UNIVERSITY OF CAPE COAST

STABILITY OF SOLUTIONS OF A SYSTEM OF FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH FINITE DELAY

BY

ISHMAEL BESING. KARADAAR

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DECLARATION

Candidate's Declaration

I hereby declare that this thesis is the result of my own original research and that no part of it has been presented for another degree in this university or elsewhere.

Candidate's Signature .... Date ....

Name: Ishmael Besing Karadaar

Supervisor's Declaration

I hereby declare that the preparation and presentation of the thesis were supervised in accordance with the guidelines on supervision of thesis laid down by the University of Cape Coast.

Supervisor's Signature .... Date ....

Name: Prof. Ernest Yankson

ABSTRACT

This thesis is concerned with the stability of solutions of a system of ordinary differential equations with finite delay. Fixed point theory is used in this thesis as the main mathematical tool to investigate the stability of solutions of a system of ordinary differential equations with finite delay. In particular, the Banach fixed point theorem is used. In the process the system of equations are inverted to obtain an equivalent integral equation. The result of the inversion is used to define a suitable mapping which is then used to derive the stability properties of the zero solution of the system of ordinary differential equations with finite delay. Sufficient conditions that guarantee that the zero solutions of a system of ordinary differential equations with finite delay is asymptotically stable are obtained.

KEY WORDS

Asymptotic Stability

Finite Delay

Fixed Point Theorem

Ordinary Differential Equations

Partial Differential Equations

Stability

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DEDICATION

To my family

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LIST OF ABBREVIATIONS

DE	Differential Equation
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation

CHAPTER ONE

INTRODUCTION

This chapter is made up of the background of the study, the statement of the problem, the objectives of this study as well as the organization of the chapters in the thesis.

Background to the Study

Mathematical modeling has transformed to become a significant tool for understanding the mechanisms of real-life phenomena, most notably in biological and medical sciences. These mathematical models normally produce equations that have derivatives of unknown functions. These type of equations are known as differential equations.

Differential equations have been used in the past few centuries in the study of population dynamics, ecology, epidemiology, malaria prevention and so forth. It has been a major branch of pure and applied mathematics since its inception in the 17th century. Whiles the history of differential equations has been well studied, it remains a vital field of on-going investigation, with the emergence of new connections with other parts of mathematics.

According to Sasser (2005), differential equations began with Leibniz, Newton, the Bernoulli brothers and others from the 1675, not long after Newton's fluxional equations in 1670. Applications were made mainly to geometry and mechanic. Most 18th century development consolidated the Leibnization tradition, extending its multi-variate form, thus leading to partial differential equations. Generalization of isoperimetrical problems led to the calculus of variations. New figures appeared, especially Euler, Daniel Bernoulli, Lagrange and Laplace. Development of the general theory of solutions include singular ones, functional solutions and those by infinite series. Many applications were made to mechanics, especially to astronomy and continuous media.

In the 19th century, the general theory of differential equations was enriched by the development of the understanding of general and particular solutions and of existence theorems.

Several types of equations and their solutions appeared, for instance, Fourier analysis and special functions. Among new figures, Cauchy stands out. Applications were now made not only to classical mechanics but also to heat theory, optics, electricity and magnetism, especially with the impact of Maxwell.

Later Poincar'e introduced the recurrence theorems, initially in connection with the three-body problems. In the 20th century, the general theory of defferential equations was influenced by the arrival of set theory in mathematical analysis; with consequences for theorisation, including further topological aspects. New applications were made to quantum mathematics, dynamical systems and relativity theory.

Statement of the Problem

The study of the stability properties of ordinary differential equations have drawn the attention of several mathematicians lately. For instance, Burton (2003) proved that the zero solution of the equation

$$x' = -a(t)x(t-\tau)$$

is asymptotically stable, by means of fixed point theory.

Also, according to Chicone (1999) the system

$$x' = Ax(t),$$

where A is an $n \times n$ matrix, the sign of the real part of the eigenvalues of matrix A can be used to determine the stability properties of its zero solution. However, the stability results obtained by Burton (2003) does not hold for the system of ordinary differential equation

$$\frac{d}{dt}x(t) = A(t)x(t-\tau), \tag{1.1}$$

where A is an $n \times n$ matrix.

Moreover, the eigenvalue technique for systems of ordinary differential equations with constant coefficients cannot be applied to obtain stability results of Equation (1.1).

Research Objectives

The objectives of the thesis are to obtain sufficient conditions for the zero solution of the system of first order ordinary differential equations

$$\frac{d}{dt}x(t) = A(t)x(t-\tau),$$

with finite delay, τ, to be,

1. stable; and
2. asymptotically stable.

Significance of the Study

The results obtained in the study generalises some results in the literature.

Delimitation

The study determined stability of solutions of a system of first order ODE with finite delay. Results concerning the asymptotic stability of the zero solution of the ODE was obtained. The results cannot be easily generalised for all first order ordinary differential equations.

Limitaion

Liapunov's direct method has been very effective in establishing stability properties for a wide variety of DEs. However, fixed point theorems is employed in this study, this is because there is a large set of problems for which the Liapunov's direct method has been ineffective. Several researchers have examined particular problems which have offered great difficulties for that theory and have presented solutions by means of various fixed point theorems over the years.

Definition of Terms

Some definitions and concepts that will be used in this study are provided in this section.

Definition 1 (Ordinary Differential Equation)
An ordinary differential equation (ODE) is an equation involving derivatives of an unknown function with one variable.

The general first order ordinary differential equation in $R^n, n \geq 1$ is given by

$$x'(t) = f(t, x(t)), \qquad x(t_0) = x_0 \tag{1.2}$$

on a domain $D \subset R \times R^n$, where $(t_0, x_0) \in D$ and $f(t, x)$ is a function from $D \subset R \times R^n$ into R^n.

Definition 2 (Delay Ordinary Differential Equation)
A delay ordinary differential equation is an ordinary differential equation where the unknown function appears with delayed argument.

For example,

$$\frac{dx}{dt} = f(t, x(t), x(t-\tau)), \tag{1.3}$$

where $x(t) \in R^n$ and the delay $\tau > 0$ is a constant.

Definition 3 (Stability)
The zero solution is said to be stable if for any $\epsilon \geq 0$ there exists $\delta(\epsilon, t_0) \geq 0$ such that $|x_0| \leq \delta$ implies that $|x(t)| \leq \varepsilon$ for $t \geq t_0$.

Definition 4 (Asymptotic Stability)

The zero solution is said to be asymptotically stable if it is stable and in addition there exists $r(t_0) \geq 0$ such that $|x_0| \leq r(t_0)$ implies that the $\lim |x(t)| = 0$, $t \longrightarrow \infty$.

Organisation of the Study

Chapter One of the thesis presented a history of the study of ordinary differential equations and its role in modelling physical phenomena. The Chapter also included, problem statement, research objectives, significance of the study, limitation and delimitation as well as the organisation of the study. Chapter Two of the study reviewed relevant related literature of stability properties of ODEs.

The Chapter Three of the sudy dealt with the tool used in the discussion of the stability properties of the ordinary differential equation considered in the study.

Chapter Four provides the main results established in the study. The results were obtained based on the objectives of the study.

Chapter Five of the study dealt with the summary of the results obtained in the study as well as the conclusions.

CHAPTER TWO

LITERATURE REVIEW

Introduction

In this chapter, existing literature which is related and also relevant and significant to the study are reviewed.

Stability by Banach Theory

The study of the behaviour of solutions of differential equations began in the nineteenth century and became a subject of intense research since 1940. The first direct reference is the work of Poincare (1899). Since then, there has been an intensified interest among researchers to explore its richness.

Lyapunov (1892) proposed a basic method for studying the problem of stability, boundedness and periodic solutions of ordinary differential equations by constructing a function known as the Lyapunov functions. This function is often presented as $V(t,x)$ defined in some regions or the whole state phase that contains the unperturbed solution $x = 0$ for all $t > 0$ and which together with its derivative $V'(t,x)$ satisfy some sign definiteness.

The Lyapunov method is by far the most general method for dealing with stability, boundedness and the periodic solution of functional differential equations. However, several difficulties with the theory and application to problems persist and therefore new methods are needed to address those challenges. There is the problem of constructing an appropriate Lyapunov functional and also the problem with the types of conditions which are typically imposed on the functions in the differential equations.

Burton & Furumochi (2001) discovered that most of the difficulties with the use of the Lyapunov's direct method disappeared using fixed point theorems. The authers pointed out that, not only do the fixed point conditions emerge as averages, but in one step the existence, uniqueness, and boundedness of solutions of problems which have challenged researchers for decades are proved.

They continued this work in Burton & Furumochi(2001), where delay equations which may be unstable when the delay is zero were considered. In particular, asymptotic stability results were proved by Schauder's and Banach's fixed point theorems and Schaefer's fixed point theorem was also used to prove that there is a periodic solution when a periodic forcing function is added to that equation.

The study of the stability properties of ordinary differential equations have attracted the attention of many mathematicians lately. For instance, according to Chicone (1999) the system

$$x' = Ax(t),$$

where A is an $n \times n$ matrix, the sign of the real part of the eigenvalues of matrix A can be used to determine the stability properties of its zero solution. Burton (2006) proved that the zero solution of the equation

$$x' = -a(t)x(t - \tau)$$

is asymptotically stable, by means of fixed point theory.

Again, Burton (2006) proved that the equation

$$x' = -a(t)x(t) + b(t)x(t - r(t))$$

is asymptotically stable by means of fixed point theory.

Delay Differential Equations

When modeling a system using a differential equation where the fundamental assumption is that the time rate at time t, given as $x'(t)$, depends only on the current status at time t, given as $f(t, x(t))$ results in the differential equation

$$x'(t) = f(t, x(t)), \qquad x(t_0) = x_0, \qquad t \geq t_0, \qquad x(t) \in R^n. \tag{2.1}$$

Moreover, the initial condition is given in the form $x(t_0) = x_0$. In applications, this assumption and initial condition should be improved so as to model the situations more accurately and therefore derive better results.

One improvement of equation (2.1) is to assume that the time rate depends not only on the current status, but also on the status in the past; that is , the past history will contribute to the future development, or, there is a time delay effect. For example, for a university, its current population will affect its population growth, however, its population in the past may also affect its population growth. In fact, in his study of predator-prey models, Volterra (1928) had investigated the equation

$$x'(t) = x(t)[a - by(t) - \int_{-r}^{0} F_1(s)y(t+s)ds],$$

$$x'(t)y'(t) = y(t)[a - bx(t) - \int_{-r}^{0} F_1(s)x(t+s)ds], \tag{2.2}$$

where x and y are the number of preys and predators, respectively, and all constants and functions are nonegative and r is a positive constant. In $\int_{-r}^{0} F_1(s)y(t+s)ds$, the variable s varies in the interval $[-r, 0]$, thus $y(t+s)$ is a function defined on the interval $[t-r, t]$. This says that for equation (2.2), the time rate at t, $\mid x'(t), y'(t) \mid^T$, depends not only on the status of $x(t)$ and $y(t)$ at t, but also on the past status of $x(t+s)$ and $y(t+s)$ defined on the interval $[t-r, t]$. That is, the history on the interval $[t-r, t]$ will affect the growth rates of the preys and predators at time t.

