UNIVERSITY OF CAPE COAST

COMPUTATION OF PARTIAL CORRELATION COEFFICIENT: VARIANCE-COVARIANCE MATRIX APPROACH

EMMANUEL BLANKSON

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BY

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Thesis submitted to the Department of Statistics of the School of Physical Sciences, College of Agriculture and Natural Sciences, of the University of Cape Coast, in partial fulfilment of the requirements for the award of Master of Philosophy degree in Statistics

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DECLARATION

Candidate's Declaration

I hereby declare that this thesis is the result of my own original study, and that no part of it has ever been submitted for any other degree at this university or elsewhere.

Candidate's Signature:.. Date:..........................

Name: Emmanuel Blankson

Supervisor's Declaration

I hereby declare that the thesis preparation and presentation have been supervised in compliance with the thesis supervision requirements established by the University of Cape Coast.

Supervisor's Signature:.................................. Date:........................

Name: Prof. Nathaniel K. Howard

ABSTRACT

This thesis examined variance-covariance matrix approach of computing orders of partial correlation coefficients. The main objective of this thesis is to explore further if the partial correlation coefficients beyond the first order can be computed using the method of variance-covariance matrix approach. Statistical tests were performed on the datasets used for the fundamental partial correlation assumptions, namely linearity, normality, and the lack of outliers. In order to account for the effects of one or more extra random variables, the thesis provided a logical investigation into the linear connection between two random variables. To achieve this, the study determines the appropriate dataset structure and partitioning, as well as the key matrices that allow us to acquire the theoretical conclusion. Practical examples and R syntax were used to clearly illustrate the computation of higher order partial correlation coefficients. It was found that the orders of partial correlation coefficient may be achieved by normalizing the conditional variance-covariance matrix results. The study demonstrates that, if the partial correlation assumptions are met, the variance-covariance matrix technique may compute partial correlation coefficients of any order. Finally, the study recommends that future researchers adopt the method of variance-covariance matrix technique to generate higher orders of partial correlation coefficients since the method is trustworthy, and comprehensible.

KEY WORDS

Conditional Distribution

Correlation Coefficient

Multivariate Dataset

Multivariate Normal Distribution

Partial Correlation Coefficient

Variance-Covariance Matrix

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DEDICATION

To my dear children, Josiah, Jochebed, and Jehosheba, and my wonderful wife, Mary Koomson.

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LIST OF ABBREVIATIONS

CD	Conditional Distribution
MLE	Maximum Likelihood Estimator
MND	Multivariate Normal Distribution
OLS	Ordinary Least Square
OR	Ordinal Correlation
PC	Partial Correlation
PCC	Partial Correlation Coefficient
PCNN	Partial Correlation Neutral Network
PCIS	Partial Correlation Interaction Screening
SEM	Structural Equation Modeling
VCM	Variance-Covariance Matrix

CHAPTER ONE

INTRODUCTION

One of the most effective ways to identify relationships and examine patterns in huge data sets is through multivariate analysis. If a structured study design is used, it is very helpful in minimizing bias. However, because of the method's intricacy, less experienced research enthusiasts choose to use it. As a result, the techniques excel at finding relationships in difficult situations, despite the fact that the analysis of the dataset and the design of the study are arduous procedures.

Techniques that look at the simultaneous effects of several variables are referred to as multivariate statistics. It involves observing, outlining, and displaying simultaneous complicated occurrences and distributions. The method of looking at a lot of variables to determine the degrees of association while holding one or more of these variables constant is called multivariate partial correlation (Francois et al., 2010).

Before understanding partial correlation analysis, we need to have a better understanding of correlation. The statistical link between two or more variables that shows how they could be connected to one another is referred to as correlation. It is a simple and popular technique for describing relationships without giving a cause and effect explanation. There is no assumption of causation in correlation computations, thus one variable is not necessarily causing the other to change, even though the variables may change in some way simultaneously. However, the methods for calculating correlation and regression using independent and dependent variables are comparable. Simple linear correlation evaluates the strength of the link between the variables and determines whether it is positive (i.e. as one variable rises, the other rises) or negative (i.e. as one variable rises, the other decreases). Coefficient of correlation or correlation value can range from -1 to $+1$, with $+1$ signifying a perfect positive correlation (all data points lie exactly on a straight line with a positive slope in the *X*-*Y* plane.) and -1 signifying a perfect negative correlation (all data points lie exactly on a straight line with a negative slope in the *X*-*Y* plane.). If the correlation coefficient is zero, it indicates that there is either no correlation, or no link at all between the variables, or that the relationship is not linear. It's critical to keep in mind that the absence of a linear relationship does not exclude the possibility that the variables are connected. If the correlation coefficient does not adequately describe the link between the variables, there may be a non-linear relationship or another kind of relationship.

In a scenario where the interest variables are continuous, five popular correlation coefficients that are widely explored and accepted by researchers to measure association between variables includes: simple, multivariate, partial, multiple and canonical correlation coefficients. Simple correlation is the term used to quantify the relationship between two variables. It explores the simple correlation for all pair-wise combinations of the variables. A simple expansion of simple correlation to include more than two variables is multivariate correlation. When one or more extra factors are taken into consideration, a partial correlation exists between the two variables. Multiple correlation refers to the relationship that is examined when many variables are investigated together. The linear connection between two sets of multivariate variables is established using a multivariate statistical technique known as canonical correlation.

When the effects of a third variable are maintained constant, the answer to the query is provided by partial correlation. "What is the correlation coefficient between any two variables?"

1.1 Background to the Study

In many other fields of study, partial correlations regularly occur. Because the partial correlation coefficient expresses the relationship between two variables while taking additional (confounding) factors into account, it is employed. This renders it a very valuable statistical coefficient, especially in scenarios where certain factors cannot or will not be controlled experimentally.

In multivariate contexts, partial correlations are used to investigate the associations between variables after other factors' impacts have been taken into consideration. In probability theory and statistics, partial correlation evaluates the strength of the relationship between two random variables when a set of controlling random factors are taken into consideration. When attempting to discover the numerical link between the two variables of interest, the correlation coefficient of the two variables of interest will yield inaccurate findings if there is another confounding variable that is numerically connected to both of the factors of interest. By calculating the partial correlation coefficient and accounting for the confounding variable, it is possible to prevent the dissemination of this misleading information. The idea of partial correlation was developed to solve this issue. When computing the correlation between two continuous variables, partial correlation takes into account the influence of one or more additional continuous variables (commonly referred to as "control" variables or "covariate") that have an impact on both variables.

In order to identify erroneous correlations, partial correlations might be quite helpful. For instance, this happens when two variables, like *u* and *v*, are connected because both are impacted by *w*. The link between *u* and *v*, however, is lost when *w* is taken out of the equation. Age is one of the factors that frequently lead to misleading correlations, according to Pedhazur (1982). Because it is not always possible to evaluate persons of the same age, partial correlation helps researchers to determine whether age is playing a role in a false finding.

If one takes away the effects of a third (or more) control variable(s) from the connection, partial correlation can help determine if two variables are linearly connected. A partial correlation coefficient is a type of Pearson correlation coefficient that is used to identify the connection between two variables while accounting for the confounding factors. We must first ascertain the magnitude of the zero-order (bivariate) correlation coefficient between the two variables before assessing a partial correlation between them. Understanding regression is aided by partial correlations. Three variables make up the simplest partial correlation: response variable, a predictor variable, and a control variable.

When evaluating the linear relationship between two variables, the partial correlation coefficient is a statistical metric that accounts for the influence of one or more extra factors. It evaluates the strength of the correlation between two variables while also accounting for the impact of additional variables that might be possibly connected to both of them. Because no other factors are controlled and there is only one independent variable, a zero-order partial correlation is a term frequently used to describe a Pearson correlation.

The partial correlation coefficient formula has a similar component to the Pearson correlation coefficient formula, but it additionally accounts for the influence of other factors.

The following is a representation of the formula:

$$r_{ab.c} = \frac{r_{ab} - r_{ac}r_{bc}}{\sqrt{1-r_{ac}^2}\sqrt{1-r_{bc}^2}}$$

where r_{bc} is the correlation coefficient between *b* and *c*, r_{ab} is the correlation coefficient between *a* and *b*, r_{ac} is the correlation coefficient between *a* and *c*, and *a*, *b*, and *c* are variables. The partial correlation coefficient's value falls between -1 and $+1$, with -1 denoting a perfectly negative link, 0 denoting no association, and $+1$ denoting a perfectly positive association. A substantial positive partial correlation demonstrates that, while keeping the values on the control variable(s) constant, as the values on one variable rise, the values on another variable also tend to ascend. Remember that the partial correlation coefficient measures connection rather than causality. A high partial correlation coefficient indicates a substantial link between two variables after adjusting for the impact of other factors, rather than that one variable directly causes the other.

Wang (2013) states that the relationship between *X* and *Y* is described by a partial correlation coefficient, which accounts for the impact of control variable *Z*. The conditional independence of the provided random variables, *X*, *Y*, and control random variable, $Z = \{Z_1, Z_2, .., Z_n\}$, may be evaluated using this. Once more, the linear correlation between two residuals from the linear regression of *X* with *Z* and the linear regression of *Y* with *Z* is measured by the partial correlation coefficient ($\rho_{XY.Z}$). The partial correlation coefficient of order one is specifically provided for $n = 1$ (i.e. controlling for only one variable) is given by:

$$\rho_{XY.\mathbf{Z}} = \frac{\rho_{XY} - \rho_{XZ}\rho_{YZ}}{\sqrt{1-\rho_{XZ}^2}\ \sqrt{1-\rho_{YZ}^2}}$$

Higher order partials are quite legitimate for analyzing product moment correlation coefficients and working with interval or ratio data, according to Wirsing (1975). Effective partial correlation needs variables with a linear connection and interval or ratio data that are properly distributed (Korn, 1984; Waliczek, 1996). A first-order correlation is a partial correlation in which only one variable is regulated. Multiple factors can be controlled at the same time. For instance, the expression $r_{12.34}$ suggests that variables 3 and 4 are being in control and this connection is of second order.

The same partial correlation algorithm is applied when partialing out many independent variables. For instance, if a second-order partial correlation is manually generated; three first-order partials must first be calculated in order to be plugged into the calculation. The formula would look like this:

$$r_{ab.cd} = \frac{r_{ab.c} - r_{ad.c}r_{bd.c}}{\sqrt{1-r_{ad.c}^2}\sqrt{1-r_{bd.c}^2}}$$

The simple correlation coefficient (ρ_{ij}) between the two variables, X_i and X_j, is defined as follows:

$$\rho_{X_i\ X_j} = \frac{\sigma_{i\,j}}{\sqrt{\sigma_{ii}}\sqrt{\sigma_{j\,j}}} \tag{1.1}$$

where the $(i, j)th$ member of the data's cross-product and sum of squares matrix is the value σ_{ij}.

When accounting for the additional variables, $Y = (Y_1, Y_2, ...Y_t)'$, the equation for the population partial correlation coefficient between X_i and X_j is provided as

$$\rho_{X_i X_j .\mathbf{Y}} = \frac{\sigma_{X_i X_j .\mathbf{Y}}}{\sqrt{\sigma_{X_i X_i .\mathbf{Y}}}\sqrt{\sigma_{X_j X_j .\mathbf{Y}}}} \qquad (1.2)$$

The element $\sigma_{X_i X_j .\mathbf{Y}}$ is the $(i, j)th$ item in the variance-covariance matrix technique and is determined by:

$$\mathbf{\Sigma}_{XX} - \mathbf{\Sigma}_{XY}\mathbf{\Sigma}_{YY}^{-1}\mathbf{\Sigma}_{YX} \qquad (1.3)$$

Specifically, if three variables V_1, V_2 and V_3 are chosen at random, the partial correlation coefficient of order one between V_1 and V_2 regulatory for V_3 is given by:

$$\rho_{V_1V_2.V_3} = \frac{\sigma_{V_1V_2} - (\sigma_{V_1V_3})(\sigma_{V_2V_3})}{\sqrt{1-\sigma_{V_1V_3}^2}\sqrt{1-\sigma_{V_2V_3}^2}} \qquad (1.4)$$

where $\sigma_{V_1V_2}$ is the correlation coefficient of order zero between V_1 and V_2, also known as the simple correlation coefficient. Equation (1.4) shows that controlling for V_3 lowers $\rho_{V_1V_2.V_3}$ (i.e., makes it less positive or more high negative, depending on the situation) if the values $\sigma_{V_1V_3}$ and $\sigma_{V_2V_3}$ have the same sign. Controlling for V_3 raises the value $\rho_{V_1V_2.V_3}$ if, on the other hand, the values $\sigma_{V_1V_3}$ and $\sigma_{V_2V_3}$ have the opposite signs. The implication is that the coefficient of the correlation between each of V_1 and V_2, and other factors heavily influences the coefficient of the partial correlation between V_1 and V_2 when adjusting for other variables.

In circumstances when there are several independent variables, matrices can be employed to compute the results (Neter *et al*., 1996). We mostly used the matrix principles technique to perform the statistical analysis for this thesis since we are using the conditional covariance-covariance matrix technique to generate orders of partial correlation coefficients.

1.2 Statement of the Problem

In research projects, statistics control is frequently sought after by researchers. The main focus of these researchers is variance control. When examining the effects of several independent variables on a dependent variable, control is essential. One might evaluate the degree of correlation between an independent and dependent variable while taking note of one or more additional effects by using the partial correlation approach.

Partial correlation analysis appears to assist a number of multivariate statistical methods. Only some aspects of the program output may reveal the usage of partial correlation. Due to the concept's precise mathematical nature, its applicability is not properly shown. Some findings are only mentioned in the text and have not been systematically presented in the literature because of the concept's mathematical complexity.

Four distinct approaches to calculate the partial correlation coefficient were studied by Ogunleye *et al*. in 2022. These approaches include regression residual's approach, the traditional technique, the variance-covariance matrix technique, and the ordinary least square (OLS) method. Each of these techniques was completely explained with use-case examples and R syntax. With examples, the discussion of each method's applicability was conducted. Each method's advantages and disadvantages are thoroughly described. After running statistical tests on the given datasets, they discovered that there was no violation of any of the essential requirements for partial correlation, including normality, linearity, and the lack of outliers. With at least one variable held constant, the study offers the most efficient approach or methods for calculating partial correlation coefficients. Although the traditional method was recommended as being the least time-consuming, all four of the outlined methods may calculate the partial correlation coefficient of the first order. However, the variance-covariance matrix method is dropped from the four methods that produced the result of the partial correlation coefficient of order two after their practical example-based illustrations. They made it obvious that the variance-covariance matrix method was only suitable for handling partial correlation coefficients of order one. Furthermore, they emphasized that, to the best of their knowledge, no new methodology (variance-covariance matrix approach) has been developed to handle partial correlation coefficients higher than the first order.

The purpose of this thesis is to look at further if the partial correlation coefficients beyond the first order can be computed or performed using the variance-covariance matrix approach.

1.3 Objectives of the Study

The purpose of the research is to compute orders of partial correlation coefficient using variance-covariance matrix approach. The specific objectives are:

1. To setup variance-covariance matrix to serves as the foundation for calculating the orders of the partial correlation coefficient.

2. To compute correlation coefficient matrix from variance-covariance matrix.
3. To estimate the maximum likelihood estimator of the partial correlation coefficient.
4. To compute orders of the partial correlation coefficient from variance-covariance matrix.

1.4 Structure of Dataset

Since the study is largely theoretical, it is necessary to use a pertinent dataset for the purpose of providing a concrete example. The information for our statistical analysis was taken from the study "Cross-Sectional Analysis of Methods of Computing Partial Correlation Coefficients: A Self-Explained Note with R Syntax" by Ogunleye *et al*., (2022). Nine variables are covered by the secondary data, $\mathbf{V} = (\mathbf{V}_1, \mathbf{V}_2, ...\mathbf{V}_9)$ with twenty-two observations. The variables are

$\mathbf{V}_1$ — Consumption

$\mathbf{V}_2$ — Corporate Profit

$\mathbf{V}_3$ — Private Wage Bill

$\mathbf{V}_4$ — Investment

$\mathbf{V}_5$ — Previous Year's Capital Stock

$\mathbf{V}_6$ — Gross National Product

$\mathbf{V}_7$ — Government Wage Bill

$\mathbf{V}_8$ — Government Expenditure

$\mathbf{V}_9$ — Taxes

The datasets are part of a package known as Applied Econometrics with R (AER), and they offer the original dataset utilized in our investigation, Klein's Model I data for the US Economy (1920–1941). It has nine variables with twenty two observations and is called 'KleinI' in R package software.

As a result of the content of this data, it is deemed appropriate to divide the data into two parts composing of $\mathbf{V}' = (\mathbf{V}^1 \vdots \mathbf{V}^2)$, where sub-vector $\mathbf{V}^1 = (\mathbf{V}_1, \mathbf{V}_2)$ and sub-vector $\mathbf{V}^2 = (\mathbf{V}_3, \mathbf{V}_{4,} \mathbf{V}_5, \mathbf{V}_6, \mathbf{V}_7, \mathbf{V}_8, \mathbf{V}_9)$. Generally, if $\mathbf{V}$ can be partitioned into $\mathbf{V}^1$ of dimension *s* and $\mathbf{V}^2$ of dimension *t*, then the mean vector $\boldsymbol{\mu}$ and variance-covariance matrix $\boldsymbol{\Sigma}_V$ will consequently be partitioned into

$$\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \hdashline \boldsymbol{\mu}_2 \end{pmatrix}, \quad \boldsymbol{\Sigma} = \left(\begin{array}{c:c} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \hdashline \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{array}\right)$$

By this structure, we have the following:

$\boldsymbol{\mu}_1$ is a $s \times 1$ mean vector of sub-vector $\mathbf{V}^1$;

$\boldsymbol{\mu}_2$ is a $t \times 1$ mean vector of sub-vector $\mathbf{V}^2$;

$\boldsymbol{\Sigma}_{11}$ is a $s \times s$ variance-covariance matrix of the sub-vector $\mathbf{V}^1$;

$\boldsymbol{\Sigma}_{22}$ is a $t \times t$ variance-covariance matrix of the sub-vector $\mathbf{V}^2$;

$\boldsymbol{\Sigma}_{12}$ is a $s \times t$ covariance matrix of the sub-vectors $\mathbf{V}^1$ and $\mathbf{V}^2$; and

$\boldsymbol{\Sigma}'_{12} = \boldsymbol{\Sigma}_{21}$

Furthermore, it is necessary for this study that the random vector $\mathbf{V}$ partitioning be insightful. Since the link between pairs of variables is the study's main focus after adjusting for the impact of additional factors are taken into consideration, the variables must meaningfully relate to or be linear to one another for the study to be worthwhile.

1.5 Significant of the Study

The research will give a focused and coherent explanation of the notion of partial correlation analysis utilizing the method of variance-covariance matrix approach by presenting a class of mathematical tools that are necessary to provide a sufficient grasp of the concepts. Our understanding of the variance-covariance matrix technique to computing partial correlation coefficient ordering will be improved by the study. Future researchers will also benefit from the study since it will inform them of the connection between conditional variance-covariance matrix strategy and partial correlation coefficient.

1.6 Delimitation of the Study

The main focused of the thesis is to calculate higher order partial correlation coefficient. There are several ways for estimating partial correlation coefficients, including the regression residual's approach, the conventional (traditional) method, the variance-covariance matrix approach, the recursive method, ordinary least squares method, and the matrix inversion method. We focused on using variance-covariance matrix strategy for computing higher order partial correlation coefficient.

1.7 Limitations of the Study

The research objectives are the exclusive focus of the thesis. This study project has several limitations. Due to the study's reliance on a single dataset, some of these challenges include a lack of resources and time restrictions.

1.8 Organisation of the Study

There are five chapters in the thesis. The first chapter explains the issue statement, the background of the investigation, the goals of the study, how the example dataset is organized, and the significance of the research. In the second chapter, pertinent literature on the subject is reviewed. The theoretical and empirical research of earlier authors is taken into consideration throughout the literature review. In Chapter Three, we explore the methodology and approach used to carry out the study. Data analysis and findings are presented in Chapter Four. Detailed analysis, discussion, and presentation of the secondary data are provided. The study's findings and a summary are presented in the last chapter. In addition to suggestions for additional research in the subject, recommendations are provided for effective and efficient decisions.

1.9 Chapter Summary

This chapter identifies the statement of the partial correlation analysis problem and explains the rationale for the investigation. The chapter describes the data's necessary organization for the study. A representative real-world example that would be used as an example in the research has been provided to assist us computes the orders of partial correlation coefficient.

CHAPTER TWO

LITERATURE REVIEW

2.0 Introduction

The pertinent works on partial correlation coefficient will be examined in this chapter. Theoretical ideas will be covered, and various empirical investigations on partial correlation coefficient analysis that have been investigated over time will be examined.

2.1 Theoretical Study

According to Timm and Carlson (1975), Sir Francis Galton brought the idea of a simple correlation into statistics in a number of studies written in the 1880s. His ideas on correlation, however, were largely ignored until the publication of his book Natural Inheritance in 1889. Partial and multiple correlation were developed by Pearson (1896, 1898) as a result of his inspiration from Galton's attempt to develop an exact mathematical theory of correlation (Yule, 1897, 1907).

Fieller et al. (1957), Fisher (1924), and Yule (1907) all state that a correlation coefficient should not be more than one $(+1)$ or lower than one (-1). On rare occasions, the coefficient will be zero (0), signifying that there is no correlation between the variables. Consequently, all correlation coefficients ought to lie within the range of -1 and $+1$.

According to Turney (2023), however there are differences in how different disciplines perceive the relationship strength (also known as impact size) of correlation coefficient. Some common guidelines for assessing correlation coefficients are shown in Table 1:

Table 1: Guidelines for Interpreting Correlation Coefficient

Correlation coefficient (r) value	Strength	Direction
r is bigger than 0.5 $(r > 0.5)$	Strong	Positive
r lies between 0.3 and 0.5 $(0.3 < r \leq 0.5)$	Moderate	Positive
r lies between 0.0 and 0.3 $(0.0 < r \leq 0.3)$	Weak	Positive
r is 0.0 $(r = 0)$	None	None
r lies between 0.0 and –0.3 $(0.0 < r \leq -0.3)$	Weak	Negative
r lies between –0.3 and –0.5 $(-0.3 < r \leq -0.5)$	Moderate	Negative
r is less than –0.5 $(r < 0.5)$	Strong	Negative

Source: Turney (2023)

2.1.1 The Concept of Partial Correlation Coefficient

Zhang *et al*. (2021) state that partial correlation is a technique that takes into consideration the impact of one or more additional continuous variables in order to construct a linear connection between two continuous variables. The degree to which two variables are linearly related is also evaluated using partial correlation, which also takes into consideration the influence of additional variables (Lonas, 2020).

