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Asymptotic behaviour of boundary condition functions for a fourth-order boundary value problem

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Abstract

In this paper, we prove that the Boundary Condition Constants for the Boundary Value Problem

$$\pi : L\phi \equiv \phi^{(4)}(x) + P_2(x)\phi^{(2)}(x) + P_3(x)\phi^{(1)}(x) + P_4(x)\phi(x) = \lambda\phi(x)$$

$$U_r\phi \equiv \sum_{s=1}^{4}\left\{m_{rs}\phi^{(s-1)}(a) + n_{rs}\phi^{(s-1)}(b)\right\} = 0 \qquad (1 \le r, s \le 4)$$

can be replaced by Boundary Condition Functions and that the Boundary Condition Functions are asymptotically equivalent for large values of $|\lambda|$, to the Boundary Condition Functions for the corresponding Fourier problem π, given by

$$\pi_F :\ \phi^{(4)}(x) = \lambda\phi(x)$$

$$U_r\phi \equiv \sum_{s=1}^{4}\left\{m_{rs}\phi^{(s-1)}(a) + n_{rs}\phi^{(s-1)}(b)\right\} = 0 \qquad (1 \le r, s \le 4)\ .$$

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1 Introduction

Over the years quite a good number of mathematicians have studied boundary condition functions. The use of boundary condition functions for boundary value problems was first considered by Kodaira in [1].

In this paper Kodaira considered the replacement of boundary condition constants of separated boundary conditions , associated with real differential equations of arbitrary even order , by solutions of the differential equation. In [2] E. C. Titchmarsh proved that, the self-adjoint boundary value problem:

$$-\phi^{(2)}(x)+q(x)\phi(x)=\lambda\phi(x) \qquad (-\infty < a \le x \le b < \infty; \lambda = \sigma + i\tau)$$

$$\left.\begin{aligned}\varphi(a)\cos\alpha+\varphi^{(1)}(a)\sin\alpha=0\\ \varphi(b)\cos\beta+\varphi^{(1)}(b)\sin\beta=0\end{aligned}\right\}$$

is equivalent asymptotically, for suitably large values of $|\lambda|$, to the corresponding Fourier problem:

$$-\phi^{(2)}(x)=\lambda\phi(x)$$

$$\left.\begin{aligned}\varphi(a)\cos\alpha+\varphi^{(1)}(a)\sin\alpha=0\\ \varphi(b)\cos\beta+\varphi^{(1)}(b)\sin\beta=0\end{aligned}\right\}$$

The coefficient q and the constants α, β are real valued and $q \in C[a,b]$. W. N. Everitt in [3] also worked on self-adjoint boundary value problems. D. N. Offei in [4] extended the use of boundary condition functions to non-self adjoint boundary value problems with complex-valued coefficients and constants and with boundary conditions separated or otherwise.

In [5], D. N. Offei proved that the boundary condition functions, for the boundary value problem:

$$L\phi \equiv i^3\phi^{(3)}(x) + P_2(x)\phi^{(1)}(x) + P_3(x)\phi(x) = \lambda\phi(x)$$

$$\phi(a) = \phi(b) = \phi^{(1)}(b) = 0$$

are asymptotically equivalent, for suitably large values of $|\lambda|$, to the corresponding functions, for the corresponding Fourier problem. In [9], M. B. Osei showed that the boundary condition functions of the second order boundary value problem are asymptotically equivalent to the boundary condition functions of the corresponding Fourier problem of the boundary value problem.

2 Notation

In this section we give some properties of the linear differential expression L and some notations used in subsequent sections of this paper.

1. For a suitable set of functions $\phi_r(x),\ (1 \le r \le 4)$, the symbol $\mathbf{\Phi}(x)$ denotes the 4 x 4 matrix $[\phi_r^{(s-1)}(x)]\ (1 \le r, s \le 4)$.

   Thus,

$$\mathbf{\Phi}(x) \equiv \begin{bmatrix} \phi_1(x) & \phi_2(x) & \phi_3(x) & \phi_4(x) \\ \phi_1^{(1)}(x) & \phi_2^{(1)}(x) & \phi_3^{(1)}(x) & \phi_4^{(1)}(x) \\ \phi_1^{(2)}(x) & \phi_2^{(2)}(x) & \phi_3^{(2)}(x) & \phi_4^{(2)}(x) \\ \phi_1^{(3)}(x) & \phi_2^{(3)}(x) & \phi_3^{(3)}(x) & \phi_4^{(3)}(x) \end{bmatrix}.$$

   Also $\hat{g}(x)$ represents the column vector with components $g(x), g^{(1)}(x), \ldots, g^{(n-1)}(x)$.

2. The symbol $\mathbf{B}^*(x)$ denotes the conjugate transpose of the matrix $\mathbf{B}(x)$ whilst $\hat{b}^*(x)$ denote the row vector with components $\overline{b}(x), \overline{b}^{(1)}(x) \ldots \overline{b}^{(n-1)}(x)$.

3. Given the linear expression L defined by

$$L\phi \equiv P_0(x)\phi^{(4)}(x) + P_1(x)\phi^{(3)}(x) + P_2(x)\phi^{(2)}(x) + P_3(x)\phi^{(1)}(x) + P_4(x)\phi(x)$$

$(a \le x \le b)$. The Lagrange adjoint of L is denoted by L^+ and defined as

$$L^+\psi \equiv (-1)^4(\overline{P}_0\psi)^{(4)} + (-1)^3(\overline{P}_1\psi)^{(3)} + (-1)^2(\overline{P}_2\psi)^{(2)} + (-1)(\overline{P}_3\psi)^{(1)} + \overline{P}_4(x)\psi .$$

4. (a) For suitable pairs of functions f and g

$$\int_a^b \left\{ \overline{g}Lf - f\,\overline{L^+g} \right\} dx = [fg](b) - [fg](a) .$$

Here $[fg](x)$ is a bilinear form in $(f, f^{(1)}, f^{(2)}, f^{(3)})$ and $(\overline{g}, \overline{g}^{(1)}, \overline{g}^{(2)}, \overline{g}^{(3)})$ given by

$$\left[fg\right](x) = \sum_{j=1}^{4}\sum_{k=1}^{4} B_{jk}(x)\overline{g}^{(j-1)}(x) f^{(k-1)}(x) = \hat{g}^*(x)\mathbf{B}(x)\hat{f}(x)$$

where,

$$\mathbf{B}(x) = \begin{bmatrix} P_3^{(3)}(x) - P_2^{(1)}(x) + P_1^{(2)}(x) - P_0^{(2)}(x) - P_0^{(3)}(x) & P_2(x) - P_1^{(1)}(x) + P_0^{(2)}(x) & P_1(x) - P_0^{(1)}(x) & P_0(x) \\ -P_2(x) + 2P_1^{(1)}(x) - 2P_0^{(2)}(x) & -P_1(x) - 2P_0^{(1)}(x) & -P_0(x) & 0 \\ P_1(x) - 3P_0^{(1)}(x) & P_0(x) & 0 & 0 \\ -P_0(x) & 0 & 0 & 0 \end{bmatrix}$$

(b) If $P_1(x)$, $P_2(x)$ and $P_3(x)$ are identically zero in some neighbourhood of a and b and P_0 is a constant independent of x then

$$\mathbf{B}(a) = \mathbf{B}(b) = \begin{bmatrix} 0 & 0 & 0 & P_0 \\ 0 & 0 & -P_0 & 0 \\ 0 & P_0 & 0 & 0 \\ -P_0 & 0 & 0 & 0 \end{bmatrix}.$$

(c) The Lagrange adjoint of L^+ is L and for suitable pair of functions g and f

$$\int_a^b \left\{ \bar{f}L^+g - g\,\overline{Lf} \right\} dx = \{gf\}(b) - \{gf\}(a)$$

where

$$\{gf\}(\mathrm{x}) = \sum_{j=1}^{4}\sum_{k=1}^{4} A_{jk}(x)\bar{f}^{(j-1)}(x) g^{(k-1)}(x) = \hat{f}^*(x)\mathbf{A}(x)\hat{g}(x) .$$

The A_{jk} are dependent on the coefficients of the differential expression L^{+} and $\mathbf{A}(x) = [A_{jk}]$.