Other physical procedures that possess such time-delay properties include blood moving through arteries, relaxation of materials with memory from bending and signals traveling through mediums.

Differential equations incorporating delay effect, or using information from the past, are called delay differential equations. They include finite delay differential equations and infinite delay differential equations.

Consider the delay differential equation below with $x \in R^n$

$$x'(t) = f(t, x(t), x(t-\tau)), \quad \tau > 0, \tag{2.3}$$

with

$$x(t) = \Phi_0(t), \quad t_0 - \tau \leq t \leq t_0. \tag{2.4}$$

Here $\Phi_0 : R \longrightarrow R^n$ is a known function, usually taken to be continuous. $\Phi_0(t)$ is called the initial function for equation (2.3), t_0 the initial instant and $[t_0 - \tau, t_0]$ the initial set.

Definition 5

A function $x : [t_0 - \tau, t_0 + T] \longrightarrow R^n$, where $T > 0$ is a constant, is said to be a solution of equation (2.3) and equation (2.4) on $[t_0 - \tau, t_0 + T]$ if $x(t_0) = \Phi_0$, $x(t)$ is differentiable on $[t_0, t_0 + T]$, and satisfies equation(2.3) for $t \in [t_0, t_0 + T]$.

Definition 6

A function $f(t, x)$ on a domain $D \subset R \times R^n$is said to satisfy a local Lipschitz condition with respect to x on D if for any $(t_1, x_1) \in D$, there exists a domain D_1 such that $(t_1, x_1) \in D_1 \subset D$ and that $f(t, x)$ satisfies a Lipschitz condition with respect to x on D_1. That is, there exists a positive constant k_1 such that

$$\mid f(t, x) - f(t, y) \mid \leq k_1 \mid x - y \mid \text{ for } (t, x), (t, y) \in D_1.$$

Theorem 2.1.1. [Driver (1977)] If $f(t, x, y)$ is continuous with respect to t and y and locally Lipschitz with respect to x in some neighbourhood of $(t_0, \Phi_0(t_0))$ and Φ_0 is continuous with respect to t in some neighbourhood of t_0, then there exists a unique solution to equations (2.3)−(2.4) in a neighbourhood of $(t_0, \Phi_0(t_0))$.

Systems with Bounded Delays-General Framework

In section 2.2 the existence and uniqueness of solutions for some delay differential equations of specific forms were considered. To consider arbitrary delay differential equations, there is the need for a more general mathematical framework in which to work. This is the subject of the current section.

For $\tau > 0$, let $C = C([-\tau, 0], R^n)$ be the space of continuous functions mapping $[-\tau, 0]$ into R^n Let $\Phi \in C$. Assume the norm of this space to be $\| \Phi \|_\tau = sup_{-\tau \leq \theta \leq 0} \| \Phi(\theta) \|$, where $\| \,.\, \|$ is the usual Euclidean norm on R^n. With this norm,C is a Banach space. Further, for $D \subseteq R^n$ let $C_D = C([-\tau, 0], D)$ be the set of continuous functions mapping $[-\tau, 0]$ into D.

Definition 7

If x is a function defined at least on $[t - \tau, t] \longrightarrow R^n$ then define a new function $x_t : [-\tau, 0] \longrightarrow R^n$ by

$$x_t(\theta) = x(t + \theta), \qquad -\tau \leq \theta \leq 0. \tag{2.5}$$

Clearly, if x is continuous on $[t - \tau, t]$, then x_t is continuous on $[-\tau, 0]$. In the following, unless otherwise stated, take $J \subseteq R$ and $D \subseteq R^n$ to be open set

Definition 8

If $F : J \times C_D \longrightarrow R^n$ is a given functional, then call the relation

$$x'(t) = F(t, x_t) \tag{2.6}$$

a delay differential equation on $J \times C_D$.

It must be noted that equation (2.6) includes the following.

(a). Ordinary differential equations (if $\tau = 0$): $x'(t) = F(t, x(t))$.

(b). Differential equations with constant delays:

$$\begin{aligned} x'(t) &= f(t, x(t-\tau_1), ..., x(t-\tau_m)) \\ &= f(t, x_t(t-\tau_1), ..., x(t-\tau_m)) \\ &= F(t, x_t) \end{aligned}$$

Here $\tau_j \geq 0$ is constant and $\tau = max_{1 \leq j \leq m} \tau_j$.

(c). Differential equations with bounded, variable delays:

$$\begin{aligned} x'(t) &= f(t, x(t-\tau_1(t)), ..., x(t-\tau_m(t))) \\ &= f(t, x(t-\tau_1(t)), ..., x(t-\tau_m(t))) \\ &= F(t, x_t) \end{aligned}$$

Here $0 \leq \tau_j \leq \tau, j = 1, ..., m, t \in J$..

(d). Differential equations with a distribution of delays:

$$\begin{aligned} x'(t) &= \int_{-\tau}^{0} f(t, \theta, x(t+\theta)) d\theta \\ &= \int_{-\tau}^{0} f(t, \theta, x_t(\theta)) d\theta \\ &= F(t, x_t) \end{aligned}$$

A more precise definition of a solution of a delay differential equation is given below.

Definition 9

Let $F : J \times C_D \longrightarrow R^n$. A function $x(t)$ is said to be a solution of equation (2.6) on $[t_0 - \tau, \beta)$ if there are $t_0 \in R$and $\beta > t_0$ such that

(i) $x \in C([t_0 - \tau, \beta), D)$

(ii) $[t_0, \beta) \subset J$

(iii) $x(t)$ satisfies equation (2.6) for $t \in [t_0, \beta)$.

For a given $t_0 \in R$ and $\Phi_0 \in C_D$, the initial value problem associated with the delay differential equation (2.6) is

$$\begin{cases} x'(t) = F(t, x_t), \quad t \geq t_0 \\ x_t 0 = \Phi_0, \end{cases} \tag{2.7}$$

or

$$\begin{cases} x'(t) = F(t, x_t), \quad t \geq t_0 \\ x_{t0} = \Phi_0(t - t0), \quad t_0 - \tau \leq t \leq t_0. \end{cases} \tag{2.8}$$

The following lemmas will be useful when discussing the properties of solutions.

Lemma 2.1.1. If x is continuous on $[t_0 - \tau, t_0 + \gamma]$ then x_t is a continuous function of t for $t \in [t_0, t_0 + \gamma]$.

Proof. Since x is continuous on $[t_0 - \tau, t_0 + \gamma]$ it is uniformly continuous. Thus for any $\varepsilon > 0$ there is a $\delta > 0$ such that $\| x(t) - x(s) \| < \varepsilon$ if $s, t \in [t_0 - \tau, t_0 + \gamma]$ and $| t - s | < \delta$. Consequently, for $s, t \in [t_0 - \tau, t_0 + \gamma]$ with $| t - s | < \delta$,there is $\| x(t + \theta) - x(s + \theta) \| < \varepsilon$ for all $\theta \in [-\tau, 0]$.

Lemma 2.1.2. [Driver (1977)] Let $F : J \times C_D \longrightarrow R^n$ be continuous and let $t_0 \in J$ and $\Phi_0 \in C_D$ be given. Then x is a solution of the initial value problem (2.8) on $[t_0 - \tau, \beta)$ if and only if $[t_0, \beta) \subset J, x \in C([t_0 - \tau, \beta), D)$ and x satisfies

$$\begin{cases} x_{t0} = \Phi_0 \\ x(t) = \Phi_0(0) + \int_{t_0}^{t} F(s, x_s)ds, \quad t_0 \leq t \leq \beta. \end{cases} \tag{2.9}$$

Definition 10

Let $F : J \times C_D \longrightarrow R^n$ and let $U \subset J \times C_D$. Then F is Lipschitz on U if there exists $K \geq 0$ such that

$$\| (t\varphi) - F(t, \Psi) \| \leq K \| \varphi - \Psi \|,$$

whenever (t, φ) and $(t, \Psi) \in U$.

Lemma 2.1.3. (Generalized Gronwall's inequality) Let c and k be given nonnegative continuous functions on an interval $J = [t_0, \beta)$ and let c be differentiable on J. Then if v : J → [0,∞) is continuous and

$$v(t) \leq c(t) + \int_{t_0}^{t} k(s)v(s)ds$$

Then

$$v(t) \leq c(t_0)e^{\int_{t_0}^{t} k(s)}ds + \int_{t_0}^{t} c'(s)e^{\int_{s}^{t} k(u)du}ds.$$

Proof. Let $R(t) = \int_{t_0}^{t} k(s)v(s)ds$. Then

$$R'(t) = k(t)v(t) \leq k(t)c(t) + k(t)\int_{t_0}^{t} k(s)v(s)ds$$

Thus $R'(t) - k(t)R(t) \leq k(t)c(t)$. Multiplying through by the integrating factor $e^{\int_{t_0}^{t} k(s)ds}$ yields

$$[e^{-\int_{t_0}^{t} k(s)ds}]^t \leq k(t)c(t)e^{-\int_{t_0}^{t} k(s)ds}.$$

Integrating from t_0 to t gives

$$e^{-\int_{t_0}^{t} k(s)ds} R(t) - R(t_0) \leq \int_{t_0}^{t} k(s)c(s)e^{-\int_{t_0}^{t} k(s)ds}.$$

Noting that $R(t_0) = 0$ and integrating by parts on the right hand side gives

$$e^{-\int_{t_0}^{t} k(s)ds} R(t) \leq c(t_0) - c(t)e^{-\int_{t_0}^{t} k(u)du} R(t) + \int_{t_0}^{t} c'(s)e^{-\int_{t_0}^{t} k(u)du} ds.$$

Thus

$$R(t) \leq -c(t) + c(t_0)e^{-\int_{t_0}^{t} k(s)ds} + \int_{t_0}^{t} c'(s)e^{-\int_{s}^{t} k(u)du} ds.$$

Using $v(t) \leq c(t) + R(t)$, we obtain the result.

Lemma 2.1.4.[Reid's Lemma, Driver (1977)] Let C be a given constant and k a given nonnegative continuous function on an interval J. Let $t_0 \in J$. Then if $v : J \longrightarrow [0, \infty)$ is continuous and

$$v(t) \leq C + \left| \int_{t_0}^{t} k(s)v(s)ds \right| \tag{2.10}$$

for all $t \in J$, it follows that

$$v(t) \leq Ce^{\left| \int_{t_0}^{t} k(s)v(s)ds \right|}$$

for all $t \in J$.

