UNIVERSITY OF CAPE COAST

ASYMPTOTIC STABILITY OF SOLUTIONS OF A SYSTEM OF DIFFERENCE EQUATIONS WITH FINITE DELAY

BY

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DECLARATION

Candidate's Declaration

I hereby declare that this thesis is the result of my own original research and that no part of it has been presented for another degree in this university or elsewhere.

Candidate's Signature .... Date ....

Name: Victor Kingsford Egyir

Supervisor's Declaration

I hereby declare that the preparation and presentation of the thesis were supervised in accordance with the guidelines on supervision of thesis laid down by the University of Cape Coast.

Supervisor's Signature .... Date ....

Name: Prof. Ernest Yankson

ABSTRACT

This thesis is concerned with the stability of solutions of a system of difference equations with finite delay.

Fixed point theory is used in this thesis to investigate the stability of solutions of a system of difference equations with finite delay.

In particular, the Banach fixed point theorem is used in the thesis. In the process the system of equations are inverted to obtain an equivalent summation equations. The result of the inversion is used to define a suitable mapping which is then used to discuss the stability properties of solutions of the system of difference equations with finite delay.

Sufficient conditions that guarantee that the zero solution of a system of difference equations with finite delay are asymptotically stable are obtained.

KEY WORDS

Asymptotic Stability

Contraction Principle

continuous mapping

Difference Equation

Fixed Point Theory

Stability solution

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DEDICATION

To my beautiful wife Cecilia Aidoo and the family

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CHAPTER ONE

INTRODUCTION

Background to the Study

A difference equation is a mathematical equality which involves the differences between successive values of a function of discrete variable. A discrete variable refers to values that differ by some finite amount, usually constant and often 1. Difference equations have many applications in a variety of disciplines such as economics, mathematical biology and physics. In 1202, Fibonacci formulated his famous rabbit problem that led to the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13,... . However, it appears that the corresponding difference equation

$$F_n = F_{n-2} + F_{n-1}$$

was first written down by Albert Girard around 1634 and was solved by De Moivré in 1730. Bombelli studied the equation

$$y_n = 2 + \frac{1}{y_{n-1}}$$

in 1572, which is similar to the equation

$$z_n = 1 + \frac{1}{z_{n-1}}$$

which is satisfied by ratios of Fibonacci numbers, in order to approximate $\sqrt{2}$. Fibonacci also gave a rough definition for the concept of continued fractions that is intimately associated with difference equations.

Difference equations describe real life situations associated with statistical problems, geometry, stochastic time series etc. In the construction of mathematical models of physical systems it is usually assumed that all of the independent variables, such as time and space are continuous. This assumption normally leads to a realistic and justified approximation of real variables of the systems. However, we regularly encounter systems for which this continuous variable assumption cannot be made. Systems in which one or more variables are inherently discrete are in areas such as population growth, digital communication networks and delay feedback oscillation as in lasser emission pulsation Rabinovich (1980). Due to their discrete character, these systems must be modelled by the use of difference equations.

In recent times, difference equations have attracted a lot of attention that it deserves which can be attributed to the introduction of computers where approximate difference equation formulation are adopted to find the solutions of differential equations. The qualitative theory of difference equation was born at the end of the 19th century with the works of Henri Poncaré (1857-1912) and Hedrih (2007).

Stability plays a major role in the qualitative analysis of solutions of difference equations. A solution of a difference equation is said to be stable if small changes in initial conditions causes only a small change in the future behaviour of the solution. Stability techniques for difference equations can be used to study the convergence of multistep methods for ordinary differential equations. The idea of using difference equations to approximate solutions of differential equations originated in 1769 with Euler's polygonal method for which the proof of convergence was given by Cauchy around 1840. The subject seems to have languished until almost the end of the nineteenth century, when Lipschitz, Runge and Kutta developed improved procedures. The urgent need for numerical approximations during World War I greatly stimulated research in this area, and the number of publications later explored with the development of the digital computer. Lakshmitankhan & Donato (2003) initiated the modern theory of the convergence of multistep methods as in Dahlquist (1985).

The efficient application of linear difference equations to the computation of special functions originated in 1952 with Miller's algorithm for Bessel functions. Such computations must be done with care because of the possibility of explosive round-off error, as illustrated by the cautionary example of Gautschi (1972). Wimp (1984) discusses the development of this method and related algorithms due to Olver (1967), Gautschi (1972), Wimp (1969) as well as some examples of computation with nonlinear difference equations. Further development of the theory of linear difference equations has brought the subject to a state comparable to that of linear differential equations which is illustrated by Hartman (1978) and Peterson (1998). Since the introduction of Lyaponuv's work 100 years ago, Lyaponuv direct method has been the main tool until recently in dealing with stability problems in various types of dynamical model equations. However, the construction of the right Lyaponuv functions prove to be technical, and are not applicable to all situations. The application of Lyaponuv's direct method to problems of stability in difference equations with delay has encountered serious difficulties if the delay is unbounded or if the equation has unbounded terms which were identified by the authors in Islam and Raffoul (2007) and Islam and Yankson (2005).

The foundation for a thorough study of the asymptotic properties of solutions of linear difference equations was laid in the 1880's by Poincare, who formalized the concept of asymptotic series and also showed that under favourable conditions the ratio of consecutive values of a solution must approach a characteristic root.

Statement of the Problem

The study of stability of difference equations have attracted the attention of several mathematicians in recent times. Raffoul (2006) obtained sufficient conditions for the zero solution of the difference equation

$$\Delta x(n) = a(n)x(n-\tau),$$

to be asymptotically stable. This equation however is a scalar equation and so the stability results do not apply to the system of difference equations

$$\Delta x(n) = A(n)x(n-\tau),$$

where, $A(n)$ is an $s \times s$ matrix.

Purpose of the Study

The purpose of this study is to determine asymptotic stability of solutions of a system of difference equations with finite delay.

Research Objectives

The study sought to determine the sufficient conditions under which the zero solution of the system of difference equations

$$\Delta x(n) = A(n)x(n-\tau),$$

with finite delay is

1. Stable;
2. Asymptotically stable.

Significant of the Study

The study generalises some results in stability of a system of difference equation with finite delay and hence add to literature which can be used by researchers in the area of stability of difference equation.

Delimitations

The study considers a system of difference equation equation with finite delay and does not include a system of difference equation with variable or multiple delays.

Limitations

The fixed point theory is the main tool used in this reasearch because it is less complex in obtaining the stability solution of difference equation with delays than construction of Lyapunov Direct Method which is sophisficated and difficult to use.

Organisation of the Study

The thesis consist of five chapters that are organised as follows:
Chapter One presents the background of the study, statement of problem, the objectives of the thesis as well as the organization of the thesis.
Chapter Two examines the previous work related to the thesis.
Chapter Three is all about the relevant mathematical background, various theorems and methodology. Chapter Four consists of results and discussions of the study. Chapter Five, which is the final chapter consists of the summary and conclusions to the study.

CHAPTER TWO

LITERATURE REVIEW

Introduction

The stability of solutions of difference equations has been studied extensively in academic literature. The focus of this chapter is to review literature on stability of difference equations related to the work in this thesis.

Relevant Literature

There are a number of researchers working on stability theory of difference equations, with or without delay, and this has resulted in the establishment of many interesting results about the stability of solutions of difference equations. For instance, M. Islam & Yankson (2005), showed that the zero solution of the difference equation with variable delay

$$x(n+1) = b(n)x(n) + a(n)x(n-\tau(n))$$

is asymptotically stable with an assumption that

$$\prod_{s=0}^{n=1} b(s) \to 0 \text{ as } n \to \infty.$$

In Raffoul (2006), the author considered the finite delay difference equation

$$\Delta x(n) = a(n)x(n-\tau),$$

which is the same as

$$x(n+1) = x(n) + a(n)x(n-\tau) \tag{2.1}$$

where $b(n) = 1$. Thus, the condition

$$\prod_{s=0}^{n=1} b(s) \to 0 \text{ as } n \to \infty$$

obtained by M. Islam & Yankson (2005) does not hold for Equation (2.1) since $b(n) = 1$, for all $n \in Z$. Raffoul (2006), obtained sufficient conditions for the zero solution of the difference equation with constant delay

$$\Delta x(n) = a(n)x(n - \tau),$$

to be asymptotically stable by fixed point theory. In addition periodicity of solutions was also proved.

Moreover,Yankson (2009), obtained sufficient conditions for the zero solution of the equation

$$\Delta x(n) = -\sum_{j=1}^{N} a_j(n)x\Big(n - \tau_j(n)\Big)$$

to be asymptotically stable by fixed point theory. The results obtained by Yankson (2009) improves and generalizes that of Raffoul (2006). Moreover, Yankson (2015), also proved the existence of a unique periodic solution of the system of difference equations

$$\Delta x(n) = A(n)x(n - \tau),$$

where, $A(n) \in R^{s\times s}$ is a non singular matrix and τ is a positive constant by means of fixed point theory.