5. If $\varphi(x,\lambda)$ is a solution of $L\varphi = \lambda\varphi$ and $\psi(x,\lambda)$ is a solution of $L^{+}\psi = \overline{\lambda}\psi$ then

$$[\varphi\psi](x_2) - [\varphi\psi](x_1) = \int_{x_1}^{x_2} \left\{ \overline{\psi} L\varphi - \varphi \overline{L^{+}\psi} \right\} dx \qquad (a \le x_1 \le x_2 \le b)$$

$$= \int \left\{ \overline{\psi}\lambda\varphi - \varphi\lambda\overline{\psi} \right\} dx$$

$$= 0$$

and hence,

$$[\phi\psi](x_2) = [\phi\psi](x_1).$$

Thus, $[\phi(x,\lambda)\psi(x,\lambda)](x)$ is independent of $x \in [a,b]$.

Similarly $\{\psi(x,\lambda)\phi(x,\lambda)\}(x)$ is independent of $x \in [a,b]$. This implies that $[\phi(x,\lambda)\psi(x,\lambda)](x)$ and $\{\psi(x,\lambda)\phi(x,\lambda)\}(x)$ may be denoted by $[\phi\psi]$ and $\{\psi\phi\}$, respectively.

6. (a) If there is a constant K such that $|f(x)| \le K\phi(x)$ for $x \ge x_0$ we write $f = O(\phi)$.

(b) If $\dfrac{f(x)}{\phi(x)} \to l$, $x \to \infty$ where $l \ne 0$ we write $f \sim l\phi$.

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3 Preliminaries

The boundary value problem to be considered is of the form

$$\pi : L\phi \equiv \phi^{(4)}(x) + P_2(x)\phi^{(2)}(x) + P_3(x)\phi^{(1)}(x) + P_4(x)\phi(x) = \lambda\phi(x) \qquad (1)$$

$$U_r\phi \equiv \sum_{s=1}^{4} \left\{ m_{rs}\phi^{(s-1)}(a) + n_{rs}\phi^{(s-1)}(b) \right\} = 0 \qquad (1 \le r, s \le 4), \qquad (2)$$

where the functions $P_2(x), P_3(x)$, $P_4(x)$, the constants m_{rs} and n_{rs} and the parameter λ are complex- valued. The functions $P_r(x)$ (r = 2, 3, 4) are of the class $C^{(4-r)}$ on the closed bounded interval [a, b] and $P_2(x)$, $P_3(x)$ are identically zero in a neighbourhood of both a and b .

The corresponding Fourier boundary value problem for π is given by

$$\pi_F \ : \phi^{(4)}(x) = \lambda\phi(x) \tag{3}$$

$$U_r\phi \equiv \sum_{s=1}^{4}\left\{m_{rs}\phi^{(s-1)}(a) + n_{rs}\phi^{(s-1)}(b)\right\} = 0 \qquad (1 \le r, s \le 4)\,. \tag{4}$$

Let $\{\psi_r(a/x,\lambda);\chi_r(b/x,\lambda)\}$ $(1 \le r \le 4)$ be the boundary condition functions for π and $\{\psi_{Fr}(a/x,\lambda);\chi_{Fr}(b/x,\lambda)\}$ $(1 \le r \le 4)$ the boundary condition functions for π_F. Then $\psi_r(a/x,\lambda)$ and $\chi_r(b/x,\lambda)$ are solutions of $L^{+}\psi = \bar{\lambda}\psi$ such that

$$\mathbf{\Psi}(a) = \mathbf{B}(a)\mathbf{M}^* \quad \text{and} \quad \mathbf{X}(b) = \mathbf{B}(b)\mathbf{N}^*,$$

where

$$\mathbf{B}(a) = \mathbf{B}(b) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathrm{M}=\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix},$$

and $\mathrm{N}=\begin{bmatrix} n_{11} & n_{12} & n_{13} & n_{14} \\ n_{21} & n_{22} & n_{23} & n_{24} \\ n_{31} & n_{32} & n_{33} & n_{34} \\ n_{41} & n_{42} & n_{43} & n_{44} \end{bmatrix}.$

Likewise $\psi_{Fr}(a/x,\lambda)$ and $\chi_{Fr}(b/x,\lambda)$ are solutions of $\psi^{(4)}(x) = \bar{\lambda}\psi(x)$ such that

$$\mathbf{\Psi}(a) = \mathbf{B}(a)\mathbf{M}^* \quad \text{and} \quad \mathbf{X}(b) = \mathbf{B}(b)\mathbf{N}^*.$$

Let $\{f_r(a/x,\lambda), g_r(b/x,\lambda)\}$ $(1 \le r \le 4)$ be the boundary condition functions for the boundary value problem

$$\phi^{(4)}(x) = \lambda\phi(x), \text{ and } \quad U\phi = I_4\hat{\phi}(a) + I_4\hat{\phi}(b) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

where I_4 is the 4 x 4 unit matrix. Then $f_r(a/x,\lambda), g_r(b/x,\lambda)$ are solutions of $\psi^{(4)}(x) = \bar{\lambda}\psi(x)$ such that

$$\mathbf{F}(a) = \mathbf{B}(a)I_4 = \mathbf{B}(a) \quad \text{and} \quad \mathbf{G}(b) = \mathbf{B}(b)I_4 = \mathbf{B}(b).$$

4 Proof of Theorems

We now prove five Theorems that will enable us to prove our main results in Theorem 6.

Theorem 1

(i) $\psi_{Fr}(a/x,\lambda) = \sum_{s=1}^{4} \overline{m}_{rs} f_s(a/x,\lambda)$.

(ii) $\chi_{Fr}(b/x,\lambda) = \sum_{s=1}^{4} \overline{n}_{rs} g_s(b/x,\lambda)$.

(iii) *Let* $f_s(x) = f_s(a/x,\lambda)$, $g_s(x) = f_s(a/x,\lambda)$.