Proof. Suppose $t \geq t_0$ and $t \in J$. Then (2.10) becomes

$$v(t) \leq C + \int_{t_0}^{t} k(s)v(s)ds$$

or

$$k(t)v(t) - k(t)[C + \int_{t_0}^{t} k(s)v(s)ds] \leq 0.$$

Let $Q(t) = C + \int_{t_0}^{t} k(s)v(s)ds$, then $Q'(t) - k(t)Q(t) \leq 0$. Multiplying through by the integrating factor $e^{-\int_{t_0}^{t} k(s)ds}$ we obtain

$$\frac{d}{dt}\left[Q(t)e^{-\int_{t_0}^{t} k(s)ds}\right] \leq 0.$$

Integrating from t_0 to t and noting that $Q(t_0) = C$, yields

$$Q(t)e^{-\int_{t_0}^{t} k(s)ds} - C \leq 0.$$

or

$$Q(t) \leq Ce^{-\int_{t_0}^{t} k(s)ds}.$$

Substituting this into inequality (2.10) yields

$$v(t) \leq Q(t) \leq Ce^{-\int_{t_0}^{t} k(s)ds}.$$

The proof for $t < t_0$ is similar.

Theorem 2.1.2. (Uniqueness) Let $F : [t_0, \alpha) \times C_D \longrightarrow R^n$ be continuous and locally Lipschitz on its domain. Then, given any $\varphi_0 \in C_D$ and $\beta \in (t_0, \alpha]$, there is at most one solution of the initial value problem (2.8) on $[t_0 - \tau, \beta)$.

Proof. Suppose (for contradiction) that for some $\beta \in (t_0, \alpha]$ there are two solutions $x(t)$ and $y(t)$ mapping $[t_0 - \tau, \beta)$ into D with $x \neq y$. Let $t_1 = inf\{t \in (t_0, \beta) : x(t) \neq y(t)\}$. Then $t_0 < t_1 < \beta$ and $x(t) = y(t)$ for $t_0 - \tau \leq t \leq t_1$. Since $(t_1, x_1) \in [t_0, \beta) \times C_D$ and F is locally Lipschitz, there exist numbers $a > 0$ and $b > 0$ such that the set $U = [t_1, t_1 + a] \times \Psi \in C :\| \Psi - x_{t1} \| r \leq b$ is contained in $[t_0, \alpha) \times C_D$ and F is Lipschitz on U (with Lipschitz constant K).

By Lemma (2.1.1) there exists $\delta \in (0, a]$ such that $(t, x_t) \in U$ and $(t, y_t) \in U$ or $t_1 \leq t < t_1 + \delta$. Thus for $t_1 \leq t < t_1 + \delta$,

$$\| x - y \| = \left\| \int_{t_0}^{t} [F(s, x_s) - F(s, y_s)] d \right\|$$

$$\leq \int_{t_1}^{t} K \left\| x_s - y_s \right\|_{\tau} ds$$

Now since the right hand side is an increasing function of t and since $\| x(t) - y(t) \| = 0f$ or $t_1 - \tau \leq t \leq t_1$,

$$\| x - y \|_{\tau} \leq \int_{t_1}^{t} K \left\| x_s - y_s \right\|_{\tau} ds$$

for $t_1 \leq t < t_1 + \delta$. From this and the Generalized Gronwall's Lemma it follows that $x(t) = y(t)$ on $[t_1, t_1 + \delta)$ contradicting the definition of t_1.

Theorem 2.1.3. (Local Existence) Let $F : [t_0, \alpha) \times C_D \longrightarrow R^n$ be continuous and locally Liptschitz. Then, for each $\Phi_0 \in C_D$, the initial value problem (2.8) has a unique solution on $[t_0 - \tau, t_0 + \triangle)$ for some $\triangle > 0$.

Proof. Choose any $a > 0$ and $b > 0$ sufficiently small so that

$$U = [t_0, t_0 + a] \times \{\Psi \in C : \| \Psi - \Phi_0 \|_{\tau} \leq b\}$$

is a subset of $[t_0, \alpha) \times C_D$ and F is Lipschitz on U, with Lipschitz constant K. Define a continuous function χ^* on $[t_0 - \tau, t_0 + a] \longrightarrow R^n$ by

$$\chi^* = \begin{cases} \Phi_0(t - t_0), & t_0 - \tau \leq t \leq t_0 \\ \Phi_0(0), & t_0 < t \leq t_0 + a. \end{cases}$$

Then $F(t, \chi_t^*)$ depends continuously on t, and hence $\| F(t, \chi_t^*) \| \leq B_1$ on $[t_0, t_0 + a]$ for some constant B_1. Define $B = Kb + B_1$. Choose $a_1 \in (0, a]$ such that $\| \chi_t^* - \Phi_0 \|_{\tau} = \{\chi_t^* - \chi_{t_0}^* \|_{\tau} \leq b$ for $t_0 \leq t \leq t_0 + a_1$.

Choose $\triangle > 0$ such that $\triangle \leq min\{a_1, b/B\}$. Let S be the set of all continuous functions $\chi : [t_0 - \tau, t_0 + \triangle] \longrightarrow R^n$ such that $\chi(t) = \Phi_0(t - t_0)$ for $t_0 - \tau \leq t \leq t_0$ and $\| \chi(t) - \Phi_0(0) \| \leq b$ for $t_0 \leq t \leq t_0 + \triangle$. Note that if $\chi \in S$ and $t \in [t_0, t_0 + \triangle]$ then $\| \chi_t - \chi_t^* \|_\tau \leq b$ so that

$$\begin{aligned}
\| F(t, \chi_t) \| &\leq \| F(t, \chi_t) - F(t, \chi_t^*) \| + \| F(t, \chi_t^*) \| \\
&\leq K \| \chi_t - \chi_t^* \|_\tau + B_1 \\
&\leq B.
\end{aligned}$$

For each $\chi \in S$ define a function $T\chi$ on $[t_0 - \tau, t_0 + \triangle]$ by

$$(T\chi)(t) = \begin{cases} \Phi_0(t - t_0), & t_0 - \tau \leq t \leq t_0 \\ \Phi_0(0) + \int_{t_0}^{t} F(s, \chi_s) ds, & t_0 \leq t \leq t_0 + \triangle. \end{cases}$$

Then $T\chi$ is continuous and, since $\| F(s, \chi_s) \| < B$, $|(T\chi)(t) - \Phi_0(0)| \leq B \triangle \leq b$ for $t_0 \leq t \leq t_0 + \triangle$. Thus $T\chi \in S$, that is, $T : S \longrightarrow S$. Choose $x_{(0)} \in S$ and construct the successive approximations $x_{(1)} = Tx_{(0)}, x_{(2)} = Tx(1), \ldots$. Note that for each l, $x_{(l)}(t) = \Phi_0(t - t_0)$ on $[t_0 - \tau, t_0]$. Now prove that the sequence $x_l(t)$ converges. For each $l = 0, 1, 2, \ldots$ when $t_0 \leq t \leq t_0 + \triangle$

$$\begin{aligned}
\| x_{(l+2)}(t) - x_{(l+1)}(t) \| &= \left\| \int_{t_0}^{t} [F(s, x_{(l+1)s}) - F(s, x_{(l)s})] ds \right\| \\
&\leq \int_{t_0}^{t} K \left\| x_{(l+1)s} - x_{(l)s} \right\|_\tau ds
\end{aligned}$$

Note that $\| x_{(1)}(t) - x_{(0)(t)} \| \leq 2b$ on $[t_0 - \tau, t_0 + \triangle]$. Thus $\| x_{(1)t} - x_{(0)t} \|$

$\tau \leq 2b$ on $[t_0, t_0 + \triangle]$ and

$$\begin{aligned} \| x_2(t) - x_1(t) \| &\leq \int_{t_0}^{t} K \Big\| x_{(1)s} - x_{(0)s} \Big\| ds \\ &\leq 2bK(t - t_0) \end{aligned}$$

on $[t_0, t_0 + \triangle]$, which further implies that $\| x_{(2)t} - x_{(1)t} \|_\tau \leq 2bK(t - t_0)$ on $[t_0, t_0 + \triangle]$. This leads to

$$\begin{aligned} \| x_3(t) - x_2(t) \| &\leq \int_{t_0}^{t} K \Big\| x_{(2)s} - x_{(1)s} \Big\| ds \\ &\leq 2b \frac{K^2 (t - t_0)^2}{2} \end{aligned}$$

Using induction it can be shown that

$$\| x_{(l+1)}(t) - x_{(t)} \| \leq 2b \frac{K^l (t - t_0)^l}{l!}$$

on $[t_0, t_0 + \triangle]$. This together with $x_{(l+1)}(t) = x_{(l)}(t)$ on $[t_0 - \tau, t_0]$ gives

$$\| x_{(l+1)}(t) - x_{(t)} \| \leq 2b \frac{K^l \triangle^l}{l!}$$

on $[t_0 - \tau, t_0 + \triangle]$. Now the series

$$x_{(0)}(t) + \sum_{p=0}^{\infty} [x_{(p+1)}(t) - x_{(p)}(t)]$$

converges uniformly on $[t_0 - \tau, t_0 + \triangle]$ by the Weierstrass M$-$Test, but

$$x_{(l)}(t) = x_{(0)}(t) + \sum_{p=0}^{l-1} [x_{(p+1)}(t) - x_{(p)}(t)],$$

and so the sequence $x_{(l)}(t)$ converges uniformly on $[t_0 - \tau, t_0 + \triangle]$.