Partial correlation, according to Brown and Hendrix (2014), is a method for determining the relationship between two variables while accounting for the influence of a third variable. It is comparable to the relationship between the two variables' residual scores after analytical regression on the control variable. They made it clearly clear that several other useful statistical methods, like stepwise multiple regression, factor analysis, route analysis, structural equation modeling, and so on, conceptually relate to and depend on partial correlation as a major element.

According to Explorable.com (2010), partial correlation investigation is the examination of a linear connection between two different variables after one or more independent factors have been taken into consideration. A first order coefficient indicates the partial correlation between two variables, which may be examined by holding the third variable constant while altering the other two. In a comparable manner, a second order coefficient might be defined, and so on. The computation of this coefficient is based on the simple correlation coefficient. The partial correlation analysis is crucial when there are several variables impacting the event being studied. The ability to alter the variables and look at each variable's impact independently makes this especially true in the physical and experimental sciences. This approach is particularly helpful in a variety of experimental designs where several related phenomena are being studied.

For the purpose of identifying the partial link between the indices, we examined scenarios in which one variable was kept constant while the others were altered, according to Morison (2007). Multivariate partial correlation analysis is highly useful when the system contains several variables and factors that affect them. This is clear from the agricultural, commercial, physical, and experimental sciences, where the aim is to find ways to control and assess the impact of different components on their own.

2.1.2 Computational Methods of Partial Correlation Coefficient

We examine many known approaches for computing partial correlation coefficients for statistical analysis.

Wikipedia (2022) describes three ways for computing partial correlation coefficient: The linear regression formula, the recursive formula, and the matrix inversion formula.

Since the first approach, linear regression, is predicated on the concept of partial regression, it provides us with a comprehension of the calculations we make when we compute the partial correlation. In this scenario, we would like to determine the partial correlation between *X* and *Y* after correcting for the influence of *Z*. The plan is to compute *X*'s linear regression with regard to *Z* first, and then identify the residual. We do the linear regression of *Y* with respect to *Z* again in order to obtain the residual. After that, we ascertain the two residuals' association. It is easy to estimate the sample partial correlation for a given set of data by solving the two related linear regression problems. The correlation between the sample partial correlation and the residuals is found using the following formula:

$$\hat{r}_{XY.Z} = \frac{n\sum_{i=1}^{n} e_{X,i} e_{Y,i} - \sum_{i=1}^{n} e_{X,i} \sum_{i=1}^{n} e_{Y,i}}{\sqrt{n\sum_{i=1}^{n} e_{X,i}^2 - (\sum_{i=1}^{n} e_{X,i})^2}\sqrt{n\sum_{i=1}^{n} e_{Y,i}^2 - (\sum_{i=1}^{n} e_{Y,i})^2}} = \frac{n\sum_{i=1}^{n} e_{X,i}\, e_{Y,i}}{\sqrt{n\sum_{i=1}^{n} e_{X,i}^2}\sqrt{n\sum_{i=1}^{n} e_{Y.i}^2}} \quad (2.1)$$

The three terms (i.e., $\sum_{i=1}^{n} e_{X,i} \sum_{i=1}^{n} e_{Y,i}$, $(\sum_{i=1}^{n} e_{X,i})^2$, and $(\sum_{i=1}^{n} e_{Y,i})^2$) that follow minus signs in Equation (2.1) are all equal to zero due to the fact that every term in Equation (2.1) reflects the total of the residuals from a regression using conventional least squares.

Recursive formula approach is the second method. Solving linear regression problems can be computationally costly. In actuality, partial correlations of the three $(n-1)^{th}$ order may be used to quickly create the partial correlation (with $|Z| = n$) of the n th order. The regular correlation coefficient ρ_{XY} is used to define the partial correlation $\rho_{XY.\phi}$ of the zeroth order. For any $Z_0 \in \mathbf{Z}$, it is true that

$$\rho_{XY.\mathbf{Z}} = \frac{\rho_{XY.\mathbf{Z}\setminus\{Z_0\}} - \rho_{XZ_0.\mathbf{Z}\setminus\{Z_0\}}\rho_{Z_0Y.\mathbf{Z}\setminus\{Z_0\}}}{\sqrt{1-\rho^2_{XZ_0.\mathbf{Z}\setminus\{Z_0\}}}\sqrt{1-\rho^2_{Z_0Y.\mathbf{Z}\setminus\{Z_0\}}}} \quad (2.2)$$

The computation has an exponential time complexity when it is implemented as a recursive algorithm. However, because of the feature of overlapping sub problems, this calculation has a complexity of $\varphi(n^3)$ regardless of whether dynamic programming is used or just storing the outcomes of the recursive calls. Equation (2.2), as we can see, becomes simpler when Z is a single variable:

$$\rho_{XY.Z} = \frac{\rho_{XY} - \rho_{XZ}\rho_{ZY}}{\sqrt{1-\rho^2_{XZ}}\sqrt{1-\rho^2_{ZY}}}$$

The joint precision matrix may also be used to express the partial correlation step of the third technique, which involves matrix inversion. let $\mathbf{U} = X_1, X_2, ..., X_n$ of cardinality n, be a set of random variables we consider .Given all other variables, we wish to know the coefficient of partial correlation between two variables X_i and X_j given others, that is $\mathbf{U}$. Assume $\mathbf{\Sigma} = (\sigma_{ij})$, the (joint/full) covariance matrix, is invertible because it is positive definite. If $\Omega = (\rho_{ij}) = \Sigma^{-1}$ is the definition of the precision matrix, then

$$\rho_{X_iX_j.\mathbf{U}} = -\frac{\rho_{ij}}{\sqrt{\rho_{ii}\rho_{jj}}} \quad (2.3)$$

It takes time $\varphi(n^3)$ to compute Σ, the inverse of the covariance matrix Σ^{-1}, in order to compute the aforementioned assertion. The sample covariance matrix is used to generate a sample partial correlation. Note that the extraction of all the partial correlations between variable pairs in $\mathbf{U}$ may be accomplished with a single matrix inversion.

In addition, Altman, (1991) was the one who first put up the concept of partial correlation in 1991. He described it as a technique for establishing a link between two variables while limiting the impact of one or more extra factors. In terms of multiple regression, the partial correlation concept is best understood. He continued by saying that the actual multiple regression is a more effective tool for examining this kind of link between three or more variables. The partial correlation coefficient from a multiple regression can be computed in the following general form:

$$r_k = \frac{t_k}{\sqrt{t_k^2 + \text{residual } d.f}} \tag{2.4}$$

where the degrees of freedom are denoted by d.f. and t_k is the Student t statistic for the *kth* term in the linear model.

The partial correlation coefficient, according to Mathematics Encyclopedia (2012), is a measurement of the linear dependency of a pair of random variables from a group of random variables when the influence of the other components is taken into consideration. In particular, take into account the likelihood that the random variables $\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_\mathrm{n}$ may jointly distribute in R^n, and let $X^*_{1;3,\ldots,n}, X^*_{2;3,\ldots,n}$ represent the best linear proxies, based on $X_3, \ldots, X_n$, for the variables X_1 and X_2. Subsequent that, the coefficient of the partial correlation between X_1 and X_2, represented by $\rho_{1;2.3,\ldots,n}$ is defined as the ordinary correlation coefficient value between the random variables $Y_1 = X_1 - X^*_{1;3,...,n}$ and $Y_2 = X_2 - X^*_{2;3,...,n}$ is provided by:

$$\rho_{12/3,...,n} = \frac{E\{(Y_1 - EY_1)(Y_2 - EY_2)\}}{\sqrt{DY_1 DY_2}} \quad (2.5)$$

It follows from the definition that $-1 \leq \rho_{12/3,...,n} \leq 1$

According to Kriz (1973), indicated that a more comprehensive general formula that may be used when someone wants to controlling for more than one uncontrolled variables and is given as follows:

$$r_{12/34,...,n} = \frac{r_{12.34,...,(n-1)} - r_{1n.34,...,(n-1)} r_{2n.34,...,(n-1)}}{\sqrt{1 - r^2_{1n.34,...,(n-1)}}\sqrt{1 - r^2_{2n.34,...,(n-1)}}} \quad (2.6)$$

It is possible to obtain the partial correlation coefficient in any order from Equation (2.6) by laboriously calculating several correlation orders.

An assessment of the link between two variables after accounting for the impact of one or more independent factors is known as partial correlation, according to Verma (2016). The following is the partial correlation of X_1 and X_3, adjusted for X_2:

$$r_{13.2} = \frac{r_{13} - r_{12} r_{32}}{\sqrt{(1 - r^2_{12})(1 - r^2_{32})}} \quad (2.7)$$

The bounds of partial correlation are -1 to +1, much as the correlation coefficient. The number of independent variables whose effects are controlled determines the order of a partial correlation. First-order partial correlation is one kind of partial correlation in which the impact of just one variable is regulated. For *(n-2)th* order partial correlation, the generalized formula is as follows:

$$r_{12.34,...,n} = \frac{r_{12.34,...,(n-1)} - r_{1n.34,...,(n-1)} r_{2n.34,...,(n-1)}}{\sqrt{1 - r^2_{1n.34,...,(n-1)}}\sqrt{1 - r^2_{2n.34,...,(n-1)}}}$$

The partial correlation coefficient is used to show the relationship between two variables when additional factors are taken into consideration (OriginLab, n.d.). This is an expression for the method of variance-covariance matrix when two sets of variables, $\mathbf{X}$ and $\mathbf{Y}$, are joined with a set of n_x random variables $\mathbf{X}$ and n_y controlling random $\mathbf{Y}$ variables:

$$\begin{pmatrix} \Sigma_{XX} & \Sigma_{XY} \\ \Sigma_{YX} & \Sigma_{YY} \end{pmatrix}$$

When Y variables are taken into account, the variance-covariance matrix of X variables is as follows:

$$\Sigma_{XX.\mathbf{Y}} = \Sigma_{XX} - \Sigma_{XY}\Sigma_{YY}^{-1}\Sigma_{YX} \tag{2.8}$$

Through normalizing the entries of the result of conditional variance-covariance matrix, using Equation (2.4), one can get the partial correlation coefficient as:

$$\rho_{XX.\mathbf{Y}} = \frac{\sigma_{XX.\mathbf{Y}}^{(2,1)}}{\sqrt{\sigma_{XX.\mathbf{Y}}^{(2,2)}}\sqrt{\sigma_{XX.\mathbf{Y}}^{(1,1)}}} \tag{2.9}$$

2.2 Empirical Study of Partial Correlation Coefficient

The relationship between several physical performance measurements and the overall, regional, and quality muscle mass and quality in older persons living in the community was investigated by Monjo et al. (2023). One hundred ninety-five local seniors—61 men and 134 women—participated in the research. Grip strength, 10-meter maximum walking test (MWT), 30-second standing test (CS30), the vertical jump test (VJT), and timed up-and-go test (TUG) were used to assess physical performance. Partial correlation analysis revealed after adjusting for age, sex, and body mass index, that CS30 was substantially connected with the thickness and intensity of the quadriceps femoris muscle.

Liu *et al*., (2023) released their work titled "Cerebellar gray matter (GM) alterations predict deep brain stimulation (DBS) outcomes in Meige syndrome (MS)". They compared regional and lobular gray matter alterations in the cerebellum between 52 healthy human controls (HCs) and 47 MS patients, as well as between 10 DBS non-responders and 31 DBS responders, using lobule-based and voxel-based morphometric. The Spatially Unbiased Infratentorial Toolbox (SUIT) was used for all volumetric investigations. Additionally, they conducted a partial correlation exploration to look at the connection between changes in clinical ratings and cerebellar GM. Studies of partial correlation revealed a favorable connection between GM volume of the relevant regions/lobules and the frequency of symptom relief following DBS surgery.

Partial correlation analysis was used by Wang et al., (2023), to evaluate the unmanned driving vehicle vibration response in 3D tire-pavement interaction. Autonomous driving will be the main emphasis of transportation in the future. As unmanned driving develops, sensor monitoring and machine control will become increasingly crucial. The results of partial correlation analysis suggested that pavement roughness should be given more consideration in order to guarantee passenger comfort and both surface roughness and vehicle speed should be taken into account simultaneously to ensure cargo safety and road friendliness.

According to Kim *et al*., (2023), the prostate-specific antigen (PSA) level rises following therapy, which is when biochemical recurrence (BCR) of prostate cancer takes place. The success of prostate cancer therapy depends on accurate BCR prediction. They created a model that makes use of a partial correlation neural network (PCNN) to forecast the BCR of prostate cancer. One thousand and twenty-one prostate cancer patients who underwent radical prostatectomy at a tertiary facility provided information for the research. BCR was the result variable, and there were nine input variables. The PCNN was built using feature-sensitive and partial correlation analysis. The neutral network (NN) architecture of the partial correlation neural network (PCNN) is particularly created for BCR prediction. In BCR prediction, the suggested partial correlation neural network (PCNN) performed better than previous machine learning methods, with specificity, accuracy, and sensitivity values of 85.62%, 87.16%, and 90.80%, respectively. The elimination of pointless prediction components through the correlation of the input variables is what makes partial correlation neural networks (PCNN) function better. The results of the investigation indicate that partial correlation neural networks (PCNN) might be used in prostate treatment throughout the clinical phase.

Correlation networks are a common method of displaying financial data because of their readability and simplicity. Sidorov *et al.*, (2022) carried out a study that was mostly focused on comparing the asset returns from the Russian stock market with the partial correlation on network-based and the growth of the Pearson correlation network of the Russian stock market. They selected to focus on data from the Russian financial markets between the years of 2012 and 2022. However, it was discovered that the asset returns of two enterprises with the same cause were improperly linked. They suggested doing their investigation using a partial correlation to avoid this.

As long as product moment correlation coefficients are calculated and interval or ratio data are employed, Conover (1971) proved that higher order partials are completely reliable. Furthermore, Waliczek (1996) states that partial correlations are permitted only in cases when the suitable model fits the pattern of interactions between the variables. A researcher should be aware of some of the limitations of partial correlations, as discussed by Korn (1984). He emphasized the requirement for a constant distribution of the data and an approximately linear connection between the variables. Thus, only data that are approximately normally distributed should be utilized with it. The multivariate normality assumption also seems to be sensitive to Pearson's partial correlation. However, if the correlations between the variables are not linear, any partial correlation might produce false findings. Pedhazur (1982) asserts that changing variables without taking into account theoretical ideas about how they interact with one another might sum up to a misunderstanding of facts and produce either false or nonsensical results. The researcher has to be completely aware of the dependent variable under evaluation after accounting for the impact of one or more independent factors.

The work by Syazali *et al*., (2019) titled "Partial correlation analysis using multiple linear regression: Impact on business environment of interest in digital marketing in the era of industrial revolution 4". Due to resource constraints and other market considerations, Small and Medium Enterprises (SMEs) find it challenging to apply technology in business at the age of four. The marketing of a product usually relies on a number of variables, such as price, product quality, and brand, to convince customers to buy it. The researchers looked at how cost, product quality, and product brand affected buyers' interest in making purchases. The results show that brand and product pricing have the strongest partial relationships with customers' propensities to buy.

Horwitz and Rapoport (1988) employed the partial correlation analysis approach to describe the functional connections between various brain areas. Coefficients of correlation between two sets of local glucose metabolism have attracted a lot of attention because they can show patterns of connectivity across different brain areas in both humans and animals. Partial correlation coefficients, which contrast the regional and global metabolic rates, or correlations between reference rates, which partially eliminate the global metabolic rate to some extent, are two strategies the researchers recommend using to lessen the confounding effect of systematic intra-subject variability in glucose intake.

Analysis of Partial correlation was used in Aloe's (2013) study to synthesize partial effect estimates. Semi-partial, partial, and standardized slope partial effect sizes were discovered for the connection. According to Aloe, partial correlation is especially helpful for meta-analyses when the original study that provided bivariate correlations were not included in the regression models or when it was crucial to isolate the impact of other components. Partial correlation has been utilized extensively.

Ha and Sun, (2014) addressed the challenge of creating a gene co-expression network in a study by utilizing a partial correlation matrix strategy. Partial correlation is also used for financial marketing tactics by Kenett *et al.*, (2015). However, dependence and links between the various firms in the analyzed sample have been affected by employing partial correlation in the analysis of the dependency network approach.

Li (2018) conducted an investigation and assessment of the prediction accuracy of three distinct approaches (Conditional mutual information, partial correlation, and Pearson's correlation coefficient) and concluded that all interaction effects were considered. Because they ignore the relationships between primary effects and interaction effects, direct interaction screening (DIS) techniques like Pearson's Correlation Coefficient approach frequently screen interaction terms wrongly. He suggested employing the Conditional Mutual Information Interaction Screening (CMIIS) technique and the Partial Correlation Interaction Screening (PCIS) methodology as two distinct interaction screening approaches to accomplish this. When partial correlation (PC) is employed, one may assess the link between two variables while reducing the influence of one or more other factors.

Chidiebere (2015) published article title "Multivariate approach to partial correlation analysis." Professional statisticians and users of statistical software have reportedly expressed concern over the partial correlation coefficient of one or more independent variables. He created a variance-covariance matrix using a multivariate technique. In order to look for multivariate partial correlations, he separated the variance covariance matrices while maintaining one or more variables constant. Given the difficulties in analyzing and processing complex data, he used matrices to calculate correlation coefficients between these variables and variance covariance matrices to establish the strength of the relationships between the variables. A normal diagonal matrix was demonstrated to be the representation of a partial correlation coefficient.

Artner *et al*., (2022) examined the creation of partial correlation matrices as part of their work into partial correlation analysis. It is common practice to quickly describe the correlational structure of a group of variables using the pairwise partial correlations because they have the advantage of being clear markers of conditional linear independence while yet preserving the same data as the Pearson correlations. For mathematical convenience, there are several matrix representations of pairwise partial correlations in the literature, despite the fact that their properties have not been thoroughly examined. The authors have constructed necessary and sufficient requirements for the eigenvalues of widely defined partial correlation matrices, so guaranteeing the validity of the correlation structure in this work. We will concentrate on the computational cost of constructing correlation structures with partial correlation matrices after these conditions are satisfied. Additionally, they contrast the legitimate Pearson correlation structure space with the legitimate partial correlation structure space. Given that these spaces are similar in terms of rotation and volume for all dimensions, it is possible to use the present methods for the creation and approximation of correlation matrices to produce valid partial correlation matrices using a straightforward formula. They then provide straightforward partial correlation criteria for often taken for granted sparse structures.

Jia *et al*., (2021) released a paper title "Research on dynamic response of subsea control system based on partial correlation analysis." This article proposes simulation models for closed loop and closed non-loop circuits for a particular underwater hydraulic system. Subsequently, it utilizes a single element analysis to examine the external and internal factors, such as the return distance and water depth, in addition to the actuators, underwater accumulators, and pipeline damping settings. The partial correlation theory was used to evaluation the link between the variables influencing the underwater hydraulic system's control response based on simulated data from a single factor. It is possible to enhance the underwater hydraulic system's responsiveness by knowing the relative importance of the key variables influencing the control response.

Zhang et al. (2021) used ordinary correlation (OR) and partial correlation (PC) research to examine the link between subjective well-being (SWB) and ecological footprint (EF) and to gauge how people's enjoyment is impacted by their surroundings. Control factors include the GDP, the proportion of wage and salaried workers (WSW), the rate of urbanization (UR), the literacy rate (LR), the youth life expectancy (YLE), the political stability (PS), and voice accountability (VA). Ecological cropland footprints (ECL), total biocapacity (TBC), ecological built-up land footprints (EBL), and ecological grazing land footprints (EGL) all have a substantial good and bad influence on subjective well-being (SWB). Subjective well-being (SWB) and the ecological carbon footprint (ECF) show a significant negative relationship in affluent countries. They concluded that a knowledge gap is closed and our understanding of pleasure is enhanced by using partial correlation to examine the link between EF and SWB.

According to Shan *et al*., (2020), researchers are collecting repeated data more often in order to study the trajectory of measurement change across time. A novel therapeutic target may be identified by establishing a link between a recurrent measurement and another that is regarded to be a biomarker for the advancement of a disease. When it is related to one of the two measures, partial correlation with the impact of the third variable eliminated can yield a reliable estimate of association, in contrast to the present raw correlation for repeated data. They suggest utilizing linear regression models to determine residuals, which show the link between each measurement and a third variable. After fitting a linear mixed model with the estimated residuals, the partial correlation for the replication data is obtained.

The Pearson correlation coefficient or Spearman correlation coefficient, which show marginal correlations, are more prone to unexpected results when analyzing the effect of genetic interaction on yeast. Because of this, Roverato and Castelo (2017) substitute partial correlation for marginal correlations in their analysis of the level of gene co-expression.

Jung and Chang, (2016) conducted a partial correlation research on the Korean stock market (KOSPI). Showing that the link between stock returns is generally strengthened by the market return, Pearson correlation coefficients that are conditional on market return are often bigger than those of partial correlation. Examined is the distinction between partial correlation and Pearson correlation.

Vargha *et al.*, (2012) released article with the title "Interpretation problems of the partial correlation with nonnormally distributed variables." They asserted that partial correlation is a widely used metric for assessing the bivariate correlation of two quantitative variables after taking into consideration the influence of one or more additional factors. The correlation that would result if the variables to be eliminated were fixed, that is, they could not change and have an impact on the other variables, is what is known as the partial correlation in statistical literature. This is how the partial correlation is typically conceptualized. Their study shows, both theoretically and through practical examples, that when the multivariate normality assumption is broken (due to nonlinear relationships between the variables under investigation, for example), there will be a fundamental error in the conventional interpretation of the partial correlation coefficient. Under some conditions, the conditional correlation may have a very strong negative value while the partial correlation coefficient may have a value that is exceptionally high and close to 1. If nonlinear interactions are anticipated, the article recommends partialing out a certain function (often the square) of the variables whose effects must be avoided in order to handle this issue.

It is mentioned by Marrelec *et al.* (2017) that partial correlation has been studied to enhance structural equation modeling of functional magnetic resonance imaging data. Functional magnetic resonance imaging (fMRI) data is used in effective connectivity studies to evaluate the inter-regional interactions between various brain regions. Structural equation modeling (SEM) is the main technique used to evaluate effective connectivity. In their study, they present a technique that does a partial correlation analysis given a number of areas. This method offers a data-driven approach to connection as it doesn't require prior understanding of the anatomical or functional relationships. Their goal was to demonstrate the usefulness of partial correlation analysis for effective connection inquiry by reanalyzing data that had previously been published by Bullmore, Horwitz, Honey, Brammer, Williams, and Sharma (2000). In particular, they show the wide range of applications for which partial correlation analysis may be applied. It can make recommendations for which connections are organizing interactions properly in a pre-processing stage and which ones are having minimal impact on the connectivity pattern. The efficiency of SEM optimization strategies may be swiftly and simply evaluated, and it can be used to show which model assumptions are accurate and which ones require more work to better fit the data.

