp\left(-\tau \int_{t_0}^{\left(t-\frac{(\alpha-1)}{\alpha}\tau\right)} (\sum_{i=1}^{n} A_i(s,s)\, ds)\right) \\
&\leq 2\Big(\frac{2\alpha-1}{\alpha-1}\Big)V(t_0)\exp\Big(-\tau\gamma\Big(t-\frac{(\alpha-1)}{\alpha}\tau - t_0\Big)\Big) \\
&= 2\Big(\frac{2\alpha-1}{\alpha-1}\Big)V(t_0)e^{\left(\tau^2\gamma\frac{(\alpha-1)}{\alpha}\right)}e^{-\tau\gamma(t-t_0)}
\end{aligned}$$

This completes the proof.

Criteria for Instability

In this section, we focus on instability of the zero solution of Equation (4.1). To help us achieve instability, we vary the conditions used for the stability and define a new Lyapunov functional given in (4.10).

Lemma 4.6 *We assume* (4.2) *and there exist a positive constant* $\lambda > \tau$ *such that*

$$\lambda \sum_{i=1}^{n} A_i(t,t) \leq Q(t) \tag{4.25}$$

for all $t \geq 0$. *Then, we define the functional* $V(t)$ *for* $x \in C[-\tau, \infty)$ *by*

$$V(t) = \left(x(t) - \sum_{i=1}^{n} \int_{t-\tau_i}^{t} A_i(t,s)x(s)\, ds\right)^2 - \lambda \sum_{i=1}^{n} \int_{t-\tau_i}^{t} A_i^2(t,z)x^2(z)\, dz \tag{4.26}$$

then along the solutions of Equation (4.1) *we have*

$$V'(t) \geq Q(t)V(t). \tag{4.27}$$

Proof.

Let $x(t) = x(t, t_0, \varphi)$ be a solution of Equation (4.1) and define $V(t)$ by (4.26). Differentiating (4.26) with respect to t along the solutions of Equation (4.1) we have

$$
\begin{aligned}
V'(t) &= 2\left(x(t) - \sum_{i=1}^{n}\int_{t-\tau_i}^{t} A_i(t,s)x(s)\,ds\right)\left(x'(t) - \frac{d}{dt}\sum_{i=1}^{n}\int_{t-\tau_i}^{t} A_i(t,s)x(s)\,ds\right) \\
&\quad - \lambda\sum_{i=1}^{n} A_i^2(t,t)x^2(t) - \lambda\sum_{i=1}^{n}\int_{t-\tau_i}^{t} 2A_i(t,z)\frac{\partial A_i(t,z)}{\partial t}x^2(z)\,dz \\
&= 2\left(x(t) - \sum_{i=1}^{n}\int_{t-\tau_i}^{t} A_i(t,s)x(s)\,ds\right)\Bigg(Q(t)x(t) \\
&\quad + \frac{d}{dt}\sum_{i=1}^{n}\int_{t-\tau_i}^{t} A_i(t,s)x(s)\,ds \\
&\quad - \frac{d}{dt}\sum_{i=1}^{n}\int_{t-\tau_i}^{t} A_i(t,s)x(s)\,ds\Bigg) \\
&\quad - \lambda\sum_{i=1}^{n} A_i^2(t,t)x^2(t) - \lambda\sum_{i=1}^{n}\int_{t-\tau_i}^{t} 2A_i(t,z)\frac{\partial A_i(t,z)}{\partial t}x^2(z)\,dz \\
&= 2\left(x(t) - \sum_{i=1}^{n}\int_{t-\tau_i}^{t} A_i(t,s)x(s)\,ds\right)Q(t)x(t) \\
&\quad - \lambda\sum_{i=1}^{n} A_i^2(t,t)x^2(t) - \lambda\sum_{i=1}^{n}\int_{t-\tau_i}^{t} 2A_i(t,z)\frac{\partial A_i(t,z)}{\partial t}x^2(z)\,dz \\
&\geq 2\left(x(t) - \sum_{i=1}^{n}\int_{t-\tau_i}^{t} A_i(t,s)x(s)\,ds\right)Q(t)x(t) \\
&\quad - \lambda\sum_{i=1}^{n} A_i^2(t,t)x^2(t)
\end{aligned}
$$