Let $x(t) = \lim_{l \longrightarrow \infty} x_{(l)}(t)$ for $t_0 - \tau \leq t \leq t_0 \Delta$. Clearly, $x(t)$ is continuous on $[t_0 - \tau, t_0 + \triangle]$ and $x_{t_0} = \Phi_0$. Further

$$\| x(t) - x_{(l)}(t) \| \leq 2b \sum_{p=l}^{\infty} \frac{(K \triangle)^p}{p!}$$

for $t_0 - \tau \leq t \leq t_0 \triangle$ and $\| x_t - x_{(l)t} \|_\tau \leq 2b \sum_{p=l}^{\infty} \frac{(K \triangle)^P}{p!}$ for $t_0 \leq t \leq t_0 \triangle$. Thus, for $t_0 \leq t \leq t_0 \triangle$,

$$\begin{aligned} \| x(t) - x_{(l)}(t) \| &\leq \| x(t) - x_{(l)}(t) \| + \| x_{(l)}(t) - \Phi_0(0) \| \\ &\leq 2b \sum_{p=l}^{\infty} \frac{(K \triangle)^p}{p!} + b \\ &\leq b, \end{aligned}$$

and $x_t \in C_D$. Finally for $t \in [t_0, t_0 + \triangle]$

$$\begin{aligned} \| x(t) - \Phi_0(0) - \int_{t_0}^{t} F(s, x_s) ds \| &\leq \| x(t) - x_{(l)}(t) \| \\ &+ \int_{t_0}^{t} \| F(s, x_{(l-1)_s}) - F(s, x_s) \| ds \\ &\leq 2b \sum_{p=l}^{\infty} \frac{(K \triangle)^p}{p!} + K \triangle\, 2b \sum_{p=l-1}^{\infty} \frac{(K \triangle)^p}{p!} \end{aligned}$$

Taking the limit as $l \longrightarrow \infty$ of this inequality then gives

$$\| x(t) - \Phi_0(0) - \int_{t_0}^{t} F(s, x_s) ds \| = 0$$

that is, $x(t)$ satisfies (2.9). Uniqueness follows from Theorem .

Theorem 2.1.4. (Continuous Dependence on Initial Conditions) Let $F : [t_0, \alpha] \times C_D \longrightarrow R^n$ be continuous and globally Lipschitz constant K. Let $\Phi_0 \in C_D$ and $\Phi_0^* \in C_D$ be given and let x and x^* be unique solutions of (2.6) with $x_{t_0} = \Phi_0$ and $x^*_{t_0}$, respectively. If x and x^* are both valid on $[t_0 - \tau, \beta)$, the

$$\| x(t) - x^*(t) \| \leq \| \Phi_0 - \Phi_0^* \|_\tau \, e^{K(t-t_0)}$$

for $t_0 \leq t < \beta$.

Proof. Since x and x^* are solutions of the given initial value problems, x satisfies (2.9) and x^* satisfies

$$\begin{cases} x^*(t_0) = \Phi_0^* \\ x^*(t) = \Phi_0(0) + \int_{t_0}^{t} F(s, x_s^*)ds, \quad t_0 \leq t < \beta. \end{cases}$$

Thus for $t_0 \leq t < \beta$

$$\begin{aligned} \| x(t) - x^*(t) \| &= \left\| \Phi_0(0) - \Phi_0^*(0) + \int_{t_0}^{t} [F(s, x_s) - F(s, x_s^*)]ds \right\| \\ &\leq \left\| \Phi_0(0) - \Phi_0^*(0) \right\| + \int_{t_0}^{t} \left\| [F(s, x_s) - F(s, x_s^*)] \right\| ds \\ &\leq \left\| \Phi_0(0) - \Phi_0^*(0) \right\|_\tau + \int_{t_0}^{t} K \left\| [x_s - x_s^*] \right\|_\tau ds \end{aligned}$$

for $t_0 \leq t \leq \beta$. Since $\| x(t) - x^*(t) \| \leq \| \Phi_0 - \Phi_0^* \|_\tau$ on $[t_0 - \tau, t_0]$, it follows that

$$\| x_t - x_t^* \| \leq \left\| \Phi_0(0) - \Phi_0^*(0) \right\|_\tau + \int_{t_0}^{t} K \left\| [x_s - x_s^*] \right\|_\tau ds$$

for $t_0 \leq t \leq \beta$.

Applying the generalized Gronwall's Lemma with $C =\| \Phi_0 - \Phi_0^* \|_\tau$ and $k(s) = K$ yields

$$\begin{aligned} \| x(t) - x^*(t) \| &\leq \| x_t - x_t^* \|_\tau \\ &\leq \| \Phi_0 - \Phi_0^* \|_\tau e^{K(t-t_0)}, \end{aligned}$$

for $t_0 \leq t \leq \beta$.

Theorem 2.1.5. (Continuous Dependence on F) Let $F, F^* : [t_0, \alpha) \times C_D \longrightarrow R^n$ be continuous, and let F be globally Lipschitz with Lipschitz constant K. Given $\Phi_0, \Phi^*0 \in C_D$, let $x(t)$ and $x^*(t)$ be the unique solutions of (2.8) and

$$\begin{cases} (x^*)'(t) = F(t, x_t^*), & t \leq t_0 \\ x^*(t) = \Phi_0^*(t - t_0), & t_0 - \tau \leq t \leq t_0. \end{cases} \tag{2.11}$$

respectively. If x and x^* are both valid on $[t_0 - \tau, \beta)$ and $\| F(t, \Psi) - F^*(t, \Psi) \| \leq \mu$ for all $t \in [t_0, \alpha), \Psi \in C_D$ then

$$\| x(t) - x^*(t) \| \leq \| \Phi_0 - \Phi_0^* \|_\tau e^{K(t-t_0)} + \frac{\mu}{K}[e^{K(t-t_0)} - 1],$$

for $t_0 \leq t < \beta$.

Proof. $x(t)$ and $x^*(t)$ must satisfy the integral equations (2.9) and

$$\begin{cases} x_{(t_0)}^* = \Phi_0^* \\ x^*(t) = \Phi_0^*(0) + \int_{t_0}^t F^*(s, x_s^*) ds, & t_0 \leq t < \beta. \end{cases}$$

Thus on $[t_0 - \tau, t_0]$

$$\begin{aligned} \| x(t) - x^*(t) \| &\leq \| \Phi_0(t - t_0) - \Phi_0^*(t - t_0) \| \\ &\leq \| \Phi_0 - \Phi_0^* \|_\tau \end{aligned}$$

on $[t_0, \beta)$

$$\begin{aligned} \| x(t) - x^*(t) \| &\leq \left\| \Phi_0(0) - \Phi_0^*(0) \right\| + \int_{t_0}^{t} \left\| [F(s, x_s) - F^*(s, x_s^*)] \right\| ds \\ &\leq \left\| \Phi_0 - \Phi_0^* \right\|_\tau + \int_{t_0}^{t} \left\| [F(s, x_s) - F^*(s, x_s^*)] \right\| ds \\ &\quad + \int_{t_0}^{t} \left\| [F(s, x_s^*) - F^*(s, x_s^*)] \right\| ds \\ &\leq \left\| \Phi_0 - \Phi_0^* \right\|_\tau + \int_{t_0}^{t} K \left\| [x_s - x_s^*] \right\|_\tau ds + \int_{t_0}^{t} \mu ds. \end{aligned}$$

Since the right hand side of this last inequality is an increasing function of t, it follows that

$$\| x(t) - x^*(t) \|_\tau \leq \left\| \Phi_0 - \Phi_0^* \right\|_\tau + \mu(t - t_0) + \int_{t_0}^{t} K \left\| [x_s - x_s^*] \right\|_\tau ds,$$

for $t_0 \leq t < \beta$.

Applying the generalized Gronwall's Lemma with $c(t) = \| \Phi_0 - \Phi^* 0 \|_\tau$ $+\mu(t - t_0)$ and $k(t) = K$ yields the result.

For $y \in C^{n-1}([-\tau, 0], R)$ define the function y_t' on $[-\tau, 0]$ as follows

$$\begin{aligned} y_t'(\theta) &= y'(t-\theta) \\ &= \lim_{h \longrightarrow 0^+} \frac{y(t+\theta+h) - y(t+\theta)}{h} \\ &= \lim_{h \longrightarrow 0^+} \frac{y(\theta+h) - y(t+\theta)}{h} - \tau \leq \theta \leq 0. \end{aligned}$$

Now define the functions $y_t'', y_t''', ..., y_t^{(n-1)}$ on $[-\tau, 0]$ in a similar manner. Then for $J \subset R$ and $G : J \times [C([-\tau, 0], R]^n$ consider the scalar nth order delay differential equation

$$y^{(n)}(t) = G(t, y_t, y_t', y_t'', ..., y_t^{(n-1)}) \tag{2.12}$$

with initial conditions

$$\begin{cases} y_{(t_0)} = \Phi_0 \\ y'_{(t_0)} = \Phi_1 \\ \\ y^{(n-1)}_{(t_0)} = \Phi_{n-1} \end{cases} \tag{2.13}$$

where $t_0 \in J$and $\Phi_j \in C([-\tau, 0], R)$. Solutions of (2.12) will be $(n-1)$ times differentiable functions. The initial value problem (2.12)-(2.13) can be reduced to a delay differential equation on $J \times C$ in the usual way, that is, by defining $x \in R^n$

$$x = [y, y', y'', y''', ..., y^{(n-1)}]^T.$$

Fundamental Matrix

In this section Some definitions connected to the fundamental matrix solution are provided. These definitions are found in Chicone (1999).

Definition 11

An $n \times n$ matrix function $t \to \Phi(t)$, defined on an open interval J, is called a matrix solution of the homogeneous linear system (4.1) if each of its columns is a (vector) solution.

Definition 12

A set of n solutions of the homogeneous linear differential equation (4.1), all defined on the same open interval J, is called a fundamental set of solutions on J if the solutions are linearly independent functions on J.

Definition 13

A matrix solution is called a fundamental matrix solution if its columns form a fundamental set of solutions. In addition, a fundamental matrix solution $t \to \Phi(t)$ is called the principal fundamental matrix solution at $t_0 \in J$ if $\Phi(t_0) = I$, where I denotes the $n \times n$ identity matrix.

Definition 14

The state transition matrix for the homogeneous linear system (4.1) on the open interval J is the family of fundamental matrix solutions $t \to \Phi(t, r)$ parametrized by $r \in J$ such that $\Phi(r, r) = I$.

Proposition 1.

If $t \to \Phi(t)$ is a fundamental matrix solution for the system (4.1) on J, then $\Phi(t, r) := \Phi(t)\Phi^{-1}(r)$ is the state transition matrix. Also, the state transition matrix satisfies the Chapman-Kolmogorov identities

$$\Phi(r, r) = I, \Phi(t, s)\Phi(s, r) = \Phi(t, r),$$

and the identities

$$\Phi(t,s)^{-1} = \Phi(s,t), \quad \frac{\delta\Phi(t,s)}{\delta s} = -\Phi(t,s)A(s).$$

Chapter Summary

In this chapter, relevant and related literature was reviewed on stability behaviour of solutions of differential equations by fixed point theorems. This was drawn from the work of other researchers which are published in journals and scholarly works.

CHAPTER THREE

METHODOLOGY

Introduction

This chapter deals with the main tool used in achieving the objectives in the thesis.

Fixed-Point Theory

Many different kinds of problems can be solved by means of fixed point theory. Generally, to solve a problem with fixed point theory is to find:

1. a set S consisting of points which would be acceptable solutions.
2. a mapping $P : S \to S$ with the property that a fixed point solves the problem.
3. a fixed point theorem stating that this mapping on this set will have a fixed point.