It was made clear by Ogunleye *et al.* (2022) that the method of variance-covariance matrix could only compute partial correlation coefficients up to order one. Furthermore, they emphasized that, to the best of their knowledge, no new methodology (variance-covariance matrix approach) has been developed to handle partial correlation coefficients higher than the first order. We are able to conclude that Ogunleye *et al.* had difficulties estimating partial correlation coefficients beyond first order because they did not partition the variance-covariance matrix using the appropriate approach.

2.3 Other Methods in Relation to Partial Correlation

As per the research conducted by Brown and Hendrix (2014), partial correlation is associated with and serves as the basis for other statistical approaches that are now gaining attention, such as causal modeling and structural equation modeling (SEM). Partial correlation has a long history as a tool for identifying the nature of causal links between observable variables, according to Lazarsfeld (1955), Simon (1957), and Blalock (1961, 1963). Path analysis and partial correlation have a conceptual and mathematical link [see Edwards (1979)]. Wright (1934, 1954, 1960a, 1960b) [see also Tukey (1954)] invented path analytic approaches in the 1930s, and they are also the basis of the advancements that allow structural equations modeling. It appears from this that the idea of partial correlation mathematics forms the foundation for minimizing the effects of every predictor in a stepwise regression to determine the additional predictive contribution of the remaining variables. Partial correlation approaches have several interesting applications, one of which is mediation demonstration [Baron & Kenny (1986), Judd & Kenny (1981). To put it another way, partial correlation approaches are employed to demonstrate that a third variable really mediates the causal effects of one variable on another, and that the lack of this third variable causes the correlative relationship to evaporate or, at the very least, severely weaken. Therefore, one may evaluate causal hypotheses and provide statistical control by using partial correlation.

2.4 Multivariate Normal Distribution

Baba *et al.* (2004) reported that when random variables are concurrently distributed as the multivariate normal, there is compatibility between the conditional correlation and the partial correlation.

According to Rencher (2001), the cornerstone for the great majority of multivariate operations is the multivariate normal distribution. This makes normal distribution knowledge valuable for the study of multivariate techniques. Mean, variance, and covariance are the only variables required to completely describe the multivariate normal distribution. Its bivariate plot shows linear trends. It is also important to remember that two or more variables in a multivariate normal distribution are independent if they are uncorrelated. The multivariate normal may still be a useful approximation even if the data are not multivariate normal, according to the central limit theorem, which claims that the samples mean vector is approximately multivariate normal. This is especially valid when drawing conclusions based on the samples mean vector. Although multivariate normality may not always accurately represent real data, it is frequently a good approximation to the underlying distribution.

The multivariate density function is as follows:

$$h(v) = \frac{1}{(\sqrt{2\pi})^p |\Sigma|^{\frac{1}{2}}} e^{-(v-\mu)'\Sigma^{-1}(v-\mu)/2}$$

where Σ is the variance-covariance matrix, μ is the mean vector, and v is the multivariate normal distribution variable.

The expression $(v-\mu)'\Sigma^{-1}(v-\mu)$ in the multivariate density function's exponent is well recognized as the Mahalanobis distance, which is often referred to as the squared generalized distance between v to μ.

v values are considered to be concentrated toward the mean if σ^2 in the univariate normal is small. In a similar manner, a small value of $|\Sigma|$ in a multivariate scenario denotes multicollinearity among the variables or that in p-space, the values of $v's$ are concentrated around μ. The term "multicollinear" refers to highly correlated variables, which means that their effective dimensionality is smaller than p.

According to Rencher (2001) the multivariate normal random variable has the following properties.

Using a multivariate normal distribution $N_p(\mu, \Sigma)$ and a random $p \times 1$ vector v:

1. If a is a vector of constants, then the function $av' = a_1v_1 + a_2v_2 + ... + a_pv_p$ is univariate normal and the linear combination of the vector v is normal. As a result, $a'v$ is $N(a'\mu, \Sigma)$ if v is $N_p(\mu, \Sigma)$. Additionally, a multivariate normal distribution may be found in the linear combinations of Av if A is a constant $p \times p$ matrix of rank, where $q \le p$.
2. A standardized vector

$$G = (T')^{-1}(v - \mu)$$

where the Cholesky method is used to factorize $\Sigma = T'T$. Therefore

$$G = (\Sigma^{1/2})^{-1}(v - \mu)$$

where $\Sigma^{1/2}$ is the squared root matrix of Σ that is symmetric. All means, all variance, and all correlation are equivalent to zero in the standardized vector corresponding to the random variable.

3. The square of p independent standard normal random variables yields a chi-square random variable with p degrees of freedom. As a result, if G is the standard vector, then $\sum_{j=1}^{p} G_j^2 = G'G$ has an χ^2 −distribution with p degrees of freedom, indicated by $\chi^2(p)$. If p is $N_p(\mu,\Sigma)$, then $(v-\mu)'\Sigma^{-1}(v-\mu)$ is χ_p^2.

2.5 Chapter Summary

The literature demonstrates that the topic of partial correlation analysis has been investigated for a long time. Some writers have developed the subject extensively, and publications have also offered applications to data analysis. A wide range of fields, most notably biology, medicine, economics, accounting, engineering, and other related fields, have embraced the usage of partial correlation.

The study is apparently motivated by an assertion by Ogunleye *et al*., (2022). The purpose of the research therefore is to discover via variance-covariance matrix technique how this difficulty may be resolved.

CHAPTER THREE

RESEARCH METHODS

3.0 Introduction

This thesis employs multivariate application to create a variance-covariance matrix as its statistical technique. The theory of partial correlation coefficients will be established and proven using the conditional variance-covariance matrix strategy. To assess the degree of link between variables after holding some variable(s) constant, the effects of independent variables will be examined concurrently. We analysed the secondary data obtained using R software programme and Minitab 19 for realistic demonstrations of the theory of variance-covariance matrix for computing orders of partial correlation coefficients.

3.1 Datasets

The original dataset given by Ogunleye *et al.*, (2022) was used in conjunction with the method of variance-covariance matrix strategy to compute partial correlation coefficients orders. Only five (5) of the nine (9) variables were used in their research, and partial correlation coefficients were computed using four distinct approaches.

3.2 Organization of Multivariate data

Let's use the notation v_{kl} to denote a specific lth variable value that was seen on the kth item. That is, v_{kl} = measurement of the lth variable on the kth item. Therefore, the following may be used to display n measurements on p variables:

$$\begin{array}{lccccccc} & \text{variable}\,1 & \text{variable}\,2 & \cdots & \text{variable}\,l & \cdots & \text{variable}\,p \\ item\,1 & v_{11} & v_{12} & \cdots & v_{1l} & \cdots & v_{1p} \\ item\,2 & v_{21} & v_{22} & \cdots & v_{2l} & \cdots & v_{2p} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ item\,k & v_{k1} & v_{k2} & \cdots & v_{kl} & \cdots & v_{kp} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ item\,n & v_{n1} & v_{n2} & \cdots & v_{nl} & \cdots & v_{np} \end{array}$$

An alternate representation of the multivariate data given above would be a rectangular array represented by the symbol $\mathbf{V}$, which has n rows and p columns:

$$\mathbf{V} = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1l} & \cdots & v_{1p} \\ v_{21} & v_{22} & \cdots & v_{2l} & \cdots & v_{2p} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{k1} & v_{k2} & \cdots & v_{kl} & \cdots & v_{kp} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{n1} & v_{n2} & \cdots & v_{nl} & \cdots & v_{np} \end{bmatrix} \tag{3.1}$$

The data is then added to the array $\mathbf{V}$, including all observations on all variables. Additionally, the number of measures (n) represents a subset of all possible measurements.

3.3 Multivariate Normal Distribution

A univariate normal density generalization to two or more variables (that is, $p \geq 2$ dimensions) is the multivariate normal density. The squared statistical distance in standard deviation units between μ and v is provided by:

$$\left(\frac{v-\mu}{\sigma}\right)^2 = (v-\mu)(\sigma^2)^{-1}(v-\mu)$$

We have $\mathbf{v}_{p\times 1}$, and parameters $\boldsymbol{\mu}_{p\times 1}$ and $\boldsymbol{\Sigma}_{p\times p}$. Given below is the multivariate normal exponent term

$$(\mathbf{v}-\boldsymbol{\mu})'(\boldsymbol{\Sigma})^{-1}(\mathbf{v}-\boldsymbol{\mu}) \qquad -\infty < v_k < \infty$$

v's joint density is determined by:

$$f(v) = \prod_{k=1}^{p} f(v_k)$$

$$f(\mathbf{v}) = \prod_{k=1}^{p} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{1}{2}\left(\frac{v_k - \mu_k}{\sigma}\right)^2 \right\}$$
$$= \frac{1}{(2\pi)^{p/2} \left|\sigma^2 \mathbf{I}_p\right|^{1/2}} \exp\left\{ -\frac{1}{2}(v-\mu)'(\sigma^2 \mathbf{I}_p)^{-1}(v-\mu) \right\}$$

An independent multivariate normal distribution's joint density function is shown here, written as $V \sim N_p(\mu, \sigma^2 \mathbf{I}_p)$.

The independent multivariate normal density is often adapted to the multivariate normal distribution by replacing $\sigma^2 \mathbf{I}_p$ with a positive definite variance-covariance matrix $\mathbf{\Sigma}$ as

$$f(\mathbf{v}) = \frac{1}{(2\pi)^{p/2} |\mathbf{\Sigma}|^{1/2}} \exp\left\{ -\frac{1}{2}(\mathbf{v}-\mu)'(\mathbf{\Sigma})^{-1}(\mathbf{v}-\mu) \right\}$$

3.3.1 Multivariate Parameters and Statistics

3.3.1.1 Population Mean Vector

It is possible to create a matrix using the means and covariances of the $p \times 1$ random vector $\mathbf{V}$. The symmetric variance-covariance matrix $\mathbf{\Sigma} = E(\mathbf{V}-\boldsymbol{\mu})(\mathbf{V}-\boldsymbol{\mu})'$ clearly indicates the p variances σ_{jj} and the $p(p-2)/2$ unique covariances $\sigma_{jl}\,(j<l)$. The vector of means $\boldsymbol{\mu} = E(\mathbf{V})$ also includes the anticipated value for each element.

Specifically, the population mean vector is given by

$$E(\mathbf{v}_{p\times 1}) = E\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_p \end{bmatrix} = \begin{bmatrix} E(v_1) \\ E(v_2) \\ \vdots \\ E(v_p) \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{bmatrix} = \boldsymbol{\mu}$$

3.3.1.2 Population Variance-Covariance Matrix

A variance-covariance matrix $(\boldsymbol{\Sigma})$ is a square matrix where the off-diagonal components represent the covariance and the diagonal members represent the variance. The spread of data from a dataset's mean is known as variance, which is a measure of dispersion, whereas covariance quantifies the combined variation of two variables. The variance-covariance matrix is represented mathematically by

$$\begin{aligned}
\boldsymbol{\Sigma} &= E(\mathbf{V}-\mu)(\mathbf{V}-\mu)' \\
&= E\left[\begin{pmatrix} V_1-\mu_1 \\ V_2-\mu_2 \\ \vdots \\ V_p-\mu_p \end{pmatrix}\begin{pmatrix} V_1-\mu_1 & V_2-\mu_2 & \cdots & V_p-\mu_p \end{pmatrix}\right] \\
&= E\begin{bmatrix} (V_1-\mu_1)^2 & (V_1-\mu_1)(V_2-\mu_2) & \cdots & (V_1-\mu_1)(V_p-\mu_p) \\ (V_2-\mu_2)(V_1-\mu_1) & (V_2-\mu_2)^2 & \cdots & (V_2-\mu_2)(V_p-\mu_p) \\ \vdots & \vdots & \ddots & \vdots \\ (V_p-\mu_p)(V_1-\mu_1) & (V_p-\mu_p)(V_2-\mu_2) & \cdots & (V_p-\mu_p)^2 \end{bmatrix} \\
&= \begin{bmatrix} E(V_1-\mu_1)^2 & E(V_1-\mu_1)(V_2-\mu_2) & \cdots & E(V_1-\mu_1)(V_p-\mu_p) \\ E(V_2-\mu_2)(V_1-\mu_1) & E(V_2-\mu_2)^2 & \cdots & E(V_2-\mu_2)(V_p-\mu_p) \\ \vdots & \vdots & \ddots & \vdots \\ E(V_p-\mu_p)(V_1-\mu_1) & E(V_p-\mu_p)(V_2-\mu_2) & \cdots & E(V_p-\mu_p)^2 \end{bmatrix}
\end{aligned}$$

The population variance-covariance matrix of a random vector **V** is therefore defined as

$$\boldsymbol{\Sigma} = E(\mathbf{V}-\mu)(\mathbf{V}-\mu)'$$

$$Cov(\mathbf{V}) = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1P} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2P} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{P1} & \sigma_{P2} & \cdots & \sigma_{PP} \end{bmatrix}$$

3.3.2 Mean Vector and Variance-Covariance Matrix

Since the parameters $\boldsymbol{\Sigma}$ and $\boldsymbol{\mu}$ are unknown, the maximum likelihood estimation approach may be employed to estimate them. Equation (3.1) may be used to create an unbiased sample variance-covariance matrix and sample mean vector as follows:

$$\bar{v} = \frac{1}{n}\sum_{k=1}^{n} v_k$$

$$S = \frac{1}{n-1}\sum_{k=1}^{n}(v_k - \bar{v})(v_k - \bar{v})' \tag{3.2}$$

Given the sum of squares and sum of products matrix, the Wishart Distribution matrix is obtained as follows:

$$\mathbf{H} = \sum_{k=1}^{n}(\mathrm{v}_k - \bar{\mathrm{v}})(\mathrm{v}_k - \bar{\mathrm{v}})' \tag{3.3}$$

Equation (3.3) may be substituted into Equation (3.2) to create an unbiased sample variance-covariance matrix, which is provided by

$$\mathbf{S} = \frac{1}{\mathrm{n}-1}\mathbf{H} \tag{3.4}$$

Theorem 3.1

In a random sampling of n observation vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ from $N_p(\mu, \Sigma)$, the sample mean vector $\bar{v} = \frac{1}{n}\sum_{k=1}^{n} v_k$ is the maximum likelihood estimator of μ and $\hat{\Sigma} = \frac{1}{n}\sum_{k=1}^{n}(v_k - \bar{v})(v_k - \bar{v})'$ is the maximum likelihood estimator of Σ.

Proof:

The probability density function for variable v_k is defined

$$f(v_k) = \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}}\exp\left\{-\frac{1}{2}(v_k - \mu)'\Sigma^{-1}(v_k - \mu)\right\}$$

so that the likelihood function (L) of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ may be easily written as

$$
\begin{aligned}
L &= f(v_1, v_2, \ldots, v_n) \\
&= \prod_{k=1}^{n} f(v_k) \\
&= \prod_{k=1}^{n} \frac{1}{(2\pi)^{p/2} |\boldsymbol{\Sigma}|^{1/2}} \exp\left\{-\frac{1}{2}(v_k - \mu)' \boldsymbol{\Sigma}^{-1} (v_k - \mu)\right\} \\
&= \frac{1}{(2\pi)^{np/2} |\boldsymbol{\Sigma}|^{n/2}} \exp\left\{-\frac{1}{2}\sum_{k=1}^{n}(v_k - \mu)' \boldsymbol{\Sigma}^{-1} (v_k - \mu)\right\}
\end{aligned}
$$

Taking natural log both sides and this gives

$$
\ln L = -\frac{np}{2}\ln(2\pi) - \frac{n}{2}\ln|\boldsymbol{\Sigma}| - \frac{1}{2}\sum_{k=1}^{n} \boldsymbol{\Sigma}^{-1}(v_k - \mu)^2 \qquad (3.5)
$$

3.3.3 Maximum Likelihood Estimate of Mean Vector

When μ is taken into account, equation (3.5) is differentiated and equals zero

$$
\begin{aligned}
\frac{d}{d\mu}(\ln L) &= \frac{d}{d\mu}[-\frac{np}{2}\ln(2\pi) - \frac{n}{2}\ln|\Sigma| - \frac{1}{2}\sum_{k=1}^{n} \Sigma^{-1}(v_k - \mu)^2 \\
&= -\sum_{k=1}^{n} \Sigma^{-1}(v_k - \mu) = 0
\end{aligned}
$$

Since Σ^{-1} is non-singular matrix, we have

$$
\sum_{k=1}^{n}(v_k - \mu) = 0
$$

$$
\Rightarrow \sum_{k=1}^{n} v_k = n\mu
$$

$$
\therefore \hat{\mu} = \frac{1}{n}\sum_{k=1}^{n} v_k = \bar{v} \qquad (3.6)
$$

The population mean vector is estimated using Equation (3.6), which is an unbiased maximum likelihood estimator.

3.3.4 Variance-Covariance Matrix Maximum Likelihood Estimate

From Equation (3.5), we apply the concept of trace of a matrix

$$\ln L = -\frac{np}{2}\ln(2\pi) + \frac{n}{2}\ln\left|\Sigma^{-1}\right| - \frac{1}{2}\left(\sum_{k=1}^{n} tr[(v_k - \mu)'\Sigma^{-1}(v_k - \mu)]\right)$$

$$= -\frac{np}{2}\ln(2\pi) + \frac{n}{2}\ln\left|\Sigma^{-1}\right| - \frac{1}{2}tr\left(\sum_{k=1}^{n}(v_k - \mu)(v_k - \mu)'\Sigma^{-1}\right)$$

We differentiate the above equation with regard to Σ^{-1} and equate it to zero

$$\frac{d}{d\Sigma^{-1}}(\ln L) = \frac{d}{d\Sigma^{-1}}\left[-\frac{np}{2}\ln(2\pi) + \frac{n}{2}\ln\left|\Sigma^{-1}\right| - \frac{1}{2}tr\left(\sum_{k=1}^{n}(v_k - \mu)(v_k - \mu)'\Sigma^{-1}\right)\right]$$

$$= \frac{n}{2}\ln\frac{d}{d\Sigma^{-1}}\left|\Sigma^{-1}\right| - \frac{1}{2}tr\left(\sum_{k=1}^{n}(v_k - \mu)(v_k - \mu)'\frac{d}{d\Sigma^{-1}}(\Sigma^{-1})\right)$$

We know that $\left|A^{-1}\right| = \frac{1}{|A|}$ and from matrix definition of $\ln\frac{d}{dA}|A| = (A^{-1})'$,

therefore we deduced that $\ln\frac{d}{dA}\left|\boldsymbol{\Sigma}^{-1}\right| = \frac{1}{(\boldsymbol{\Sigma}^{-1})'} = \boldsymbol{\Sigma}$. That is,

$$\frac{d}{d\Sigma^{-1}}(\ln L) = \frac{n}{2}\Sigma - \frac{1}{2}\left(\sum_{k=1}^{n}(v_k - \mu)(v_k - \mu)'\right)$$

Equating to zero and re-arranging, gives

$$0 = \frac{n}{2}\Sigma - \frac{1}{2}\left(\sum_{k=1}^{n}(v_k - \mu)(v_k - \mu)'\right)$$

$$\Rightarrow \frac{n}{2}\Sigma = \frac{1}{2}\left(\sum_{k=1}^{n}(v_k - \mu)(v_k - \mu)'\right)$$

$$\Rightarrow \hat{\Sigma} = \frac{1}{n}\left(\sum_{k=1}^{n}(v_k - \mu)(v_k - \mu)'\right) \tag{3.7}$$

Equation (3.7) is the maximum likelihood estimator of the variance-covariance matrix (Σ) and is biased estimator Σ.

Theorem 3.2

In a random sampling of n observation vectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ from $N_p(\mu, \Sigma)$, the estimator $\bar{v} = \frac{1}{n}\sum_{k=1}^{n} v_k$ is an unbiased estimator of μ and $S = \frac{1}{n-1}H = \frac{1}{n-1}\sum_{k=1}^{n}(v_k - \bar{v})(v_k - \bar{v})'$ is an unbiased estimator of Σ.

Proof

We can rewrite Equation (3.3) as

$$\begin{aligned} H &= \sum_{k=1}^{n} (v_k - \bar{v})(v_k - \bar{v})' \\ &= \sum_{k=1}^{n} [(v_k - \mu) + (\mu - \bar{v})]\,[(v_k - \mu) + (\mu - \bar{v})]' \\ &= \sum_{k=1}^{n} (v_k - \mu)(v_k - \mu)' + \sum_{k=1}^{n} (v_k - \mu)(\mu - \bar{v})' + \sum_{k=1}^{n} (\mu - \bar{v})(v_k - \mu)' + \sum_{k=1}^{n} (\mu - \bar{v})(\mu - \bar{v})' \end{aligned}$$

We take expectation on both sides.

But $\sum_{k=1}^{n} E(\mu - \bar{v})(v_k - \mu)' + \sum_{k=1}^{n} E(\mu - \bar{v})(\mu - \bar{v})' = 0$

$$\begin{aligned} E(H) &= \sum_{k=1}^{n} E(v_k - \mu)(v_k - \mu)' + \sum_{k=1}^{n} E(\mu - \bar{v})(\mu - \bar{v})' \\ &= \sum_{k=1}^{n} E(v_k - \mu)(v_k - \mu)' - \sum_{k=1}^{n} E(\bar{v} - \mu)(\bar{v} - \mu)' \\ &= nE(v_k - \mu)(v_k - \mu)' - nE(\bar{v} - \mu)(\bar{v} - \mu)' \\ &= n\,\mathrm{var}(v_k) - n\,\mathrm{var}(\bar{v}) \end{aligned}$$

We substitute $\mathrm{var}(\bar{v}) = \frac{\Sigma}{n}$ and $\mathrm{var}(v_k) = \Sigma$ into the above Equation and obtain

$$\begin{aligned} E(H) &= n\Sigma - \Sigma \\ \Sigma &= E\left(\frac{H}{n-1}\right) = E(S) \end{aligned} \tag{3.8}$$

For the variance-covariance matrix (Σ), Equation (3.8) provides an unbiased estimate.