Then

$$f_s(x) = (-1)^{(s-1)} f_1^{(s-1)}(x) \qquad 2 \le s \le 4$$

$$g_s(x) = (-1)^{(s-1)} g_1^{(s-1)}(x) \qquad 2 \le s \le 4.$$

Proof. (i) and **(ii)** $f_s(a/x,\lambda)$ and $g_s(b/x,\lambda)$ are solutions of

$$\psi^{(4)}(x) = \bar{\lambda}\psi(x) \tag{5}$$

such that

$$\mathbf{F}(a) = \mathbf{B}(a)\mathbf{I}_4 = \mathbf{B}(a) \tag{6}$$

i.e., $$\begin{bmatrix} f_1(a) & f_2(a) & f_3(a) & f_4(a) \\ f_1^{(1)}(a) & f_2^{(1)}(a) & f_3^{(1)}(a) & f_4^{(1)}(a) \\ f_1^{(2)}(a) & f_2^{(2)}(a) & f_3^{(2)}(a) & f_4^{(2)}(a) \\ f_1^{(3)}(a) & f_2^{(3)}(a) & f_3^{(3)}(a) & f_4^{(3)}(a) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

Similarly $\mathbf{G}(b) = \mathbf{B}(b)\mathbf{I}_4 = \mathbf{B}(b)$.

Let $h_r(x,\lambda) = \sum_{s=1}^{4} \overline{m}_{rs} f_s(a/x,\lambda)$, then $h_1(x,\lambda), h_2(x,\lambda), h_3(x,\lambda)$ and $h_4(x,\lambda)$ are solutions of (5) such that

$$\mathbf{H}(a) = \begin{bmatrix} h_1(a) & h_2(a) & h_3(a) & h_4(a) \\ h_1^{(1)}(a) & h_2^{(1)}(a) & h_3^{(1)}(a) & h_4^{(1)}(a) \\ h_1^{(2)}(a) & h_2^{(2)}(a) & h_3^{(2)}(a) & h_4^{(2)}(a) \\ h_1^{(3)}(a) & h_2^{(3)}(a) & h_3^{(3)}(a) & h_4^{(3)}(a) \end{bmatrix}$$

$$= \begin{bmatrix} f_1(a) & f_2(a) & f_3(a) & f_4(a) \\ f_1^{(1)}(a) & f_2^{(1)}(a) & f_3^{(1)}(a) & f_4^{(1)}(a) \\ f_1^{(2)}(a) & f_2^{(2)}(a) & f_3^{(2)}(a) & f_4^{(2)}(a) \\ f_1^{(3)}(a) & f_2^{(3)}(a) & f_3^{(3)}(a) & f_4^{(3)}(a) \end{bmatrix} \begin{bmatrix} \overline{m}_{11} & \overline{m}_{21} & \overline{m}_{31} & \overline{m}_{41} \\ \overline{m}_{12} & \overline{m}_{22} & \overline{m}_{32} & \overline{m}_{42} \\ \overline{m}_{13} & \overline{m}_{23} & \overline{m}_{33} & \overline{m}_{43} \\ \overline{m}_{14} & \overline{m}_{24} & \overline{m}_{34} & \overline{m}_{44} \end{bmatrix}.$$

This implies that

$$\mathbf{H}(a) = \mathbf{F}(a)\mathbf{M}^* .$$

But from (6) $\mathbf{F}(a) = \mathbf{B}(a)$, therefore

$$\mathbf{H}(a) = \mathbf{B}(a)M^* .$$

Now $\psi_{Fr}(a/x,\lambda)$ $(1 \le r \le 4)$ are solutions of the same (5) such that

$$\Psi(a) = \mathbf{B}(a) M^* .$$

Hence we have $\psi_{Fr}(a/x,\lambda) = h_r(x,\lambda) = \sum_{s=1}^{4} \overline{m}_{rs} f_s(a/x,\lambda)$ $(1 \le r \le 4)$.

Similarly if

$$q_r(x,\lambda) = \sum_{s=1}^{4} \overline{n}_{rs} g_s(b/x,\lambda), \tag{7}$$

then $q_r(x,\lambda)$ $(1 \le r \le 4)$ are solutions of (5) such that **Q**(b) = **B**(b)**N*** and so

$$\chi_{Fr}(b/x,\lambda) = q_r(x,\lambda) = \sum_{s=1}^{4} \overline{n}_{rs} g_s(b/x,\lambda).$$

This proves (i) and (ii).

(iii) $f_r(a/x,\lambda)$ $(1 \le r \le 4)$ are solutions of $\psi^{(4)}(x) = \overline{\lambda}\,\psi(x)$ such that

$$\mathbf{F}(a) = \mathbf{B}(a)\mathbf{I}_4 = \mathbf{B}(a)$$

$$\begin{bmatrix} f_1(a) & f_2(a) & f_3(a) & f_4(a) \\ f_1^{(1)}(a) & f_2^{(1)}(a) & f_3^{(1)}(a) & f_4^{(1)}(a) \\ f_1^{(2)}(a) & f_2^{(2)}(a) & f_3^{(2)}(a) & f_4^{(2)}(a) \\ f_1^{(3)}(a) & f_2^{(3)}(a) & f_3^{(3)}(a) & f_4^{(3)}(a) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \tag{8}$$

The general solution of the equation $\psi^{(4)}(x) = \overline{\lambda}\,\psi(x)$ can be obtained as follows:

Let $\rho^4 = \overline{\lambda} = r(\cos\theta + i\sin\theta)$ for $-\pi < \theta \le \pi$, $\rho = \sigma + i\tau$.

Then,

$$\rho_k = r^{\frac{1}{4}}\left[\cos\left(\frac{\theta + 2\pi k}{4}\right) + i\sin\left(\frac{\theta + 2\pi k}{4}\right)\right] \quad (\text{k} = 0, 1, 2, 3).$$

Hence, for $k = 0$ we have

$$\rho_0 = r^{\frac{1}{4}}\left[\cos\left(\frac{\theta}{4}\right) + i\sin\left(\frac{\theta}{4}\right)\right] \tag{9}$$

Since the fourth roots of unity are 1, -1, i and –i we see that the four roots of the equation $\rho^4 = \overline{\lambda}$ are ρ_0 ,- ρ_0 , -$i\,\rho_0$ and $i\,\rho_0$. Put $P = \rho_0$ then the four roots are *P*, -*P* , *iP* and –*iP*. The general solution of the equation

$$\psi^{(4)}(x) - \rho^4\psi(x) = 0 \quad (\rho_0{}^4 = \overline{\lambda}) \tag{10}$$

is thus given by $\psi(x) = A_1 e^{p(x-a)} + A_2 e^{-p(x-a)} + A_3 e^{ip(x-a)} + A_4 e^{-iP(x-a)}$.

