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BOUNDEDNESS AND STABILITY IN NONLINEAR DELAY DIFFERENCE EQUATIONS EMPLOYING FIXED POINT THEORY

MUHAMMAD N. ISLAM AND ERNEST YANKSON

Abstract. In this paper we study stability and boundedness of the nonlinear difference equation

$$x(t+1) = a(t)x(t) + c(t)\Delta x(t-g(t)) + q\big(x(t), x(t-g(t))\big).$$

In particular we study equi-boundedness of solutions and the stability of the zero solution of this equation. Fixed point theorems are used in the analysis.

1. Introduction

Liapunov's method is normally used to study the stability properties of the zero solution of differential and difference equations. Certain difficulties arise when Liapunov's method is applied to equations with unbounded delay or equations containing unbounded terms [11], [20]. It has been found that some of these difficulties can be eliminated if fixed point theory is used instead [3]. In the present paper we study certain type of boundedness and stability properties of solutions of linear and nonlinear difference equations with delay using fixed point theory as the main mathematical tool.

In particular we study equi-boundedness of solutions and stability of the zero solution of

$$x(t+1) = a(t)x(t) + c(t)\Delta x(t-g(t)) + q\big(x(t), x(t-g(t))\big) \tag{1.1}$$

where $a, c : \mathbb{Z} \to \mathbb{R}$, $q : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{Z} \to \mathbb{Z}^+$. The operator Δ is defined as

$$\Delta x(t) = x(t+1) - x(t).$$

We assume that a and c are bounded discrete functions whereas g can be unbounded. Continuous versions of (1.1) are generally known as neutral differential equations. Neutral differential equations have many applications. For example, they arise in the study of two or more simple oscillatory systems with some interconnections between them [4, 22], and in modelling physical problems such as vibration of masses attached to an elastic bar [22].

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In section 2, we consider a linear difference equation and obtain the asymptotic stability of the zero solution employing Liapunov's method. We then point out some difficulties and restrictions that arise when the method of Liapunov is used. In sections 3 and 4 we study equi-boundedness of solutions and stability of the zero solution of (1.1) employing the contraction mapping principle.

The work in this paper is related to the work done in [17] and [20]. The equations that we consider in the current paper are the discrete cases of the differential equations studied in [20]. In [17], the authors studied the existence of periodic solution of (1.1). For more on stability of ordinary and functional differential equations we refer to [1], [2], [8], [10], [12], [14], [23], and for difference equations we refer to [5], [6], [7], [8], [9], [13], [15], [18], [19], and [21].

If g is bounded and the maximum of g is k, then for any integer $t_0 \geq 0$, we define $\mathbb{Z}_0$ to be the set of integers in $[t_0 - k, t_0]$. If g is unbounded then $\mathbb{Z}_0$ will be the set of integers in $(-\infty, t_0]$.

Let $\psi : \mathbb{Z}_0 \to \mathbb{R}$ be an initial discrete bounded function.

Definition 1.1. We say $x(t) := x(t, t_0, \psi)$ is a solution of (1.1) if $x(t) = \psi(t)$ on $\mathbb{Z}_0$ and satisfies (1.1) for $t \geq t_0$.

Definition 1.2. The zero solution of (1.1) is Liapunov stable if for any $\epsilon > 0$ and any integer $t_0 \geq 0$ there exists a $\delta > 0$ such that $|\psi(t)| \leq \delta$ on $\mathbb{Z}_0$ implies $|x(t, t_0, \psi)| \leq \epsilon$ for $t \geq t_0$.

Definition 1.3. The zero solution of (1.1) is asymptotically stable if it is Liapunov stable and if for any integer $t_0 \geq 0$ there exists $r(t_0) > 0$ such that $|\psi(t)| \leq r(t_0)$ on $\mathbb{Z}_0$ implies $|x(t, t_0, \psi)| \to 0$ as $t \to \infty$.

Definition 1.4. A solution $x(t, t_0, \psi)$ of (1.1) is said to be bounded if there exist a $B(t_0, \psi) > 0$ such that $|x(t, t_0, \psi)| \leq B(t_0, \psi)$ for $t \geq t_0$.

Definition 1.5. The solutions of (1.1) are said to be equi-bounded if for any t_0 and any $B_1 > 0$, there exists a $B_2 = B_2(t_0, B_1) > 0$ such that $|\psi(t)| \leq B_1$ on $\mathbb{Z}_0$ implies $|x(t, t_0, \psi)| \leq B_2$ for $t \geq t_0$.

We refer the readers to [16] for definitions of boundedness of solutions for general differential equations.

2. Stability of Linear Difference Equation Employing Liapunov's Method

Our objective in this section is to illustrate the difficulties and restrictions that arise when Liaponuv's method is used to study the stability of the zero solution of difference equations with delay. To show this, we consider the linear difference equation

$$x(t+1) = a(t)x(t) + b(t)x(t-g(t)), \tag{2.1}$$

where a, b and g are as defined above.