3.3.5 Sample Mean Vector

The sample mean vector is shown as follows and is defined as the average of the n observation vectors or the average of each of the p variables separately:

$$\bar{v} = \frac{1}{n}\sum_{k=1}^{n} v_k = \begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \vdots \\ \bar{v}_p \end{pmatrix}$$

Consequently, $\bar{v}_1$ denotes the first variable's mean of the n observations, $\bar{v}_2$ the second variable's mean, and so on. The observation vectors should be entered as rows rather than columns since n is typically bigger than p, making it easier to tabulate the data. It was pointed out that the first subscript, k, stands for units (objects), while the second subscript, l, stands for variables.

The sample mean vector $\bar{v}$ can also be determined from the data matrix $\mathbf{V}$ by using matrix notation as shown below:

$$\bar{\mathbf{v}}_{p\times 1} = \begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \vdots \\ \bar{v}_p \end{bmatrix} = \begin{bmatrix} \frac{\sum_{k=1}^{n} v_{k1}}{n} \\ \frac{\sum_{k=1}^{n} v_{k2}}{n} \\ \vdots \\ \frac{\sum_{k=1}^{n} v_{kp}}{n} \end{bmatrix}$$

$$= \frac{1}{n}\begin{bmatrix} v_{11} + v_{21} + \cdots + v_{n1} \\ v_{12} + v_{22} + \cdots + v_{n2} \\ \vdots \\ v_{1p} + v_{2p} + \cdots + v_{np} \end{bmatrix}$$

$$= \frac{1}{n}\begin{bmatrix} v_{11} & v_{21} & \cdots & v_{n1} \\ v_{12} & v_{22} & \cdots & v_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ v_{1p} & v_{2p} & \cdots & v_{np} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$\bar{\mathbf{v}}_{p\times 1} = \frac{1}{n}\mathbf{V}'\mathbf{j} \tag{3.9}$$

where $\mathbf{V}'$ is a $p \times n$ matrix and **j** is a vector of one's with order $n \times 1$.

3.3.6 Sample Variance-Covariance Matrix

The sample variance-covariance matrix $\mathbf{S} = (s_{jl})$ denotes the matrix of sample variances and covariances for the p variables:

$$\mathbf{S} = (s_{jl}) = \begin{pmatrix} s_{11} & s_{12} & \cdots & s_{1p} \\ s_{21} & s_{22} & \cdots & s_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ s_{p1} & s_{p2} & \cdots & s_{pp} \end{pmatrix}$$

While every possible pairwise sample covariance in $\boldsymbol{S}$ is off the diagonal, the sample variances of the p variables are on the diagonal.

The lth variable's sample variance is provided as:

$$\begin{aligned} s_{ll} = s_l^2 &= \frac{1}{n-1}\sum_{k=1}^{n}(v_{kl} - \bar{v}_l)^2 \\ &= \frac{1}{n-1}\left(\sum_{k=1}^{n} v_{kl}^2 - n\bar{v}_l^2\right) \end{aligned}$$

The following is the sample covariance for the variables j and l:

$$\begin{aligned} s_{jl} &= \frac{1}{n-1}\sum_{k=1}^{n}(v_{kj} - \bar{v}_j)(v_{kl} - \bar{v}_l) \\ &= \frac{1}{n-1}\left(\sum_{k=1}^{n} v_{kj}v_{kl} - n\bar{v}_j\bar{v}_l\right) \end{aligned}$$

The sample variance-covariance matrix $\boldsymbol{S}$ may alternatively be described by the observation vectors in the following way:

$$\begin{aligned} S &= \frac{1}{n-1}\sum_{k=1}^{n}(v_k - \bar{v})(v_k - \bar{v})' \\ &= \frac{1}{n-1}\left(\sum_{k=1}^{n} v_k v'_k - n\bar{v}\bar{v}'\right) \end{aligned}$$

Matrix notation may also be used to directly extract the sample variance-covariance matrix **S** from the data matrix **V**. To compute the variance-covariance matrix, we centre each column of **V** and write it as follows:

$$\begin{pmatrix} v_{11}-\bar{v}_1 & v_{12}-\bar{v}_2 & \cdots & v_{1p}-\bar{v}_p \\ v_{21}-\bar{v}_1 & v_{22}-\bar{v}_2 & \cdots & v_{2p}-\bar{v}_p \\ \vdots & \vdots & \ddots & \vdots \\ v_{n1}-\bar{v}_1 & v_{n2}-\bar{v}_2 & \cdots & v_{np}-\bar{v}_p \end{pmatrix} = \mathbf{V} - \frac{1}{n}\mathbf{JV}$$

$$\begin{aligned} \mathbf{S} &= \frac{1}{n-1}(\mathbf{V} - \frac{1}{n}\mathbf{JV})'(\mathbf{V} - \frac{1}{n}\mathbf{JV}) \\ &= \frac{1}{n-1}\left[\mathbf{V'V} - \mathbf{V'}\left(\frac{1}{n}\mathbf{J}\right)\mathbf{V}\right] \\ &= \frac{1}{n-1}\mathbf{V'}\left(\mathbf{I} - \frac{1}{n}\mathbf{J}\right)\mathbf{V} \end{aligned} \tag{3.10}$$

where **J** is $n \times n$ matrix with one's and **I** is $n \times n$ identity matrix. Due to the direct usage of the multivariate data matrix $\mathbf{V}_{(n\times p)}$ and the fact that the matrix $\mathbf{I} - \mathbf{J}/n$ of dimension $n \times n$, the Expression (3.10) gives a useful representation of **S**

3.3.7 Population Correlation Coefficient Matrix - ρ_{jl}

"Population correlation," also known as "zero order correlation," is a statistical measure of the correlation between two variables (the independent and dependent variables) without accounting for the impact of other variables. This suggests that a Pearson correlation and a zero order correlation are equivalent.

Mathematically, population correlation coefficient (ρ_{jl}) or zero order correlation coefficient is defined by the variances σ_{jj} and σ_{ll}, and covariances σ_{jl} as:

$$\rho_{jl} = \frac{\sigma_{jl}}{\sqrt{\sigma_{jj}}\sqrt{\sigma_{ll}}} = \frac{\operatorname{cov}(V_j, V_l)}{\sqrt{\operatorname{var}(V_j)}\sqrt{\operatorname{var}(V_l)}} \quad (3.11)$$

where $-1 \le \rho_{jl} \le 1$

Equation (3.11) provides the variance-covariance matrix of a $p \times p$ symmetric matrix, from which we may get the population correlation matrix as follows:

$$\rho = \begin{bmatrix} \frac{\sigma_{11}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{11}}} & \frac{\sigma_{12}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{22}}} & \cdots & \frac{\sigma_{1p}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{pp}}} \\ \frac{\sigma_{21}}{\sqrt{\sigma_{22}}\sqrt{\sigma_{11}}} & \frac{\sigma_{22}}{\sqrt{\sigma_{22}}\sqrt{\sigma_{22}}} & \cdots & \frac{\sigma_{2p}}{\sqrt{\sigma_{22}}\sqrt{\sigma_{pp}}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\sigma_{p1}}{\sqrt{\sigma_{pp}}\sqrt{\sigma_{11}}} & \frac{\sigma_{p2}}{\sqrt{\sigma_{pp}}\sqrt{\sigma_{22}}} & \cdots & \frac{\sigma_{pp}}{\sqrt{\sigma_{pp}}\sqrt{\sigma_{pp}}} \end{bmatrix} = \begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1p} \\ \rho_{21} & 1 & \cdots & \rho_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{p1} & \rho_{p2} & \cdots & 1 \end{bmatrix} \quad (3.12)$$

The variance-covariance matrix may be used to derive the correlation matrix, and vice versa. It is possible to write equation (3.11) as follows:

$$\rho_{jl} = \left(\sqrt{\sigma_{jj}}\right)^{-1} \sigma_{jl} \left(\sqrt{\sigma_{ll}}\right)^{-1} \quad (3.13)$$

we substitute $\sigma_{jl} = \mathbf{\Sigma}$, and $\sqrt{\sigma_{jj}} = \sqrt{\sigma_{ll}} = \mathbf{M}^{1/2}$, where $\mathbf{M}^{1/2}$ is a special $p \times p$ diagonal standard deviation matrix as define by:

$$\mathbf{M}^{1/2} = \begin{bmatrix} \sqrt{\sigma_{11}} & 0 & \cdots & 0 \\ 0 & \sqrt{\sigma_{11}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\sigma_{11}} \end{bmatrix}$$

From Equation (3.13) we obtain the population correlation matrix ($\boldsymbol{\rho}$) as given by:

$$\rho = (\mathbf{M}^{1/2})^{-1}\boldsymbol{\Sigma}(\mathbf{M}^{1/2})^{-1} \tag{3.14}$$

It is abundantly evident from Equation (3.14) that, to derive the Pearson correlation or population correlation matrix (ρ), the variance covariance matrix ($\boldsymbol{\Sigma}$) may be used.

3.3.8 Sample Correlation Matrix

By replacing the population variance-covariance matrix ($\boldsymbol{\Sigma}$) with the sample correlation matrix ($\mathbf{S}$) in Equation (3.14), we can get the sample correlation matrix from the sample variance-covariance matrix. Consequently, we defined our sample correlation matrix as follows:

$$\mathbf{R} = (\mathbf{M}^{1/2})^{-1}\mathbf{S}(\mathbf{M}^{1/2})^{-1} \tag{3.15}$$

3.4 Conditional Distribution

A conditional distribution that is obtained from the multivariate normal distribution must first be introduced in order to comprehend the idea of partial correlations.

Suppose the multivariate normal vector $\mathbf{V} = (\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_p)$ is partitioned as $\mathbf{V}' = \left(\mathbf{V}^{(1)} \,\vdots\, \mathbf{V}^{(2)}\right)$ with $\mathbf{V}^{(1)} = (\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_s)$ and $\mathbf{V}^{(2)} = (\mathbf{V}_{s+1}, \mathbf{V}_{s+2}, \ldots, \mathbf{V}_p)$, $t = p - s$. It is also assumed that the mean vector and variance-covariance matrix are

$$\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \hline \boldsymbol{\mu}_2 \end{pmatrix}, \qquad \boldsymbol{\Sigma} = \left(\begin{array}{c|c} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \hline \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{array}\right) \tag{3.16}$$

where

$\boldsymbol{\mu}_1$ is a $s \times 1$ mean vector of sub-vector $\mathbf{V}^{(1)}$;

$\boldsymbol{\mu}_2$ is a $t \times 1$ mean vector of sub-vector $\mathbf{V}^{(2)}$;

$\mathbf{\Sigma}_{11}$ is a $s \times s$ variance-covariance matrix of the sub-vector $\mathbf{V}^{(1)}$;

$\mathbf{\Sigma}_{22}$ is a $t \times t$ variance-covariance matrix of the sub-vector $\mathbf{V}^{(2)}$;

$\mathbf{\Sigma}_{12}$ is a $s \times t$ variance-covariance matrix of sub-vector $\mathbf{V}^{(1)}$ and $\mathbf{V}^{(2)}$; and

$\mathbf{\Sigma}_{21}$ is a $t \times s$ variance-covariance matrix sub-vector $\mathbf{V}^{(2)}$ and $\mathbf{V}^{(1)}$.

We wish to determine the distribution of $\mathbf{V}^{(1)} | \mathbf{V}^{(2)} = \mathbf{c}$, where $\mathbf{c} = (c_1, c_2, \ldots c_{p-s})$

Theorem 3.3

Let $\mathbf{V}$ have a non-singular multivariate normal distribution, and let $\mathbf{V}^{(1)}$ and $\mathbf{V}^{(2)}$ be as defined. If

$$\mathbf{W}_1 = \mathbf{V}^{(1)} - \boldsymbol{\beta}\mathbf{V}^{(2)} \quad \text{and} \quad \mathbf{W}_2 = \mathbf{V}^{(2)}$$

then

(i) $\mathbf{W} = \begin{pmatrix} \mathbf{W}_1 \\ \mathbf{W}_2 \end{pmatrix}$ is also non-singular multivariate normal;

(ii) $\mathbf{W}_1$ and $\mathbf{W}_2$ are independent iff $\boldsymbol{\beta} = \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}$

Proof

We re-write the transformation as

$$\mathbf{W} = \begin{pmatrix} \mathbf{V}^{(1)} - \boldsymbol{\beta}\mathbf{V}^{(2)} \\ \mathbf{V}^{(2)} \end{pmatrix} = \begin{pmatrix} 1 & -\boldsymbol{\beta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{V}^{(1)} \\ \mathbf{V}^{(2)} \end{pmatrix} = \mathbf{AV}$$

Where $\mathbf{A}$ is non-singular ($\det(\mathbf{A}) \neq 0$).

$\mathbf{W}$ is therefore a non-singular linear transformation of $\mathbf{V}$.

(iii) $\mathbf{W}_1$ and $\mathbf{W}_2$ are independent if $\operatorname{cov}(\mathbf{W}_1, \mathbf{W}_2) = 0$

Now,

$$\begin{aligned} \operatorname{cov}(\mathbf{W}_1, \mathbf{W}_2) &= \operatorname{cov}\left(\mathbf{V}^{(1)} - \boldsymbol{\beta}\mathbf{V}^{(2)},\ \mathbf{V}^{(2)}\right) \\ &= \operatorname{cov}\left(\mathbf{V}^{(1)}, \mathbf{V}^{(2)}\right) - \boldsymbol{\beta}\operatorname{cov}\left(\mathbf{V}^{(2)}, \mathbf{V}^{(2)}\right) \\ &= \mathbf{\Sigma}_{12} - \boldsymbol{\beta}\mathbf{\Sigma}_{22} \end{aligned}$$

For independence, $\mathbf{\Sigma}_{12} - \boldsymbol{\beta}\mathbf{\Sigma}_{22} = 0$. This implies that

$$\boldsymbol{\beta} = \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}, \qquad (3.17)$$

as stated.

It follows that the distribution $\mathbf{W}_1 | \mathbf{V}^{(2)} = \mathbf{c}$ is the same for all values of $\mathbf{c}$ and the marginal distribution of $\mathbf{W}_1$ is exactly the same.

From the transformation,

$$\begin{aligned} E(\mathbf{W}_1) &= E\left(\mathbf{V}^{(1)} - \boldsymbol{\beta}\mathbf{V}^{(2)}\right) \\ &= \boldsymbol{\mu}_1 - \boldsymbol{\beta}\boldsymbol{\mu}_2 \\ &= \boldsymbol{\mu}_1 - \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\boldsymbol{\mu}_2 \end{aligned}$$

Let D denote the variance-covariance matrix

$$\begin{aligned} D(\mathbf{W}_1) &= D\left(\mathbf{V}^{(1)} - \boldsymbol{\beta}\mathbf{V}^{(2)}\right) \\ &= \operatorname{cov}\left(\mathbf{V}^{(1)} - \boldsymbol{\beta}\mathbf{V}^{(2)},\ \mathbf{V}^{(1)} - \boldsymbol{\beta}\mathbf{V}^{(2)}\right) \\ &= \operatorname{cov}\left(\mathbf{V}^{(1)}, \mathbf{V}^{(1)}\right) - \operatorname{cov}\left(\boldsymbol{\beta}\mathbf{V}^{(2)}, \mathbf{V}^{(1)}\right) - \operatorname{cov}\left(\mathbf{V}^{(1)}, \boldsymbol{\beta}\mathbf{V}^{(2)}\right) + \operatorname{cov}\left(\boldsymbol{\beta}\mathbf{V}^{(2)}, \boldsymbol{\beta}\mathbf{V}^{(2)}\right) \\ &= \mathbf{\Sigma}_{11} - \boldsymbol{\beta}\mathbf{\Sigma}_{21} - \mathbf{\Sigma}_{12}\boldsymbol{\beta}' + \boldsymbol{\beta}\mathbf{\Sigma}_{22}\boldsymbol{\beta}' \end{aligned}$$

Since $\boldsymbol{\beta} = \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}$, denoting $D(\mathbf{W}_1) = \mathbf{\Sigma}_{\mathbf{W}_1}$, we have

$$\mathbf{\Sigma}_{\mathbf{W}_1} = \mathbf{\Sigma}_{11} - \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21} \qquad (3.18)$$

Hence, the distribution of $\mathbf{W}_1$ is therefore

$$\mathbf{W}_1 \sim MN\left(\boldsymbol{\mu}_1 - \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\boldsymbol{\mu}_2,\ \mathbf{\Sigma}_{11} - \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21}\right) \qquad (3.19)$$

Having identified the distribution of $\mathbf{W}_1$, the distribution of $\mathbf{V}^{(1)}$ may be deduced.

From $\mathbf{W}_1 = \mathbf{V}^{(1)} - \boldsymbol{\beta}\mathbf{V}^{(2)}$, we have

$$\mathbf{V}^{(1)} = \mathbf{W}_1 + \boldsymbol{\beta}\mathbf{V}^{(2)}$$

The distribution $\mathbf{V}^{(1)}\big|\mathbf{V}^{(2)}=\mathbf{c}$ is therefore the translation of Equation (3.19) through the vector $\boldsymbol{\beta}\mathbf{c}$. That is, by denoting the conditional expectation of $\mathbf{V}^{(1)}\big|\mathbf{V}^{(2)}$ by $\mu_{1\cdot\mathbf{c}}$, we have

$$\begin{aligned}\mu_{1\cdot\mathbf{c}} &= E(\mathbf{W}_1)+\boldsymbol{\beta}\mathbf{c}\\ &= \boldsymbol{\mu}_1+\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{c}-\boldsymbol{\mu}_2)\end{aligned} \tag{3.20}$$

By noting that $D\left(\mathbf{V}^{(1)}\big|\mathbf{V}^{(2)}\right)=D(\mathbf{W}_1)$, which we denote by $\boldsymbol{\Sigma}_{11\cdot\mathbf{c}}$, is the variance-covariance matrix of the conditional vector $\mathbf{V}^{(1)}\big|\mathbf{V}^{(2)}$ is given by:

$$\boldsymbol{\Sigma}_{11.c}=\boldsymbol{\Sigma}_{11}-\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21} \tag{3.21}$$

The partial correlation coefficients between $\mathbf{V}^{(1)}$ for a given components of $\mathbf{V}^{(2)}$ are determined using equation (3.21).

Hence, the distribution of $\mathbf{V}^{(1)}\big|\mathbf{V}^{(2)}$ is therefore

$$\mathbf{V}^{(1)}\big|\mathbf{V}^{(2)}\sim MN\left(\boldsymbol{\mu}_1+\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{c}-\boldsymbol{\mu}_2),\ \boldsymbol{\Sigma}_{11}-\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}\right) \tag{3.22}$$

Thus, the conditional expectation is given as

$$\boldsymbol{\mu}_{1\cdot\mathbf{c}}=\boldsymbol{\mu}_1+\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\left(\mathbf{X}^{(2)}-\boldsymbol{\mu}_2\right)$$

Provides the multivariate multiple linear regression model by establishing the regression function of $\mathbf{V}^{(1)}\big|\mathbf{V}^{(2)}$. In this procedure, the $s\times t$ matrix $\boldsymbol{\beta}=\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}$ represents the regression coefficients of $\mathbf{V}^{(1)}$ on $\mathbf{V}^{(2)}$. The jth row of $\boldsymbol{\beta}$ is the regression coefficients of $\mathbf{V}^{(2)}$ in the linear regression given by $E\left(\mathbf{V}_j^{(1)}\big|\mathbf{V}^{(2)}\right)$ for the jth component of $\mathbf{V}^{(1)}$, $j=1, 2, \ldots, s.$

3.5 Joint Density

By substituting $\mathbf{v}^{(1)}-\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\mathbf{v}^{(2)}$ for $\mathbf{w}^{(1)}$ and $\mathbf{v}^{(2)}$ for $\mathbf{w}^{(2)}$ into Equation (3.2), the density of $\mathbf{V}^{(1)}$ and $\mathbf{V}^{(2)}$ can be achieved.

As a result, the joint density of $\mathbf{V}^{(1)}$ and $\mathbf{V}^{(2)}$ is as follows:

$$\begin{aligned} f(v^{(1)},v^{(2)}) &= f(v^{(1)}).f(v^{(2)}) \\ &= \frac{1}{(2\pi)^{s/2}|\mathbf{\Sigma}_{11.2}|^{1/2}}\exp\{-\frac{1}{2}\left[(\mathbf{v}^{(1)}-\mu^{(1)})-\mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}(\mathbf{v}^{(2)}-\mu^{(2)})\right]' \\ &.\mathbf{\Sigma}_{11.2}^{-1}\left[(\mathbf{v}^{(1)}-\mu^{(1)})-\mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}(\mathbf{v}^{(2)}-\mu^{(2)})\right]\}\times\frac{1}{(2\pi)^{\frac{p-s}{2}}|\mathbf{\Sigma}_{22}|^{1/2}} \\ &\exp\{-\frac{1}{2}\left(\mathbf{v}^{(2)}-\mu^{(2)}\right)'\mathbf{\Sigma}_{22}^{-1}(\mathbf{v}^{(2)}-\mu^{(2)})\} \end{aligned} \tag{3.23}$$

where $\mathbf{\Sigma}_{11.2} = \mathbf{\Sigma}_{11} - \mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{21}$

3.6 Conditional Density

The conditional density of $\mathbf{V}^{(1)}$ given that $\mathbf{V}^{(2)} = \mathbf{v}^{(2)}$ is the quotient of Equation (3.23) and the marginal density of $\mathbf{V}^{(2)}$. The resulting conditional density of $\mathbf{V}^{(1)}$ given $\mathbf{V}^{(2)}$ is given by:

$$\begin{aligned} f(v^{(1)}/v^{(2)}) &= \frac{1}{(2\pi)^{s/2}|\mathbf{\Sigma}_{11.2}|^{1/2}}\exp\{-\frac{1}{2}\left[(\mathbf{v}^{(1)}-\boldsymbol{\mu}^{(1)})-\mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}(\mathbf{v}^{(2)}-\boldsymbol{\mu}^{(2)})\right]'. \\ &\mathbf{\Sigma}_{11.2}^{-1}\left[(\mathbf{v}^{(1)}-\boldsymbol{\mu}^{(1)})-\mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}(\mathbf{v}^{(2)}-\boldsymbol{\mu}^{(2)})\right]\} \end{aligned} \tag{3.24}$$

3.7 Maximum Likelihood Estimators

We take into consideration the issue of estimate $N(\boldsymbol{\mu},\mathbf{\Sigma})$ using a sample of n. It is established that the maximum likelihood estimate of $\mathbf{\Sigma}$ is given by

$$\hat{\mathbf{\Sigma}} = \frac{1}{n}H$$

where H is given as

$$\begin{aligned} H &= \sum_{k=1}^{n}(v_k-\bar{v})(v_k-\bar{v})' \\ &= \sum_{k=1}^{n}\begin{pmatrix} v_k^{(1)}-\bar{v}^{(1)} \\ v_k^{(2)}-\bar{v}^{(2)} \end{pmatrix}\left(v_k^{(1)'}-\bar{v}^{(1)'},\quad v_k^{(2)'}-\bar{v}^{(2)'}\right) \\ &= \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \end{aligned} \tag{3.25}$$

and $\bar{v} = \frac{1}{n}\sum_{k=1}^{n}v_k = (\bar{v}^{(1)'} \quad \bar{v}^{(2)'})'$

Theorem 3.4

Let $v_1, v_2, \ldots, v_n$ be sample from $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are partitioned as in (3.16). We define $\mathbf{H}$ by (3.25) and $(\bar{v}^{(1)'} \quad \bar{v}^{(2)'}) = \frac{1}{n}\sum_{k=1}^{N}(\bar{v}_k^{(1)'} \quad \bar{v}_k^{(2)'})$. Therefore, the maximum likelihood estimators of $\boldsymbol{\mu}^{(1)}$, $\boldsymbol{\mu}^{(2)}$, $\boldsymbol{\Sigma}_{11}$, $\boldsymbol{\Sigma}_{22}$ $\boldsymbol{\beta}$, and $\boldsymbol{\Sigma}_{11.2}$, are given by $\hat{\boldsymbol{\mu}}^{(1)} = \bar{\mathbf{v}}^{(1)}$, $\hat{\boldsymbol{\mu}}^{(2)} = \bar{\mathbf{v}}^{(2)}$, $\hat{\boldsymbol{\Sigma}}_{11} = \frac{1}{n}(\mathbf{H}_{11})$, $\hat{\boldsymbol{\Sigma}}_{22} = \frac{1}{n}(\mathbf{H}_{22})$ $\hat{\boldsymbol{\beta}} = \mathbf{H}_{12}\mathbf{H}_{21}^{-1}$, and $\hat{\boldsymbol{\Sigma}}_{11.2} = \frac{1}{n}\left(\mathbf{H}_{11} - \mathbf{H}_{12}\mathbf{H}_{22}^{-1}\mathbf{H}_{21}\right)$ respectively.