Now $f_r(a/x,\lambda)$ is a solution of (10) such that **F**(a) = **B**(a), and so

$$f_1(a/x,\lambda) = A_1 e^{p(x-a)} + A_2 e^{-p(x-a)} + A_3 e^{ip(x-a)} + A_4 e^{-iP(x-a)} \ . \tag{11}$$

Since from (8) $f_1(a) = 0$; $f_1^{(1)}(a) = 0$; $f_1^{(2)}(a) = 0$; $f_1^{(3)}(a) = -1$ we see that the constants A_k (k = 1, 2, 3, 4) in (11) are given by

$$A_1 + A_2 + A_3 + A_4 = 0$$

$$A_1 p - A_2 p + ipA_3 - ipA_4 = 0$$

$$A_1 p^2 + A_2 p^2 - A_3 p^2 - A_4 p^2 = 0$$

$$A_1 p^3 - A_2 p^3 - ip^3 A_3 + ip^3 A_4 = -1.$$

Solving gives

$$A_1 = -\frac{1}{4} p^{-3}; \quad A_2 = \frac{1}{4} p^{-3}; \quad A_3 = \frac{1}{4i} p^{-3} = -\frac{i}{4} p^{-3}; \quad A_4 = -\frac{1}{4i} p^{-3} = \frac{i}{4} p^{-3}.$$

Substituting into (11) we find that

$$\left.\begin{aligned}
f_1(a/x,\lambda) &= \frac{1}{4} p^{-3} \left\{ -e^{p(x-a)} + e^{-p(x-a)} - ie^{ip(x-a)} + ie^{-iP(x-a)} \right\} \\
f_1^{(1)}(a/x,\lambda) &= \frac{1}{4} p^{-2} \left\{ -e^{p(x-a)} - e^{-p(x-a)} + e^{ip(x-a)} + e^{-iP(x-a)} \right\} \\
f_1^{(2)}(a/x,\lambda) &= \frac{1}{4} p^{-1} \left\{ -e^{p(x-a)} + e^{-p(x-a)} + ie^{ip(x-a)} - ie^{-iP(x-a)} \right\} \\
f_1^{(3)}(a/x,\lambda) &= \frac{1}{4} \left\{ -e^{p(x-a)} - e^{-p(x-a)} - e^{ip(x-a)} - e^{-iP(x-a)} \right\} \\
f_1^{(4)}(a/x,\lambda) &= \frac{1}{4} p \left\{ -e^{p(x-a)} + e^{-p(x-a)} - ie^{ip(x-a)} + ie^{-iP(x-a)} \right\} \\
&= \frac{1}{4} p \left\{ f_1(a/x,\lambda)\, 4\, p^3 \right\}
\end{aligned}\right\} \qquad (12)$$

hence

$$f_1^{(4)}(a/x,\lambda) = \mathrm{p}^4 \; f_1(a/x,\lambda) \; .$$

Next we prove that

$$f_2(x) = -f_1^{(1)}(x); \;\; f_3(x) = f_1^{(2)}(x); \;\; f_4(x) = -f_1^{(3)}(x)$$

Let $R_1(x) = -f_1^{(1)}(x)$. Then $R_1(x)$ is a solution of (5) such that

$$\begin{bmatrix} R_1(a) \\ R_1^{(1)}(a) \\ R_1^{(2)}(a) \\ R_1^{(3)}(a) \end{bmatrix} = \begin{bmatrix} -f_1^{(1)}(a) \\ -f_1^{(2)}(a) \\ -f_1^{(3)}(a) \\ -f_1^{(4)}(a) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}. \tag{13}$$

We note that (13) is obtained by substituting $x = a$ into (12). From (8)

$$\begin{bmatrix} f_2(a) \\ f_2^{(1)}(a) \\ f_2^{(2)}(a) \\ f_2^{(3)}(a) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ and so, } \begin{bmatrix} R_1(a) \\ R_1^{(1)}(a) \\ R_1^{(2)}(a) \\ R_1^{(3)}(a) \end{bmatrix} = \begin{bmatrix} -f_1(a) \\ -f_1^{(1)}(a) \\ -f_1^{(2)}(a) \\ -f_1^{(3)}(a) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} f_2(a) \\ f_2^{(1)}(a) \\ f_2^{(2)}(a) \\ f_2^{(3)}(a) \end{bmatrix}$$

which implies that $R_1(x) = f_2(a/x, \lambda)$.

Again, $R_1(x) = -f_1^{(1)}(a/x, \lambda)$ implies that

$$f_2(a/x, \lambda) = -f_1^{(1)}(a/x, \lambda). \tag{14}$$

Let $R_2(x) = f_1^{(2)}(x)$, then $R_2(x)$ is a solution of (5) such that

$$\begin{bmatrix} R_2(a) \\ R_2^{(1)}(a) \\ R_2^{(2)}(a) \\ R_2^{(3)}(a) \end{bmatrix} = \begin{bmatrix} f_1^{(2)}(a) \\ f_1^{(3)}(a) \\ f_1^{(4)}(a) \\ f_1^{(5)}(a) \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} f_3(a) \\ f_3^{(1)}(a) \\ f_3^{(2)}(a) \\ f_3^{(3)}(a) \end{bmatrix}$$

and so $R_2(x) = f_3(a/x, \lambda)$. Thus, $R_2(x) = f_1^{(2)}(a/x, \lambda)$ implies that

$$f_3(a/x, \lambda) = f_1^{(2)}(a/x, \lambda). \tag{15}$$

Next we let $R_3(x) = -f_1^{(3)}(x)$, so that $R_3(x)$ is a solution of (5) implies that

$$\begin{bmatrix} R_3(a) \\ R_3^{(1)}(a) \\ R_3^{(2)}(a) \\ R_3^{(3)}(a) \end{bmatrix} = \begin{bmatrix} -f_1^{(3)}(a) \\ -f_1^{(4)}(a) \\ -f_1^{(5)}(a) \\ -f_1^{(6)}(a) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} f_4(a) \\ f_4^{(1)}(a) \\ f_4^{(2)}(a) \\ f_4^{(3)}(a) \end{bmatrix}.$$

Therefore $R_3(x) = f_4(a/x, \lambda)$. Finally, $R_3(x) = -f_1^{(3)}(x)$ implies that

$$f_4(a/x, \lambda) = -f_1^{(3)}(x). \tag{16}$$

It follows from (14), (15) and (16) that

$$f_s(x) = (-1)^{(s-1)} f_1^{(s-1)}(x) \qquad (2 \le s \le 4).$$

Similarly $g_s(x) = (-1)^{(s-1)} g_1^{(s-1)}(x)$ $(2 \le s \le 4)$. The proof of the theorem is complete. □

Theorem 2 *Let*

$$R(t/x,\lambda) = \overline{P}_4(t) f_1(t/x,\lambda) - \overline{P}_3(t) f_1^{(1)}(t/x,\lambda) + \overline{P}_2(t) f_1^{(2)}(t/x,\lambda) \tag{17}$$

$$S(t/x,\lambda) = \overline{P}_4(t) g_1(t/x,\lambda) - \overline{P}_3(t) g_1^{(1)}(t/x,\lambda) + \overline{P}_2(t) g_1^{(2)}(t/x,\lambda).$$

Then the boundary condition functions for the boundary value problem (1) *and* (2) *are given by*

$$\psi_r(a/x,\lambda) = \psi_{Fr}(a/x,\lambda) + \int_a^x R(t/x,\lambda)\psi_r(t)dt \qquad (1 \le r \le 4)$$

$$\chi_r(b/x,\lambda) = \chi_{Fr}(b/x,\lambda) + \int_a^x S(t/x,\lambda)\chi_r(t)dt \qquad (1 \le r \le 4).$$