Difference Calculus

Many of the computations used in solving and analyzing difference equations can be simplified by use of the difference calculus, a collection of mathematical tools quite similar to the differential calculus.

The Difference Operator

Just as the differential operator plays the central role in the differential calculus, the difference operator is the basic component of calculations involving finite differences.

Definition 1

Let $y(n)$ be a function of a real or complex variable n. The difference operator Δ is defined by

$$\Delta y(n) = y(n+1) - y(n).$$

For the most part, we will take the domain of y to be a set of consecutive integers such as the natural numbers $N = 1, 2, 3, \ldots$. Occasionally we will apply the difference operator to a function of two or more variables. In this case, a subscript will be used to indicate which variable is to be shifted by one unit. Higher order differences are defined by composing the difference operator with itself. The second order difference is

$$\begin{aligned}
\Delta^2 x(n) &= \Delta\Big(\Delta x(n)\Big) \\
&= \Delta\Big(x(n+1) - x(n)\Big) \\
&= \Big(x(n+2) - x(n+1)\Big) - \Big(x(n+1) - x(n)\Big) \\
&= x(n+2) - 2x(n+1) + x(n).
\end{aligned}$$

The following formula for the n^{th} order difference is readily verified by induction:

$$\begin{aligned}\Delta^n y(t) &= y(t+n) - ny(t+n-l) + \frac{n(n-1)}{2!}y(t+n-2) \\ &+ ... + (-1)^n y(t) \\ &= \sum_{k=0}^{n} (-1)^k \binom{n}{y} y(t+n-k).\end{aligned}$$

These calculations can be verified just as in algebra since the composition of the operators I and E has the same properties as the multiplication of numbers. In much the same way, we have

$$E^n y(t) = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \Delta y(t)$$

An elementary operator that is often used in conjunction with the difference operator is the shift operator.

Definition 2 (**Shift Operator**)
The shift operator E is defined by

$$Ex(t) = x(t+1).$$

If I denotes the identity operator, that is, $Ix(t) = x(t)$ then we have

$$\Delta = E - I. \tag{2.2}$$

In fact, Equation (2.2) is similar to the Binomial Theorem from algebra:

$$\begin{aligned}
\Delta^n x(t) &= (E-I)^n x(t) \\
&= \sum_{k=0}^{n} \binom{n}{k} (-I)^k E^{n-k} x(t) \\
&= \sum_{k=0}^{n} \binom{n}{k} (-1)^k x(t+n-k)
\end{aligned}$$

The fundamental properties of Δ are given in the following theorem.

Theorem 2 [Walter & Peterson (1991)]

(i) $\Delta^a(\Delta^b w(n)) = \Delta^{(a+b)} w(n)$ for all positive integers a and b.

(ii) $\Delta(x(n)+y(n)) = \Delta x(n) + \Delta y(n)$.

(iii) $\Delta\Big(Cx(n)\Big) = C\Delta x(n)$ if C is a constant.

(iv) $\Delta(x(n)y(n)) = x(n)\Delta y(n) + Ex(n)\Delta y(n)$.

(v) $\Delta\left(\frac{x(n)}{y(n)}\right) = \frac{y(n)\Delta x(n) - x(n)\Delta y(n)}{y(n)Ey(n)}$.

Proof.

ii. By the definition of the difference operator,

$$\begin{aligned}
\Delta[x(n)+y(n)] &= x(n+1)+y(n+1)-(x(n)y(n)) \\
&= x(n+1)+y(n+1)-x(n)-y(n) \\
&= x(n+1)-x(n)+y(n+1)-y(n) \\
&= \Delta x(n)+\Delta y(n)
\end{aligned}$$

iii. By the definition of the difference operator,

$$\begin{aligned} \Delta(by(n)) &= by(n+1) - by(n) \\ &= b[y(n+1) - y(n)] \\ &= b\Delta y(n) \end{aligned}$$

iv. Applying the difference operator,

$$\begin{aligned} \Delta(x(n)y(n)) &= x(n+1)y(n+1) - x(n)y(n) \\ &= x(n+1)y(n+1) - x(n+1)y(n) + x(n+1)y(n) - x(n)y(n) \\ &= x(n+1)[y(n+1) - y(n)] + [x(n+1) - x(n)]y(n) \\ &= Ex(n)\Delta y(n) + [\Delta x(n)]y(n). \end{aligned}$$

v. Using the definition of the difference operator we obtain,

$$\begin{aligned} \Delta\frac{x(n)}{y(n)} &= \frac{x(n+1)}{y(n+1)} - \frac{x(n)}{y(n)} \\ &= \frac{x(n+1)y(n) - x(n)y(n+1)}{y(n)y(n+1)} \\ &= \frac{y(n)(\Delta x(n) + x(n)) - x(n)(\Delta y(n) + y(n))}{y(n)Ey(n)} \end{aligned}$$

$$= \frac{y(n)\Delta x(n) + x(n)y(n) - x(n)y(n) - x(n)\Delta y(n)}{y(n)Ey(n)}$$

$$= \frac{y(n)\Delta x(n) - x(n)\Delta y(n)}{y(n)Ey(n)}$$

In addition to the general properties for computing differences, there are other fundamental properties. Theorems give is a list for some functions and their differences.

Theorem 3[Walter & Peterson (1991)]

Let a be a constant. Then

a. $\Delta w^t = (w-1)w^t$

b. $\Delta \sin bt = 2\sin\left(\frac{b}{2}\right)\cos b\Big(t+\frac{1}{2}\Big)$

c. $\Delta \cos bt = -2\sin(\frac{b}{2})\sin b\Big(t+\frac{1}{2}\Big)$

d. $\Delta \log(bt) = \log\left(1+\frac{1}{t}\right)$

e. $\Delta \log \Gamma t = \log(t)$

Proof.

a. Using the definition of the difference operator we obtain

$$\begin{aligned} \Delta w^t &= w^{t+1} - w^t \\ &= (w-1)w^t \end{aligned}$$

b. Applying the diffference operator

$$
\begin{aligned}
\Delta \sin(bt) &= \sin b(t+1) - \sin b(t) \\
&= \sin b\Big(t + \frac{1}{2} + \frac{1}{2}\Big) - \sin b\Big(t + \frac{1}{2} - \frac{1}{2}\Big) \\
&= \sin b\Big(t + \frac{1}{2}\Big)\cos\Big(\frac{b}{2}\Big) + \sin\Big(\frac{b}{2}\Big)\cos b\Big(t + \frac{1}{2}\Big) \\
&\quad -\Big[\sin\Big(b\Big(t+\frac{1}{2}\Big)\Big)\cos\Big(\frac{b}{2}\Big) \\
&\quad - \sin\Big(\frac{b}{2}\Big)\cos\Big(b\Big(t+\frac{1}{2}\Big)\Big)\Big] \\
&= 2\sin\Big(\frac{b}{2}\Big)\cos\Big(b\Big(t+\frac{1}{2}\Big)\Big)
\end{aligned}
$$

c. Applying the difference operator

$$
\begin{aligned}
\Delta \cos(bt) &= \cos(b(t+1)) - \cos(bt) \\
&= \cos b\Big(t + \frac{1}{2} + \frac{1}{2}\Big) - \cos\Big(b\Big(t + \frac{1}{2} - \frac{1}{2}\Big)\Big) \\
&= \cos b\Big(t + \frac{1}{2}\Big)\cos\Big(\frac{b}{2}\Big) - \sin\Big(\frac{b}{2}\Big)\sin\Big(b\Big(t + \frac{1}{2}\Big)\Big) \\
&\quad -\Big[\cos\Big(b\Big(t+\frac{1}{2}\Big)\Big)\cos\Big(\frac{b}{2}\Big) + \sin\Big(\frac{b}{2}\Big)\sin\Big(b\Big(t+\frac{1}{2}\Big)\Big)\Big] \\
&= -2\sin\Big(\frac{b}{2}\Big)\sin\Big(b\Big(t+\frac{1}{2}\Big)\Big)
\end{aligned}
$$

d. Using the definition of difference operator

$$\begin{aligned}
\Delta \log(bt) &= \log b(t+1) - \log at \\
&= \log \frac{b(t+1)}{bt} \\
&= \log \frac{t+1}{t} \\
&= \log \left(1 + \frac{1}{t}\right)
\end{aligned}$$

e. Using the definition of difference operator

$$\begin{aligned}
\Delta \log \Gamma(t) &= \log \Gamma(t+1) - \log \Gamma(t) \\
&= \log \left(\frac{\Gamma(t+1)}{\Gamma(t)}\right) \\
&= \log \frac{t\Gamma(t)}{\Gamma(t)} \\
&= \log(t).
\end{aligned}$$

The "falling factorial power" $t^{\underline{r}}$: (read "t to the r falling") is defined as follows, according to the value of r.

a. If $r = 1, 2, 3, ...,$ then $t^{\underline{r}} = (t-1)(t-2)(t-3)...(t-r+1)$

b. If $r = 0$ then $t^{\underline{0}} = 1$

c. If $r = -1, -2, -3, ...,$ then $t^{\underline{r}} = \frac{1}{(t-1)(t-2)(t-3)...(t-r)}$

d. If r is not an integer, then $t^{\underline{r}} = \frac{\Gamma(t+1)}{\Gamma(t-r+1)}$

Theorem 4 [**Walter & Peterson (1991)**]

Let

$$p(n) = a_0 n^k + a_1 n^{k-1} + ... + a_k$$

be a polynomial of degree k. Then

$$\Delta p(n) = a_0 k n^{k-1} + ... \text{ terms of degree lower than } (k-1).$$

Proof.