Formulation of Fixed Point Problems in Differential Equations

This section is an elementary introduction to the formulations of fixed point problems in differential equations.

Consider an ordinary differential equation

$$x'(t) = g(t, x(t)), \tag{3.1}$$

where $g : [0, \infty) \times R^n \longrightarrow R^n$ is contiuous. Perhaps the most basic problem concerning equation (3.1) is to find a solution through a given point $(t_0, x_0) \in [0, \infty) \times R^n$ defined on some interval $[t_0, t_0 + \gamma]$ and satisfying equation (3.1) on that interval.

For this problem, a guess would be that the set S should consist of differentiable functions $\phi : [t_0, t_0 + \gamma] \longrightarrow R^n$ with $\phi(t_0) = x_0$.

Next, the simplest way to find a mapping is to formally integrate equation (3.1) and obtain

$$x(t) = x_0 + \int_{t_0}^{t} g(s, x(s))ds,$$

so that the mapping P on S is defined by

$$(P\phi)x(t) = x_0 + \int_{t_0}^{t} g(s, \phi(s))ds.$$

A fixed point will certainly satisfy the equation. Since the mapping is given by an integral, the second approximation to S is the set of continuous functions; differentiability will be automatic. There is now a vast array of fixed point theorems which will yield a fixed point of that mapping and satisfy our initial value problem. The contraction mapping principle is used for this problem. In the given example, it is easiest to complete the solution by asking that g satisfy a global Lipschitz condition of the form

$$|g(t, x) - g(t, y)| \leq K|x - y|,$$

for $t \geq t_0$, $K > 0$, and for all $x, y \in R^n$, where $| \cdot |$ is any norm on R^n. This will allow us to give a contraction mapping argument. For any fixed interval $[t_0, t_0 + \gamma]$, the set S with the supremum metric is a complete metric space and $P : S \longrightarrow S$. Checking the contraction requirement, gives

$$\begin{aligned} &| (P\phi_1)(t) - (P\phi_2)(t) | \\ \leq &\int_{t_0}^{t} K|\phi_1(s) - \phi_2(s)|ds \\ \leq &K\gamma\|\phi_1 - \phi_2\|. \end{aligned}$$

Definition 15 (Metric Space)

A pair (S, ρ) is a metric space if S is a set and $\rho : \mathbb{S} \times \mathbb{S} \rightarrow [0, \infty)$ such that when y, z, and u are in S then

1. $\rho(y, z) \geq 0, \rho(y, y) = 0$ and $\rho(y, z) = 0$ implies $y = z$,
2. $\rho(y, z) = \rho(z, y)$, and
3. $\rho(y, z) \leq \rho(y, u) + \rho(u, z)$.

Definition 16 (Fixed Point)

A fixed point of a function $T : X \rightarrow X$ is a point $x \in X$ such that $Tx = x$.

Definition 17 (Vector Space)

A vector space $(V, +, .)$ is a normed vector space if for each $x, y \in V$ there is a nonnegative real number $\|x\|$ called the norm of x, such that

1. $\|x\| = 0$ if and only if $x = 0$,
2. $\|\alpha x\| = |\alpha|\|x\|$ for each $\alpha \in \mathbb{R}$
3. $\|x + y\| \leq \|x\| + \|y\|$.

Definition 18 (Banach Space)

A Banach space is a complete normed space.

Next the contraction mapping principle which generally goes under the name Banach-Caccioppoli Theorem, or Banach's (1932) Contraction Mapping Principle is defined. A proof can be found in many literature such as Smart (1974) or Burton (1985). It gains more respect every day. The real power of the result lies in its application with cleverly chosen metrics.

Definition 19 (Contraction Mapping Principle)

Let (X, d) be a nonempty complete metric space and $T : X \rightarrow X$ is a contraction mapping, if there exist a constant ρ with $0 \leq \rho < 1$ such that $d(T(x), T(y)) \leq \rho d(x, y)$ for all x, $y \in X$, then T has a unique fixed point x such that $T(x) = x$.

Chapter Summary

This chapter presented the method that was used in conducting the research. It focused on the fixed point theorem specifically the Banach fixed point theorem.

CHAPTER FOUR

RESULTS AND DISCUSSION

Introduction

This chapter covers the results of the study. In particular, the results of stability of solutions of a system of first order ordinary differential equations with finite delay are presented and discussed. The results are presented based on the objectives of the study.

Preliminary Results

Let $\Phi(t)$ denote the fundamental matrix solution of the system

$$x' = G(t)x(t), \tag{4.1}$$

where G(t) is a nonsingular $n \times n$ matrix with continuous real-valued functions as its elements.

In this chapter the system of ordinary differential equations

$$\frac{d}{dt}x(t) = A(t)x(t - \tau) \tag{4.2}$$

where $A(t)$ is an $n \times n$ non-singular matrix and τ is a positive constant is considered.

Let $\psi : [-\tau, \infty) \longmapsto R^n$ denote the initial function for equation (4.2). For $x \in R^n$ define

$$\| x \| = \sup_{t \in [-\tau, \infty)} | x(t) |$$

In Lemma 4.1.1, an equivalent form of Equation (4.2) is provided which will be used extensively in the rest of the work.

Lemma 4.1.1. Let $G(t)$ be an $n \times n$ nonsingular continuous matrix. Then equation (4.2) is equivalent to the equation

$$\frac{d}{dt}x(t) = G(t)x(t) - \frac{d}{dt}\int_{t-\tau}^{t} G(s)x(s)ds + \Big[A(t) - G(t-\tau)\Big]x(t-\tau) \tag{4.3}$$

Proof. Consider the integral form in equation (4.3) gives

$$\frac{d}{dt}\int_{t-\tau}^{t} G(s)x(s)ds = G(t)x(t) - G(t-\tau)x(t-\tau).$$

Thus equation (4.3) becomes,

$$\begin{aligned}
\frac{d}{dt}x(t) &= G(t)x(t) - \Big[G(t)x(t) - G(t-\tau)x(t-\tau)\Big] \\
&\quad + A(t)x(t-\tau) - G(t-\tau)x(t-\tau) \\
&= G(t)x(t) - G(t)x(t) + G(t-\tau)x(t-\tau) \\
&\quad + A(t)x(t-\tau) - G(t-\tau)x(t-\tau) \\
&= A(t)x(t-\tau) + G(t)x(t) \\
&\quad - G(t)x(t) + G(t-\tau)x(t-\tau) - G(t-\tau)x(t-\tau) \\
&= A(t)x(t-\tau)
\end{aligned}$$

This completes the proof.

In Lemma 4.1.2 an equivalent integral equation to equation (4.3) is obtained. This result will be used to define a mapping.

Lemma 4.1.2. Suppose the hypothesis of Lemma 4.1.1 hold. Then $x(t)$ is a solution of equation (4.3) if and only if

$$\begin{aligned} x(t) &= \Phi(t,t_0)\Big[x(t_0) + \int_{t_0-\tau}^{t_0} G(u)x(u)du\Big] - \int_{t-\tau}^{t} G(u)x(u)du \\ &+ \int_{t_0}^{t} \Phi(t,u)\Big(\Big[A(u) - G(u-\tau)\Big]x(u-\tau) \\ &- G(u)\int_{u-\tau}^{u} G(s)x(s)ds\Big)du \end{aligned}$$

Proof. Let x be the solution of Equation (4.3) and $\Phi(t)$ be a fundamental matrix solution of Equation (4.1). Rewrite Equation (4.3) as

$$\begin{aligned} \frac{d}{dt}\Big[x(t) + \int_{t-\tau}^{t} G(s)x(s)ds\Big] &= G(t)x(t) \\ &+ \Big[A(t) - G(t-\tau)\Big]x(t-\tau). \quad (4.4) \end{aligned}$$

Define a new function z by $z(t) = \Phi^{-1}(t)\Big[x(t) + \int_{t-\tau}^{t} G(s)x(s)ds\Big]$. This implies that,

$$\begin{aligned} \frac{d}{dt}z(t) &= \frac{d}{dt}\Phi^{-1}(t)\Big[x(t) + \int_{t-\tau}^{t} G(s)x(s)ds\Big] \\ &+ \Phi^{-1}(t)\frac{d}{dt}\Big[x(t) + \int_{t-\tau}^{t} G(s)x(s)ds\Big]. \end{aligned}$$

Since, $\Phi(t)\Phi^{-1}(t) = I$,

$$\frac{d}{dt}\Big(\Phi(t)\Phi^{-1}(t)\Big) = 0$$

This implies that,

$$\frac{d}{dt}\Big(\Phi(t)\Big)\Phi^{-1}(t) + \Phi(t)\frac{d}{dt}\Big(\Phi^{-1}(t)\Big) = 0$$

Which gives

$$\Big(G(t)\Phi(t)\Big)\Phi^{-1}(t) + \Phi(t)\frac{d}{dt}\Big(\Phi^{-1}(t)\Big) = 0$$

Thus

$$G(t) + \Phi(t)\frac{d}{dt}\Big(\Phi^{-1}(t)\Big) = 0 \tag{4.5}$$

Multiplying through equation (4.5) by Φ^{-1} gives,

$$\frac{d}{dt}\Phi^{-1}(t) = -\Phi^{-1}(t)G(t). \tag{4.6}$$

Hence,

$$\begin{aligned}\frac{d}{dt}z(t) &= -\Phi^{-1}(t)G(t)\Big[x(t) + \int_{t-\tau}^{t} G(s)x(s)ds\Big] \\ &\quad + \Phi^{-1}(t)\frac{d}{dt}\Big[x(t) + \int_{t-\tau}^{t} G(s)x(s)ds\Big]\end{aligned}$$

Which implies that

$$\begin{aligned}&\frac{d}{dt}z(t) + \Phi^{-1}(t)G(t)\Big[x(t) + \int_{t-\tau}^{t} G(s)x(s)ds\Big] \\ &= \Phi^{-1}(t)\frac{d}{dt}\Big[x(t) + \int_{t-\tau}^{t} G(s)x(s)ds\Big]. \end{aligned} \tag{4.7}$$

Multiplying through equation (4.7) by $\Phi(t)$ yields,

$$\Phi(t)\frac{d}{dt}z(t) + G(t)\Big[x(t) + \int_{t-\tau}^{t} G(s)x(s)ds\Big]$$

$$= \frac{d}{dt}\Big[x(t) + \int_{t-\tau}^{t} G(s)x(s)ds\Big].$$

Therefore,

$$\begin{aligned}\frac{d}{dt}\Big[x(t) + \int_{t-\tau}^{t} G(s)x(s)ds\Big] \\ = G(t)\Big[x(t) + \int_{t-\tau}^{t} G(s)x(s)ds\Big] \\ + \Phi(t)\frac{d}{dt}z(t).\end{aligned} \tag{4.8}$$