Proof. As a function of *x*, $f_1(t/x,\lambda)$ $(x, t \in [a,b])$ is a solution of $\psi^{(4)}(x) = \overline{\lambda}\psi(x)$ such that from (8)

$$f_1(x_0/x_0,\lambda) = f_1^{(1)}(x_0/x_0,\lambda) = f_1^{(2)}(x_0/x_0,\lambda) = 0 \tag{18}$$

and

$$f_1^{(3)}(x_0/x_0,\lambda) = -1 \text{ where } x_0 \in [a,b]. \tag{19}$$

It follows from (17), (18) and (19) that $R(t/x,\lambda)$ is a solution of $\psi^{(4)}(x) = \overline{\lambda}\psi(x)$ such that

$$\left.\begin{aligned} &R(x_0/x_0,\lambda) = 0 \\ &R^{(1)}(x_0/x_0,\lambda) = -\overline{P}_2(x_0) \\ &R^{(2)}(x_0/x_0,\lambda) = \overline{P}_3(x_0) \\ &R^{(3)}(x_0/x_0,\lambda) = -\overline{P}_4(x_0) \end{aligned}\right\} \tag{20}$$

If $\psi_r(a/x,\lambda) = \psi_{Fr}(a/x,\lambda) + \int_a^x R(t/x,\lambda)\psi_r(t)dt \qquad (1 \le r \le 4).$ (21)

Then using the formula

$\frac{d}{dx}\int_a^x F(x,t)dt = \int \frac{\delta F}{\delta x}(x,t) + F(x,x)$ and (20) we obtain the following equations from (21)

$$\psi_r^{(1)}(a/x,\lambda) = \psi_{Fr}^{(1)}(x) + \int_a^x R^{(1)}(t/x,\lambda)\psi_r(t)dt + R(x/x,\lambda)\psi_r(x)$$

$$\psi_r^{(2)}(a/x,\lambda) = \psi_{Fr}^{(2)}(x) + \int_a^x R^{(2)}(t/x,\lambda)\psi_r(t)dt + R^{(1)}(x/x,\lambda)\psi_r(x)$$
$$= \psi_{Fr}^{(2)}(x) + \int_a^x R^{(2)}(t/x,\lambda)\psi_r(t)dt \; - \overline{P}_2(x)\psi_r(x)$$

$$\psi_r^{(3)}(a/x,\lambda) = \psi_{Fr}^{(3)}(x) + \int_a^x R^{(3)}(t/x,\lambda)\psi_r(t)dt + R^{(2)}(x/x,\lambda)\psi_r(x) \; - [\overline{P}_2(x)\psi_r(x)]^{(1)}$$
$$= \psi_{Fr}^{(3)}(x) + \int_a^x R^{(3)}(t/x,\lambda)\psi_r(t)dt + \overline{P}_3(x)\psi_r(x) \; - [\overline{P}_2(x)\psi_r(x)]^{(1)}$$

$$\psi^{(4)}(a/x,\lambda) = \psi_{Fr}^{(4)}(x) + \int_a^x R^{(4)}(t/x,\lambda)\psi_r(t)dt + R^{(3)}(x/x,\lambda)\psi_r(x)$$
$$+ \overline{P}_3(x)\psi_r(x) - [\overline{P}_2(x)\psi_r(x)]^{(1)}$$
$$= \psi_{Fr}^{(4)}(x) + \int_a^x R^{(4)}(t/x,\lambda)\psi_r(t)dt - \overline{P}_4(x)\psi_r(x)$$
$$+ [\overline{P}_3(x)\psi_r(x)]^{(1)} - \left[\overline{P}_2(x)\psi_r(x)\right]^{(2)}.$$

By definition, $R(t/x,\lambda)$ being a solution of $\psi^{(4)}(x) = \overline{\lambda}\,\psi(x)$ implies that $R^{(4)}(t/x,\lambda) = \overline{\lambda}\,R(t/x,\lambda)$ and so

$$\psi_r^{(4)}(x) = \psi_{Fr}^{(4)}(x) + \overline{\lambda}\int_a^x R^{(4)}(t/x,\lambda)\psi_r(t)dt$$
$$- \overline{P}_4(x)\psi_r(x) + [\overline{P}_3(x)\psi_r(x)]^{(1)} - \left[\overline{P}_2(x)\psi_r(x)\right]^{(2)} \tag{22}$$

Substituting (21) into (22) we have

$$\psi_r^{(4)}(x) = \psi_{Fr}^{(4)}(x) + \overline{\lambda}[\psi_r(x) - \psi_{Fr}(x)] - \overline{P}_4(x)\psi_r(x) + [\overline{P}_3(x)\psi_r(x)]^{(1)} - \left[\overline{P}_2(x)\psi_r(x)\right]^{(2)}.$$

But, $\psi^{(4)}(x) = \overline{\lambda}\,\psi(x)$ implies that $\psi_{Fr}^{(4)}(x) = \overline{\lambda}\,\psi_{Fr}(x)$ and hence,

$$\psi_r{}^{(4)}(x)=\bar{\lambda}\psi_{Fr}{}^{(4)}(x)+\bar{\lambda}[\psi_r(x)-\psi_{Fr}(x)]-\bar{P}_4(x)\psi_r(x)$$
$$+[\bar{P}_3(x)\psi_r(x)]^{(1)}-\left[\bar{P}_2(x)\psi_r(x)\right]^{(2)}$$

$$\psi_r{}^{(4)}(x) = \bar{\lambda}\psi_r(x) - \bar{P}_4(x)\psi_r(x) + [\bar{P}_3(x)\psi_r(x)]^{(1)} - \left[\bar{P}_2(x)\psi_r(x)\right]^{(2)} \qquad (23)$$

$\psi_r{}^{(4)}(x) + \bar{P}_4(x)\psi_r(x) - [\bar{P}_3(x)\psi_r(x)]^{(1)} + \left[\bar{P}_2(x)\psi_r(x)\right]^{(2)} = \bar{\lambda}\psi_r(x)$, hence

$\psi_r(x)\,(1\le r\le 4)$ are solutions of (23) such that $\mathbf{\Psi}(a)=\mathbf{B}(a)\mathbf{M}^*$.

Similarly $\chi_r(x)$ $(1\le r\le 4)$ are solutions of (23) such that $\mathbf{X}(b)=\mathbf{B}(b)\mathbf{N}^*$.

Hence we conclude that $\{\psi_r(a/x,\lambda),\chi_r(b/x,\lambda)\}$ are the set of boundary conditions of π . □

Theorem 3

$\psi_{Fr}{}^{(s-1)}(a/x,\lambda)=O\left(|P|^{(s-1)}e^{\sigma(x-a)}\right)$ as $|\lambda|\to\infty$ $(1\le r,s\le 4)$

$\chi_{Fr}{}^{(s-1)}(b/x,\lambda)=O\left(|P|^{(s-1)}e^{\sigma(b-x)}\right)$ as $|\lambda|\to\infty$ $(1\le r,s\le 4)$

Proof.

$$\psi_{Fr}(a/x,\lambda)=\sum_{s=1}^{4}\bar{m}_{rs}f_s(a/x,\lambda)$$
$$=\bar{m}_{ri}f_1(a/x,\lambda)+\bar{m}_{r2}f_2(a/x,\lambda)+\bar{m}_{r3}f_3(a/x,\lambda)+\bar{m}_{r4}f_4(a/x,\lambda)$$
$$=\bar{m}_{ri}f_1(a/x,\lambda)-\bar{m}_{r2}f_1^{(1)}(a/x,\lambda)+\bar{m}_{r3}f_1^{(2)}(a/x,\lambda)-\bar{m}_{r4}f_1^{(3)}(a/x,\lambda).$$