We now apply the difference operator on the polynomial function of degree k to obtain:

$$\begin{aligned} \Delta p(n) &= [a_0(n+1)^k + a_1(n+1)^{k-1} + ... + a_k] \\ &\quad - [a_0(n)^k + a_1 n^{k-1} + ... + a_k] \\ &= a_0 k (n)^{k-1} + ...\text{terms of degree lower than } (k-1) \end{aligned}$$

Carrying out this process k times, one obtains

$$\Delta^k p(n) = a_0 k!.$$

Summation

To make effective use of the difference operator, we introduce in this section its right inverse operator, which is sometimes called the "indefinite sum." The discrete analogue of the indefinite integral in calculus is the anti-difference operator Δ^{-1}, defined as follows. If

$$\Delta P(n) = 0$$

then

$$\Delta^{-1}(0) = P(n) = c$$

for some arbitrary constant c. Moreover, if

$$\Delta P(n) = p(n)$$

then

$$\Delta^{-1} P(n) = p(n) + c$$

for some arbitrary constant c. Thus

$$\Delta\Delta^{-1} P(n) = p(n).$$

This implies that

$$\Delta^{-1}\Delta P(n) = p(n) + c$$

and

$$\Delta\Delta^{-1} = I$$

but

$$\Delta^{-1}\Delta \neq I$$

Therefore,

$$\Delta\Big(\sum x(n)\Big) = x(n)$$

for all n in the domain of x.

Theorem 5 [**Walter & Peterson (1991)**]

The anti-difference operator Δ^{-1} is linear.

Proof.

Let $a, b \in R$, then

$$\begin{aligned}
\Delta^{-1}\Big[ax(n)+by(n)\Big] &= \sum_{i=0}^{n-1}\Big[ax(i)+by(i)\Big]+c \\
&= a\sum_{i=0}^{n-1}x(i)+b\sum_{i=0}^{n-1}y(i)+c \\
&= a\Delta^{-1}x(n)+b\Delta^{-1}x(n).
\end{aligned}$$

Theorem 6 [**Walter & Peterson (1991)**]

If $z(n)$ is an indefinite sum of $x(n)$, then every indefinite sum of $x(n)$ is given by

$$\sum q(n) = z(n) + C(n),$$

where, $C(n)$ has the same domain as y and $\Delta C(n) = 0$,

says that $C(n+1) = C(n)$ for all real n, which means that C can be any periodic function having period one. In the theory of difference equations it is convenient to use the convention

$$\sum_{k=a}^{b} y(k) = 0$$

whenever $a > b$. Observe that for m fixed and $n \geq m$,

$$\Delta_n\Big(\sum_{k=m}^{n-1} y_k\Big) = y_n,$$

and for p fixed and $p \geq h$,

$$\Delta_n\Big(\sum_{k=m}^{p} y(k)\Big) = -y_n,$$

The following theorem contains important formula for calculating definite sums, which is analogous to the fundamental theorem of calculus.

Theorem 7 [**Walter & Peterson (1991)**]

If $w(n)$ is an indefinite sum of $h(n)$, then

$$\sum_{k=m}^{n-1} h(k) = \Big[w(k)\Big]_m^n = w(n) - w(m).$$

Chapter Summary

The chapter focused on a review of relevant literature such Raffoul (2006) which was later improved by Yankson (2009)on the stability of difference equations with delay.

CHAPTER THREE

RESEARCH METHODS

Introduction

The chapter discusses the methods, definitions and theorems used in achieving the objectives of the research work.

Fixed Point Theory

This section contains an elementary set of definitions and theorems relevant to our study. The stability property we discuss are formulated in complete metric spaces. This research has a Banach space $(B, \| \,.\, \|)$ in the background. A subset S of B is selected and $(S, \| \,.\, \|)$ is the complete metric space in this work, where the metric on S is defined by the norm inherited from the Banach space. Thus the notation almost always suggests a norm $\| \,.\, \|$ instead of a metric ρ. Even if the zero function, say δ, is not in S, then for $\Phi \in S$, $||\delta||$ is interpreted as $\rho(\Phi, \delta) = ||\Phi - \delta||$.

Definition 3 (**Metric Space**)

A pair (S, ρ) is a metric space if S is a set and $\rho : S \times S \to [0, \infty)$ such that when y, z, and u are in S then

(i) $\rho(y, z) \geq 0$, $\rho(y, y) = 0$ and $\rho(y, z) = 0$ implies $y = z$,

(ii) $\rho(y, z) = \rho(z, y)$, and

(iii) $\rho(y, z) \leq \rho(y, u) + \rho(u, z)$.

The metric space is complete if every Cauchy sequence in (S, ρ) has a limit in that space. A sequence $\{x_n\}_{n\geq 1} \subset S$ is a Cauchy sequence if for each $\epsilon > 0$ there exists N such that n, $m > N$ imply $\rho(x_n, x_m) < \epsilon$.

Stating the contraction mapping principle which generally goes under the name Banach Caccioppoli Theorem, or Banach's (1932) Contraction

Mapping Principle. A proof can be found in Burton (1985). It gains more respect every day. The real power of the result lies in its application with cleverly chosen metrics.

Theorem 1 (The Contraction Mapping Principle)

The contraction mapping principle states that if (X, d) is a complete metric space and $T : X \to X$ is a mapping such that $d(T_x, T_y) < \lambda d(x, y)$ for all $x, y \in X$ where $0 \leq \lambda < 1$, then there exists a unique $x \in X$ such that $T(x) = x$.

Definition 4 (**Stability**)

The zero solution of a difference equation is said to be stable if for any $\epsilon \geq 0$ there exists $\delta(\epsilon, n_0) \geq 0$ such that $||x_0|| \leq \delta$ implies that $||x|| \leq \epsilon$ for $n \geq n_0$.

Definition 5 (**Asymptotic Stability**)

The zero solution of a difference equation is said to be asymptotically stable if it is stable and in addition for each $n_0 \geq 0$ there is an $\eta(n_0) > 0$ such that $||\psi|| < \eta(n_0)$ implies that $x(n) \to 0$ as $n \to \infty$.

The Fundamental Matrices

In this section we consider some of the properties of systems of difference equations with variable coefficients

$$x(n+1) = A(n)x(n) + P(n), \tag{3.1}$$

and the corresponding homogeneous system

$$x(n+1) = A(n)x(n), \tag{3.2}$$

where the matrix function $A(n)$ is assumed to be non-singular for all integers n.With this assumption, initial value problems for Equation (3.2) will have a unique solutions defined on the set of all integers.

Definition 6

The solutions $x_1(n), x_2(n), ..., x_k(n)$ of Equation (3.2) are said to be linearly independent for $n \geq n_0 \geq 0$ if whenever

$$c_1x_1(n) + c_2x_2(n) + ... + c_kx_k(n) = 0$$

for all $n \geq n_0$, then

$$c_i = 0, 1 \leq i \leq k.$$

Let $\Phi(n)$ be a $k \times k$ matrix whose columns are solutions of Equation (3.2).

We write

$$\Phi(n) = [x_1(n), x_2(n), ..., x_k(n)].$$

Now,

$$\begin{aligned}
\Phi(n+1) &= [A(n)x_1(n), A(n)x_2(n), ..., A(n)x_k(n)] \\
&= A(n)[x_1(n), x_2(n), ..., x_k(n)] \\
&= A(n)\Phi(n).
\end{aligned}$$

Hence, $\Phi(n)$ satisfies the matrix difference equation

$$\Phi(n+1) = A(n)\Phi(n). \tag{3.3}$$

Furthermore, the solutions $x_1(n), x_2(n), ..., x_k(n)$ are linearly independent for $n \geq n_0$ if and only if the matrix $\Phi(n)$ is non singular ($\det \Phi(n) \neq 0$) for all $n \geq n_0$. This actually leads to the next definition.