$$
\begin{aligned}
&= Q(t)V(t) + Q(t)\Big[- \left(\sum_{i=1}^{n} \int_{t-\tau_i}^{t} A_i(t,s)x(s)\,ds \right)^2 \\
&\quad + \lambda \sum_{i=1}^{n} \int_{t-\tau_i}^{t} A_i^2(t,z)x^2(z)\,dz \Big] \\
&\quad + \left[Q(t) - \lambda \sum_{i=1}^{n} A_i^2(t,t) \right] x^2(t)
\end{aligned}
$$

$$
\begin{aligned}
&\geq Q(t)V(t) + Q(t)\Big[- \tau \left(\sum_{i=1}^{n} \int_{t-\tau_i}^{t} A_i^2(t,s)x^2(s)\,ds \right) \\
&\quad + \lambda \sum_{i=1}^{n} \int_{t-\tau_i}^{t} A_i^2(t,z)x^2(z)\,dz \Big] \\
&\quad + \Big[Q(t) - \lambda \sum_{i=1}^{n} A_i^2(t,t) \Big] x^2(t)
\end{aligned}
$$

$$
\begin{aligned}
&\geq Q(t)V(t) + Q(t)(-\tau + \lambda) \sum_{i=1}^{n} \int_{t-\tau_i}^{t} A_i^2(t,s)x^2(s)\,ds \\
&\quad + \left[Q(t) - \lambda \sum_{i=1}^{n} A_i^2(t,t) \right] x^2(t) \\
&\geq Q(t)V(t).
\end{aligned}
$$

This completes the proof.

In Theorem 4.7 we provide conditions for the zero solution of Equation (4.1) to be exponentially unstable.

Theorem 4.7 *Suppose the hypothesis of Lemma* 4.6 *is true. Then the zero solution of Equation* (4.1) *is unstable, provided that;*

$$
\sum_{i=1}^{n} \int_{t_0}^{\infty} A_i^2(s,s)\,ds = \infty.
$$

Proof.

Integrating expression (4.27) given as

$$V'(t) \geq Q(t)V(t)$$

from t_0 to t, we obtain

$$\ln V(s)\mid_{t_0}^{t} \geq \int_{t_0}^{t} Q(s)\,ds$$

which gives

$$V(t) \geq V(t_0)e^{\int_{t_0}^{t} Q(s)\,ds} \tag{4.28}$$

or

$$V(t) \geq V(t_0)e^{\int_{t_0}^{t}[p(s)-\sum_{i=1}^{n} A_i(s,s)]\,ds} \tag{4.29}$$

Expanding the expression of $V(t)$ be given by Equation (4.10) we obtain

$$\begin{aligned} V(t) &= x^2(t) + 2x(t)\sum_{i=1}^{n}\int_{t-\tau_i}^{t} A_i(t,s)x(s)\,ds \\ &+ \left(\sum_{i=1}^{n}\int_{t-\tau_i}^{t} A_i(t,s)x(s)\,ds\right)^2 \\ &- \lambda\sum_{i=1}^{n}\int_{t-\tau_i}^{t} A_i^2(t,z)x^2(z)\,dz \end{aligned} \tag{4.30}$$

Now for the subsequent work we let $\beta = \lambda - \tau$, then it follows from the expression

$$\left(\frac{\sqrt{\tau}}{\sqrt{\beta}}a - \frac{\sqrt{\beta}}{\sqrt{\tau}}b\right)^2 \geq 0 \tag{4.31}$$

that

$$\left(\frac{\sqrt{\tau}}{\sqrt{\beta}}a\right)^2 - 2\frac{\sqrt{\tau}}{\sqrt{\beta}}\frac{\sqrt{\beta}}{\sqrt{\tau}}ab + \left(\frac{\sqrt{\beta}}{\sqrt{\tau}}b\right)^2 \geq 0$$
$$\Longrightarrow \frac{\tau}{\beta}a^2 - 2ab + \frac{\beta}{\tau}b^2 \geq 0.$$