For r=1 $r=1$ and using (12) we have

$$\psi_{F1}(a/x,\lambda)=\frac{\bar{m}_{11}}{4}p^{-3}\left\{-e^{p(x-a)}+e^{-p(x-a)}-ie^{ip(x-a)}+ie^{-iP(x-a)}\right\}$$
$$-\frac{\bar{m}_{12}}{4}p^{-2}\left\{-e^{p(x-a)}-e^{-p(x-a)}+e^{ip(x-a)}+e^{-iP(x-a)}\right\}$$
$$+\frac{\bar{m}_{13}}{4}p^{-1}\left\{-e^{p(x-a)}+e^{-p(x-a)}+ie^{ip(x-a)}-ie^{-iP(x-a)}\right\}$$
$$-\frac{\bar{m}_{14}}{4}\left\{-e^{p(x-a)}-e^{-p(x-a)}-e^{ip(x-a)}-e^{-iP(x-a)}\right\}$$

Furthermore,

$$
\begin{aligned}
|\ \psi_{F1}(a/x,\lambda)\ | &\leq K_1|P|^{-3}\left|-e^{p(x-a)}+e^{-p(x-a)}-ie^{ip(x-a)}+ie^{-iP(x-a)}\right| \\
&\quad + K_2|P|^{-2}\left|-e^{p(x-a)}-e^{-p(x-a)}+e^{ip(x-a)}+e^{-iP(x-a)}\right| \\
&\quad + K_3|P|^{-1}\left|-e^{p(x-a)}+e^{-p(x-a)}+ie^{ip(x-a)}-ie^{-iP(x-a)}\right| \\
&\quad + K_4\left|-e^{p(x-a)}-e^{-p(x-a)}-e^{ip(x-a)}-e^{-iP(x-a)}\right| \\
&\leq K_1|P|^{-3}\left\{ \left|e^{p(x-a)}\right| + \left|e^{-p(x-a)}\right| + \left|e^{ip(x-a)}\right| + \left|e^{-iP(x-a)}\right| \right\} \\
&\quad + K_2|P|^{-2}\left\{ \left|e^{p(x-a)}\right| + \left|e^{-p(x-a)}\right| + \left|e^{ip(x-a)}\right| + \left|e^{-iP(x-a)}\right| \right\} \\
&\quad + K_3|P|^{-1}\left\{ \left|e^{p(x-a)}\right| + \left|e^{-p(x-a)}\right| + \left|e^{ip(x-a)}\right| + \left|e^{-iP(x-a)}\right| \right\} \\
&\quad + K_4\left\{ \left|e^{p(x-a)}\right| + \left|e^{-p(x-a)}\right| + \left|e^{ip(x-a)}\right| + \left|e^{-iP(x-a)}\right| \right\} \\
&\leq \left(K_4+K_3|P|^{-1}+K_2|P|^{-2}+K_1|P|^{-3}\right)\cdot \\
&\qquad \cdot\left\{ \left|e^{p(x-a)}\right| + \left|e^{-p(x-a)}\right| + \left|e^{ip(x-a)}\right| + \left|e^{-iP(x-a)}\right| \right\} \qquad (24)
\end{aligned}
$$

Since $p = \rho = \sigma + i\tau$ we see that

$$
\left|e^{p(x-a)}\right| = \left|e^{\sigma(x-a)}\right|; \quad \left|e^{-p(x-a)}\right| = \left|e^{-\sigma(x-a)}\right|; \quad \left|e^{ip(x-a)}\right| = \left|e^{-\tau(x-a)}\right|;
$$

$$
\left|e^{-ip(x-a)}\right| = \left|e^{-\tau(x-a)}\right|. \qquad (25)
$$

From (24) and (25) we find that

$$
\begin{aligned}
&|\ \psi_{F1}(a/x,\lambda)\ | \\
&\leq \left(K_4+K_3|P|^{-1}+K_2|P|^{-2}+K_1|P|^{-3}\right)\left\{ \left|e^{\sigma(x-a)}\right| + \left|e^{-\sigma(x-a)}\right| + \left|e^{-\tau(x-a)}\right| + \left|e^{\tau(x-a)}\right| \right\} \\
&\leq \left(K_4+K_3|P|^{-1}+K_2|P|^{-2}+K_1|P|^{-3}\right)\left\{ \left|e^{|\sigma|(x-a)}\right| + \left|e^{|\sigma|(x-a)}\right| + \left|e^{|\tau|(x-a)}\right| + \left|e^{|\tau|(x-a)}\right| \right\} \\
&\leq 2\left(K_4+K_3|P|^{-1}+K_2|P|^{-2}+K_1|P|^{-3}\right) \quad \left\{ \left|e^{\sigma(x-a)}\right| + \left|e^{|\tau|(x-a)}\right| \right\}
\end{aligned}
$$

as $|\lambda| \to \infty$, $|P| \to \infty$ and so $|P|^{-2} \to 0$; $|P|^{-1} \to 0$; $|P|^{-3} \to 0$.

Hence, $|\ \psi_{F1}(a/x,\lambda)\ | \leq K\left(e^{\sigma(x-a)} + e^{|\tau|(x-a)}\right)$, where $K = 2K_4$. But from (9) $\sigma = r^{\frac{1}{4}}\cos\frac{\theta}{4}$ and $\tau = r^{\frac{1}{4}}\sin\frac{\theta}{4}$ and so $\sigma > \tau$ and $\sigma \geq 0$. Thus, $|\ \psi_{F1}(a/x,\lambda)\ | \leq K\left(e^{\sigma(x-a)}\right)$ as $|\lambda| \to \infty$ and so $\psi_{F1}(a/x,\lambda) = O\left(e^{\sigma(x-a)}\right)$ as $|\lambda| \to \infty$.

By similar argument

$$\psi_{F1}^{(1)}(a/x,\lambda) = O\left(|p| e^{\sigma(x-a)}\right) \text{ as } |\lambda| \to \infty \quad (1 \le s \le 4)$$

and so,

$$\psi_{F1}^{(s-1)}(a/x,\lambda) = O\left(|P|^{(s-1)} e^{\sigma(x-a)}\right) \text{ as } |\lambda| \to \infty \quad (1 \le s \le 4).$$

Also,

$$\psi_{F2}^{(s-1)}(a/x,\lambda) = O\left(|P|^{(s-1)} e^{\sigma(x-a)}\right) \text{ as } |\lambda| \to \infty \quad (1 \le s \le 4)$$

Similarly,

$$\chi_{F1}^{(s-1)}(b/x,\lambda) = O\left(|P|^{(s-1)} e^{\sigma(b-x)}\right) \text{ as } |\lambda| \to \infty \quad (1 \le s \le 4)$$

$$\chi_{F2}(b/x,\lambda) = O\left(|P|^{(s-1)} e^{\sigma(b-x)}\right) \text{ as } |\lambda| \to \infty \quad (1 \le s \le 4).$$

By combining all the result we have

$$\psi_{Fr}^{(s-1)}(a/x,\lambda) = O\left(|P|^{(s-1)} e^{\sigma(x-a)}\right) \text{ as } |\lambda| \to \infty \quad (1 \le s \le 4) \tag{26}$$

$$\chi_{Fr}^{(s-1)}(b/x,\lambda) = O\left(|P|^{(s-1)} e^{\sigma(b-x)}\right) \text{ as } |\lambda| \to \infty \quad (1 \le s \le 4).$$