Definition 7

If $\Phi(n)$ is a matrix that is non-singular for all $n \geq n_0$ and satisfies Equation (3.3), then it is said to be a fundamental matrix for system Equation (3.2).

Note that if $\Phi(n)$ is a fundamental matrix and C is any non-singular matrix, then $\Phi(n)C$ is also a fundamental matrix. Thus there are infinitely many fundamental matrices for a given system. However, there is one fundamental matrix that we already know, namely,

$$\Phi(n,m) = \prod_{i=n_0}^{n-1} A(i), \text{ with } \Phi(n_0) = I.$$

In the autonomous case when A is a constant matrix,

$$\Phi(n) = A^{n-n_0}, \text{ and if } n_0 = 0,$$

then

$$\Phi(n) = A^n.$$

We may add here that starting with any fundamental matrix $\Phi(n)$, the fundamental matrix $\Phi(n)\Phi^{-1}(n_0)$ is also a matrix. This special fundamental matrix is denoted by $\Phi(n, n_0)$ and is referred to as the state transition matrix. One may, in general, write

$$\Phi(n,m) = \Phi(n)\Phi^{-1}(m)$$

for any two positive integers n, m with $n \geq m$. The fundamental matrix $\Phi(n,m)$ has some agreeable properties that we ought to list here. Observe first that $\Phi(n,m)$ is a solution of the matrix difference equation

$$\Phi(n+1,m) = A(n)\Phi(n,m).$$

The following are the agreeable properties of fundamental matrix:

i. $\Phi^{-1}(n,m) = \Phi(m,n)$;

ii. $\Phi(n,m) = \Phi(n,r)\Phi(r,m)$;

iii. $\Phi(n,m) = \prod_{i=m}^{n-1} A(i)$;

iv. $\Phi(n,n) = I$

v. $\Phi(n,n_1)\Phi^{-1}(n_0,n_1) = \Phi(n,n_0)$

First Order Linear Difference Equations

Let $p(n)$ and $r(n)$ be given functions with $p(n) \neq 0$ for all n. The first order linear difference equation is

$$y(n+1) - p(n)x(n) = r(n). \tag{3.4}$$

Equation (3.4) is said to be of the first order because it involves the values of y at n and only $n+1$ only, as in the first order difference of $x(n)$ that is

$$\Delta y(n) = y(n+1) - y(n)$$

If $p(n) = 1$ for all n, then Equation (3.4) is simply

$$\triangle x(n) = r(n),$$

whose solution is given by

$$x(n) = \sum r(n) + C(n),$$

where $\triangle C(n) = 0$. For simplicity, we assume that the domain of interest is a discrete set $n = a, a+1, a+2, ...$ Consider first the equation

$$u(n+1) = p(n)u(n), \tag{3.5}$$

which is easily solved by iteration as follows Evaluate Equation (3.5) at

$n = a$ to obtain

$$u(a+1) = p(a)u(a)$$

Thus,

$$\begin{aligned} u(a+2) &= p(a+1)u(a+1) \\ &= p(a+1)p(a)u(a), \end{aligned}$$

Hence,

$$u(a+h) = u(a)\prod_{k=0}^{n-1} p(a+k).$$

We can write the solution in the more convenient form

$$u(n) = u(a)\prod_{s=a}^{n-1} p(s), \ (n = a, a+1, ...),$$

where it is understood that $\prod_a^{a-1} p(s) = 1$, and for $n \geq a+1$, the product is taken over $a, a+1, ..., n-1$.

Methodology

In this thesis fixed point theory was the main method by which the results of this thesis was obtained. To solve a problem with fixed point theory is to find:

i. A set X consisting of points which would be acceptable solutions;

ii. A mapping $T : X \to X$ with the property that a fixed point solves the problem;

iii. A fixed point theorem stating that this mapping on this set will have a fixed point;

iv. The mapping T will be obtained by inverting the difference equations;

The fixed point theorem that was used in this research is the contraction mapping principle.

Chapter Summary

This chapter dealt with some of the theorems and definitions such as metric space, stability, difference calculus, difference operator, shift operator, indefinite sum and orders of difference equation which are relevant in this research and also methodology of the research.

CHAPTER FOUR

RESULTS AND DISCUSSION

Introduction

This chapter deals with the results concerning the asymptotic stability of a certain class of system of difference equations with finite delay.

Preliminaries Result

In this chapter we consider the system of difference equations

$$\Delta x(n) = A(n)x(n-\tau), \tag{4.1}$$

where $A(n)$ is an $s \times s$ non singular matrix and τ is a positive constant. Let $\psi(n)$ defined on $[-\tau, n_0] \cap \mathbb{Z}$ denote the initial function for Equation (4.1). For $x \in \mathbb{R}^s$ define $||x|| = \max_{n \in [-\tau, \infty) \cap \mathbb{Z}} |x(n)|$ where $|.|$ denotes the infinity norm for $x \in \mathbb{R}^s$. Define the norm of an $s \times s$ matrix A by

$$|A| = \max_{1 \leq i \leq s} \sum_{j=1}^{s} |a_{ij}|.$$

In this thesis we make the following assumptions:

(H1) Suppose that there exists a non-singular $s \times s$ matrix $G(n)$ such that

$$\begin{aligned}\Delta x(n) &= G(n)x(n) - \Delta_n \sum_{k=n-\tau}^{n-1} G(k)x(k) \\ &\quad + \big[A(n) - G(n-\tau)\big]x(n-\tau). \end{aligned} \tag{4.2}$$

(H2) Let $\Phi(n, n_0)$ denote the fundamental matrix solution of the equation

$$\Delta x(n) = G(n)x(n). \tag{4.3}$$

Lemma 1

Suppose that (H1) hold. Then Equation (4.1) is equivalent to

$$\Delta x(n) = G(n)x(n) - \Delta_n \sum_{k=n-\tau}^{n-1} G(k)x(k) + \big[A(n) - G(n-\tau)\big]x(n-\tau).$$

Proof.

By taking the difference with respect to n of the summation term in Equation (4.2) we obtain

$$\begin{aligned}
\Delta x(n) &= G(n)x(n) - \Big[\sum_{k=n+1-\tau}^{n} G(k)x(k) - \sum_{k=n-\tau}^{n-1} G(k)x(k)\Big] \\
&\quad + \big[A(n) - G(n-\tau)\big]x(n-\tau) \\
&= G(n)x(n) - \Big[G(n)x(n) + \sum_{k=n+1-\tau}^{n-1} G(k)x(k) \\
&\quad - \sum_{k=n+1-\tau}^{n-1} G(k)x(k) - G(n-\tau)x(n-\tau)\Big] \\
&\quad + \big[A(n) - G(n-\tau)\big]x(n-\tau) \\
&= G(n)x(n) - \Big[G(n)x(n) - G(n-\tau)x(n-\tau)\Big] \\
&\quad + \Big[A(n) - G(n-\tau)\Big]x(n-\tau)
\end{aligned}$$

$$
\begin{aligned}
&= G(n-\tau)x(n-\tau) - G(n-\tau)x(n-\tau \\
&\quad + A(n)x(n-\tau) \\
&= A(n)x(n-\tau).
\end{aligned}
$$

This completes the proof.

In Lemma 2 we provide an equivalent summation equation for Equation (4.1) which will be used to obtain the mappings in this thesis.

Lemma 2

Suppose (H2) hold. Then $x(n)$ is a solution of Equation (4.1) if and only if

$$
\begin{aligned}
x(n) &= -\sum_{k=n-\tau}^{n-1} G(k)x(k) + \Phi(n,n_0)\Big(x(n_0) + \sum_{k=n_0-\tau}^{n_0-1} G(k)x(k)\Big) \\
&\quad + \sum_{s=n_0}^{n-1} \Phi(n,s)\Big(G(s)B^{-1}(s) - I\Big)\Big[G(s)\sum_{k=s-\tau}^{s-1} G(k)x(k) \\
&\quad - \big[A(s) - G(s-\tau)\big]x(s-\tau)\Big]
\end{aligned}
\tag{4.4}
$$

where $B(n) = I + G(n)$.

Proof.