From equtions (4.7) and (4.8),

$$\begin{aligned}G(t)x(t) + \Big[A(t) - G(t-\tau)\Big]x(t-\tau) \\ = G(t)\Big[x(t) + \int_{t-\tau}^{t} G(s)x(s)ds\Big] \\ + \Phi(t)\frac{d}{dt}z(t)\end{aligned}$$

This implies that

$$\begin{aligned}\Phi(t)\frac{d}{dt}z(t) &= \Big[A(t) - G(t-\tau)\Big]x(t-\tau) \\ &\quad - G(t)\int_{t-\tau}^{t} G(s)x(s)ds\end{aligned}$$

This gives,

$$\begin{aligned}\frac{d}{dt}z(t) &= \Phi^{-1}(t)\Big(\Big[A(t)-G(t-\tau)\Big]x(t-\tau) \\ &- G(t)\int_{t-\tau}^{t} G(s)x(s)\Big)ds \qquad (4.9)\end{aligned}$$

An integration of Equation (4.9) from t_0 to t yields

$$\begin{aligned}\int_{t_0}^{t}\frac{d}{ds}z(s)ds &= \int_{t_0}^{t}\Phi^{-1}(u)\Big(\Big[A(u)-G(u-\tau)\Big]x(u-\tau) \\ &- G(u)\int_{u-\tau}^{u} G(s)x(s)ds\Big)du\end{aligned}$$

This implies that

$$\begin{aligned}z(t)-z(t_0) &= \int_{t_0}^{t}\Phi^{-1}(u)\Big(\Big[A(u)-G(u-\tau)\Big]x(u-\tau) \\ &- G(u)\int_{u-\tau}^{u} G(s)x(s)ds\Big)du\end{aligned}$$

Thus,

$$\begin{aligned}&\Phi^{-1}(t)\Big[x(t)+\int_{t-\tau}^{t}G(u)x(u)du\Big] \\ &= \Phi^{-1}(t_0)\Big[x(t_0)+\int_{t_0-\tau}^{t_0}G(u)x(u)du\Big] \\ &+ \int_{t_0}^{t}\Phi^{-1}(u)\Big(\Big[A(u)-G(u-\tau)\Big]x(u-\tau) \\ &- G(u)\int_{u-\tau}^{u}G(s)x(s)ds\Big)du \qquad (4.10)\end{aligned}$$

Multiplying equation (4.10) by $\Phi(t)$ gives

$$\begin{aligned} x(t) &= \Phi(t)\Phi^{-1}(t_0)\Big[x(t_0) + \int_{t_0-\tau}^{t_0} G(u)x(u)du\Big] - \int_{t-\tau}^{t} G(u)x(u)du \\ &+ \int_{t_0}^{t} \Phi(t)\Phi^{-1}(u)\Big(\Big[A(u) - G(u-\tau)\Big]x(u-\tau) \\ &- G(u)\int_{u-\tau}^{u} G(s)x(s)ds\Big)du \end{aligned}$$

Therefore,

$$\begin{aligned} x(t) &= \Phi(t,t_0)\Big[x(t_0) + \int_{t_0-\tau}^{t_0} G(u)x(u)du\Big] \\ &- \int_{t-\tau}^{t} G(u)x(u)du \\ &+ \int_{t_0}^{t} \Phi(t,u)\Big(\Big[A(u) - G(u-\tau)\Big]x(u-\tau) \\ &- G(u)\int_{u-\tau}^{u} G(s)x(s)ds\Big)du \end{aligned}$$

Conversely, suppose that

$$\begin{aligned} x(t) &= \Phi(t,t_0)\Big[x(t_0) + \int_{t_0-\tau}^{t_0} G(u)x(u)du\Big] \\ &- \int_{t-\tau}^{t} G(u)x(u)du \\ &+ \int_{t_0}^{t} \Phi(t,u)\Big(\Big[A(u) - G(u-\tau)\Big]x(u-\tau) \\ &- G(u)\int_{u-\tau}^{u} G(s)x(s)ds\Big)du \end{aligned}$$

Replacing $\Phi(t,t_0)$ with $\Phi(t)\Phi^{-1}(t_0)$ and $\Phi(t,u)$ with $\Phi(t)\Phi^{-1}(u)$ gives

$$x(t) + \int_{t-\tau}^{t} G(u)x(u)du$$

$$= \Phi(t)\Phi^{-1}(t_0)\Big[x(t_0) + \int_{t_0-\tau}^{t_0} G(u)x(u)du\Big]$$

$$+ \int_{t_0}^{t} \Phi(t)\Phi^{-1}(u)\Big(\Big[A(u) - G(u-\tau)\Big]x(u-\tau)$$

$$- G(u)\int_{u-\tau}^{u} G(s)x(s)ds\Big)du. \tag{4.11}$$

Multiplying through equation (4.11) by $\Phi^{-1}(t)$ yields

$$\Phi^{-1}(t)\Big[x(t) + \int_{t-\tau}^{t} G(u)x(u)du\Big]$$

$$= \Phi^{-1}(t_0)\Big[x(t_0) + \int_{t_0-\tau}^{t_0} G(u)x(u)du\Big]$$

$$+ \int_{t_0}^{t} \Phi^{-1}(u)\Big(\Big[A(u) - G(u-\tau)\Big]x(u-\tau)$$

$$- G(u)\int_{u-\tau}^{u} G(s)x(s)ds\Big)du. \tag{4.12}$$

Differentiating equation (4.12) with respect to t, gives

$$\frac{d}{dt}\Phi^{-1}(t)\Big[x(t)+\int_{t-\tau}^{t}G(u)x(u)du\Big]$$

$$=\frac{d}{dt}\Phi^{-1}(t_0)\Big[x(t_0)+\int_{t_0-\tau}^{t_0}G(u)x(u)du\Big]$$

$$+\frac{d}{dt}\Big[\int_{t_0}^{t}\Phi^{-1}(u)\Big(\Big[A(u)-G(u-\tau)\Big]x(u-\tau)$$

$$-G(u)\int_{u-\tau}^{u}G(s)x(s)ds\Big)du\Big]$$

Thus,

$$\frac{d}{dt}\Phi^{-1}(t)\Big[x(t)+\int_{t-\tau}^{t}G(u)x(u)du\Big]$$

$$=\frac{d}{dt}\Big[\int_{t_0}^{t}\Phi^{-1}(u)\Big(\Big[A(u)-G(u-\tau)\Big]x(u-\tau)$$

$$-G(u)\int_{u-\tau}^{u}G(s)x(s)ds\Big)du\Big].$$

Hence,

$$\frac{d}{dt}\Phi^{-1}(t)\Big[x(t) + \int_{t-\tau}^{t} G(s)x(s)ds\Big]$$

$$+ \Phi^{-1}(t)\frac{d}{dt}\Big[x(t) + \int_{t-\tau}^{t} G(s)x(s)ds\Big]$$

$$= \Phi^{-1}(t)\Big(\Big[A(t) - G(t-\tau)\Big]x(t-\tau)$$

$$- G(t)\int_{t-\tau}^{t} G(s)x(s)\Big)ds$$

Applying equation (4.6) gives

$$-\Phi^{-1}(t)G(t)\Big[x(t) + \int_{t-\tau}^{t} G(s)x(s)ds\Big]$$

$$+ \Phi^{-1}(t)\frac{d}{dt}\Big[x(t) + \int_{t-\tau}^{t} G(s)x(s)ds\Big]$$

$$= \Phi^{-1}(t)\Big(\Big[A(t) - G(t-\tau)\Big]x(t-\tau)$$

$$- G(t)\int_{t-\tau}^{t} G(s)x(s)\Big)ds \qquad (4.13)$$

Multiplying through equation (4.13) by $\Phi^{-1}(t)$ gives

$$-G(t)\Big[x(t)+\int_{t-\tau}^{t}G(s)x(s)ds\Big]+\frac{d}{dt}\Big[x(t)+\int_{t-\tau}^{t}G(s)x(s)ds\Big]$$

$$=\Big[A(t)-G(t-\tau)\Big]x(t-\tau)-G(t)\int_{t-\tau}^{t}G(s)x(s)ds$$

$$-G(t)x(t)-G(t)\int_{t-\tau}^{t}G(s)x(s)ds+\frac{d}{dt}\Big[x(t)+\int_{t-\tau}^{t}G(s)x(s)ds\Big]$$

$$=\Big[A(t)-G(t-\tau)\Big]x(t-\tau)-G(t)\int_{t-\tau}^{t}G(s)x(s)ds$$

Thus,

$$-G(t)x(t)+\frac{d}{dt}\Big[x(t)+\int_{t-\tau}^{t}G(s)x(s)ds\Big]$$

$$=\Big[A(t)-G(t-\tau)\Big]x(t-\tau)$$

$$\frac{d}{dt}\Big[x(t)+\int_{t-\tau}^{t}G(s)x(s)ds\Big]$$

$$=G(t)x(t)+\Big[A(t)-G(t-\tau)\Big]x(t-\tau)$$

Hence,

$$\frac{d}{dt}x(t) + \frac{d}{dt}\int_{t-\tau}^{t} G(s)x(s)ds$$

$$= G(t)x(t) + \Big[A(t) - G(t-\tau)\Big]x(t-\tau)$$

$$\begin{aligned}\frac{d}{dt}x(t) &= G(t)x(t) - \frac{d}{dt}\int_{t-\tau}^{t} G(s)x(s)ds \\ &+ \Big[A(t) - G(t-\tau)\Big]x(t-\tau)\end{aligned}$$

This completes the proof.

Main Results

In Theorem 4.2.1 the stability results for the zero solution of equation (4.2) is given.

Theorem 4.2.1. Suppose the hypotheses of Lemma 4.1.1 and Lemma 4.1.2 hold. If $\alpha \in [0, 1)$ such that

$$\tau \mid G \mid + \int_{t_0}^{t} |\Phi|\Big[|A| + \mid G \mid + \tau \mid G \mid^2\Big]du \leq \alpha, \quad t \geq t_0 \tag{4.14}$$

then the zero solution of Equation (4.2) is stable.

Proof. Let $\varepsilon > 0$ be given. Choose $\delta > 0$ such that

$$|\Phi|\delta(1 + \tau \mid G \mid) + \alpha\varepsilon \leq \varepsilon. \tag{4.15}$$

Define $S = \{\varphi : R \to R^n, \varphi(t) = \psi(t)$ if $t \in [-\tau, t_0]$ and for $t \geq t_0$, $\|\varphi\| \leq \varepsilon\}$. Then $(S, \|\cdot\|)$ is a complete metric space, where $\|\cdot\|$ is supremum norm.