□

Theorem 4

$$R^{(s-1)}(t/x,\lambda) = O\left(|P|^{(s-2)} e^{\sigma(x-t)}\right) \text{ as } |\lambda| \to \infty \quad (1 \le s \le 4)$$

$$S^{(s-1)}(t/x,\lambda) = O\left(|P|^{(s-2)} e^{\sigma(t-x)}\right) \text{ as } |\lambda| \to \infty \quad (1 \le s \le 4) \text{ and } p = \sigma + i\tau.$$

Proof.

$$R(t/x,\lambda) = \overline{P}_4(t) f_1(t/x,\lambda) - \overline{P}_3(t) f_1^{(1)}(t/x,\lambda) + \overline{P}_2(t) f_1^{(2)}(t/x,\lambda) \tag{27}$$

From (12)

$$f_1(a/x,\lambda) = \frac{1}{4} p^{-3} \left\{ -e^{p(x-a)} + e^{-p(x-a)} - ie^{ip(x-a)} + ie^{-iP(x-a)} \right\}$$
$$= \frac{1}{4} p^{-3} \left\{ -e^{(\sigma+i\tau)(x-a)} + e^{-(\sigma+i\tau)(x-a)} - ie^{i(\sigma+i\tau)(x-a)} + ie^{-i(\sigma+i\tau)(x-a)} \right\}$$

$$\left| f_1(t/x,\lambda) \right| \le \frac{|P|^{-3}}{4} \left\{ \left| e^{(\sigma+i\tau)(x-a)} \right| + \left| e^{-(\sigma+i\tau)(x-a)} \right| + \left| ie^{i(\sigma+i\tau)(x-a)} \right| + \left| ie^{-i(\sigma+i\tau)(x-a)} \right| \right\}$$

If $z = x + iy$ then $\left|e^{z}\right| = e^{x}$. Hence,

$$\left| f_1(t/x,\lambda)\right| \le \frac{|P|^{-3}}{4}\left\{ e^{\sigma(x-t)} + e^{\sigma(x-t)} + e^{|\tau|(x-t)} + e^{|\tau|(x-t)} \right\}$$

$$\le \frac{|P|^{-3}}{4}\left\{ 2e^{\sigma(x-t)} + 2e^{|\tau|(x-t)} \right\}$$

$$\le \frac{|P|^{-3}}{2}\left\{ e^{\sigma(x-t)} + e^{|\tau|(x-t)} \right\} \text{ as } |\lambda| \to \infty.$$

$$\left| f_1(t/x,\lambda)\right| \le \frac{|P|^{-3}}{2} e^{\sigma(x-t)} \quad \text{as } |\lambda| \to \infty.$$

Similarly

$$\left| f^{(1)}{}_1(t/x,\lambda)\right| \le \frac{|P|^{-2}}{2} e^{\sigma(x-t)} \quad \text{as } |\lambda| \to \infty$$

and

$$\left| f^{(2)}{}_1(t/x,\lambda)\right| \le \frac{|P|^{-1}}{2} e^{\sigma(x-t)} \quad \text{as } |\lambda| \to \infty$$

Since $\overline{P}_4(t)$, $\overline{P}_3(t)$ and $\overline{P}_2(t)$ are continuous in [a,b] they are bounded in [a,b] and so

$$\left|\overline{P}_4(t)\right| \le K_1, \quad \left|\overline{P}_3(t)\right| \le K_2, \quad \left|\overline{P}_2(t)\right| \le K_3. \tag{28}$$

Substituting all the above into (27) we have

$$\left|R(t/x,\lambda)\right| \le \left\{ \frac{K_1|P|^{-3}}{2} - \frac{K_2|P|^{-2}}{2} + \frac{K_3|P|^{-1}}{2} \right\} e^{\sigma(x-t)}$$

$$\left|R(t/x,\lambda)\right| \le K|P|^{-1} e^{\sigma(x-t)} \quad \text{as } |\lambda| \to \infty. \tag{29}$$

or

$$R(t/x,\lambda) = O\left(|P|^{-1} e^{\sigma(x-t)}\right) \quad \text{as } |\lambda| \to \infty \tag{30}$$

From (27)

$$R^{(1)}(t/x,\lambda) = \overline{P}_4(t) f_1^{(1)}(t/x,\lambda) - \overline{P}_3(t) f_1^{(2)}(t/x,\lambda) + \overline{P}_2(t) f_1^{(3)}(t/x,\lambda).$$

By similar argument

$$\left|R^{(1)}(t/x,\lambda)\right| \le \left\{\frac{K_1|P|^{-2}}{2} - \frac{K_2|P|^{-1}}{2} + \frac{K_3|P|^{-0}}{2}\right\} e^{\sigma(x-t)} \le K|P|^{-0} e^{\sigma(x-t)}$$

or

$$R^{(1)}(t/x,\lambda) = O\left(e^{\sigma(x-t)}\right) \text{ as } |\lambda| \to \infty \tag{31}$$

Combining (30) and (31) and generalizing we have

$$R^{(s-1)}(t/x,\lambda) = O\left(|P|^{s-2} e^{\sigma(x-t)}\right) \quad \text{as } |\lambda| \to \infty \quad (1 \le s \le 4).$$

Similarly

$$S^{(s-1)}(t/x,\lambda) = O\left(|P|^{s-2} e^{\sigma(x-t)}\right) \quad \text{as } |\lambda| \to \infty \quad (1 \le s \le 4). \qquad \square$$

Theorem 5

$$\psi_r(a/x,\lambda) = O\left(e^{\sigma(x-a)}\right) \text{ as } |\lambda| \to \infty$$

$$\chi_r(b/x,\lambda) = O\left(e^{\sigma(b-x)}\right) \text{ as } |\lambda| \to \infty.$$

Proof. Let

$$\psi_r(a/x,\lambda) = F_r(a/x,\lambda)\left(e^{\sigma(x-a)}\right) \quad \text{as } |\lambda| \to \infty \tag{32}$$

then

$$\psi_r(t) = F_r(t)\left(e^{\sigma(t-a)}\right) \quad \text{as } |\lambda| \to \infty \tag{33}$$

and

$$F_r(a/x,\lambda) = e^{-\sigma(x-a)}\psi_r(a/x,\lambda) \tag{34}$$

Next we substitute $\psi_r(a/x,\lambda) = \psi_{Fr}(a/x,\lambda) + \int_a^x R(t/x,\lambda)\psi_r(t)dt$ into (34) to obtain

$$F_r(a/x,\lambda) = e^{-\sigma(x-a)}\left\{\psi_{Fr}(a/x,\lambda) + \int_a^x R(t/x,\lambda)\psi_r(t)dt\right\}. \tag{35}$$

Similarly, substituting (33) into (35) we have

$$F_r(a/x,\lambda) = e^{-\sigma(x-a)}\,\psi_{Fr}(a/x,\lambda) + \int_a^x R(t/x,\lambda)\; e^{\sigma(t-x)}\, F_r(t)dt \quad \text{as } |\lambda| \to \infty.$$