Let $x(n)$ be a solution of Equation (4.2) and $\Phi(n,n_0)$ be a fundamental matrix solution of (4.1). Rewriting Equation (4.2) as

$$
\begin{aligned}
&\Delta\Big[x(n) + \sum_{k=n-\tau}^{n-1} G(k)x(k)\Big] \\
&\qquad = G(n)x(n) + \Big[A(n) - G(n-\tau)\Big]x(n-\tau)
\end{aligned}
$$

$$
\begin{aligned}
&= G(n)x(n) + G(n)\sum_{k=n-\tau}^{n-1} G(k)x(k) - G(n)\sum_{k=n-\tau}^{n-1} G(k)x(k) \\
&\quad + \Big[A(n) - G(n-\tau)\Big]x(n-\tau) \\
&= G(n)\Big[x(n) + \sum_{k=n-\tau}^{n-1} G(k)x(k)\Big] - G(n)\sum_{k=n-\tau}^{n-1} G(k)x(k) \\
&\quad + \Big[A(n) - G(n-\tau)\Big]x(n-\tau). \qquad (4.5)
\end{aligned}
$$

In view of the fact that $\Phi(n,n_0)\Phi^{-1}(n,n_0) = I$ and applying Theorem 2 (vi) we have that

$$
\begin{aligned}
0 &= \Delta\Big[\Phi(n,n_0)\Phi^{-1}(n,n_0)\Big] \\
&= \Delta(\Phi(n,n_0))E\Phi^{-1}(n,n_0) + \Phi(n,n_0)\Delta(\Phi^{-1}(n,n_0)) \\
&= G(n)\Phi(n,n_0)\Phi^{-1}(n,n_0)B^{-1}(n) + \Phi(n,n_0)\Delta(\Phi^{-1}(n,n_0)) \\
&= G(n)B^{-1}(n) + \Phi(n,n_0)\Delta(\Phi^{-1}(n,n_0)). \qquad (4.6)
\end{aligned}
$$

Pre-multiplying Equation (4.6) by $\Phi^{-1}(n,n_0)$ we obtain

$$
\begin{aligned}
0 &= \Phi^{-1}(n,n_0)G(n)B^{-1}(n) \\
&\quad + \Phi^{-1}(n,n_0)\Phi(n,n_0)\Delta(\Phi^{-1}(n,n_0)) \\
&= \Phi^{-1}(n,n_0)G(n)B^{-1}(n) + \Delta(\Phi^{-1}(n,n_0))
\end{aligned}
$$

This implies that

$$\Delta\Phi^{-1}(n,n_0) = -\Phi^{-1}(n,n_0)G(n)B^{-1}(n).$$

If $x(n)$ is a solution of Equation (4.1), then by Theorem 2(iv) we obtain

$$\begin{aligned}
\Delta & \left\{\Phi^{-1}(n,n_0)\Big(x(n) + \sum_{k=n-\tau}^{n-1} G(k)x(k)\Big)\right\} \\
= & \ \Delta\Phi^{-1}(n,n_0)E\Big(x(n) + \sum_{k=n-\tau}^{n-1} G(k)x(k)\Big) \\
& + \Phi^{-1}(n,n_0)\Delta\Big(x(n) + \sum_{k=n-\tau}^{n-1} G(k)x(k)\Big).
\end{aligned} \tag{4.7}$$

But

$$E\Big(x(n) + \sum_{k=n-\tau}^{n-1} G(k)x(k)\Big)$$

is given by

$$\begin{aligned}
E\Big[x(n) + \sum_{k=n-\tau}^{n-1} G(k)x(k)\Big] &= G(n)\Big[x(n) + \sum_{k=n-\tau}^{n-1} G(k)x(k)\Big] \\
&\quad + x(n) + \sum_{k=n-\tau}^{n-1} G(k)x(k) \\
&\quad - G(n)\sum_{k=n-\tau}^{n-1} G(k)x(k) \\
&\quad + \Big[A(n) - G(n-\tau)\Big]x(n-\tau) \\
&= \Big(G(n) + I\Big)\Big(x(n) + \sum_{k=n-\tau}^{n-1} G(k)x(k)\Big) \\
&\quad - G(n)\sum_{k=n-\tau}^{n-1} G(k)x(k)] \\
&\quad + \Big[A(n) - G(n-\tau)\Big]x(n-\tau)
\end{aligned}$$

$$
\begin{aligned}
&= B(n)\Big[x(n) + \sum_{k=n-\tau}^{n-1} G(k)x(k)\Big] \\
&\quad - G(n)\sum_{k=n-\tau}^{n-1} G(k)x(k) \\
&\quad + \Big[A(n) - G(n-\tau)\Big]x(n-\tau) \qquad (4.8)
\end{aligned}
$$

Thus Equation (4.7) becomes

$$
\begin{aligned}
&\Delta\Big\{\Phi^{-1}(n,n_0)\Big(x(n) + \sum_{k=n-\tau}^{n-1} G(k)x(k)\Big)\Big\} \\
&\qquad = -\Phi^{-1}(n,n_0)G(n)B^{-1}(n) \\
&\qquad\quad \times \Big[B(n)\Big(x(n) + \sum_{k=n-\tau}^{n-1} G(k)x(k)\Big) \\
&\qquad\quad - G(n)\sum_{k=n-\tau}^{n-1} G(k)x(k) \\
&\qquad\quad + \Big(A(n) - G(n-\tau)\Big)x(n-\tau)\Big] \\
&\qquad\quad + \Phi^{-1}(n,n_0)\Big[G(n)\Big(x(n) + \sum_{k=n-\tau}^{n-1} G(k)x(k)\Big) \\
&\qquad\quad - G(n)\sum_{k=n-\tau}^{n-1} G(k)x(k) \\
&\qquad\quad + \big[A(n) - G(n-\tau)\big]x(n-\tau)\Big]
\end{aligned}
$$

$$
\begin{aligned}
= & -\Phi^{-1}(n,n_0)G(n)\Big[x(n)+\sum_{k=n-\tau}^{n-1}G(k)x(k)\Big] \\
& +\Phi^{-1}(n,n_0)G(n)B^{-1}(n)G(n)\sum_{k=n-\tau}^{n-1}G(k)x(k) \\
& -\Phi^{-1}(n,n_0)G(n)B^{-1}(n)\Big[A(n)-G(n-\tau)\Big]x(n-\tau)\Big] \\
& +\Phi^{-1}(n,n_0)G(n)\Big[x(n)+\sum_{k=n-\tau}^{n-1}G(k)x(k)\Big] \\
& -\Phi^{-1}(n,n_0)G(n)\sum_{k=n-\tau}^{n-1}G(k)x(k) \\
& +\Phi^{-1}(n,n_0)\Big[\big[A(n)-G(n-\tau)\big]x(n-\tau)\Big] \\
= & \ \Phi^{-1}(n,n_0)G(n)B^{-1}(n)\Big[G(n)\sum_{k=n-\tau}^{n-1}G(k)x(k) \\
& -\big[A(n)-G(n-\tau)\big]x(n-\tau)\Big] \\
& -\Phi^{-1}(n,n_0)\Big[G(n)\sum_{k=n-\tau}^{n-1}G(k)x(k) \\
& -\big[A(n)-G(n-\tau)\big]x(n-\tau)\Big] \\
= & \ \Phi^{-1}(n,n_0)\Big(G(n)B^{-1}(n)-I\Big)\Big[G(n)\sum_{k=n-\tau}^{n-1}G(k)x(k) \\
& -\big[A(n)-G(n-\tau)\big]x(n-\tau)\Big]. \qquad (4.9)
\end{aligned}
$$

Summing Equation (4.9) from n_0 to $n-1$ gives

$$\sum_{s=n_0}^{n-1}\Big\{ \quad \Delta \quad \Big[\Phi^{-1}(s,n_0)\Big(x(s)+\sum_{k=s-\tau}^{s-1} G(k)x(k)\Big)\Big]\Big\}$$

$$= \sum_{s=n_0}^{s-1} \Phi^{-1}(s,n_0)\Big(G(s)B^{-1}(s)-I\Big)$$

$$\times \Big[G(s)\sum_{k=s-\tau}^{s-1} G(k)x(k)$$

$$-\big[A(s)-G(s-\tau)\big]x(s-\tau)\Big] \tag{4.10}$$

Thus applying Theorem 7 on Equation (4.10) gives

$$\Big[\quad \Phi^{-1}(s,n_0) \quad \Big(x(s)+\sum_{k=s-\tau}^{s-1} G(k)x(k)\Big)\Big]_{s=n_0}^{n}$$

$$= \sum_{s=n_0}^{n-1} \Phi^{-1}(s,n_0)\Big(G(s)B^{-1}(s)-I\Big)\Big[G(s)\sum_{k=s-\tau}^{s-1} G(k)x(k)$$

$$-\big[A(s)-G(s-\tau)\big]x(s-\tau)\Big]$$

Thus

$$\Phi^{-1}(n,n_0)\Big(x(n) \quad +\sum_{k=n-\tau}^{n-1} G(k)x(k)\Big)$$

$$= \quad \Phi^{-1}(n_0,n_0)\Big(x(n_0)+\sum_{s=n_0-\tau}^{n_0-1} G(s)x(s)\Big)$$