Proof

The maximum likelihood estimator of $\boldsymbol{\mu}^{(1)}$ and $\boldsymbol{\mu}^{(2)}$

The mean proof of theorem 3.1 may be used to simply establish the proof of the maximum likelihood estimators of $\boldsymbol{\mu}^{(1)}$ and $\boldsymbol{\mu}^{(2)}$.

The maximum likelihood estimator of $\boldsymbol{\Sigma}_{11}$ *and* $\boldsymbol{\Sigma}_{22}$

The maximum likelihood estimator of $\boldsymbol{\Sigma}$ is $\hat{\boldsymbol{\Sigma}} = \frac{1}{n}(\mathbf{H})$, as we may infer from Equation (3.7). Therefore, the proof easily follows that $\hat{\boldsymbol{\Sigma}}_{11} = \frac{1}{n}(\mathbf{H}_{11})$ and $\hat{\boldsymbol{\Sigma}}_{22} = \frac{1}{n}(\mathbf{H}_{22})$

The maximum likelihood estimator of $\boldsymbol{\beta}$

We know that $\boldsymbol{\beta} = \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}$

$$\hat{\beta} = \frac{\hat{\Sigma}_{12}}{\hat{\Sigma}_{22}}$$
$$= \frac{\frac{1}{n}(\mathbf{H}_{12})}{\frac{1}{n}(\mathbf{H}_{22})}$$
$$= \mathbf{H}_{12}\mathbf{H}_{22}^{-1}$$

The maximum likelihood estimator of $\Sigma_{11.2}$

$$\Sigma_{11.2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$
$$\hat{\Sigma}_{11.2} = \hat{\Sigma}_{11} - \hat{\Sigma}_{12}\hat{\Sigma}_{22}^{-1}\hat{\Sigma}_{21}$$
$$= \frac{1}{n}(\mathbf{H}_{11}) - \frac{1}{n}(\mathbf{H}_{12}\mathbf{H}_{22}^{-1}\mathbf{H}_{21})$$
$$= \frac{1}{n}(\mathbf{H}_{11} - \mathbf{H}_{12}\mathbf{H}_{22}^{-1}\mathbf{H}_{21})$$

3.7.1 Maximum Likelihood Estimator of Partial Correlation Coefficient

Utilizing the maximum likelihood estimator of the variance-covariance matrix, one may approximate the maximum likelihood estimators of the partial correlation coefficients.

Theorem 3.5

Let $v_1, v_2, ..., v_N$ be sample N from $N_p(\mu, \Sigma)$. The maximum likelihood estimators of $\rho_{jl.s+1,..,t}$, the partial correlations of the first s components conditional on the last $p-s$ components, are given by

$$\hat{\rho}_{jl.s+1,...,t} = \frac{\hat{\sigma}_{jl.s+1,...,t}}{\sqrt{\hat{\sigma}_{jj.s+1,...,t}}\sqrt{\hat{\sigma}_{ll.s+1,...,t}}} \qquad j,l = 1,....,s$$

Where the estimator $\hat{\rho}_{jl.s+1,...,t}$ is known as the population partial correlation coefficient between V_j and V_l holding $V_{s+1,..,}$ V_t constant. Also, $\hat{\sigma}_{jl.s+1,...,t}$ is the $(j,l)th$ entry of $\hat{\Sigma}_{11.2}$

Proof

The proof of Theorem 3.5 is as easy as normalizing the results of Equation (3.21) to obtain the partial correlation coefficient's maximum likelihood estimator.

3.7.2 Maximum likelihood Estimator of Population Partial Correlation

The greatest likelihood estimate of the population partial correlation coefficient is known as the sample partial correlation coefficient. Because the population characteristics are unknown, sample partial correlation may be determined by substituting the sample variance-covariance matrix for the population variance-covariance matrix.

Theorem 3.6

Let $v_1, v_2, \ldots, v_n$ be sample n from $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. The maximum likelihood estimators of $\rho_{jl.s+1,\ldots,t}$, the partial correlations of the first s components conditional on the last $p-s$ components, are given by

$$r_{jl.s+1,\ldots,t} = \frac{\hat{a}_{jl.s+1,\ldots,t}}{\sqrt{\hat{a}_{jj.s+1,\ldots,t}}\sqrt{\hat{a}_{ll.s+1,\ldots,t}}} \qquad j,l = 1,\ldots,s$$

The sample partial correlation coefficient between V_j and V_l having taken explanation of $V_{s+1,\ldots,}$ V_t fixed, is given by the estimator $r_{jl.s+1,\ldots,t}$,

Proof

The proof of theorem 3.3 makes the proof of theorem 3.6 simple to follow, and it yields the sample conditional variance-covariance matrix as provided by:

$$\mathbf{S}_{11.2} = \mathbf{S}_{11} - \mathbf{S}_{12}\mathbf{S}_{22}^{-1}\mathbf{S}_{21} \qquad (3.26)$$

where the maximum likelihood estimator of Equation (3.22) is represented by Equation (3.26).

Using Equation (3.26), the outcome is

$$\mathbf{S}_{11.2} = \hat{\mathbf{A}} = \begin{pmatrix} \hat{a}_{jj.s+1,...,t} & \hat{a}_{jl.s+1,...,t} \\ \hat{a}_{lj.s+1,...,t} & \hat{a}_{ll.s+1,...,t} \end{pmatrix} \tag{3.27}$$

By normalizing the matrix Equation (3.27) mentioned above, the maximum likelihood estimator for the partial correlation coefficient may be produced as follows:

$$r_{jl.s+1,...,t} = \frac{\hat{a}_{jl.s+1,...,t}}{\sqrt{\hat{a}_{jj.s+1,...,t}}\sqrt{\hat{a}_{ll.s+1,...,t}}}$$

3.8 Computational Procedures for Computing Sample Partial Correlation

Using the variance-covariance matrix method, the sample partial correlation coefficients are calculated. The detailed steps for calculating sample partial correlation coefficients are as follows:

- Using the sample raw dataset, we build up the sample variance-covariance matrix.
- The estimated sample variance-covariance matrix is partitioned according to the number of variables we wish to control and the number of variables between which we are interested in finding partial correlation.
- The partitioned sample variance-covariance matrix components or entries are substituted into the conditional variance-covariance matrix Equation (3.26).

The conditional sample variance-covariance matrix will produce a matrix comprising partial variances and partial covariances as follows:

$$\mathbf{S}_{11.2} = \begin{bmatrix} \hat{a}_{11.s+1,...,t} & \hat{a}_{12.s+1,...,t} & \cdots & \hat{a}_{1s.s+1,...,t} \\ \hat{a}_{21.s+1,...,t} & \hat{a}_{22.s+1,...,t} & \cdots & \hat{a}_{2s.s+1,...,t} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{a}_{s1.s+1,...,t} & \hat{a}_{s2.s+1,...,t} & \cdots & \hat{a}_{ss.s+1,...,t} \end{bmatrix}$$

- The off-diagonal element of the conditional sample variance-covariance matrix based on the two variables between which we wish to detect partial correlation is then normalized, and this allows us to calculate the sample partial correlation coefficients.

For example, we obtain the sample partial correlation between V_1 and V_3 holding V_2 *and* V_4 constant as given by:

$$r_{13.24} = \frac{\hat{a}_{13.24}}{\sqrt{\hat{a}_{11.24}}\sqrt{\hat{a}_{33.24}}}$$

where $\hat{a}_{13.24}$ is the partial covariance element, $\hat{a}_{11.24}$ and $\hat{a}_{33.24}$ are the partial variances element of $\mathbf{S}_{11\cdot 2}$.

3.9 Sample Partial Correlation Matrix

To obtain the sample partial correlation coefficients for every column in a matrix, utilize the sample partial correlation matrix. In other words, the sample partial correlation between columns 1 and 2 of the original matrix is matched by rows 1 and 2 of the partial correlation matrix. Even after accounting for the impacts of the other columns, there is still a correlation between columns 1 and 2 in this sample partial correlation. The diagonal components of the sample partial correlation matrix, which display the sample partial correlation of a column with itself, will always equal 1. Because there is no difference in the sample partial correlation between columns 1 and 2 and 2 and 1, the sample partial correlation matrix is symmetric. The following is a representation of a first-order and second-order sample partial correlation matrix:

$$\mathbf{R} = r_{jl.y} = \begin{bmatrix} 1 & r_{12.3} & r_{13.2} \\ r_{21.3} & 1 & r_{23.1} \\ r_{31.2} & r_{32.1} & 1 \end{bmatrix}$$

$$\mathbf{R} = r_{jl.yz} = \begin{bmatrix} 1 & r_{12.34} & r_{13.24} & r_{14.23} \\ r_{21.34} & 1 & r_{23.14} & r_{24.13} \\ r_{31.24} & r_{32.14} & 1 & r_{34.12} \\ r_{41.23} & r_{42.13} & r_{43.12} & 1 \end{bmatrix}$$

3.10 Testing for partial correlation coefficient

Fisher (1924) states that when utilizing the t statistic to determine whether a simple correlation coefficient is equal to zero, the degrees of freedom for error are reduced by one for each variable that is removed, and the simple correlation coefficient is replaced with a partial correlation coefficient. This is accomplished in order to confirm that a partial correlation coefficient equals 0 under normal circumstances.

Hypothesis testing

To compare it with the alternative, which maintains that a partial correlation is not equal to zero, we first look at testing the null hypothesis, which states that a partial correlation is equal to zero. This is stated as follows:

$$H_0 : \rho_{jl.\mathbf{U}} = 0$$

$$H_1 : \rho_{jl.\mathbf{U}} \neq 0$$

Test statistic is shown below:

$$t = r_{jl.\mathbf{U}} \sqrt{\frac{n-2-k}{1-r_{jl.\mathbf{U}}^2}} \sim t_{n-2-k}$$

where k denotes how many variables we are controlling or conditioning for.

We define the critical value as follows:

$$|t| = t_{n-2-k,\alpha/2}$$

Under the null hypothesis, the test statistic will approximately correspond to a t-distribution, with $n-2-k$ degrees of freedom. We would reject the null hypothesis (H_0) in the event that the test statistic's absolute value surpassed the t-table's critical threshold, which was determined at $\alpha/2$.

3.11 Confidence Interval for the partial correlation ($\rho_{jl.\mathbf{U}}$)

The confidence interval for partial correlation is the region where the partial correlation of a population is most likely to be observed. The degree of certainty that it is likely to fall inside that range is indicated by the confidence level.

These steps were taken for the purpose to calculate the confidence interval for the partial correlation coefficient:

- We compute the partial correlation's Fisher's transformation using the following formula:

$$Z_{jl} = \frac{1}{2}\ln\left(\frac{1+r_{jl.\mathbf{U}}}{1-r_{jl.\mathbf{U}}}\right)$$

This Fisher transformation variable will be probably normally distributed in this situation for a large n. For this partial correlation, the mean is equal to the Fisher transformation for the population value, and the variance is equal to $1/(n-3)$

$$Z_{jl} \sim N\left(\frac{1}{2}\ln\frac{1+\rho_{jl.\mathbf{U}}}{1-\rho_{jl.\mathbf{U}}}, \quad \frac{1}{n-3}\right)$$

- For the Fisher transformation correlation, we determine a $(1-\alpha)100\%$ confidence interval. The following is an illustration of this:
- $\frac{1}{2}\ln\frac{1+\rho_{jl.\mathbf{U}}}{1-\rho_{jl.\mathbf{U}}}$

The above formula provides the boundaries Z_a and Z_b as:

$$\left(Z_{jl} - \frac{Z_{\alpha/2}}{\sqrt{n-3}}, \quad Z_{jl} + \frac{Z_{\alpha/2}}{\sqrt{n-3}} \right)$$

where $\sqrt{\frac{1}{n-3}}$ is the standard error, $Z_{\alpha} = Z_{jl} - \frac{Z_{\alpha/2}}{\sqrt{n-3}}$ is the lower limit and

$Z_b = Z_{jl} + \frac{Z_{\alpha/2}}{\sqrt{n-3}}$ is the upper limit.

- To get the required confidence interval for the partial correlation ($\rho_{jl.\mathbf{U}}$), we transform back as follows:

$$\left(\frac{e^{2Z_a} - 1}{e^{2Z_a} + 1}, \quad \frac{e^{2Z_b} - 1}{e^{2Z_b} + 1} \right)$$

3.12 Assumptions of partial correlation coefficient

Before choosing to study data using partial correlation, we must first determine whether the data we intend to research can really be investigated using the partial correlation approach. This is necessary because a partial correlation can only be utilized if the data "passes" five assumptions; otherwise, the data cannot provide us a trustworthy conclusion. We'll explore the following five assumptions:

3.12.1 Assumption one (Dependent and independent variables)

There should be one of each type of dependent and independent variable, and they both need to be measured on a continuous scale (i.e., an interval or ratio scale).

3.12.2 Assumption two (controlling variables)

Covariates, also known as control variables, are important to add since they merely serve to influence the correlation between the other two variables. A continuous scale must also be used to measure these controlling variables.

3.12.3 Assumption three (Linearity)

There must be a linear relationship between at least three of the variables. That is, every possible pair of variables must be connected linearly. To do this, a scatterplot matrix may commonly be visually examined.

3.12.4 Assumption four (Normality)

The distribution of the variables must approximate normal. To assess the statistical significance of the partial correlation, we need bivariate normality for each pair of variables, although it might be difficult to validate this assumption. Consequently, a more straightforward strategy that evaluates the distribution of each variable is more frequently employed. The Shapiro-Wilk test for normality is used for this.

3.12.5 Assumption five (No significant Outlier)

No noticeable outliers should exist. All that constitutes an outlier in a set of data is a single data point that deviates from the norm. Since outliers might affect partial correlation, it might be difficult to interpret the data correctly when outliers have a significant effect on the correlation coefficient and the line of best fit. Therefore, it is ideal if the dataset has no outliers. We employed the interquartile range (IQR) strategy and the Dixon's test for confirmation to detect outliers.

The validity of the findings we obtain while executing or computing a partial correlation coefficient depends on whether assumptions 3, 4, or 5 are not violated.

3.12.6 Shapiro-Wilk Test for Normality

To find out if a random sample V_k, $k = 1,2,...,n$ is taken from a normal Gaussian probability distribution with true mean (μ) and variance (σ^2), one can apply the Shapiro-Wilk goodness-of-fit test. That is, $N \sim (\mu, \sigma^2)$. Therefore, we want to investigate the following hypothesis:

Null Hypothesis (H_0): The population from which the sample data is drawn is normally distributed.

Alternate Hypothesis (H_1): The sample is not drawn from a normally distributed population.

We employ the Shapiro-Wilk test statistic, which is provided by, to examine the above hypothesis.

$$S.W = \frac{\left(\sum_{k=1}^{n} c_k v_{(k)}\right)^2}{\sum_{k=1}^{n} \left(v_k - \bar{v}\right)^2}$$

where $v_{(k)}$ denotes the ordered sample statistics and c_k denotes the tabulated coefficients:

$$(c_1, c_2, \ldots\ldots, c_n) = \frac{k' s^{-1}}{\sqrt{(k' s^{-1} s^{-1} k)}}$$

where s stands for the variance-covariance matrix for the ordered statistics and $k = (k_1, k_2, \ldots\ldots, k_n)^2$ stands for the expected values of the ordered statistics, which are independent random variables with identical distributions that satisfy the standard normal $N(0, 1)$.

A critical value (S.W Table) is compared to the test statistic (S.W). The null hypothesis is accepted if the estimated S.W is larger than the S.W critical value; otherwise, it is rejected. Additionally, we can accept or reject the null hypothesis using the probability value (p-value). The null hypothesis is rejected if the p-value is less than the significant threshold value; otherwise, it is not.

3.12.7 Checking for outliers

The method of statistical outlier detection involves using statistical tests or approaches to locate extreme results. We applied the interquartile range (IQR) strategy to find outliers. The center half of our dataset is represented by the interquartile range. We used the IQR to create "fences" around our data, and any values that fell outside of those fences were considered outliers.

Step by step procedures to detect outlier using interquartile range method

- Data arranged in ascending order
- Find the median, first quartile (Q_1), and third quartile (Q_3).
- Determine the $IQR = Q_3 - Q_1$.
- We determine our upper limit using the formula $Q_3 + (1.5 * IQR)$.
- The lower limit is determined by the formula $Q_1 - (1.5 * IQR)$.

In order to find outliers, or values that are outside the boundaries, we used fences.

3.12.8 Dixon's Q Test for outliers

To assess if a single lowest or highest value in the dataset is an outlier, we apply the Dixon's Q test. If more than one outlier is detected, the test must be run on each suspected outlier separately. Dixon tests are particularly beneficial for small sample sizes, typically $n \leq 25$. To use Dixon's Q test, we first ensure that our data set is normally distributed (i.e., we compute a Shapiro-Wilk test). If our data set still does not match the normality assumption after a test, we should not perform Dixon's Q Test.

The following is a definition of Dixon's Q test for the proposed hypothesis:

H_0 :The lowest or highest value is not an outlier

H_1 :The lowest or highest value is an outlier

Dixon's Q test procedure for finding outliers:

Step 1: Ascending order is used to organize the data.

Step 2: We select the value (lowest or highest) that we wish to examine for an outlier.

Step 3: We compute the Q statistic using the following formula:

- If sample size $3 \leq n \leq 7$, we use

$$Q(R_{10}) = \frac{x_2 - x_1}{x_n - x_1}$$

where:

x_1 is the smallest (suspect) value

x_2 is the second smallest value

x_n is the largest value.

- If sample size $8 \leq n \leq 10$, we use

$$Q(R_{11}) = \frac{x_2 - x_1}{x_{n-1} - x_1}$$

- If sample size $11 \leq n \leq 13$, we use

$$Q(R_{21}) = \frac{x_3 - x_1}{x_{n-1} - x_1}$$

- If sample size $n \geq 14$, we use

$$Q(R_{22}) = \frac{x_3 - x_1}{x_{n-2} - x_1}$$

Step 4: We calculate the Q-table value at the specified level of significance.

Step 5: The null hypothesis (H_0) is rejected if the expected Q statistic is larger than the Q-table value; otherwise, the null hypothesis (H_0) is not rejected. If the null hypothesis (H_0) is not accepted, the accompanying data point is regarded as an outlier.

3.12.9 Checking for linearity

When two or more variables tend to change at the same pace, the relationship between them is said to be linear. Visual examination (graphical approach) was used to test the linearity assumption between pairs of variables.

The scatter plot method was used because it is the most efficient way of displaying the linearity assumption between pairs of variables.

3.13 Chapter Summary

We looked at the partial correlation and the theoretical basis of the conditional variance-covariance matrix in this chapter. We demonstrated how the variance-covariance matrix will be computed from the raw datasets. Furthermore, we specifically provided the maximum likelihood estimator, which can be used to compute orders of partial correlation coefficient, as well as how partial correlation coefficients significantly differ from zero. The assumptions that must be made for partial correlation coefficients to be valid were also illustrated with practical examples. The results of applying the variance-covariance matrix technique to real datasets to figure out the partial correlation coefficients' ordering are presented in the next chapter, along with practical illustrations.

CHAPTER FOUR

RESULTS AND DISCUSSIONS

4.0 Introduction

There are three sections in this chapter. These comprise the dataset's descriptive statistics, the partial correlation coefficients' computational process, and the validation of the partial correlation coefficients' underlying assumptions. In this chapter, the fundamental assumptions that must be met in order to compute the orders of partial correlation coefficient are presented in detail. The variance-covariance matrix and the correlation coefficient matrix are generated using real-world data sets. Furthermore, the partitioning of variance-covariance matrices to get the ordering of partial correlation coefficients is explained in detail with examples from practical applications.

4.1 Descriptive Statistics

Table 2: Descriptive Statistics on Klein's US Economy Model I (1920-1941)

Variable	N	Mean	SE Mean	StDev	Min.	Q1	Median	Q3	Max.
v_1	22	53.35	1.57	7.35	39.80	48.15	53.80	57.73	69.70
v_2	22	16.70	0.90	4.21	7.00	12.63	17.45	19.88	23.50
v_3	22	36.02	1.36	6.36	25.50	30.28	36.10	39.65	53.30
v_4	22	1.33	0.74	3.48	-6.20	-1.45	2.05	4.38	5.60
v_5	22	199.57	2.26	10.61	180.10	191.95	200.55	207.22	216.70
v_6	22	59.37	2.31	10.85	44.30	50.00	60.95	64.63	88.40
v_7	22	4.99	0.43	2.01	2.20	3.18	4.50	6.88	8.50
v_8	22	4.69	0.51	2.38	2.40	3.30	4.05	5.23	13.80
v_9	22	6.65	0.45	2.11	3.40	4.58	6.90	7.85	11.60

Source: Field work (2023)

Table 2 presents the sample data obtained for our computational analysis of partial correlation coefficient. On a continuous scale, all variables were evaluated. The dependent variable, consumption (v_1), had a mean of 53.35 and a standard deviation of 7.35, whereas the independent variable, corporate profit (v_2), had a mean of 16.70 and a standard deviation of 4.21. Furthermore, we discovered the remaining variables that influence our study, that serve as the controlling variables, reported their mean, standard deviation, and other measures of variability. In addition, we observed that previous year's capital stock (v_5) had the highest mean value of 199.57, followed by the gross national product (v_6) which had a mean value of 59.37 and investment (v_4) had the lowest mean value of 1.33. Finally, we discovered that only the investment variable (v_4) recorded negative values for the standard deviation and the lower quartile.