Thus,

$$2ab \leq \frac{\tau}{\beta}a^2 + \frac{\beta}{\tau}b^2 \tag{4.32}$$

By inequality (4.32), the second term on the right hand side of (4.30) can be expressed as

$$\begin{aligned}
&2x(t)\sum_{i=1}^{n}\int_{t-\tau_i}^{t} A_i(t,s)x(s)\,ds \\
&\leq 2 \mid x(t) \mid\mid \sum_{i=1}^{n}\int_{t-\tau_i}^{t} A_i(t,s)x(s)\,ds \mid \\
&\leq \frac{\tau}{\beta}x^2(t) + \frac{\beta}{\tau}\left(\sum_{i=1}^{n}\int_{t-\tau_i}^{t} A_i(t,s)x(s)\,ds\right)^2 \\
&\leq \frac{\tau}{\beta}x^2(t) + \frac{\beta}{\tau}\tau\sum_{i=1}^{n}\int_{t-\tau_i}^{t} A_i^2(t,s)x^2(s)\,ds \\
&= \frac{\tau}{\beta}x^2(t) + \beta\sum_{i=1}^{n}\int_{t-\tau_i}^{t} A_i^2(t,s)x^2(s)\,ds
\end{aligned} \tag{4.33}$$

Substituting inequality (4.33) into (4.30), we obtain

$$\begin{aligned}
V(t) \;\leq\; & x^2(t) + \frac{\tau}{\beta}x^2(t) + \beta\sum_{i=1}^{n}\int_{t-\tau_i}^{t} A_i^2(t,s)x^2(s)\,ds \\
& + \left(\sum_{i=1}^{n}\int_{t-\tau_i}^{t} A_i(t,s)x(s)\,ds\right)^2 \\
& - \lambda\sum_{i=1}^{n}\int_{t-\tau_i}^{t} A_i^2(t,z)x^2(z)\,dz
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(1+\frac{\tau}{\beta}\right)x^2(t) + \beta\sum_{i=1}^{n}\int_{t-\tau_i}^{t} A_i^2(t,s)x^2(s)\,ds \\
&\quad + \tau\sum_{i=1}^{n}\int_{t-\tau_i}^{t} A_i^2(t,s)x^2(s)\,ds \\
&\quad - \lambda\sum_{i=1}^{n}\int_{t-\tau_i}^{t} A_i^2(t,s)x^2(s)\,ds \\
&\leq (\frac{\beta+\tau}{\beta})x^2(t) \\
&\quad + (\beta+\tau-\lambda)\sum_{i=1}^{n}\int_{t-\tau_i}^{t} A_i^2(t,s)x^2(s)\,ds \\
&\leq \left(\frac{\beta+\tau}{\beta}\right)x^2(t).
\end{aligned}
$$

Therefore,

$$
\mid x(t)\mid^2 \geq \left(\frac{\lambda-\tau}{\lambda}\right)V(t) \tag{4.34}
$$

Substituting the expression for $V(t)$ in (4.28) into (4.34), we obtain

$$
\begin{aligned}
\mid x(t)\mid^2 &\geq \left(\frac{\lambda-\tau}{\lambda}\right)V(t_0)e^{\left(\sum\limits_{i=1}^{n}\int\limits_{t_0}^{t} Q(s)\,ds\right)} \\
&= \left(\frac{\lambda-\tau}{\lambda}\right)V(t_0)exp\left(\sum_{i=1}^{n}\int_{t_0}^{t}[p(s)-\sum_{i=1}^{n}A_i(s,s)]\,ds\right) \\
&\geq \left(\frac{\lambda-\tau}{\lambda}\right)V(t_0)exp\left(\tau\int_{t_0}^{t}\sum_{i=1}^{n}A_i^2(s,s)\,ds\right)
\end{aligned}
$$

This completes the proof.