$$+\sum_{s=n_0}^{n-1} \Phi^{-1}(s,n_0)\Big(G(s)B^{-1}(s)-I\Big)$$

$$\times \Big[G(n) \sum_{k=s-\tau}^{n-1} G(k)x(k)$$

$$- \big[A(s) - G(s-\tau)\big]x(s-\tau)\Big]$$

By property (iv) of fundamental matrices we obtain

$$\Phi^{-1}(n,n_0)\Big(\quad x(n) \quad + \sum_{k=n-\tau}^{n-1} G(k)x(k)\Big)$$

$$= \quad \Big(x(n_0) + \sum_{s=n_0-\tau}^{n_0-1} G(s)x(s)\Big)$$

$$+ \sum_{s=n_0}^{n-1} \Phi^{-1}(s,n_0)\Big(G(s)B^{-1}(s) - I\Big)$$

$$\times \Big[G(n) \sum_{k=s-\tau}^{n-1} G(k)x(k)$$

$$- \big[A(s) - G(s-\tau)\big]x(s-\tau)\Big]. \tag{4.11}$$

Pre-multiplying both sides of Equation (4.11) by $\Phi(n,n_0)$ gives

$$\Big\{\Phi(n,n_0)\Phi^{-1}(n,n_0)\Big(\quad x(n) \quad + \sum_{k=n-\tau}^{n-1} G(k)x(k)\Big)\Big\}$$

$$= \quad \Phi(n,n_0)\Big\{\Big(x(n_0) + \sum_{s=n_0-\tau}^{n_0-1} G(s)x(s)\Big)$$

$$+ \sum_{s=n_0}^{n-1} \Phi^{-1}(s,n_0)\Big(G(s)B^{-1}(s) - I\Big)$$

$$\times \Big[G(s) \sum_{k=s-\tau}^{s-1} G(k)x(k)$$

$$- \big[A(s) - G(s-\tau)\big]x(s-\tau)\Big]\Big\}$$

This implies that,

$$\begin{aligned}\Big(x(n) + \sum_{k=n-\tau}^{n-1} G(k)x(k)\Big) &= \Phi(n,n_0)\Big\{\Big(x(n_0) + \sum_{s=n_0}^{n_0-1} G(s)x(s)\Big) \\ &+ \sum_{s=n_0}^{n-1} \Phi^{-1}(s,n_0)\big(G(s)B^{-1}(s) - I\big) \\ &\times \Big[G(s)\sum_{k=s-\tau}^{s-1} G(k)x(k) \\ &- \big[A(s) - G(s-\tau)\big]x(n-\tau)\Big]\Big\}\end{aligned}$$

Thus we obtain

$$\begin{aligned}x(n) &= -\sum_{k=n-\tau}^{n-1} G(k)x(k) + \Phi(n,n_0)\Big(x(n_0) + \sum_{s=n_0-\tau}^{n_0-1} G(k)x(k)\Big) \\ &+ \Phi(n,n_0)\sum_{s=n_0}^{n-1} \Phi^{-1}(s,n_0)\Big(G(s)B^{-1}(s) - I\Big) \\ &\times \Big[G(s)\sum_{k=s-\tau}^{s-1} G(k)x(k) - \big[A(s) - G(s-\tau)\big]x(s-\tau)\Big]\end{aligned}$$

Therefore by property(v) of fundamental matrix we obtain

$$\begin{aligned}x(n) &= -\sum_{k=n-\tau}^{n-1} G(k)x(k) + \Phi(n,n_0)\Big(x(n_0) + \sum_{s=n_0}^{n_0-1} G(k)x(k)\Big) \\ &+ \sum_{s=n_0}^{n-1} \Phi(n,s)\Big(G(s)B^{-1}(s) - I\Big) \\ &\times \Big[G(s)\sum_{k=s-\tau}^{s-1} G(k)x(k) \\ &- \big[A(s) - G(s-\tau)\big]x(s-\tau)\Big] \qquad (4.12)\end{aligned}$$

This complete the proof.

Main Results

Stability of the zero solution

In this section we obtain conditions for the zero solution of Equation (4.1) to be stable.

Theorem 4.1

Suppose that there exist a non-singular $s \times s$ matrix $G(n)$ and a constant $\alpha \in (0,1)$ such that

$$\sum_{k=n-\tau}^{n-1} |G| + \sum_{s=n_0}^{n-1} |\Phi| \Big(1 + |G||B^{-1}| \Big)$$

$$\times \Big[|G| \sum_{k=s-\tau}^{s-1} |G| + \big[|A| + |G| \big] \Big] \leq \alpha. \quad (4.13)$$

Then the zero solution of Equation (4.1) is stable.

Proof.

Let $\epsilon > 0$ be given. Choose $\delta > 0$ such that if $||\psi|| \leq \delta$ then

$$\alpha\epsilon + \delta |\Phi| \Big(1 + \sum_{k=n_0-\tau}^{n_0-1} |G| \Big) < \epsilon.$$

Define

$$S = \{ \varphi \in C(\mathbb{Z}, \mathbb{R}^s) : \ \varphi(n) = \psi(n) \text{ if } n \in [-\tau, n_0] \cap \mathbb{Z} \text{ and for } n \geq n_0$$
$$||\varphi|| \leq \epsilon \}$$

where $||\varphi|| = \max_{n \in [-\tau, \infty) \cap \mathbb{Z}} |\varphi(n)|$ with $|.|$ being the infinity norm for $\varphi \in \mathbb{R}^s$.

Then $(S, \| \,.\, \|)$ is a complete metric space. Define the mapping $P : S \to S$ by

$$(P\varphi)(n) = \psi(n) \text{ for } n \in [-\tau, n_0] \cap \mathbb{Z},$$

and

$$\begin{aligned}(P\varphi)(n) &= -\sum_{k=n-\tau}^{n-1} G(k)\varphi(k) \\ &\quad + \Phi(n,n_0)\Big(\psi(n_0) + \sum_{k=n_0-\tau}^{n_0-1} G(k)\psi(k)\Big) \\ &\quad + \sum_{s=n_0}^{n-1} \Phi(n,s)\Big(G(s)B^{-1}(s) - I\Big) \\ &\quad \times \Big[G(s) \sum_{k=s-\tau}^{s-1} G(k)\varphi(k) \\ &\quad - \big[A(s) - G(s-\tau)\big]\varphi(s-\tau)\Big] \end{aligned} \tag{4.14}$$

Next to show that P maps S into itself. To do that consider

$$\begin{aligned}|(P\varphi)(n)| &= \Big| -\sum_{k=n-\tau}^{n-1} G(k)\varphi(k) + \Phi(n,n_0)\Big(\psi(n_0) + \sum_{k=n_0-\tau}^{n_0-1} G(k)\psi(k)\Big) \\ &\quad + \sum_{s=n_0}^{n-1} \Phi(n,s)\Big(G(s)B^{-1}(s) - I\Big)\Big[G(s) \sum_{k=s-\tau}^{s-1} G(k)\varphi(k) \\ &\quad - \big[A(s) - G(s-\tau)\big]\varphi(s-\tau)\Big]\Big|\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{k=n-\tau}^{n-1} |G||\varphi|| + |\Phi|\Big(||\psi|| + \sum_{k=n_0-\tau}^{n_0-1} |G|||\psi||\Big) \\
&\quad + \sum_{s=n_0}^{n-1} |\Phi|\Big(1 + |G||B^{-1}|\Big)\Big[|G| \sum_{k=s-\tau}^{s-1} |G|||\varphi|| \\
&\quad + \big[|A| + |G|\big]||\varphi||\Big] \\
&= \Big(\sum_{k=n-\tau}^{n-1} |G| + \sum_{s=n_0}^{n-1} |\Phi|(1 + |G||B^{-1}|) \\
&\quad \times \Big[|G| \sum_{k=s-\tau}^{s-1} |G| + [|A| + |G|]\Big]\Big)||\varphi|| \\
&\quad + |\Phi|\Big(||\psi|| + \sum_{k=n_0-\tau}^{n_0-1} |G|||\psi||\Big) \\
&\leq \alpha\epsilon + |\Phi|\Big(\delta + \sum_{k=n_0-\tau}^{n_0-1} |G|\delta\Big) \\
&\leq \alpha\epsilon + |\Phi|\delta\Big(1 + \sum_{k=n_0-\tau}^{n_0-1} |G|\Big) \\
&\leq \epsilon.
\end{aligned}
$$

This shows that P maps S into itself.