4.2 Assumptions for Partial Correlation

Assumptions imply that certain properties of datasets must be satisfied in order for statistical method outputs to be accurate. We must first examine the datasets acquired by confirming the correctness of the fundamental partial correlation assumptions before defining any order for the partial correlation coefficient. It is necessary to measure the variables being examined on a continuous scale (i.e., ratio data or interval data). Statistical tests were performed on the datasets used for the fundamental partial correlation assumptions, namely linearity, normality, and the lack of outliers. The theoretical underpinnings for verifying each of these assumptions independently are covered in depth in the section that follows:

4.2.1 Dataset for Analysis

Table: 3 Klein's US Economy Model I (1920–1941)

Year	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9
1920	39.8	12.7	28.8	2.7	180.1	44.9	2.2	2.4	3.4
1921	41.9	12.4	25.5	-0.2	182.8	45.6	2.7	3.9	7.7
1922	45.0	16.9	29.3	1.9	182.6	50.1	2.9	3.2	3.9
1923	49.2	18.4	34.1	5.2	184.5	57.2	2.9	2.8	4.7
1924	50.6	19.4	33.9	3.0	189.7	57.1	3.1	3.5	3.8
1925	52.6	20.1	35.4	5.1	192.7	61.0	3.2	3.3	5.5
1926	55.1	19.6	37.4	5.6	197.8	64.0	3.3	3.3	7.0
1927	56.2	19.8	37.9	4.2	203.4	64.4	3.6	4.0	6.7
1928	57.3	21.1	39.2	3.0	207.6	64.5	3.7	4.2	4.2
1929	57.8	21.7	41.3	5.1	210.6	67.0	4.0	4.1	4.0
1930	55.0	15.6	37.9	1.0	215.7	61.2	4.2	5.2	7.7
1931	50.9	11.4	34.5	-3.4	216.7	53.4	4.8	5.9	7.5
1932	45.6	7.0	29.0	-6.2	213.3	44.3	5.3	4.9	8.3
1933	46.5	11.2	28.5	-5.1	207.1	45.1	5.6	3.7	5.4
1934	48.7	12.3	30.6	-3.0	202.0	49.7	6.0	4.0	6.8
1935	51.3	14.0	33.2	-1.3	199.0	54.4	6.1	4.4	7.2
1936	57.7	17.6	36.8	2.1	197.7	62.7	7.4	2.9	8.3
1937	58.7	17.3	41.0	2.0	199.8	65.0	6.7	4.3	6.7
1938	57.5	15.3	38.2	-1.9	201.8	60.9	7.7	5.3	7.4
1939	61.6	19.0	41.6	1.3	199.9	69.5	7.8	6.6	8.9
1940	65.0	21.1	45.0	3.3	201.2	75.7	8.0	7.4	9.6
1941	69.7	23.5	53.3	4.9	204.5	88.4	8.5	13.8	11.6

Source: R Core Team (2020)

Table 3 illustrates the original dataset that we used for our investigation, which is Klein's Model I data for the US Economy (1920–1941). We learned from Table 3 that all of the variables are assessed on a continuous scale (i.e., on a ratio scale or interval scale). For the purpose of computational analysis, we select corporate profit (v_2) as the independent variable and consumption (v_1) as the dependent variable. Moreover, we include private wage bill (v_3), investment (v_4), previous year's capital stock (v_5), gross national product (v_6), government wage bill (v_7), government expenditure (v_8), and taxes (v_9) as controlling variables (or covariates) to enable us compute the orders of partial correlation coefficients.

4.2.2 Examining the Assumption of Normality

The Shapiro-Wilk test is a statistical technique used for determining whether a continuous variable exhibits a normal distribution. We used this test to confirm the normality test assumption of the dataset variables.

Table 4: Shapiro-Wilk Normality Test

Variable	Shapiro-Wilk Statistic	P-value	Remarks
Consumption (v_1)	0.98453	0.971300	Not violated
Corporate profit (v_2)	0.95829	0.455500	Not violated
Private wage bill (v_3)	0.95412	0.380200	Not violated
Investment (v_4)	0.91824	0.069910	Not violated
Previous year's capital stock (v_5)	0.95100	0.330500	Not violated
Gross national product (v_6)	0.93820	0.181700	Not violated
Government wage bill (v_7)	0.91626	0.063630	Not violated
Government expenditure (v_8)	**0.69559**	**0.000001**	**Violated**
Taxes (v_9)	0.95483	0.392200	Not violated

Source: Field work (2023)

Table 4 presents the findings from the Shapiro-Wilk statistical test as confirmation of the normality assumption. We observe that, with the exception of government expenditure (v_8), all of the variables' *p*-values are greater than the alpha-value (i.e. 0.05). We may conclude that the normality test assumption is solely violated by the government expenditure (v_8), which will be eliminated from our computational analysis.

4.2.3 Checking Assumption of Outliers

This assumption is used to determine whether there are any outliers that have a significant impact on the dataset. This was accomplished by applying the boxplot and Dixon's *Q* test for detecting outliers to determine the existence of outlier(s) in the dataset. The results of the boxplot and Dixon's test are shown below:

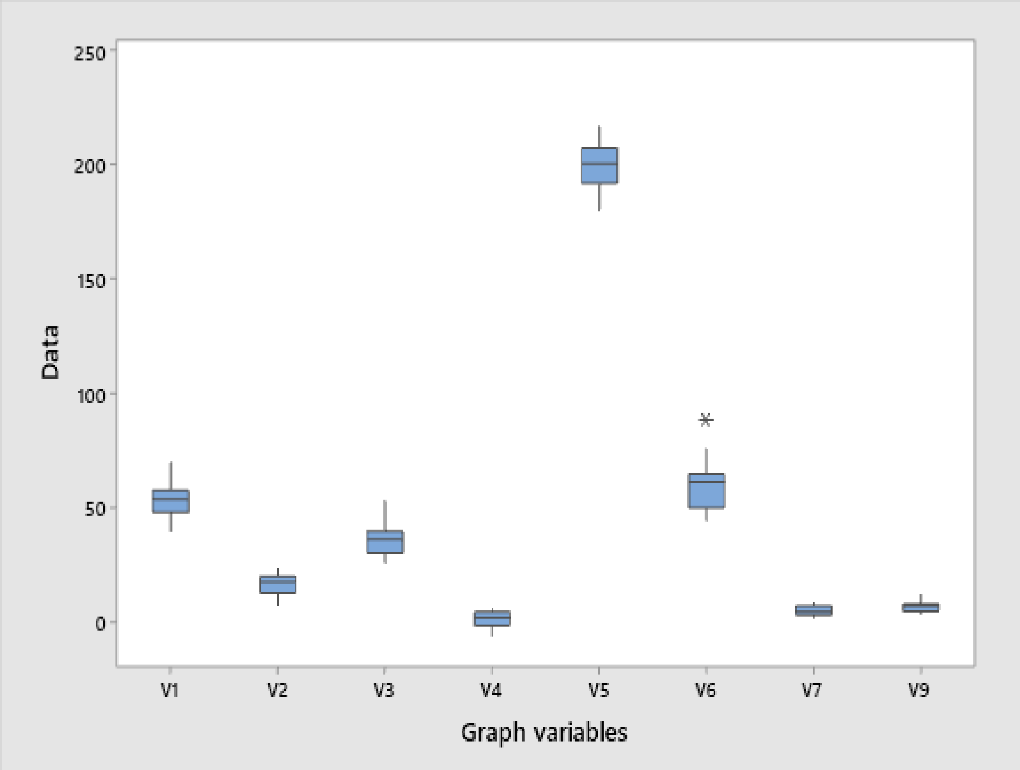

Figure 1: Boxplot of Klein's US Economy Model I (1920–1941)

Figure 1 shows that none of the other variables has an outlier, except gross domestic product (v_6). Therefore, it is important for us to find out if the reported outlier is significant or not. We employed Dixon's Q test verify if the observed outlier is influential.

4.2.3.1 Dixon's Test for Outliers

We performed Dixon's test to confirm whether Gross national product (v_6) largest data reported as outlier by the boxplot above is actually an outlier or not.

Hypothesis

H_0 The same normal population from which all data values are drawn.

H_1 The largest value is an outlier.

Significance level $\alpha = 0.05$

Table 5: Dixon's r22 Ratio Test

Variable	N	Min	x[2]	x[3]	x[N-2]	x[N-1]	Max	r22	P
v_6	22	44.30	44.90	45.10	69.50	75.70	88.40	0.44	0.045

Source: Field work (2023)

The Dixon's r22 ratio test was reported in Table 5 for only the gross national product (v_6). We discovered that the *p*-value for gross national product (v_6), 0.045 is less than the alpha value (0.05), implying that the largest value is an outlier with a substantial influence on the dataset. As a result, in order to get valid partial correlation coefficients, we eliminated the gross national product (v_6) from the dataset for computational analysis.

4.2.4 Linearity Assumption

The scatterplot matrix representation in Figure 2 below explains the linearity assumption for the remaining seven (7) variables.

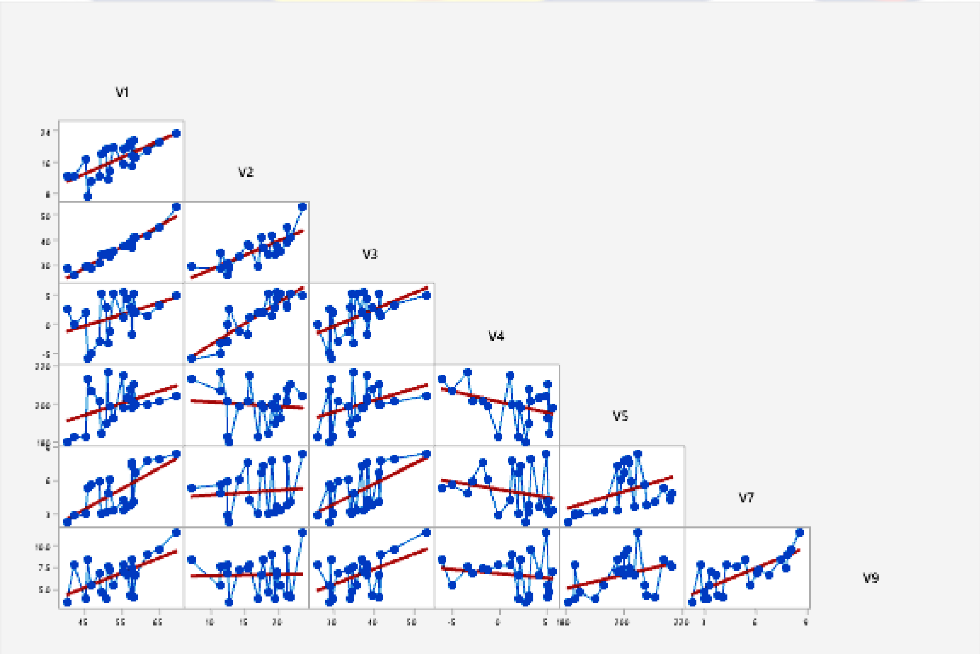

Figure 2: A Scatterplot Matrix of Klein's Model I Data

In Figure 2, the scatterplot matrix for the remaining seven variables is displayed. For each pair of variables, scatter plots are displayed in the matrix's panels. It was discovered that there are either upwards or downwards linear regression lines for each pair of variables. These straight line evidence point to linearity between the variable pairings. The twenty-one (21) distinct visualization plots that we critically analyzed demonstrate that the linearity assumption remains valid.

4.3 Computational Analysis Section

4.3.1 Computation of variance-covariance matrix

We produced the sample variance-covariance matrix from the multivariate dataset obtained using the matrix technique since the dataset came from a sample. The sample variance-covariance matrix for seven variables that satisfy the partial correlation requirements was provided by Equation (3.11), which we employed:

$$\mathbf{S} = \frac{1}{n-1}\mathbf{V}'(\mathbf{I} - \frac{1}{n}\mathbf{J})\mathbf{V}$$

$$\mathbf{S} = \frac{1}{22-1}\begin{pmatrix} 39.8 & 41.9 & \cdots & 69.7 \\ 12.7 & 12.4 & \cdots & 23.5 \\ \vdots & \vdots & \ddots & \vdots \\ 3.4 & 7.7 & \cdots & 11.6 \end{pmatrix}\left[\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} - \frac{1}{22}\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}\right]\begin{pmatrix} 39.8 & 12.7 & \cdots & 3.4 \\ 41.9 & 12.4 & \cdots & 7.7 \\ \vdots & \vdots & \ddots & \vdots \\ 69.7 & 23.5 & & 11.6 \end{pmatrix}$$

$$= \frac{1}{21}\begin{pmatrix} 39.8 & 41.9 & \cdots & 69.7 \\ 12.7 & 12.4 & \cdots & 23.5 \\ \vdots & \vdots & \ddots & \vdots \\ 3.4 & 7.7 & \cdots & 11.6 \end{pmatrix}\begin{pmatrix} \frac{21}{22} & -\frac{1}{22} & \cdots & -\frac{1}{22} \\ -\frac{1}{22} & \frac{21}{22} & \cdots & -\frac{1}{22} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{22} & -\frac{1}{22} & \cdots & \frac{21}{22} \end{pmatrix}\begin{pmatrix} 39.8 & 12.7 & \cdots & 3.4 \\ 41.9 & 12.4 & \cdots & 7.7 \\ \vdots & \vdots & \ddots & \vdots \\ 69.7 & 23.5 & & 11.6 \end{pmatrix}$$

$$= \begin{pmatrix} 53.9893 & 22.4295 & 45.1657 & 10.6341 & 33.4374 & 10.1141 & 9.0036 \\ 22.4295 & 17.7600 & 20.4424 & 12.9086 & -4.1571 & 0.8124 & 0.2619 \\ 45.1657 & 20.4424 & 40.4520 & 11.4451 & 24.7435 & 7.4398 & 6.8610 \\ 10.6341 & 12.9086 & 11.4451 & 12.1089 & -13.7061 & -1.7315 & -1.2031 \\ 33.4374 & -4.1571 & 24.7435 & -13.7061 & 112.6004 & 8.8938 & 8.6555 \\ 10.1141 & 0.8124 & 7.4398 & -1.7315 & 8.8938 & 4.0336 & 3.2021 \\ 9.0036 & 0.2619 & 6.8610 & -1.2031 & 8.6555 & 3.2021 & 4.4579 \end{pmatrix}$$

For a clearer display, we setup a table with appropriate variable and row labels of the variance-covariance matrix shown as follows in Table 6:

Table 6: Variance-Covariance Matrix

	V1	V2	V3	V4	V5	V7	V9
V1	53.98929						
V2	22.42952	17.76000					
V3	45.16571	20.44238	40.45203				
V4	10.63405	12.90857	11.44511	12.10894			
V5	33.43738	-4.15714	24.74346	-13.70608	112.60037		
V7	10.11405	0.81238	7.43978	-1.73145	8.89383	4.03361	
V9	9.00357	0.26190	6.86095	-1.20310	8.65548	3.20214	4.45786

Source: Field work (2023)

Table 6 shows the 7×7 sample variance-covariance matrix, which served as the basis for computing the orders of partial correlation coefficient.

4.3.2 Computation of correlation matrix from variance-covariance matrix

Using Equation (3.16), we calculate the correlation coefficient matrix or zero-order partial correlation matrix from the variance-covariance matrix in Table 6.

$$\mathbf{R} = (\mathbf{M}^{1/2})^{-1}\mathbf{S}(\mathbf{M}^{1/2})^{-1}$$

Where

$$(\mathbf{M}^{1/2})^{-1} = \begin{pmatrix} \frac{1}{\sqrt{5.99}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{17.76}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{40.45}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{12.11}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{112.60}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{4.03}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{4.46}} \end{pmatrix}$$

The following outcome was obtained by putting the special diagonal matrix ($\mathbf{M}^{1/2}$) and the computed sample variance-covariance matrix (**S**) into Equation (3.16) to create the correlation matrix (**R**):

$$\mathbf{R} = (\mathbf{M}^{1/2})^{-1}\mathbf{S}(\mathbf{M}^{1/2})^{-1}$$

$$= \begin{pmatrix} 1 & 0.724 & 0.996 & 0.416 & 0.429 & 0.685 & 0.580 \\ 0.724 & 1 & 0.763 & 0.880 & -0.093 & 0.096 & 0.029 \\ 0.996 & 0.763 & 1 & 0.517 & 0.367 & 0.582 & 0.511 \\ 0.416 & 0.880 & 0.517 & 1 & -0.371 & -0.248 & -.0164 \\ 0.429 & -0.093 & 0.367 & -0.371 & 1 & 0.417 & 0.386 \\ 0.685 & 0.096 & 0.582 & -0.248 & 0.417 & 1 & 0.755 \\ 0.580 & 0.029 & 0.511 & -0.164 & 0.386 & 0.755 & 1 \end{pmatrix}$$

The sample correlation matrix is shown in tabular form below. To make it easier to see, we make a table with the proper variable and row labels:

Table 7: Correlation Matrix

	V1	V2	V3	V4	V5	V7	V9
V1	1						
V2	0.724	1					
V3	0.966	0.763	1				
V4	0.416	0.880	0.517	1			
V5	0.429	-0.093	0.367	-0.371	1		
V7	0.685	0.096	0.582	-0.248	0.417	1	
V9	0.580	0.029	0.511	-0.164	0.386	0.755	1

Source: Field work (2023)

Table 7 provided the sample correlation matrix, also known as a zero-order partial correlation matrix, which by eliminating the impact of other factors, depicts the connection between each pairwise variable. The highest association was recorded between consumption (v_1) and private wage bill (v_3) of 0.966, followed by corporate profit (v_2) and investment (v_4) of 0.880, whilst corporate profit (v_2) and taxes (v_9) have the lowest correlation (0.029) of any paired variable.

4.4 Computing Orders of Partial Correlation Coefficient by R Software

The scripts (codes) from the R software programme that were employed to calculate orders of partial correlation coefficient, together with the pertinent test statistics, p-values, and confidence interval, may be found in Appendix A. Table 8 shows the findings of the orders of partial correlation coefficients as follows:

Table 8: Orders of Partial Correlation Coefficient

Orders Method	Estimation	Statistic	P-Value	Confidence interval
First-order ($r_{12.3}$) Pearson	-0.076781	-0.3357	0.740800	(-0.482762, 0.356363)
Second-order ($r_{12.34}$) Pearson	0.582622	3.0414	0.007023	(0.213444, 0.806198)
Third-order ($r_{12.345}$) Pearson	0.584107	2.9671	0.008639	(0.215591, 0.806984)
Fourth-order ($r_{12.3457}$) Pearson	0.714795	4.0884	0.008571	(0.419655, 0.873241)
Fifth-order ($r_{12.34579}$) Pearson	0.825936	5.6741	0.000044	(0.620346, 0.925326)

Source: Field work (2023)

4.5 Computing Orders of Partial Correlation Coefficient Manually

The objective is to calculate the partial correlation coefficients between corporate profit (v_2) and consumption (v_1) while regulatory the influence of one or more additional continuous variables.