Define the mapping $H : S \to S$ by $(H\varphi)(t) = \psi(t)$ for $t \in [-\tau, t_0]$ and for $t \geq t_0$.

$$
\begin{aligned}
(H\varphi)(t) \quad &= \quad \Phi(t, t_0)\Big[\psi(t_0) + \int_{t_0-\tau}^{t_0} G(u)\psi(u)du\Big] \\
&\quad - \int_{t-\tau}^{t} G(u)\varphi(u)du \\
&\quad + \int_{t_0}^{t} \Phi(t, u)\Big(\Big[A(u) - G(u-\tau)\Big]\varphi(u-\tau) \\
&\quad - G(u)\int_{u-\tau}^{u} G(s)\varphi(s)ds\Big)du. \qquad (4.16)
\end{aligned}
$$

First, it is shown that H maps from S into S. From Equation (4.16)

$$\begin{aligned}
\| (H\varphi) \| &= \Big| \Phi(t,t_0)\Big[\psi(t_0) + \int_{t_0-\tau}^{t_0} G(u)\psi(u)du\Big] \\
&\quad - \int_{t-\tau}^{t} G(u)\varphi(u)du \\
&\quad + \int_{t_0}^{t} \Phi(t,u)\Big(\Big[A(u) - G(u-\tau)\Big]\varphi(u-\tau) \\
&\quad - G(u)\int_{u-\tau}^{u} G(s)\varphi(s)ds\Big)du\Big| \\
&\leq |\Phi(t,t_0)|\Big[|\psi(t_0)| + \Big|\int_{t_0-\tau}^{t_0} G(u)\psi(u)du\Big|\Big] \\
&\quad + \Big|\int_{t-\tau}^{t} G(u)\varphi(u)du\Big| \\
&\quad + \Big|\int_{t_0}^{t} \Phi(t,u)\Big(\Big[A(u) - G(u-\tau)\Big]\varphi(u-\tau) \\
&\quad - G(u)\int_{u-\tau}^{u} G(s)\varphi(s)ds\Big)du\Big| \\
&\leq |\Phi|\Big[\|\psi\| + \Big|\int_{t_0-\tau}^{t_0} |G(u)|\|\psi\|du\Big|\Big] \\
&\quad + \int_{t-\tau}^{t} |G(u)|\|\varphi(u)\|du \\
&\quad + \int_{t_0}^{t} |\Phi|\Big(\Big[|A| + |G|\Big]\|\varphi\| \\
&\quad + |G|\int_{u-\tau}^{u} |G(s)|\|\varphi\|ds\Big)du
\end{aligned}$$

$$\leq \left|\Phi\right|\left[\delta+\tau|G|\delta\right]+\tau|G|\|\varphi\|$$

$$+\int_{t_0}^{t}|\Phi|\left[|A|+|G|+\tau|G|^2\right]\|\varphi\|du$$

$$\leq \left|\Phi\right|\delta(1+\tau\mid G\mid)$$

$$+\left(\tau\mid G\mid+\int_{t_0}^{t}|\Phi|\left[|A|+\mid G\mid+\tau\mid G\mid^2\right]du\right)\|\varphi\|$$

$$\leq \left|\Phi\right|\delta(1+\tau\mid G\mid)+\alpha\|\varphi\|$$

$$\leq \left|\Phi\right|\delta(1+\tau\mid G\mid)+\alpha\varepsilon$$

$$\leq \varepsilon.$$

This shows that H maps from S into itself.

Next the mapping defined by H is shown to be continuous. Let $\varphi, \eta \in S$. Given $\varepsilon_1 > 0$, choose $\delta = \frac{\varepsilon_1}{L}$, where

$$L=\tau\mid G\mid+\int_{t_0}^{t}|\Phi|\left[|A|+\mid G\mid+\tau\mid G\mid\right]du;$$

such that

$$\left\|\varphi-\eta\right\|<\delta.$$

Then

$$
\begin{aligned}
\|(H\varphi)-(H\eta)\| &= \Big|\Big(\Phi(t,t_0)\Big[\psi(t_0)+\int_{t_0-\tau}^{t_0}G(u)\psi(u)du\Big]-\int_{t-\tau}^{t}G(u)\varphi(u)du \\
&+\int_{t_0}^{t}\Phi(t,u)\Big(\Big[A(u)-G(u-\tau)\Big]\varphi(u-\tau) \\
&-G(u)\int_{u-\tau}^{u}G(s)\varphi(s)ds\Big)du\Big) \\
&-\Big(\Phi(t,t_0)\Big[\psi(t_0)+\int_{t_0-\tau}^{t_0}G(u)\psi(u)du\Big]-\int_{t-\tau}^{t}G(u)\eta(u)du \\
&+\int_{t_0}^{t}\Phi(t,u)\Big(\Big[A(u)-G(u-\tau)\Big]\eta(u-\tau) \\
&-G(u)\int_{u-\tau}^{u}G(s)\eta(s)ds\Big)du\Big)\Big| \\
&= \Big|\Phi(t,t_0)\Big[\psi(t_0)+\int_{t_0-\tau}^{t_0}G(u)\psi(u)du\Big]-\int_{t-\tau}^{t}G(u)\varphi(u)du \\
&+\int_{t_0}^{t}\Phi(t,u)\Big(\Big[A(u)-G(u-\tau)\Big]\varphi(u-\tau) \\
&-G(u)\int_{u-\tau}^{u}G(s)\varphi(s)ds\Big)du \\
&-\Phi(t,t_0)\Big[\psi(t_0)+\int_{t_0-\tau}^{t_0}G(u)\psi(u)du\Big]+\int_{t-\tau}^{t}G(u)\eta(u)du \\
&-\int_{t_0}^{t}\Phi(t,u)\Big(\Big[A(u)-G(u-\tau)\Big]\eta(u-\tau) \\
&+G(u)\int_{u-\tau}^{u}G(s)\eta(s)ds\Big)du\Big|
\end{aligned}
$$

$$
\begin{aligned}
= & \Big| -\int_{t-\tau}^{t} G(u)\varphi(u)du \\
& + \int_{t_0}^{t} \Phi(t,u)\Big(\Big[A(u) - G(u-\tau)\Big]\varphi(u-\tau) \\
& - G(u)\int_{u-\tau}^{u} G(s)\varphi(s)ds\Big)du \\
& + \int_{t-\tau}^{t} G(u)\eta(u)du \\
& - \int_{t_0}^{t} \Phi(t,u)\Big(\Big[A(u) - G(u-\tau)\Big]\eta(u-\tau) \\
& + G(u)\int_{u-\tau}^{u} G(s)\eta(s)ds\Big)du\Big| \\
\leq & \Big|\int_{t-\tau}^{t} G(u)\varphi(u)du - \int_{t-\tau}^{t} G(u)\eta(u)du\Big| \\
& + \Big|\int_{t_0}^{t} \Phi(t,u)\Big(\Big[A(u) - G(u-\tau)\Big]\varphi(u-\tau) \\
& - G(u)\int_{u-\tau}^{u} G(s)\varphi(s)ds\Big)du \\
& - \int_{t_0}^{t} \Phi(t,u)\Big(\Big[A(u) - G(u-\tau)\Big]\eta(u-\tau) \\
& + G(u)\int_{u-\tau}^{u} G(s)\eta(s)ds\Big)du\Big|
\end{aligned}
$$

$$\leq \Big|\int_{t-\tau}^{t} G(u)(\varphi(u)-\eta(u))du\Big|$$

$$+\Big|\int_{t_0}^{t}\Phi(t,u)\Big(\Big[A(u)-G(u-\tau)\Big](\varphi(u-\tau)-\eta(u-\tau))$$

$$-G(u)\int_{u-\tau}^{u} G(s)(\varphi(s)-\eta(s))ds\Big)du\Big|$$

$$\leq \int_{t-\tau}^{t} \mid G(u)\mid \|\varphi(u)-\eta(u)\|du$$

$$+\int_{t_0}^{t}\mid\Phi\mid\Big(\Big[\mid A\mid+\mid G\mid\Big]\|\varphi(u-\tau)-\eta(u-\tau)\|$$

$$+\mid G\mid\int_{u-\tau}^{u}\mid G\|\|\varphi(s)-\eta(s)\|ds\Big)du$$

$$\leq \tau\mid G\mid\|\varphi-\eta\|+\int_{t_0}^{t}\Big|\Phi\Big|\Big[\mid A\mid+\mid G\mid+\tau\mid G\mid^2\Big]du\|\varphi-\eta\|$$

$$\leq \Big\{\tau\mid G\mid+\int_{t_0}^{t}\Big|\Phi\Big|\Big[\mid A\mid+\mid G\mid+\tau\mid G\mid^2\Big]du\Big\}\frac{\varepsilon_1}{L}$$

$$\leq \varepsilon_1.$$

This shows that H is continuous. Next it is shown that H is a contraction under the supremum norm. Let $\varphi, \phi \in S$. Then

$$
\begin{aligned}
\|(H\varphi)-(H\phi)\| &= \Big|\Big(\Phi(t,t_0)\Big[\psi(t_0)+\int_{t_0-\tau}^{t_0} G(u)\psi(u)du\Big]-\int_{t-\tau}^{t} G(u)\varphi(u)du \\
&+ \int_{t_0}^{t}\Phi(t,u)\Big(\Big[A(u)-G(u-\tau)\Big]\varphi(u-\tau) \\
&- G(u)\int_{u-\tau}^{u} G(s)\varphi(s)ds\Big)du\Big) \\
&- \Big(\Phi(t,t_0)\Big[\psi(t_0)+\int_{t_0-\tau}^{t_0} G(u)\psi(u)du\Big]-\int_{t-\tau}^{t} G(u)\phi(u)du \\
&+ \int_{t_0}^{t}\Phi(t,u)\Big(\Big[A(u)-G(u-\tau)\Big]\phi(u-\tau) \\
&- G(u)\int_{u-\tau}^{u} G(s)\phi(s)ds\Big)du\Big)\Big| \\
&= \Big|\Phi(t,t_0)\Big[\psi(t_0)+\int_{t_0-\tau}^{t_0} G(u)\psi(u)du\Big]-\int_{t-\tau}^{t} G(u)\varphi(u)du \\
&+ \int_{t_0}^{t}\Phi(t,u)\Big(\Big[A(u)-G(u-\tau)\Big]\varphi(u-\tau) \\
&- G(u)\int_{u-\tau}^{u} G(s)\varphi(s)ds\Big)du \\
&- \Phi(t,t_0)\Big[\psi(t_0)+\int_{t_0-\tau}^{t_0} G(u)\psi(u)du\Big]+\int_{t-\tau}^{t} G(u)\phi(u)du \\
&- \int_{t_0}^{t}\Phi(t,u)\Big(\Big[A(u)-G(u-\tau)\Big]\phi(u-\tau) \\
&+ G(u)\int_{u-\tau}^{u} G(s)\phi(s)ds\Big)du\Big|
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \Big| - \int_{t-\tau}^{t} G(u)\varphi(u)du \\
& + \int_{t_0}^{t} \Phi(t,u)\Big(\Big[A(u) - G(u-\tau)\Big]\varphi(u-\tau) \\
& - G(u)\int_{u-\tau}^{u} G(s)\varphi(s)ds\Big)du \\
& + \int_{t-\tau}^{t} G(u)\phi(u)du \\
& - \int_{t_0}^{t} \Phi(t,u)\Big(\Big[A(u) - G(u-\tau)\Big]\phi(u-\tau) \\
& + G(u)\int_{u-\tau}^{u} G(s)\phi(s)ds\Big)du\Big| \\
\leq \quad & \Big|\int_{t-\tau}^{t} G(u)\varphi(u)du - \int_{t-\tau}^{t} G(u)\phi(u)du\Big| \\
& + \Big|\int_{t_0}^{t} \Phi(t,u)\Big(\Big[A(u) - G(u-\tau)\Big]\varphi(u-\tau) \\
& - G(u)\int_{u-\tau}^{u} G(s)\varphi(s)ds\Big)du \\
& - \int_{t_0}^{t} \Phi(t,u)\Big(\Big[A(u) - G(u-\tau)\Big]\phi(u-\tau) \\
& + G(u)\int_{u-\tau}^{u} G(s)\phi(s)ds\Big)du\Big|
\end{aligned}
$$