Chapter Summary

The concept of exponential stability was discussed in this chapter. We constructed suitable Lyapunov functional that helped us to obtain inequalities to achieve exponential stability and exponential instability of the zero solution of our multiple finite delay integro-differential equation. We proceeded to prove relevant theorems, lemmas and corollary to establish exponential stability and instability of the integro-differential equation.

CHAPTER FIVE

SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS

In this chapter, we provided summaries of results, conclusions, and some recommendations.

Summary

In chapter one, we provided the background to Volterra's integro-differential equations and identified some of the challenges that researchers encounter in this field of research. We provided the rationale for this thesis and set out the objectives of this thesis.

In Chapter 2, we reviewed some of the relevant literature on Volterra integro-differential equations. We explained differential equations in general and provided some examples of differential equations. We also explained the integro-differential equations and some examples of integro-differential equations. We later looked at areas in other fields of science and engineering where integro-differential equations can be applied and some methods for solving integro-differential equations.

In chapter three, we explained the Lyapunov direct method of solving integro-differential equations and how we were going to use the method in this thesis.

In Chapter 4, we constructed Lyapunov functional and used it to obtain inequalities that made it possible for us to achieve exponential stability. We also constructed another Lyapunov functional and had inequalities that ensured exponential instability.

Conclusions

In conclusion, we have constructed suitable Lyapunov functionals for establishing the exponential stability and instability of Volterra integro-differential equation with multiple finite delays.

We have also obtained sufficient conditions for the exponential stability of the zero solution of Volterra integro-differential equation with multiple finite delays.

Finally, we also obtained sufficient conditions that guarantees that the zero solution of Volterra integro-differential equation with multiple finite delays to be unstable.

Recommendations

The Lyapunov functional should be used to find the exponential stability of the zero solution of Volterra integro-differential equations.

REFERENCE

Adıvar, M., and Raffoul, Y. N. (2012). *Inequalities and exponential stability and instability in finite delay Volterra integro-differential equations*. Rendiconti del Circolo Matematico di Palermo, 61(3) 321-330.

Becker, L. C. (2009). *Uniformly Continuous L 1 Solutions of Volterra Equations and Global Asymptotic Stability*. CUBO, A Mathematical Journal, 11(3), 1-24.

Bellomo, N., Firmani, B., and Guerri, L. (1999). *Bifurcation analysis for a nonlinear system of integro-differential equations modelling tumor-immune cells competition*. Applied mathematics letters, 12(2), 39-44.

Brauer, F., Castillo-Chavez, C., and Castillo-Chavez, C. (2012). *Mathematical models in population biology and epidemiology*. New York: Springer. (Vol. 2, p.508).

Burton, T. A. (2005). *Volterra integral and Differential Equations*. Second Edition. Mathematics in Science and Engineering, 85(1), 202.

Burton, T. A., Huang, Q., and Mahfoud, W. E. (1985). *Rate of decay of solutions of Volterra equations*. Nonlinear Analysis: Theory, Methods and Applications, 9(7), 651-663.

Burton, T. A., and Mahfoud, W. E. (1982). *Stability criteria for Volterra equations*. Transactions of the American Mathematical Society, 143-174.

Burton, T. A. (1982). *Construction of Liapunov functionals for Volterra equations*. Journal of Mathematical Analysis and Applications, 85(1), 90-105.

Corduneanu, C. (2008). *Principles of differential and integral equations (Vol. 295). American Mathematical Soc*.

Corduneanu, C. (1973). *Integral equations and stability of feedback systems.* Academic Press, Inc. Mathematics in Science and Engineering 104(1), 109-216.

Dawson, H., and Lugli, A. (2015). *Molecular and pathogenetic aspects of tumor budding in colorectal cancer*. Frontiers in medicine, 2, 11.

Destexhe, A., and Sejnowski, T. J. (2009). *The Wilson–Cowan model, 36 years later*. Biological cybernetics, 101(1), 1-2.

Driver, R. D. (1962). *Existence and stability of solutions of a delay-differential system.* Archive for Rational Mechanics and Analysis, 10(1), 401-426.