4.5.1 Calculating the Partial Correlation Coefficient of Zero Order

Without adjusting for any variables, we arrive at the zero-order partial correlation coefficient between corporate profit (v_2) and consumption (v_1). The sample variance-covariance matrix generated for v_1 and v_2 presented as follows:

$$\mathbf{S} = \begin{bmatrix} 53.9893 & 22.4295 \\ 22.4295 & 17.7600 \end{bmatrix}$$

We normalized the above 2x2 matrix, as shown below:

$$r_{12} = \frac{22.4295}{\sqrt{53.9893} \times \sqrt{17.7600}}$$
$$= 0.72434$$

4.5.2 Computing the Partial Correlation Coefficient of First Order

For calculating the order one partial correlation coefficient between corporate profit (v_2) and consumption (v_1), we just account for one variable. The sample variance-covariance matrix generated for consumption (v_1), corporate profit (v_2), and the controlling variable private wage bill (v_3) was partitioned as follows:

$$\mathbf{S} = \left(\begin{array}{cc|c} 53.9893 & 22.4295 & 45.1657 \\ 22.4295 & 17.7600 & 20.4424 \\ \hline 45.1657 & 20.4424 & 40.4520 \end{array} \right)$$

$$\mathbf{S}_{11.2} = \mathbf{S}_{11} - \mathbf{S}_{12}\mathbf{S}_{22}^{-1}\mathbf{S}_{21}$$

$$\mathbf{S}_{11.2} = \begin{bmatrix} 53.9893 & 22.4295 \\ 22.4295 & 17.7600 \end{bmatrix} - \begin{bmatrix} 45.1657 \\ 20.4424 \end{bmatrix} [40.4520]^{-1} [45.1657 \quad 20.4424]$$

$$= \begin{bmatrix} 53.9893 & 22.4295 \\ 22.4295 & 17.7600 \end{bmatrix} - \begin{bmatrix} 50.4287 & 22.8244 \\ 22.8244 & 10.3305 \end{bmatrix}$$

$$= \begin{bmatrix} 3.5606 & -0.3949 \\ -0.3949 & 7.4295 \end{bmatrix}$$

It was possible to get the partial correlation coefficient of order one by normalizing the above matrix, as indicated below:

$$r_{12.3} = \frac{-0.3949}{\sqrt{3.5606} \times \sqrt{7.4295}}$$

$$= -0.07678$$

4.5.3 Computing Partial Correlation Coefficient of Order Two

The sample variance-covariance matrix created for consumption (v_1), corporate profit (v_2), the controlling private age bill (v_3), and investment (v_4) was partitioned as follows to get second-order partial correlation:

$$\mathbf{S} = \left(\begin{array}{cc|cc} 53.9893 & 22.4295 & 45.1657 & 10.6341 \\ 22.4295 & 17.7600 & 20.4424 & 12.9086 \\ \hline 45.1657 & 20.4424 & 40.4520 & 11.4451 \\ 10.6341 & 12.9086 & 11.4451 & 12.1089 \end{array} \right)$$

$$\mathbf{S}_{11.2} = \mathbf{S}_{11} - \mathbf{S}_{12}\mathbf{S}_{22}^{-1}\mathbf{S}_{21}$$

$$\mathbf{S}_{11.2} = \begin{bmatrix} 53.9893 & 22.4295 \\ 22.4295 & 17.7600 \end{bmatrix} - \begin{bmatrix} 45.1657 & 10.6341 \\ 20.4424 & 12.1089 \end{bmatrix} \begin{bmatrix} 40.4520 & 11.4451 \\ 11.4451 & 12.1089 \end{bmatrix}^{-1} \begin{bmatrix} 45.1657 & 20.4424 \\ 10.6341 & 12.9086 \end{bmatrix}$$

$$= \begin{bmatrix} 53.9893 & 22.4295 \\ 22.4295 & 17.7600 \end{bmatrix} - \begin{bmatrix} 53.9893 & 21.1019 \\ 21.1019 & 16.0531 \end{bmatrix}$$

$$= \begin{bmatrix} 3.0421 & 1.3276 \\ 1.3276 & 1.7069 \end{bmatrix}$$

The above matrix, when normalized, yields the second-order partial correlation

$$r_{12.34} = \frac{1.3276}{\sqrt{3..0421} \times \sqrt{1.7069}}$$

$$= 0.58261$$

4.5.4 Computing the Third Order Partial Correlation Coefficient

We partitioned the sample variance-covariance matrix obtained for consumption (v_1), corporate profit (v_2) and the controlling variables, private wage bill (v_3), investment (v_4), and previous year's capital stock (v_5) to determine the partial correlation coefficient of third order as follows:

$$\mathbf{S} = \left(\begin{array}{cc|ccc} 53.9893 & 22.4295 & 45.1657 & 10.6341 & 33.4374 \\ 22.4295 & 17.7600 & 20.4424 & 12.9086 & -4.1571 \\ \hline 45.1671 & 20.4424 & 40.4520 & 11.4451 & 24.7435 \\ 10.6341 & 12.9086 & 11.4451 & 12.1089 & -13.7061 \\ 33.4374 & -4.1571 & 24.7435 & -13.7061 & 112.6004 \end{array}\right)$$

$$\mathbf{S}_{12}\mathbf{S}_{22}^{-1}\mathbf{S}_{21}$$

$$\begin{bmatrix} 45.1657 & 10.6341 & 33.4374 \\ 20.4424 & 12.9086 & -4.1571 \end{bmatrix} \begin{bmatrix} 40.4520 & 11.4451 & 24.7435 \\ 11.4451 & 12.1089 & -13.7061 \\ 24.7435 & -13.7061 & 112.6004 \end{bmatrix}^{-1} \begin{bmatrix} 45.1671 & 20.4424 \\ 10.6341 & 12.9086 \\ 33.4374 & -4.1571 \end{bmatrix}$$

$$\begin{bmatrix} 50.9603 & 21.1014 \\ 21.1014 & 16.0531 \end{bmatrix}$$

$$\mathbf{S}_{11.2} = \mathbf{S}_{11} - \mathbf{S}_{12}\mathbf{S}_{22}^{-1}\mathbf{S}_{21}$$

$$\begin{aligned} \mathbf{S}_{11.2} &= \begin{bmatrix} 53.9893 & 22.4295 \\ 22.4295 & 17.7600 \end{bmatrix} - \begin{bmatrix} 50.9603 & 21.1014 \\ 21.1014 & 16.0531 \end{bmatrix} \\ &= \begin{bmatrix} 3.0290 & 1.3281 \\ 1.3281 & 1.7069 \end{bmatrix} \end{aligned}$$

Third-order partial correlation was obtained by normalizing the above matrix.

$$\begin{aligned} r_{12.345} &= \frac{1.3281}{\sqrt{3.0290} \times \sqrt{1.7069}} \\ &= 0.58409 \end{aligned}$$

4.5.5 Computation of Fourth-Order Partial Correlation Coefficient

To get the fourth-order partial correlation, the sample variance-covariance matrix for consumption (v_1), corporate profit (v_2), and the controlled variables of private wage bill (v_3), investment (v_4), previous year's capital stock (v_5), and government wage bill (v_7), were partitioned as follows:

$$\mathbf{S} = \left(\begin{array}{cc|cccc} 53.9893 & 22.4295 & 45.1657 & 10.6341 & 76.5988 & 10.1141 \\ 22.4295 & 17.7600 & 20.4424 & 12.9086 & 38.4643 & 0.8124 \\ \hline 45.1657 & 20.4424 & 40.4520 & 11.4451 & 67.7554 & 7.4398 \\ 10.6341 & 12.9086 & 11.4451 & 12.1089 & 23.1506 & -1.7315 \\ 76.5988 & 38.4643 & 67.7554 & 23.1506 & 117.8004 & 11.4543 \\ 10.1141 & 0.8124 & 7.4398 & -1.7315 & 11.4543 & 4.0336 \end{array}\right)$$

$$\mathbf{S}_{11.2} = \mathbf{S}_{11} - \mathbf{S}_{12}\mathbf{S}_{22}^{-1}\mathbf{S}_{21}$$

$$\mathbf{S}_{11.2} = \begin{bmatrix} 53.9893 & 22.4295 \\ 22.4295 & 17.7600 \end{bmatrix} - \begin{bmatrix} 45.1657 & 10.6341 & 33.4374 & 10.1141 \\ 20.4424 & 12.9086 & -4.1571 & 0.8124 \end{bmatrix} \begin{bmatrix} 40.4520 & 11.4451 & 24.7435 & 7.4398 \\ 11.4451 & 12.1089 & -13.7061 & -1.7315 \\ 24.7435 & -13.7061 & 112.6004 & 8.8938 \\ 7.4398 & -1.7315 & 8.8938 & 4.0336 \end{bmatrix}^{-1} \begin{bmatrix} 45.1657 & 20.4424 \\ 10.6341 & 12.9086 \\ 33.4374 & -4.1571 \\ 10.1141 & 0.8124 \end{bmatrix}$$

$$= \begin{bmatrix} 53.9893 & 22.4295 \\ 22.4295 & 17.7600 \end{bmatrix} - \begin{bmatrix} 52.5530 & 21.3203 \\ 21.3203 & 16.0832 \end{bmatrix}$$

$$= \begin{bmatrix} 1.4363 & 1.1092 \\ 1.1092 & 1.6768 \end{bmatrix}$$

We normalized the above matrix gives the fourth-order partial correlation coefficient:

$$r_{12.3457} = \frac{1.1092}{\sqrt{1.4363} \times \sqrt{1.6768}}$$
$$= 0.71474$$

4.5.6 Computation of Fifth-Order Partial Correlation Coefficient

We partition the sample variance-covariance matrix obtained for consumption (v_1), corporate profit (v_2) and controlling variables of private wage bill (v_3), investment (v_4), previous year's capital stock (v_5), government wage bill (v_7), and taxes (v_9) as follows to find the partial correlation coefficient of fifth order:

$$\mathbf{S} = \left(\begin{array}{cc|ccccc} 53.9893 & 22.4295 & 45.1657 & 10.6341 & 33.4374 & 10.1141 & 9.0036 \\ 22.4295 & 17.7600 & 20.4424 & 12.9086 & -4.1571 & 0.8124 & 0.2619 \\ \hline 45.1657 & 20.4424 & 40.4520 & 11.4451 & 24.7435 & 7.4398 & 6.8610 \\ 10.6341 & 12.9086 & 11.4451 & 12.1089 & -13.7061 & -1.7315 & -1.2031 \\ 33.4374 & -4.1571 & 24.7435 & -13.7061 & 112.6004 & 8.8938 & 8.6555 \\ 10.1141 & 0.8124 & 7.4398 & -1.7315 & 8.8938 & 4.0336 & 3.2021 \\ 9.0036 & 0.2619 & 6.8610 & -1.2031 & 8.6555 & 3.2021 & 4.4579 \end{array} \right)$$

$$\mathbf{S}_{11.2} = \mathbf{S}_{11} - \mathbf{S}_{12}\mathbf{S}_{22}^{-1}\mathbf{S}_{21}$$

$$\mathbf{S}_{11.2} = \begin{bmatrix} 53.9893 & 22.4295 \\ 22.4295 & 17.7600 \end{bmatrix} - \begin{bmatrix} 45.1657 & 10.6341 & 33.4374 & 10.1141 & 9.0036 \\ 20.4424 & 12.9086 & -4.1571 & 0.8124 & 0.2619 \end{bmatrix}$$

$$\begin{bmatrix} 40.4520 & 11.4451 & 24.7435 & 7.4398 & 6.8610 \\ 11.4451 & 12.1089 & -13.7061 & -1.7315 & -1.2031 \\ 24.7435 & -13.7061 & 112.6004 & 8.8938 & 8.6555 \\ 7.4398 & -1.7315 & 8.8938 & 4.0336 & 3.2021 \\ 6.8610 & -1.2031 & 8.6555 & 3.2021 & 4.4579 \end{bmatrix}^{-1} \begin{bmatrix} 45.1657 & 20.4424 \\ 10.6341 & 12.9086 \\ 33.4374 & -4.1571 \\ 10.1141 & 0.8124 \\ 9.0036 & 0.2619 \end{bmatrix}$$

$$= \begin{bmatrix} 53.9893 & 22.4295 \\ 22.4295 & 17.7600 \end{bmatrix} - \begin{bmatrix} 52.5599 & 21.2739 \\ 21.2739 & 16.3905 \end{bmatrix}$$

$$= \begin{bmatrix} 1.4294 & 1.1556 \\ 1.1556 & 1.3695 \end{bmatrix}$$

We normalized the above matrix gives the fifth-order partial correlation coefficient

$$r_{12.34579} = \frac{1.1556}{\sqrt{1.4294} \times \sqrt{1.3695}}$$
$$= 0.82594$$

4.5.7 Computation of Partial Correlation Including v_6

While computing the orders of partial correlation coefficients, we chose to utilize the gross national product (v_6) as a controlling variable, which violated the assumption of partial correlation. Table 9 shows the results of partial correlation coefficient orders comprising gross national product (v_6) as follows:

Table 9: Partial Correlation Coefficients Including v_6

Orders	Estimation
First-order ($r_{12.6}$)	-0.55360
Second-order ($r_{12.36}$)	-0.39841
Third-order ($r_{12.346}$)	0.38395
Fourth-order ($r_{12.3456}$)	0.35782
Fifth-order ($r_{12.34567}$)	0.58798
Sixth-order ($r_{12.345679}$)	Invalid

Source: Field work (2023)

To obtain the sixth order partial correlation, the conditional variance-covariance matrix result of the partitioned variance-covariance matrix is shown in Appendix B as follows:

$$\mathbf{S}_{11.2} = \begin{pmatrix} 0.45444198744 & -0.00000334948 \\ -0.00000334948 & -0.00002000029 \end{pmatrix}$$

The above matrix was normalized to get the sixth-order partial correlation

$$r_{12.345679} = \frac{-0.00000334948}{\sqrt{0.45444198744} \times \sqrt{-0.00002000029}}$$
$$= invalid$$

Is invalid due to the square root of a negative value

4.5.8 Computation of Partial Correlation Including Both v_6 and v_8

We opted to add both violating variables gross national product (v_6) and government expenditure (v_8) as controlling factors. Table 10 illustrates the results of the partial correlation coefficient ordering, which contain both gross national product (v_6) and government spending (v_8), as follows:

Table 10: Partial Correlation Coefficients Including v_6 and v_8

Orders	Estimation
Second-order ($r_{12.68}$)	-0.72127
Third-order ($r_{12.368}$)	-0.63691
Fourth-order ($r_{12.3468}$)	Invalid

Source: Field work (2023)

In order to determine the fourth order partial correlation, the conditional variance-covariance matrix result of the partitioned variance-covariance matrix is shown as follows in Appendix B:

$$\mathbf{S}_{11.2} = \begin{pmatrix} -0.000001 & -0.00000170106 \\ -0.00000170106 & 1.09940520031 \end{pmatrix}$$

We normalized the above matrix gives the fourth-order partial correlation

$$r_{12.3468} = \frac{-0.00000170106}{\sqrt{-0.000001} \times \sqrt{1.09940520031}}$$
$$= invalid$$

Square root of a negative number renders the value invalid

4.6 Testing for Partial Correlation Coefficient

The importance of a linear link between two variables may also be estimated using the partial correlation coefficient, provided that all other impacts on the set of correlated variables have been considered. The alternative, that a partial correlation is not equal to zero, is contrasted with the null hypothesis, which states that a partial correlation is equal to zero.

4.6.1 Testing forFirst Order-Partial Correlation Coefficient

Hypothesis Testing

$$H_0 : \rho_{12.3} = 0$$

$$H_1 : \rho_{12.3} \neq 0$$

Test statistic

$$t = -0.07678\sqrt{\frac{22-2-1}{1-(-0.07678)^2}}$$
$$= -0.3357$$

Critical values

$$|t| = t_{19,0.05/2} = \pm 2.093$$

Decision and conclusion

There is insufficient evidence to refute the null hypothesis at the significance level of 0.05 since the test statistic (-0.3357) is within the critical interval (± 2.093). After adjusting for the impact of the private wage bill (v_3), we may conclude that there is no significant partial connection between consumption (v_1) and corporate profit (v_2).

4.6.2 Testing for Partial Correlation Coefficient of Second Order

Hypothesis Testing

$$H_0 : \rho_{12.34} = 0$$

$$H_1 : \rho_{12.34} \neq 0$$

Test statistic

$$t = 0.58261\sqrt{\frac{22-2-2}{1-(0.58261)^2}}$$
$$= 3.0413$$

Critical values

$$|t| = t_{18,0.05/2} = \pm 2.101$$

Decision and conclusion

It is possible to reject the null hypothesis at a significance level of 0.05 since the test statistic (3.0413) is not within the critical value of (± 2.101). We conclude that there is a significant partial connection between consumption (v_1) and corporate profit (v_2) after adjusting for private wage bill (v_3) and investment (v_4).

4.7 Confidence Interval for First-Order Partial Correlation Coefficient

First-order partial correlation's null hypothesis was not rejected and did not achieve statistical significance. We show here how to generate a confidence range for the first-order partial correlation coefficient.

We calculate the Fisher's transformation partial correlation:

$$Z_{12} = \frac{1}{2}\ln\left(\frac{1+(-0.0768)}{1-(-0.0768)}\right)$$
$$= -0.07695$$

We compute the 95% confidence interval for 0.6664:

$$Z_l = -0.07695 - \frac{1.96}{\sqrt{22-3}}$$
$$= -0.5266$$

$$Z_u = -0.07695 + \frac{1.96}{\sqrt{22-3}}$$
$$= 0.3727$$

The 95% confidence interval for the first-order partial correlation coefficient ($\rho_{12.3}$) may be found by back-transforming the data.

$$\left(\frac{\exp(2\times -0.5266)-1}{\exp(2\times -0.5266)+1}, \frac{\exp(2\times 0.3727)-1}{\exp(2\times 0.3727)+1}\right)$$

$$(-0.4828, 0.3564)$$

After taking into consideration the effect of the private wage bill (v_3), we are 95% confidence that the interval $(-0.4828, 0.3564)$ includes the true partial correlation between corporate profit (v_2) and consumption (v_1).

4.7.1 Confidence Interval for Second-Order Partial Correlation Coefficient

The second-order partial correlation's null hypothesis was rejected, and it was statistically significant. We show how to compute the confidence interval for the partial correlation coefficient of second order.

We calculate the Fisher's transformation partial correlation:

$$Z_{12} = \frac{1}{2}\ln\left(\frac{1+0.58261}{1-0.58261}\right)$$
$$= 0.6664$$

We compute the 95% confidence interval for 0.6664:

$$Z_l = 0.6664 - \frac{1.96}{\sqrt{22-3}}$$
$$= 0.21675$$

$$Z_u = 0.6664 + \frac{1.96}{\sqrt{22-3}}$$
$$= 1.11605$$

Using back-transformation, the 95% confidence interval for the partial correlation coefficient of second order ($\rho_{12.34}$) is obtained:

$$\left(\frac{\exp(2\times0.21675)-1}{\exp(2\times0.21675)+1}, \frac{\exp(2\times1.11605)-1}{\exp(2\times1.11605)+1}\right)$$

$$(0.2134, 0.8062)$$

This finding leads us to the conclusion that, after adjusting for the impacts of investment (v_4) and private wage bill (v_3), we are 95% certain that the interval $(0.2134, 0.8062)$ includes the genuine partial correlation coefficient between corporate profit (v_2) and consumption (v_1).

4.8 Computation of Partial Correlation Matrices

The R software package scripts utilized to produce the partial correlation matrix ordering are shown in appendix C. We utilize the seven variables that passed the partial correlation assumptions test (i.e., v_1, v_2, v_3, v_4, v_5, v_7, and v_9) to generate partial correlation matrices between pairs of variables in v while regulating for the effect of the other factors (variables) in v as follows:

4.8.1 First-Order Partial Correlation Matrix

We generate the first-order partial correlation matrices for consumption (v_1), corporate profit (v_2), and private wage bill (v_3). As you may remember, order one partial correlation coefficient of $r_{12.3} = -0.07678$ was discovered. The findings for the other two first-order partial correlation coefficients, $r_{13.2} = 0.9285$ and $r_{23.1} = 0.3537$, are shown in Appendix D. The results for the first order partial correlation matrix for v_1, v_2, and v_3 are displayed in the following illustration:

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{pmatrix} 1 & -0.0768 & 0.9285 \\ -0.0768 & 1 & 0.3537 \\ 0.9285 & 0.3537 & 1 \end{pmatrix} \end{array}$$

An additional 34 distinct first-order partial correlation coefficient matrices may be obtained by combining the seven variables that meet partial correlation assumptions in the same manner.

4.8.2 Second Order Partial Correlation Matrix

The partial correlation coefficient matrices for second-order are created using four variables. The results of the partial correlation coefficient matrix of second order for v_1, v_2, v_3, and v_4 are given as follows:

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 1 & 0.583 & 0.925 & -0.658 \\ 0.583 & 1 & -0.305 & 0.921 \\ 0.925 & -0.305 & 1 & 0.464 \\ -0.658 & 0.921 & 0.464 & 1 \end{pmatrix} \end{array}$$

The same method may be applied to produce 34 more different second-order partial correlation coefficient matrices.

4.8.3 Third-Order Partial Correlation Matrix

To get the matrices for the partial correlation coefficients for order three, we require five variables. The following are the results of the third-order partial correlation matrix for the variables v_1, v_2, v_3, v_4, and v_5 are as follows:

$$\begin{array}{c} \\ v1 \\ v2 \\ v3 \\ v4 \\ v5 \end{array} \begin{array}{c} \begin{array}{ccccc} v1 & v2 & v3 & v4 & v5 \end{array} \\ \begin{pmatrix} 1 & 0.584 & 0.895 & -0.603 & 0.083 \\ 0.584 & 1 & -0.291 & 0.865 & -0.051 \\ 0.895 & -0.291 & 1 & 0.486 & 0.168 \\ -0.603 & 0.865 & 0.486 & 1 & -0.289 \\ 0.083 & -0.051 & 0.168 & -0.289 & 1 \end{pmatrix} \end{array}$$

The same approach may be used to generate twenty (20) more third-order partial correlation coefficient matrices in various combinations.

4.8.4 Fourth-Order Partial Correlation Matrix

Six variables are the only ones that may provide the fourth-order partial correlation coefficient matrices. The following is the result of the fourth-order partial correlation matrix for the variables v_1, v_2, v_3, v_4, v_5 and v_7 is as follows:

$$\begin{array}{c} \\ v1 \\ v2 \\ v3 \\ v4 \\ v5 \\ v7 \end{array} \begin{array}{c} \begin{array}{cccccc} v1 & v2 & v3 & v4 & v5 & v7 \end{array} \\ \begin{pmatrix} 1 & 0.715 & 0.533 & 0.239 & 0.707 & 0.805 \\ 0.715 & 1 & -0.250 & 0.236 & -0.463 & -0.520 \\ 0.533 & -0.250 & 1 & 0.373 & 0.088 & -0.002 \\ 0.239 & 0.236 & 0.373 & 1 & -0.659 & -0.639 \\ 0.707 & -0.463 & 0.088 & -0.659 & 1 & -0.846 \\ 0.805 & -0.520 & -0.002 & -0.639 & -0.846 & 1 \end{pmatrix} \end{array}$$

Similar matrix construction techniques may be used to produce an additional 6 unique combinations of fourth-order partial correlation matrices.

4.8.5 Fifth-Order Partial Correlation Matrix

To create the fifth-order partial correlation matrix, seven variables are required. For the fifth-order partial correlation matrix, the result is illustrated as follows for v_1, v_2, v_3, v_4, v_5 , v_7, and v_9 :

$$\begin{array}{c} \\ v1 \\ v2 \\ v3 \\ v4 \\ v5 \\ v7 \\ v9 \end{array} \begin{array}{c} \begin{array}{ccccccc} v1 & v2 & v3 & v4 & v5 & v7 & v9 \end{array} \\ \begin{pmatrix} 1 & 0.826 & 0.501 & 0.160 & 0.778 & 0.792 & 0.594 \\ 0.826 & 1 & -0.268 & 0.208 & -0.614 & -0.562 & -0.685 \\ 0.501 & -0.268 & 1 & 0.376 & 0.019 & -0.039 & -0.128 \\ 0.160 & 0.208 & 0.376 & 1 & -0.560 & -0.595 & 0.054 \\ 0.778 & -0.614 & 0.019 & -0.560 & 1 & -0.852 & -0.459 \\ 0.792 & -0.562 & -0.039 & -0.595 & -0.852 & 1 & -0.291 \\ 0.594 & -0.685 & -0.128 & 0.054 & -0.459 & -0.291 & 1 \end{pmatrix} \end{array}$$

There is just one conceivable combination of the seven variables in the fifth-order partial correlation matrix.

4.9 Chapter Summary

With full justifications, we have addressed both the manually and electronically computed methods for obtaining higher order partial correlation coefficients when employing the variance-covariance matrix technique.