Now

$$|F_r(a/x,\lambda)| \;\le\; \left|e^{-\sigma(x-a)}\right| |\psi_{Fr}(a/x,\lambda)| + \int_a^x |R(t/x,\lambda)|\; \left|e^{\sigma(t-x)}\right| |F_r(t)| dt. \tag{36}$$

But from (26) and (29)

$$|\psi_{Fr}(a/x,\lambda)| \le K_1 e^{\sigma(x-a)}; \qquad |R(t/x,\lambda)| \le K_2 |P|^{-1} e^{\sigma(x-t)}$$

By the mean value theorem for integrals

$$\int_a^x F_r(t)dt \;=\; F_r(a/\xi,\lambda)(x-a)$$

where a < ξ < x . But $(x - a)$ being a constant imply that

$$\int_a^x F_r(t)dt \;=\; F_r(a/x,\lambda)K$$

where $x \in [a,b]$, $K = (x - a)$. Substituting the above into (36) we have

$$|F_r(a/x,\lambda)| \le\; K_1 + K_3 |P|^{-1} |F_r(a/x,\lambda)|$$

where $K_3 = K_2 K = K_2(x-a)$. Therefore,

$$|F_r(a/x,\lambda)| -\; K_3 |P|^{-1} |F_r(a/x,\lambda)| \le K_1 .$$

Hence,

$$|F_r(a/x,\lambda)|\; |(1 - K_3 |P|^{-1}|)| \le K_1$$

and so

$$|F_r(a/x,\lambda)| \le \frac{K_1}{(1 - K_3 |P|^{-1})} \quad \text{provided that } 1 - K_3 |P|^{-1} > 0 \;. \tag{37}$$

This is true if $|P|$ is large enough. Substituting (37) into (32) we have

$$\psi_r(a/x,\lambda) \le K_4 e^{\sigma(x-a)} \quad \text{as } |\lambda| \to \infty$$

where $K_4 \;=\; \dfrac{K_1}{1 - K_3 |P|^{-1}}$ and so

$$\psi_r(a/x,\lambda) = O(e^{\sigma(x-a)}) \quad \text{as} \quad |\lambda| \to \infty. \tag{38}$$

Similarly $\chi_r(b/x,\lambda) = O(e^{\sigma(b-x)})$ as $|\lambda| \to \infty$. □

Theorem 6

$\psi_r^{(s-1)}(a/x,\lambda) \sim \psi_{Fr}^{(s-1)}(a/x,\lambda)$ as $|\lambda| \to \infty$

$\chi_r^{(s-1)}(b/x,\lambda) \sim \chi_{Fr}^{(s-1)}(b/x,\lambda)$ as $|\lambda| \to \infty$ $(1 \le r \le 4)$.

Proof.

$$\psi_r(a/x,\lambda) = \psi_{Fr}(a/x,\lambda) + \int_a^x R(t/x,\lambda)\psi_r(t)dt \tag{39}$$

$$\psi_r^{(1)}(a/x,\lambda) = \psi_{Fr}^{(1)}(a/x,\lambda) + \int_a^x R^{(1)}(t/x,\lambda)\psi_r(t)dt \tag{40}$$

From (30), (31) and (38)

$$\left.\begin{aligned} R(t/x,\lambda) &= O\left(|P|^{-1} e^{\sigma(x-t)}\right) \quad \text{as} \quad |\lambda| \to \infty \\ R^{(1)}(t/x,\lambda) &= O\left(e^{\sigma(x-t)}\right) \quad \text{as} \quad |\lambda| \to \infty \end{aligned}\right\} \tag{41}$$

$$\psi_r(t) = O\left(e^{\sigma(t-a)}\right) \text{ as } |\lambda| \to \infty$$

According to (35) there exist constants K_1, K_2 and K_3 such that

$$\left.\begin{aligned} |R(t/x,\lambda)| &\le K_1|P|^{-1} e^{\sigma(x-t)} \quad \text{as} \quad |\lambda| \to \infty \\ |R^{(1)}(t/x,\lambda)| &\le K_2 e^{\sigma(x-t)} \quad \text{as} \quad |\lambda| \to \infty \end{aligned}\right\} \tag{42}$$

$$|\psi_r(t)| \le K_3 e^{\sigma(t-a)} \text{ as } |\lambda| \to \infty$$

Substituting (42) into (39) we have

$$\left|\int_a^x R(t/x,\lambda)\psi_r(t)dt\right| \le \int_a^x K_1K_3|P|^{-1} e^{\sigma(x-a)}dt$$

$$\le K_1K_3|P|^{-1} e^{\sigma(x-a)}(x-a) \quad \text{since } \int_a^x dt = (x-a)$$

$$= K_1K_3|P|^{-1}(b-a)e^{\sigma(x-a)} \quad \text{since if } a \le x \le b, \text{ then } b-a \ge x-a$$

$$= K_4 |P|^{-1} e^{\sigma(x-a)} \text{ as } |\lambda| \to \infty$$

$$\int_a^x R(t/x,\lambda)\psi_r(t)dt = O\left(|P|^{-1} e^{\sigma(x-a)}\right) \text{ as } |\lambda| \to \infty. \quad (43)$$

$$\text{Similarly } \int_a^x R^{(1)}(t/x,\lambda)\psi_r(t)dt = O\left(e^{\sigma(x-a)}\right) \quad (44)$$

From (39) and (43)

$$\psi_r(a/x,\lambda) = \psi_{Fr}(a/x,\lambda) + O\left(|P|^{-1} e^{\sigma(x-a)}\right) \text{ as } |\lambda| \to \infty \quad (45)$$

From (40) and (44)

$$\psi_r^{(1)}(a/x,\lambda) = \psi_{Fr}^{(1)}(a/x,\lambda) + O\left(e^{\sigma(x-a)}\right) \text{ as } |\lambda| \to \infty. \quad (46)$$

Combining (45) and (46) we obtain a general formula

$$\psi_r^{(s-1)}(a/x,\lambda) = \psi_{Fr}^{(s-1)}(a/x,\lambda) + O\left(|P|^{(s-2)} e^{\sigma(x-a)}\right) \text{ as } |\lambda| \to \infty. \quad (47)$$

Similarly

$$\chi_r^{(s-1)}(b/x,\lambda) = \chi_{Fr}^{(s-1)}(b/x,\lambda) + O\left(|P|^{(s-2)} e^{\sigma(x-b)}\right) \text{ as } |\lambda| \to \infty. \quad (48)$$

If follows from theorem 3 and (47) that

$$\psi_r^{(s-1)}(a/x,\lambda) \sim \psi_{Fr}^{(s-1)}(a/x,\lambda) \quad \text{as } |\lambda| \to \infty.$$

Similarly it follows from theorem 3 and (48) that

$$\chi_r^{(s-1)}(b/x,\lambda) \sim \chi_{Fr}^{(s-1)}(b/x,\lambda) \quad \text{as } |\lambda| \to \infty \quad (1 \le r, s \le 4).$$

The proof is complete. □

5 Conclusion

We have successfully proved that the boundary condition functions for the boundary value problem in (1) and (2) are asymptotically equivalent for large values of $|\lambda|$, to the boundary condition functions for the corresponding Fourier boundary value problem for π, given by (3) and (4).

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