Next to show the continuity of the mapping. To show that P is continuous let $\varphi, \eta \in S$ such that

$$
\begin{aligned}
(P\varphi)(n) \;&=\; -\sum_{k=n-\tau}^{n-1} G(k)\varphi(k) + \Phi(n, n_0)\Big(\psi(n_0) + \sum_{k=n_0-\tau}^{n_0-1} G(k)\psi(k)\Big) \\
&\quad + \sum_{s=n_0}^{n-1} \Phi(n, s)\Big(G(s)B^{-1}(s) - I\Big)\Big[G(s) \sum_{k=s-\tau}^{s-1} G(k)\varphi(k) \\
&\quad - \big[A(s) - G(s-\tau)\big]\varphi(s-\tau)\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
(P\eta)(n) &= \sum_{k=n-\tau}^{n-1} G(k)\eta(k) + \Phi(n,n_0)\Big(\psi(n_0) + \sum_{k=n_0-\tau}^{n_0-1} G(k)\psi(k)\Big) \\
&+ \sum_{s=n_0}^{n-1} \Phi(n,s)\Big(G(s)B^{-1}(s) - I\Big)\Big[G(s)\sum_{k=s-\tau}^{s-1} G(k)\eta(k) \\
&- \big[A(s) - G(s-\tau)\big]\eta(s-\tau)\Big].
\end{aligned}
$$

Then given any $\epsilon_1 > 0$, choose $\delta = \frac{\epsilon_1}{\alpha}$ such that $||\varphi - \eta|| < \delta$. Thus,

$$
\begin{aligned}
|P\varphi(n) - P\eta(n)| &= \Big| - \sum_{k=n-\tau}^{n-1} G(k)\varphi(k) \\
&+ \Phi(n,n_0)\Big(\psi(n_0) + \sum_{k=s_0-\tau}^{s_0-1} G(k)\psi(k)\Big) \\
&+ \sum_{s=n_0}^{n-1} \Phi(n,s)\Big(G(s)B^{-1}(s) - I\Big) \\
&\times \Big[G(s)\sum_{k=s-\tau}^{s-1} G(k)\varphi(k) \\
&- \big[A(s) - G(s-\tau)\big]\varphi(s-\tau)\Big] - \Big(\sum_{k=n-\tau}^{n-1} G(k)\eta(k) \\
&+ \Phi(n,n_0)\Big(\psi(n_0) + \sum_{k=n_0-\tau}^{n_0-1} G(k)\psi(k)\Big) \\
&+ \sum_{s=h_0}^{n-1} \Phi(n,s)\Big(G(s)B^{-1}(s) - I\Big) \\
&\times \Big[G(s)\sum_{k=s-\tau}^{s-1} G(k)\eta(k) \\
&- \big[A(s) - G(s-\tau)\big]\eta(s-\tau)\Big]\Big|
\end{aligned}
$$

$$
\begin{aligned}
= & \Big| - \sum_{k=n-\tau}^{n-1} G(k)\varphi(k) - (- \sum_{k=n-\tau}^{n-1} G(k)\eta(k)) \\
& + \sum_{s=n_0}^{n-1} \Phi(n,s)\Big(G(s)B^{-1}(s) - I\Big) \\
& \times \Big[G(s) \sum_{k=s-\tau}^{s-1} G(k)\varphi(k) \\
& - \big[A(s) - G(s-\tau)\big]\varphi(s-\tau)\Big] \\
& - \sum_{s=n_0}^{n-1} \Phi(n,s)\Big(G(s)B^{-1}(s) - I\Big) \\
& \times \Big[G(s) \sum_{k=s-\tau}^{s-1} G(k)\eta(k) \\
& - \big[A(s) - G(s-\tau)\big]\eta(s-\tau)\Big]\Big| \\
= & \Big| \sum_{k=n-\tau}^{n-1} G(k)\Big(\varphi(k) - \eta(k)\Big) \\
& + \sum_{s=n_0}^{n-1} \Phi(n,s)\Big(G(s)B^{-1}(s) - I\Big) \\
& \times \Big(\Big[G(s) \sum_{k=s-\tau}^{s-1} G(k)(\varphi(k) - \eta(k)) \\
& - \big[A(s) - G(s-\tau)\big]\Big(\varphi(s-\tau) - \eta(s-\tau)\Big]\Big)\Big| \\
\leq & \sum_{k=n-\tau}^{n-1} |G||\varphi - \eta|| + \sum_{s=n_0}^{n-1} |\Phi|\Big(1 + |G||B^{-1}|\Big) \\
& \times \Big[|G| \sum_{k=s-\tau}^{s-1} |G|||\varphi - \eta|| + \Big[|A| + |G|\Big]||\varphi - \eta||\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sum_{k=n-\tau}^{n-1} |G|||\varphi - \eta|| + \sum_{s=n_0}^{n-1} |\Phi| \Big(1 + |G||B^{-1}(s)| \Big) \\
&\quad \times \Big[|G| \sum_{k=s-\tau}^{s-1} |G| + \Big[|A| + |G| \Big] \Big] ||\varphi - \eta|| \\
&\leq \Big(\sum_{k=n-\tau}^{n-1} |G(k)| + \sum_{s=n_0}^{n-1} |\Phi| \Big(1 + |G||B^{-1}| \Big) \\
&\quad \times \Big[|G| \sum_{k=s-\tau}^{s-1} |G| + \Big[|A| + |G| \Big] \Big] \Big) ||\varphi - \eta|| \\
&\leq \alpha ||\varphi - \eta|| \\
&\leq \epsilon_1 .
\end{aligned}
$$

Thus showing that P is continuous.

Now prove that P is contraction mapping. Let$\varphi_1, \varphi_2 \in S$ then

$$
\begin{aligned}
(P\varphi_1)(n) &= - \sum_{k=n-\tau}^{n-1} G(k)\varphi_1(k) + \Phi(n, n_0) \Big(\psi(n_0) + \sum_{k=n_0-\tau}^{n_0-1} G(k)\psi(k) \Big) \\
&\quad + \sum_{s=n_0}^{n-1} \Phi(n, s) \Big(G(s)B^{-1}(s) - I \Big) \Big[G(s) \sum_{k=s-\tau}^{s-1} G(k)\varphi_1(k) \\
&\quad - \big[A(s) - G(s-\tau) \big] \varphi_1(s-\tau) \Big]
\end{aligned}
$$

$$
\begin{aligned}
(P\varphi_2)(n) &= - \sum_{k=n-\tau}^{n-1} G(k)\varphi_2(k) + \Phi(n, n_0) \Big(\psi(n_0) + \sum_{k=n_0-\tau}^{n_0-1} G(k)\psi(k) \Big) \\
&\quad + \sum_{s=n_0}^{n-1} \Phi(n, s) \Big(G(s)B^{-1}(s) - I \Big) \Big[G(s) \sum_{k=s-\tau}^{s-1} G(k)\varphi_2(k) \\
&\quad - \big[A(s) - G(s-\tau) \big] \varphi_1(s-\tau) \Big].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
|P\varphi_1(n) - P\varphi_2(n)| &= \Big| - \sum_{k=n-\tau}^{n-1} G(k)\varphi_1(k) \\
&\quad + \Phi(n,n_0)\Big(\psi(n_0) + \sum_{k=n_0-\tau}^{n_0-1} G(k)\psi(k)\Big) \\
&\quad + \sum_{s=n_0}^{n-1} \Phi(n,s)\Big(G(s)B^{-1}(s) - I\Big) \\
&\quad \times \Big[G(s)\sum_{k=s-\tau}^{s-1} G(k)\varphi_1(k) \\
&\quad - \big[A(s) - G(s-\tau)\big]\varphi_1(s-\tau)\Big] \\
&\quad - - \sum_{k=n-\tau}^{n-1} G(k)\varphi_2(k) \\
&\quad - \Phi(n,n_0)\Big(\psi(n_0) + \sum_{k=n_0-\tau}^{n_0-1} G(k)\psi(k)\Big) \\
&\quad - \sum_{s=n_0}^{n-1} \Phi(n,s)\Big(G(s)B^{-1}(s) - I\Big) \\
&\quad \times \Big[G(s)\sum_{k=s-\tau}^{s-1} G(k)\varphi_2(k) \\
&\quad - \big[A(s) - G(s-\tau)\big]\varphi_2(s-\tau)\Big]\Big| \\
&= \Big| - \sum_{k=n-\tau}^{n-1} G(k)\varphi_1(k) - - \sum_{k=n-\tau}^{n-1} G(k)\varphi_2(k) \\
&\quad + \sum_{s=n_0}^{n-1} \Phi(n,s)\Big(G(s)B^{-1}(s) - I\Big)
\end{aligned}
$$