$$\leq \Big| \int_{t-\tau}^{t} G(u)(\varphi(u) - \phi(u))du \Big|$$

$$+ \Big| \int_{t_0}^{t} \Phi(t,u)\Big(\Big[A(u) - G(u-\tau)\Big](\varphi(u-\tau) - \phi(u-\tau))$$

$$- G(u) \int_{u-\tau}^{u} G(s)(\varphi(s) - \phi(s))ds\Big)du\Big|$$

$$\leq \int_{t-\tau}^{t} \mid G(u) \mid \|\varphi(u) - \phi(u)\| du$$

$$+ \int_{t_0}^{t} \mid \Phi \mid \Big(\Big[\mid A \mid + \mid G \mid\Big] \|\varphi(u-\tau) - \phi(u-\tau)\|$$

$$+ \mid G \mid \int_{u-\tau}^{u} \mid G \mid \|\varphi(s) - \phi(s)\| ds\Big)du$$

$$\leq \tau \mid G \mid \|\varphi - \phi\| + \int_{t_0}^{t} \Big|\Phi\Big|\Big[\mid A \mid + \mid G \mid + \tau \mid G \mid^2\Big]du \|\varphi - \phi\|$$

$$\leq \Big\{\tau \mid G \mid + \int_{t_0}^{t} \Big|\Phi\Big|\Big[\mid A \mid + \mid G \mid + \tau \mid G \mid^2\Big]du\Big\} \|\varphi - \phi\|$$

$$\leq \alpha \|\varphi - \eta\|.$$

Therefore,

$$\|(H\varphi) - (H\phi)\| \leq \alpha\|\varphi - \eta\|.$$

Since $\alpha \in [0, 1)$, H is a contraction. By the contraction mapping principle, H has a unique fixed point in S which solves Equation (4.2) and for any $\varphi \in S, \| H\varphi \| \leq \varepsilon$. This proves that the zero solution of Equation (4.2) is stable.

In the next theorem, the results for the zero solution of equation (4.1) to be asymptotically stable is stated.

Theorem 4.2.2. Assume the hypothesis of Theorem 4.2.1 hold. Assume further that

$$\Phi(t, t_0) \to 0 \quad as \quad t \to \infty \tag{4.17}$$

Then the zero solution of Equation (4.2) is asymptotically stable.

Proof. According to definition 4, the zero solution of a differential equation is asymptotically stable if it is stable and in addition for each $t_0 \geq 0$ there is an $\eta(t_0) > 0$ such that $\|\psi\| < \eta(t_0)$ implies that $x(t) \longrightarrow 0$ as $t \longrightarrow \infty$. The stability of the zero solution of eqution (4.2) has already been proved (theorem 4.2.1). Define $S^* = \{\varphi : R \longrightarrow R^n | \varphi(t) = \psi(t)$ if $t \in [-\tau, t_0]$ and for $t \geq t_0, \|\varphi\| \leq \varepsilon$, and $\varphi(t) \longrightarrow 0$ as $t \longrightarrow \infty\}$. Define $H : S^* \longrightarrow S^*$ by Equation (4.16). The first term on the right hand side of Equation (4.16) tends to zero in view of condition (4.17). Now the second term on the right hand side of equation (4.16) is shown to approach zero as $t \longrightarrow \infty$. To this end, let $\varphi \in S^*$. Then $\varphi(t) \longrightarrow 0$ as $t \longrightarrow \infty$. Thus by the continuity of norms, $\|\varphi\| \longrightarrow 0$ as $t \longrightarrow \infty$. Hence

$$\left| -\int_{-\tau}^{t} G(u)\varphi(u)du \right| \leq \int_{t-\tau}^{t} |G|\|\varphi\|du.$$

Finally, to show that the third term on the right hand side of equation (4.16) goes to zero as $t \longrightarrow \infty$, let $\varphi \in S^*$. Then given $\varepsilon_1 > 0$, there exists $t_1 > t_0$ such that for $t > t_1$, $|\varphi(t)| < \varepsilon_1$. Also, by condition (4.17), there exists $t_2 > t_1$ such that for $t > t_2$ implies that

$$|\Phi(t, t_2)| < \frac{\varepsilon_1}{\alpha\varepsilon}.$$

Thus for $t > t_2$,

$$\begin{aligned}
& \Big|\int_{t_0}^{t_2} \Phi(t,u)\Big(\Big[A(u) - G(u-\tau)\Big]\varphi(u-\tau) - G(u)\int_{u-\tau}^{u} G(s)\varphi(s)ds\Big)du\Big| \\
& + \Big|\int_{t_2}^{t} \Phi(t,u)\Big(\Big[A(u) - G(u-\tau)\Big]\varphi(u-\tau) - G(u)\int_{u-\tau}^{u} G(s)\varphi(s)ds\Big)du\Big| \\
\leq & \int_{t_0}^{t_2} \mid \Phi(t,u) \mid \Big(\Big[\mid A\mid + \mid G\mid\Big]\|\varphi\| + \mid G \mid \int_{u-\tau}^{u} \mid G\mid \|\varphi\| ds\Big)du \\
& + \int_{t_2}^{t} \mid \Phi(t,u) \mid \Big(\Big[\mid A\mid + \mid G\mid\Big]\|\varphi\| + \mid G \mid \int_{u-\tau}^{u} \mid G\mid \|\varphi\| ds\Big)du \\
= & \int_{t_0}^{t_2} \mid \Phi(t,t_2) \mid\mid \Phi(t_2,u) \mid \Big(\Big[\mid A\mid + \mid G\mid\Big]\|\varphi\| + \mid G\mid^2 \tau\|\varphi\|\Big)du \\
& + \int_{t_2}^{t} \mid \Phi(t,u) \mid \Big(\Big[\mid A\mid + \mid G\mid\Big]\|\varphi\| + \mid G\mid^2 \tau\|\varphi\|\Big)du \\
\leq & \ \varepsilon\Big|\Phi(t,t_2)\Big|\alpha + \alpha\varepsilon \\
< & \ \varepsilon_1 + \alpha\varepsilon.
\end{aligned}$$

Hence, $(H\varphi)(t) \longrightarrow 0$ as $t \longrightarrow \infty$. Therefore, the contraction mapping principle implies that, H has a unique fixed point in S^* which solves Equation (4.2), and goes to zero as $t \longrightarrow \infty$.

Chapter Summary

In this chapter, results concerning the asymptotic stability of first order ordinary differential equations with finite delay was established. In the process, a fundamental matrix solution was used to invert the system of ODEs into an integral system for which a fixed point mapping is derived. A suitable complete space was prudently defined. The sufficient conditions for the asymptotic stability of the zero solution of systems of ODEs, with finite delay was obtained.

CHAPTER FIVE

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

Overview

This chapter provides the summary, conclusions as well as recommendation of the study. The summary briefly presents an overview of the research problem, objectives, method and results of the study. The conclusions encompasses the overall results of the study with respect to the research objectives of the study. Some recommendation based on the work done is also presented.

Summary

In this thesis, as set out in the research objectives, the stability and asymptotic stability of a system of first order ordinary differential equations with finite delay were investigated. In the process a fundamental matrix solution was used to invert the ODE into an integral system from which a fixed point mapping was derived. A suitable complete space was then defined. Using the Banach fixed point theorem or contraction mapping principle. This mapping was used to obtain sufficient conditions for which the zero solution of a system of ODEs with finite delay is asymptotically stable.

Conclusion

Sufficient conditions for the stability of the zero solution of a system of first order ordinary differential equations with finite delay have been established.

Also, sufficient conditions for the asymptotic stability of the zero solution of the system of first order ordinary differential equations with finite delay have been established.

Recommendation

Fixed point theorems should be applied to systems of first order ordinary differential equations with finite delay for stability investigations.

REFERENCES

Burton, T. (2003). Stability by fixed point theory or liapunov's theory a comparison. *Fixed point theory*, *4*(1), 15–32.

Burton, T. (2006). Stability by fixed point theory for functional differential equations, 356 page manuscript to be published by dover publications. *New York, Fall*.

Burton, T., & Furumochi, T. (2001). Fixed points and problems in stability theory for ordinary and functional differential equations. *Dynamic systems and applications*, *10*(1), 89–116.

Burton, T. A. (1985). Stability and periodic solutions of ordinary and functional differential equations. *Orlando Academic Press.*.

Chicone, C. (1999). *Ordinary differential equations with applications* (Vol. 34). Springer Science & Business Media.

Driver, R. (1984). A neutral system with state-dependent delay. *J. Differential Equations*, *54*, 73-86.

Lyapunov, A. M. (1892). The general problem of the stability of motion. *Annals of Mathematics Studies*, *17*.

Poincaré, H., & Magini, R. (1899). The new methods of celestial method é. *Il Nuovo Cimento(1895-1900)*, *10*(1), 128–130.

Sasser, J. E. (2005). *History of ordinary differential equations: The first hundred years* (Vol. 1). Citeseer.

Smart, D. R. (1974). Fixed point theorem. *Cambridge: Cambridge University Press.*.

Volterra, V. (1928). Sur la theorie mathematique des phenom'enes hereditaires. *J. Math. Pures Appl.*, *7*, 249-298.