Funakubo, M., Hara, T., and Sakata, S. (2006). *On the uniform asymptotic stability for a linear integro-differential equation of Volterra type*. Journal of mathematical analysis and applications, 324(2), 1036-1049.

Graef, J. R., Tunc, C., and Sevgin, S. (2016). *Behavior of solutions of non-linear functional Voltera integro-differential equations with multiple delays*. Dynam. Systems Appl, 25(1-2), 39-46.

Graef, J. R., and Tunc¸, C. (2015). *Continuability and boundedness of multidelay functional integro-differential equations of the second order.* Revista de la

Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas, 109(1), 169-173.

Hatamzadeh, S., Naser-Moghadasi, M., Babolian, E., and Masouri, Z. (2008). *Numerical approach to survey the problem of electromagnetic scattering from resistive strips based on using a set of orthogonal basis functions.* Progress In Electromagnetics Research, 81, 393-412.

Jaswon, M. A., Symm, G. T., and Cruse, T. A. (1977). *Integral equation methods in potential theory and elastostatics.* 45(4), 85-115

Kheybari, S., Darvishi, M. T., and Wazwaz, A. M. (2017). *A semi-analytical algorithm to solve systems of integro-differential equations under mixed boundary conditions.* Journal of Computational and Applied Mathematics, 317, 72-89.

Kyselka, A. (1977). *Properties of systems of integro-differential equations in the statistics of polymer chains.* Polymer Science USSR, 19(11), 2852-2858.

Lakshmikantham, V and Rama Mohana Rao M. (1995) *Theory of Integro-differential equations,* 125-136.

Miller, R. K. (1970). *Asymptotic stability properties of linear Volterra integro-differential equations.* Journal of differential equations. 10(3), 485-506.

Raffoul, Y. (2013). *Exponential stability and instability in finite delay nonlinear Volterra integro-differential equations*. Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal, 20(1), 95-106.

Raffoul, Y., and Unal, M. (2014). ¨ *Stability in nonlinear delay Volterra integrodifferential systems.* Journal of Nonlinear Sciences and Applications (JNSA), 7(6).

Staffans, O. J. (1988). *A direct Lyapunov approach to Volterra integro-differential equations.* SIAM journal on mathematical analysis, 19(4), 879-901.

Tunc, C. (2017a). *Stability and boundedness in Volterra-integro differential equations with delays.* Dyn Syst Appl, 26(1), 121-130.

Tunc¸, C. (2017b). *Qualitative properties in nonlinear Volterra integro-differential equations with delay*. Journal of Taibah University for Science, 11(2), 309-314.

Tunc¸, C. (2016). *A note on the qualitative behaviors of non-linear Volterra integro-differential equation.* Journal of the Egyptian Mathematical Society, 24(2), 187-192.

Tunc¸, C., and Tunc¸, O. (2018a). *New qualitative criteria for solutions of Volterra integro-differential equations.* Arab Journal of Basic and Applied Sciences, 25(3), 158-165.

Tunc¸, C., and Tunc¸, O. (2018b). *On behaviours of functional Volterra integrodifferential equations with multiple time lags.* Journal of Taibah University for Science, 12(2), 173-179.

Tunc, C., and Mohammed, S. A. (2017). *On the stability and instability of functional Volterra integrodifferential equations of first order.* Bull. Math. Anal. Appl, 9(1), 151-160

Verdugo, A. (2018). *Linear analysis of an integro-differential delay equation model.* International Journal of Differential Equations.

Volterra, V. (1959). *Theory of Functionals and Integral and Integro-differential Equations.* Dover Publications. Inc., USA.

Wang, T. (2004). *Inequalities and stability for a linear scalar functional differential equation.* Journal of mathematical analysis and applications, 298(1), 33-44.

Wazwaz, A. M. (2015). *First Course In Integral Equations*, A. World Scientific Publishing Company. Wazwaz, A. M. (2011). Linear and nonlinear integral equations (Vol.639). Berlin: Springer.

Wazwaz, A. M. (2003). *The existence of noise terms for systems of inhomogeneous differential and integral equations.* Applied Mathematics and Computation, 146(1), 81-92.