$$\times \Big[G(s) \sum_{k=s-\tau}^{s-1} G(k)\varphi_1(k)$$

$$- \big[A(s) - G(s-\tau)\big]\varphi_1(s-\tau)\Big]$$

$$- \sum_{s=n_0}^{n-1} \Phi(n,s)\Big(G(s)B^{-1}(s) - I\Big)\Big[G(s)$$

$$\times \sum_{k=s-\tau}^{s-1} G(k)\varphi_2(k) + \Big[A(s) - G(s-\tau)\Big]\varphi_2(s-\tau)\Big]\Big|$$

$$= \Big| \sum_{k=n-\tau}^{n-1} G(k)\Big(\varphi_1(k) - \varphi_2(k)\Big)$$

$$+ \sum_{s=n_0}^{n-1} \Phi(n,s)\Big(G(s)B^{-1}(s) - I\Big)$$

$$\times \Big[G(s) \sum_{k=s-\tau}^{s-1} G(k)(\varphi_1(k) - \varphi_2(k))$$

$$-\big[A(s) - G(s-\tau)\big]\Big(\varphi_1(s-\tau) - \varphi_2(s-\tau)\Big)\Big]\Big|$$

$$\leq \sum_{k=n-\tau}^{n-1} |G|||\varphi_1 - \varphi_2|| + \sum_{s=n_0}^{n-1} |\Phi|\Big(1 + |G||B^{-1}|\Big)$$

$$\times \Big[|G| \sum_{k=s-\tau}^{s-1} |G|||\varphi_1 - \varphi_2||$$

$$+ \big[|A| + |G|\big]||\varphi_1 - \varphi_2||\Big]$$

$$\leq \Big(\sum_{k=n-\tau}^{n-1} |G| + \sum_{s=n_0}^{n-1} |\Phi|\Big(1 + |G||B^{-1}|\Big)$$

$$\times \Big[|G| \sum_{k=s-\tau}^{s-1} |G| + \big[|A| + |G|\big]\Big)||\varphi_1 - \varphi_2||$$

$$\leq \alpha||\varphi_1 - \varphi_2|| \tag{4.15}$$

Thus, showing that P is a contraction.

Therefore, by the contraction mapping principle, P has a unique fixed point in S which solves Equation (4.1) and for any $\varphi \in S$, $\|P\varphi\| \leq \epsilon$. This proves that the zero solution of Equation (4.1) is stable.

Asymptotic Stability

In this section sufficient conditions for the asymptotic stability of the zero solution of Equation (4.1)are obtained.

Theorem 4.2

Assume that the hypotheses of Theorem 2.1 hold. Also assume that

$$\Phi(n, n_0) \to 0 \text{ as } n \to \infty. \tag{4.16}$$

Then the zero solution of Equation (4.1) is asymptotically stable.

Proof.

According to definition 3, the zero solution of a difference equation is asymptotically stable if it is stable and in addition for each $n_0 \geq 0$ there is an $\eta(n_0) > 0$ such that $||\psi|| < \eta(n_0)$ implies that $x(n) \to 0$ as $n \to \infty$. We have already proved that the zero solution of Equation (4.1) is stable. Define

$$S^* = \{\varphi \in C(\mathbb{Z}, \mathbb{R}^s) : \ \varphi(n) = \psi(n) \text{ if } n \in [-\tau, n_0] \cap Z, \ \| \varphi \| \leq \epsilon$$

$$\text{for } n \geq n_0 \text{ and } \varphi(n) \to 0, \text{ as } n \to \infty\}.$$

Define the mapping by $P : S^* \to S^*$

$$(P\varphi)(n) = \psi(n) \text{ if } n \in [-\tau, n_0]$$

and

$$
\begin{aligned}
(P\varphi)(n) &= -\sum_{k=n-\tau}^{n-1} G(k)\varphi(k) + \Phi(n,n_0)\Big(\psi(n_0) + \sum_{k=n_0-\tau}^{n_0-1} G(k)\psi(k)\Big) \\
&\quad + \sum_{s=n_0}^{n-1} \Phi(n,s)\Big(G(s)B^{-1}(s) - I\Big)\Big[G(s)\sum_{k=s-\tau}^{s-1} G(k)\varphi(k) \\
&\quad - \big[A(s) - G(s-\tau)\big]\varphi(s-\tau)\Big]. \qquad (4.17)
\end{aligned}
$$

To show that $P\varphi \to 0$ as $n \to \infty$ we proceed as follows. There are three terms on the right hand side of Equation (4.17). We denote them respectively by T_1, T_2 and T_3. The second term denoted by T_2 tends to zero as $n \to \infty$ due to the fact that $\Phi(n, n_0) \to 0$. We then consider the first term and show that it also goes to zero as $n \to \infty$. Let $\varphi \in S^*$, then $\varphi(n) \to 0$ as $n \to \infty$. Thus by the continuity of norms we have $\| \varphi \| \to 0$ as $n \to \infty$.

Hence,

$$
\begin{aligned}
|T_1| &= \Big| -\sum_{k=n-\tau}^{n-1} G(k)\varphi(k)\Big| \\
&\leq \sum_{k=n-\tau}^{n-1} |G||\varphi|| \\
&\leq ||\varphi|| \sum_{k=n-\tau}^{n-1} |G| \to 0 \text{ as } n \to \infty
\end{aligned}
$$

. Now to show that the last term on the right hand side of Equation (4.17) goes to zero as $n \to \infty$. Since $\varphi(n) \to 0$ as $n \to \infty$, for $\epsilon_1 > 0$, there exists an $N_1 > n_0$ such that $n \geq N_1$ implies $|\varphi(n)| < \epsilon_1$. Thus for $n \geq N_1$, the last term, T_3 in Equation (4.17) satisfies

$$
\begin{aligned}
|T_3| &= \Big|\sum_{s=n_0}^{n-1} \Phi(n,s)\Big(G(s)B^{-1}(s) - I\Big)\Big[G(s)\sum_{k=s-\tau}^{s-1} G(k)\varphi(k) \\
&\quad - \big[A(s) - G(s-\tau)\big]\varphi(s-\tau)\Big]\Big| \\
&\le \sum_{s=n_0}^{N_1-1} |\Phi|\Big(1 + |G||B^{-1}|\Big)\Big[|G|\sum_{k=s-\tau}^{s-1} |G|\varphi(k) + \big[|A| + |G|\big]\varphi(s-\tau)\Big] \\
&\quad + \sum_{s=N_1}^{n-1} |\Phi|\Big(1 + |G|B^{-1}|\Big)\Big[|G|\sum_{k=s-\tau}^{s-1} |G||\varphi(k)| \\
&\quad + \big[|A(s)| + |G|\big]|\varphi(s-\tau)|\Big] \\
&\le \sum_{s=n_0}^{N_1-1} |\Phi|\Big(1 + |G(s)||B^{-1}(s)|\Big) \\
&\quad \times \Big[|G|\sum_{k=s-\tau}^{s-1} |G| + \big[|A| + |G|\big]\Big]||\varphi|| + \epsilon_1 \sum_{s=N_1}^{n-1} |\Phi|\Big(1 + |G|B^{-1}|\Big) \\
&\quad \times \Big[|G|\sum_{k=s-\tau}^{s-1} |G| + \big[|A| + |G|\big]\Big] \\
&\le \alpha\epsilon + \alpha\epsilon_1
\end{aligned}
$$

By the contraction mapping principle, P has a unique fixed point that solves Equation (4.1) and approaches to zero as n goes to infinity. Therefore, the zero solution of Equation (4.1) is asymptotically stable.

Chapter Summary

This chapter dealt with stability solutions of systems difference equations with finite delay through the determination of the continuity of the solution and showing that the solution contracts by the application of contraction mapping principle of the solution when it is inverted. The chapter also dealt with asymptotic stability of the zero solution of the difference equation.

CHAPTER FIVE

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

Overview

This chapter provides the summary and conclusion of the study. The summary briefly presents an overview of the research problem, objectives, methodology and results of the study.

Summary

In this thesis, as spelt out in the objectives of the research, we investigated the asymptotic stability of solutions of certain classes of systems of difference equations with finite delay. The fixed point theory was the main tool used to investigate the stability behaviour of the system of difference equations.

The difference equation has been transformed into an equivalent summation form. The summation equation was then used to define a mapping that was used to study the stability properties of the system of difference equations with finite delay. The contraction mapping principle was used since the mappings were contraction mappings. This theorem was also used to prove the asymptotic stability of the zero solution of the system of difference equations.

Conclusions

Sufficient conditions for the asymptotic stability of the zero solution of a system of difference equations with finite delay have been obtained.

Recommendations

For a system of difference equations with finite delay the fixed delay point theory should be employed obtain sufficient conditions that can guarantee the stability of its zero solutions.

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