This chapter's main objective was to explain the theory underlying the conditional variance-covariance matrix approach for obtaining the orders of partial correlation coefficients from the illustrative dataset. Because the partial correlation established satisfies the intended characteristics, the implementation certifies that the knowledge acquired on the subject is correct. The basic presumptions of the partial correlation were confirmed before constructing the partial correlation coefficient orders. Findings from the normalized conditional variance-covariance matrix were used to determine the orders of partial correlation coefficient. We were successful in creating the correlation matrix using a multivariate technique employing the given variance-covariance matrix. We discovered from our practical illustrations that variables that violated the partial correlation assumptions may still be included in the analysis for computing orders of partial correlation but would become invalid as the partial correlation coefficients orders increased. We provide practical examples that show how higher order partial correlation coefficients than first order partial correlation may be obtained using the variance-covariance matrix approach. Considering the impacts of several parameters, we were able to ascertain the significance of the partial correlation coefficient between pairs of variables and produce confidence ranges for it. Finally, for the various orders of partial correlation coefficient, we were able to construct suitable partial correlation matrices up to the fifth order.

CHAPTER FIVE

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

5.0 Overview

We give a synopsis of all results obtained from the previous chapters in this chapter. Finally, considering the study's findings, pertinent conclusions and suggestions are provided.

5.1 Summary

Partial correlation is used to take into consideration the impact of one or more extra factors when evaluating the strength of a relationship between two variables. Ogunleye *et al*., (2022) demonstrate through real-world examples that partial correlation coefficients above first-order partial correlation cannot be calculated using the variance-covariance matrix method. In order to obtain partial correlation coefficient orders beyond the partial correlation coefficient of order one from this statement problem, we would like to investigate the method of variance-covariance matrix approach further. As a result, there is a need for a research that offers partial correlation analysis computation in a more cohesive and application-friendly manner. The study's objectives were therefore to give a clear process for creating a variance-covariance matrix, which served as the foundation for determining the partial correlation coefficient orders, and then utilize the variance co-variance matrix to generate the correlation coefficient matrix. Following that, we also discovered the partial correlation coefficient's maximum likelihood estimator. Lastly, we used the variance-covariance methodology to determine the partial correlation coefficient orders and offer clarity of mathematical derivation of findings with practical applications.

The study planned to apply matrix principles to compute the orders of partial correlation coefficients. As a consequence, we relied heavily on the matrix principles to conduct the partial correlation analysis for this thesis. Real datasets were gathered in order for the study to be completed. The investigation started with typical real-world data from the literature that was discovered to be beneficial for illustration in the study and has been sufficiently explained. The R package software was utilized for data analysis, and the results were validated manually using the method of variance-covariance approach.

The conventional (traditional) method, the matrix inversion method, ordinary least squares method (OLS), the regression residual's approach, the variance-covariance matrix approach, and the recursive method are some of the approaches for estimating partial correlation coefficients that have been documented in the literature. Concepts and their theoretical applications may be found in famous text books, and numerous ways to determining partial correlation coefficients have been used. The literature indicates that there has been much research on partial correlation analysis over a lengthy period of time. Several authors have written a great deal about this topic, and published articles have also suggested applications to data analysis. Partial correlation is frequently used in a wide range of professions, most notably biology, medicine, economics, accounting, engineering, and other related fields. We demonstrated the process of computing the variance-covariance matrix from unprocessed datasets and partitioning it to obtain the results of the conditional variance-covariance matrix. Furthermore, we presented the maximum likelihood estimator, which was utilized to compute partial correlation coefficient orders. We discovered from our practical examples that variables that did not adhere to the partial correlation assumptions may still be included in the analysis to compute the partial correlation orders; however, they would become invalid when partial correlation coefficient orders increased. The findings from the data illustration indicate that the variance-covariance matrix strategy may compute partial correlation coefficients of any order as long as the partial correlation assumptions are fulfilled.

5.2 Conclusion

The thesis offered a rational investigation into the linear relationship between two random variables in order to take into consideration the influence of one or more additional random variables. An extensive understanding of the variance-covariance matrix technique to obtaining partial correlation coefficient orders necessitates the discovery of matrix principles techniques that create the requisite orders. To achieve this, the study determines the appropriate dataset structure and partitioning, as well as the key matrices that allow us to acquire the theoretical conclusion. It has been shown that normalizing the conditional variance-covariance matrix findings can yield the partial correlation coefficient ordering. Understanding partial correlation coefficients has allowed for the development of useful codes in the R software programme that construct partial correlation coefficient orders. As a result, the study proposed a more user-friendly way for manually estimating orders of partial correlation coefficient utilizing the method of variance-covariance matrix approach.

It was demonstrated how to utilize the method of variance-covariance matrix as a multivariate approach to construct the correlation matrix using the example dataset. The variance-covariance matrix method can produce higher order partial correlation coefficients based on our examples. This contradicts the claim made by Ogunleye et al., 2022. Finally, we showed how to construct appropriate partial correlation matrices for various orders of partial correlation coefficients.

5.3 Recommendation

The study makes a convincing case for the theory of partial correlation coefficient using the variance-covariance matrix approach. It also emphasizes the need of having a firm understanding of the concept in order to put the right strategies into practice and make it user-friendly. The R codes presented in the research may be beneficial for obtaining orders of partial correlation coefficient. Furthermore, the study itself offers a logical utilization of the topic in order to provide interested learners with the necessary comprehension.

As long as the partial correlation assumptions are met, the outcome shows that the variance-covariance matrix approach technique may compute partial correlation coefficients of any order. As a result, we recommend that future researchers adopt the method of variance-covariance matrix technique to generate higher orders of partial correlation coefficients since the method is trustworthy, and comprehensible.

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APPENDICES

APPENDIX A

R CODES FOR COMPUTING ORDERS OF PARTIAL CORRELATION COEFFICIENT

```
library(ppcor)
# Original Data
x1<-
c(39.8,41.9,45,49.2,50.6,52.6,55.1,56.2,57.3,57.8,55,50.9,45.6,46.5,48.7,51.3,
57.7,58.7,57.5,61.6,65,69.7)
x2<-
c(12.7,12.4,16.9,18.4,19.4,20.1,19.6,19.8,21.1,21.7,15.6,11.4,7,11.2,12.3,14,1
7.6,17.3,15.3,19,21.1,23.5)
x3<-
c(28.8,25.5,29.3,34.1,33.9,35.4,37.4,37.9,39.2,41.3,37.9,34.5,29,28.5,30.6,33.
2,36.8,41,38.2,41.6,45,53.3)
x4<-c(2.7,-0.2,1.9,5.2,3,5.1,5.6,4.2,3,5.1,1,-3.4,-6.2,-5.1,-3,-1.3,2.1,2,-
1.9,1.3,3.3,4.9)
x5<-
c(180.1,182.8,182.6,184.5,189.7,192.7,197.8,203.4,207.6,210.6,215.7,216.7,2
13.3,207.1,202,199,197.7,199.8,201.8,199.9,201.2,204.5)
x6<-
c(44.9,45.6,50.1,57.2,57.1,61,64,64.4,64.5,67,61.2,53.4,44.3,45.1,49.7,54.4,62
.7,65,60.9,69.5,75.7,88.4)
x7<-
c(2.2,2.7,2.9,2.9,3.1,3.2,3.3,3.6,3.7,4,4.2,4.8,5.3,5.6,6,6.1,7.4,6.7,7.7,7.8,8,8.5)
x9<-
c(3.4,7.7,3.9,4.7,3.8,5.5,7,6.7,4.2,4,7.7,7.5,8.3,5.4,6.8,7.2,8.3,6.7,7.4,8.9,9.6,1
1.6)
```

1st Order Partial Correlation

```
pcor_ci.test <-
 function (x1, x2, x3, method = c("pearson"), conf.level = 0.95) {
  d1 <- deparse(substitute(x1))
  d2 <- deparse(substitute(x2))
  d3 <- deparse(substitute(x3))
  data.name <- paste0(d1, " and ", d2, "; controlling: ", d3)
  method <- match.arg(method)
  Method <- paste0("Partial correlation (", method, ")")
  alternative <- "true partial correlation is not equal to 0"

  x1 <- as.vector(x1)
  x2 <- as.vector(x2)
  x3 <- as.data.frame(x3)
  df <- data.frame(x1, x2, x3)
  pcor <- ppcor::pcor(df, method = method)
  estimate <- pcor$est[1, 2]
  p.value <- pcor$p.value[1, 2]
```

```
  parameter <- c(n = pcor$n, gp = pcor$gp)
  statistic <- c(Stat = pcor$statistic[1, 2])

  fit1 <- lm(x1 ~ x3, data = df)
  fit2 <- lm(x2 ~ x3, data = df)
  cortest <- cor.test(resid(fit1), resid(fit2), method = method, conf.level =
conf.level)
  ci <- cortest$conf.int
  ht <- list(
   statistic = statistic,
   parameter = parameter,
   p.value = p.value,
   estimate = c(partial.cor = estimate),
   alternative = alternative,
   method = Method,
   data.name = data.name,
   conf.int = ci)
  class(ht) <- "htest"
  ht}
pcor_ci.test(x1, x2, x3)
```

2nd Order Partial Correlation

```
pcor_ci.test <-
 function (x1, x2, x3, x4, method = c("pearson"), conf.level = 0.95) {
  d1 <- deparse(substitute(x1))
  d2 <- deparse(substitute(x2))
  d3 <- deparse(substitute(x3))
  d4 <- deparse(substitute(x4))
  data.name <- paste0(d1, " and ", d2, "; controlling: ", d3, d4)
  method <- match.arg(method)
  Method <- paste0("Partial correlation (", method, ")")
  alternative <- "true partial correlation is not equal to 0"
  x1 <- as.vector(x1)
  x2 <- as.vector(x2)
  x3 <- as.data.frame(x3)
  x4 <- as.data.frame(x4)
  df <- data.frame(x1, x2, x3, x4)
  pcor <- ppcor::pcor(df, method = method)
  estimate <- pcor$est[1, 2]
  p.value <- pcor$p.value[1, 2]
  parameter <- c(n = pcor$n, gp = pcor$gp)
  statistic <- c(Stat = pcor$statistic[1, 2])
  fit1 <- lm(x1 ~ x3+x4, data = df)
  fit2 <- lm(x2 ~ x3+x4, data = df)
  cortest <- cor.test(resid(fit1), resid(fit2), method = method, conf.level =
conf.level)
  ci <- cortest$conf.int
  ht <- list(
   statistic = statistic,
   parameter = parameter,
   p.value = p.value,
```

```
    estimate = c(partial.cor = estimate),
    alternative = alternative,
    method = Method,
    data.name = data.name,
    conf.int = ci)
  class(ht) <- "htest"
  ht}
pcor_ci.test(x1, x2, x3, x4)
```

3rd Order Partial Correlation

```
pcor_ci.test <-
 function (x1, x2, x3, x4, x5, method = c("pearson"), conf.level = 0.95) {
  d1 <- deparse(substitute(x1))
  d2 <- deparse(substitute(x2))
  d3 <- deparse(substitute(x3))
  d4 <- deparse(substitute(x4))
  d5 <- deparse(substitute(x5))
  data.name <- paste0(d1, " and ", d2, "; controlling: ", d3, d4, d5)
  method <- match.arg(method)
  Method <- paste0("Partial correlation (", method, ")")
  alternative <- "true partial correlation is not equal to 0"
  x1 <- as.vector(x1)
  x2 <- as.vector(x2)
  x3 <- as.data.frame(x3)
  x4 <- as.data.frame(x4)
  x5 <- as.data.frame(x5)
  df <- data.frame(x1, x2, x3, x4, x5)
  pcor <- ppcor::pcor(df, method = method)
  estimate <- pcor$est[1, 2]
  p.value <- pcor$p.value[1, 2]
  parameter <- c(n = pcor$n, gp = pcor$gp)
  statistic <- c(Stat = pcor$statistic[1, 2])
  fit1 <- lm(x1 ~ x3+x4+x5, data = df)
  fit2 <- lm(x2 ~ x3+x4+x5, data = df)
  cortest <- cor.test(resid(fit1), resid(fit2), method = method, conf.level =
conf.level)
  ci <- cortest$conf.int
  ht <- list(
    statistic = statistic,
    parameter = parameter,
    p.value = p.value,
    estimate = c(partial.cor = estimate),
    alternative = alternative,
    method = Method,
    data.name = data.name,
    conf.int = ci)
  class(ht) <- "htest"
  ht}
pcor_ci.test(x1, x2, x3, x4, x5)
```

4th Order Partial Correlation

```
pcor_ci.test <-
```

```
 function (x1, x2, x3, x4, x5, x7, method = c("pearson"), conf.level = 0.95) {
  d1 <- deparse(substitute(x1))
  d2 <- deparse(substitute(x2))
  d3 <- deparse(substitute(x3))
  d4 <- deparse(substitute(x4))
  d5 <- deparse(substitute(x5))
  d6 <- deparse(substitute(x7))
  data.name <- paste0(d1, " and ", d2, "; controlling: ", d3, d4, d5, d6)
  method <- match.arg(method)
  Method <- paste0("Partial correlation (", method, ")")
  alternative <- "true partial correlation is not equal to 0"
  x1 <- as.vector(x1)
  x2 <- as.vector(x2)
  x3 <- as.data.frame(x3)
  x4 <- as.data.frame(x4)
  x5 <- as.data.frame(x5)
  x7 <- as.data.frame(x7)
  df <- data.frame(x1, x2, x3, x4, x5, x7)
  pcor <- ppcor::pcor(df, method = method)
  estimate <- pcor$est[1, 2]
  p.value <- pcor$p.value[1, 2]
  parameter <- c(n = pcor$n, gp = pcor$gp)
  statistic <- c(Stat = pcor$statistic[1, 2])
  fit1 <- lm(x1 ~ x3+x4+x5+x7, data = df)
  fit2 <- lm(x2 ~ x3+x4+x5+x7, data = df)
  cortest <- cor.test(resid(fit1), resid(fit2), method = method, conf.level =
conf.level)
  ci <- cortest$conf.int
  ht <- list(
   statistic = statistic,
   parameter = parameter,
   p.value = p.value,
   estimate = c(partial.cor = estimate),
   alternative = alternative,
   method = Method,
   data.name = data.name,
   conf.int = ci)
  class(ht) <- "htest"
  ht}
pcor_ci.test(x1, x2, x3, x4, x5, x7)
```

APPENDIX B

COMPUTATION OF PARTIAL CORRELATION INCLUDING VIOLATED VARIABLES

Computation of Sixth-Order Partial Correlation including v_6

Variance-covariance matrix achieved for v_1, v_2 and controlling variables v_3, v_4, v_5, v_6, v_7, v_9 was partitioned as follows:

$$\mathbf{S} = \left(\begin{array}{cc|cccccc} 53.98929 & 22.42952 & 45.16571 & 10.63405 & 33.43738 & 76.59881 & 10.11405 & 9.00357 \\ 22.42952 & 17.76000 & 20.44238 & 12.90857 & -4.15714 & 38.46429 & 0.81238 & 0.26190 \\ \hline 45.1657 & 20.4424 & 40.45203 & 11.44511 & 24.74346 & 67.75537 & 7.43978 & 6.8610 \\ 10.6341 & 12.9086 & 11.44511 & 12.10894 & -13.70608 & 23.15058 & -1.73145 & -1.2031 \\ 33.43738 & -4.15714 & 24.74346 & -13.70608 & 112.60037 & 29.24180 & 8.89383 & 8.65548 \\ 76.59881 & 38.46429 & 67.75537 & 23.15058 & 29.24180 & 117.80037 & 11.45431 & 11.58071 \\ 10.1141 & 0.8124 & 7.43978 & -1.73145 & 8.89383 & 11.45431 & 4.03361 & 3.2021 \\ 9.0036 & 0.2619 & 6.86095 & -1.20310 & 8.65548 & 11.58071 & 3.20214 & 4.4579 \end{array}\right)$$

$$\mathbf{S}_{11.2} = \mathbf{S}_{11} - \mathbf{S}_{12}\mathbf{S}_{22}^{-1}\mathbf{S}_{21}$$

$$\mathbf{S}_{11.2} = \begin{bmatrix} 53.98929 & 22.42952 \\ 22.42952 & 17.76000 \end{bmatrix} - \begin{bmatrix} 53.53484801256 & 22.42952334948 \\ 22.42952334948 & 17.76002000029 \end{bmatrix}$$

$$= \begin{bmatrix} 0.45444198744 & -0.00000334948 \\ -0.00000334948 & -0.00002000029 \end{bmatrix}$$

Computation of fourth-order partial correlation including v_6 and v_8

The variance-covariance matrix generated for v_1, v_2, and controlling variables v_3, v_4, v_6, v_8 was partitioned as follows:

$$\mathbf{S} = \left(\begin{array}{cc|cccc} 53.989286 & 22.429524 & 45.165714 & 10.634048 & 76.598810 & 11.975476 \\ 22.429524 & 17.760000 & 20.442381 & 12.908571 & 38.464286 & 3.126190 \\ \hline 45.165714 & 20.442381 & 40.452035 & 11.445108 & 67.755368 & 11.144545 \\ 10.634048 & 12.908571 & 11.445108 & 12.108939 & 23.150584 & 0.407597 \\ 76.598810 & 38.464286 & 67.755368 & 23.150584 & 117.800368 & 18.050974 \\ 11.975476 & 3.126190 & 11.144545 & 0.407597 & 18.050974 & 5.667900 \end{array}\right)$$

$$\mathbf{S}_{11.2} = \mathbf{S}_{11} - \mathbf{S}_{12}\mathbf{S}_{22}^{-1}\mathbf{S}_{21}$$

$$\mathbf{S}_{11.2} = \begin{bmatrix} 53.989286 & 22.429524 \\ 22.429524 & 17.760000 \end{bmatrix} - \begin{bmatrix} 53.989287 & 22.42952570106 \\ 22.42952570106 & 16.66059479969 \end{bmatrix}$$

$$= \begin{bmatrix} -0.000001 & -0.00000170106 \\ -0.00000170106 & 1.09940520031 \end{bmatrix}$$

APPENDIX C

R CODES FOR COMPUTING PARTIAL CORRELATION MATRICES

```
library(ppcor)
# Original Data
```

1st Order Partial Correlation Matrix

```
# Run for V1, V2 and V3
xmatrix<-cbind.data.frame(x1,x2,x3)
data<-data.matrix(xmatrix[,c('x1','x2','x3')])
pcor(data)
```

2nd Order Partial Correlation Matrix

```
# Run for V1, V2, V3 and V4
xmatrix<-cbind.data.frame(x1,x2,x3,x4)
data<-data.matrix(xmatrix[,c('x1','x2','x3','x4')])
pcor(data)
```

3rd Order Partial Correlation Matrix

```
# Run for V1, V2, V3, V4 and V5
xmatrix<-cbind.data.frame(x1,x2,x3,x4,x5)
data<-data.matrix(xmatrix[,c('x1','x2','x3','x4','x5')])
pcor(data)
```

4th order partial correlation matrix

```
# Run for V1, V2, V3, V4, V5 and V7
xmatrix<-cbind.data.frame(x1,x2,x3,x4,x5,x7)
data<-data.matrix(xmatrix[,c('x1','x2','x3','x4','x5','x7')])
pcor(data)
```

5th order partial correlation matrix

```
# Run for V1, V2, V3, V4, V5, V7 and V9
xmatrix<-cbind.data.frame(x1,x2,x3,x4,x5,x7,x9)
data<-data.matrix(xmatrix[,c('x1','x2','x3','x4', 'x5', 'x7', 'x9')])
pcor(data)
```

APPENDIX D

FIRST-ORDER PARTIAL CORRELATION COEFFICIENT

The variance-covariance matrix generated for consumption (V1), private wage bill (V3), and the controlling variable corporate profit (V2) was partitioned as follows:

$$\mathbf{S} = \left(\begin{array}{cc|c} 53.9893 & 45.1657 & 22.4295 \\ 45.1657 & 40.4520 & 20.4424 \\ \hline 22.4295 & 20.4424 & 17.7600 \end{array}\right)$$

$$\mathbf{S}_{11.2} = \begin{bmatrix} 53.9893 & 45.1657 \\ 45.1657 & 40.4520 \end{bmatrix} - \begin{bmatrix} 22.4295 \\ 20.4424 \end{bmatrix} [17.7600]^{-1} [22.4295 \quad 20.4424]$$

$$= \begin{bmatrix} 53.9893 & 45.1657 \\ 45.1657 & 40.4520 \end{bmatrix} - \begin{bmatrix} 28.3267 & 25.8172 \\ 25.8172 & 23.5299 \end{bmatrix}$$

$$= \begin{bmatrix} 25.6626 & 19.3485 \\ 19.3485 & 16.9221 \end{bmatrix}$$

By normalizing the matrix above, gives

$$r_{13.2} = \frac{19.3485}{\sqrt{25.6626}\sqrt{16.9221}} = 0.92847$$

The variance-covariance matrix generated for corporate profit (V2) and private wage bill (V3), and the controlling variable consumption (V1) was partitioned as follows:

$$\mathbf{S} = \left(\begin{array}{cc|c} 17.7600 & 20.4424 & 22.4295 \\ 20.4424 & 40.4520 & 45.1657 \\ \hline 22.4295 & 45.1657 & 53.9893 \end{array}\right)$$

$$\mathbf{S}_{11.2} = \begin{bmatrix} 17.7600 & 20.4424 \\ 20.4424 & 40.4520 \end{bmatrix} - \begin{bmatrix} 22.4295 \\ 45.1657 \end{bmatrix} [53.9893]^{-1} [22.4295 \quad 45.1657]$$

$$= \begin{bmatrix} 17.7600 & 20.4424 \\ 20.4424 & 40.4520 \end{bmatrix} - \begin{bmatrix} 9.3182 & 18.7638 \\ 18.7638 & 37.7842 \end{bmatrix}$$

$$= \begin{bmatrix} 8.4418 & 1.6786 \\ 1.6786 & 2.6678 \end{bmatrix}$$

Standardizing the aforementioned matrix gives:

$$r_{23.1} = \frac{1.6786}{\sqrt{8.4418}\sqrt{2.6678}} = 0.35371$$

APPENDIX E

VARIANCE-COVARIANCE MATRIX (ALL VARIABLES)

Table 11: Variance-Covariance Matrix (All variables)

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9
v_1	53.989286								
v_2	22.429524	17.760000							
v_3	45.165714	20.442381	40.452035						
v_4	10.634048	12.908571	11.445108	12.108939					
v_5	33.437381	-4.157143	24.743463	-13.706082	112.600368				
v_6	76.598810	38.464286	67.755368	23.150584	29.241797	117.800368			
v_7	10.114048	0.812381	7.439784	-1.731450	8.893831	11.454307	4.033615		
v_8	11.975476	3.126190	11.144545	0.407597	9.510498	18.050974	3.071710	5.667900	
v_9	9.003571	0.261905	6.860952	-1.203095	8.655476	11.580714	3.202143	3.780238	4.457857

Source: Field work (2023)