Theorem 2.1. Suppose

$$\triangle g(t) \leq 0 \text{ and } g(t) > 0 \text{ for all } t \in \mathbb{Z}.$$

Also, suppose there is a $\delta > 0$ such that fot all $t \in \mathbb{Z}$

$$|a(t)| + \delta < 1, \tag{2.2}$$

and

$$|b(t)| \leq \delta. \tag{2.3}$$

Then the zero solution of (2.1) is asymptotically stable.

Proof. Define the Liapunov functional $V(t, x_t)$ by

$$V(t, x_t) = |x(t)| + \delta \sum_{s=t-g(t)}^{t-1} |x(s)|$$

Then along solutions of (2.1) we have

$$\begin{aligned}
\triangle V &= |x(t+1)| - |x(t)| + \delta \sum_{s=t+1-g(t+1)}^{t} |x(s)| - \delta \sum_{s=t-g(t)}^{t-1} |x(s)| \\
&\leq |a(t)||x(t)| - |x(t)| + |b(t)|x(t-g(t))| + \delta \sum_{s=t+1-g(t)}^{t} |x(s)| - \delta \sum_{s=t-g(t)}^{t-1} |x(s)| \\
&= \Big(|a(t)| + \delta - 1\Big)|x(t)| + \Big(|b(t)| - \delta\Big)|x(t-g(t))| \\
&\leq \Big(|a(t)| + \delta - 1\Big)|x(t)| \\
&\leq -\gamma|x(t)|, \text{for some positive constant } \gamma.
\end{aligned}$$

It follows from the above relation that the zero solution of (2.1) is asymptotically stable.

Remark. The first difficulty associated with the above method is the construction of an efficient Liapunov functional. Secondly, conditions (2.2) and (2.3) in Theorem 2.1 imply that

$$|a(t)| + |b(t)| < 1 \text{ for all } t \in \mathbb{Z}$$

which is a very restrictive condition on the functions a and b.

3. Boundedness and Stability of Linear Difference Equation Employing Fixed Point Theory

In this section we begin our study of boundedness and stability using the contraction mapping principle by considering the linear difference equation with delay

$$x(t+1) = a(t)x(t) + b(t)x(t-g(t)) + c(t)\Delta x(t-g(t)) \qquad (3.1)$$

where a, b, c, g and Δ are defined above.

The use of the contraction mapping principle requires a map and a complete metric space. Thus we begin by inverting equation (3.1) to obtain the map. In the process we will require the following:
a) For any sequence $x(k)$

$$\sum_{k=a}^{b} x(k) = 0 \text{ and } \prod_{k=a}^{b} x(k) = 1 \text{ whenever } a > b.$$

b)

$$\sum \Big(Ex(t)\Delta z(t)\Big) = x(t)z(t) - \sum z(t)\Delta x(t)$$

where E is defined as $Ex(t) = x(t+1)$.

Lemma 3.1. Suppose that $a(t) \neq 0$ for all $t \in \mathbb{Z}$. Then x is a solution of equation (3.1) if and only if

$$\begin{aligned} x(t) \;=\; & \big[x(t_0) - c(t_0-1)x(t_0 - g(t_0))\big] \prod_{s=t_0}^{t-1} a(s) + c(t-1)x(t-g(t)) \\ & + \sum_{r=t_0}^{t-1} \Big([b(r) - \phi(r)]x(r-g(r)) \prod_{s=r+1}^{t-1} a(s) \Big), \; t \geq t_0 \end{aligned} \qquad (3.2)$$

where

$$\phi(r) = c(r) - c(r-1)a(r).$$

Proof. Note that equation (3.1) is equivalent to the equation

$$\Delta\Big[x(t)\prod_{s=t_0}^{t-1} a^{-1}(s)\Big] = \Big[b(t)x(t-g(t)) + c(t)\Delta x(t-g(t))\Big]\prod_{s=t_0}^{t} a^{-1}(s). \quad (3.3)$$

Summing equation (3.3) from t_0 to t-1 gives

$$\sum_{r=t_0}^{t-1}\Delta\Big[x(r)\prod_{s=t_0}^{r-1} a^{-1}(s)\Big] = \sum_{r=t_0}^{t-1}\Big[b(r)x(r-g(r)) + c(r)\Delta x(r-g(r))\Big]\prod_{s=t_0}^{r} a^{-1}(s)$$

which gives,

$$x(t)\prod_{s=t_0}^{t-1} a^{-1}(s) - x(t_0)\prod_{s=t_0}^{t_0-1} a^{-1}(s) = \sum_{r=t_0}^{t-1}\Big[b(r)x(r-g(r)) + c(r)\Delta x(r-g(r))\Big]\prod_{s=t_0}^{r} a^{-1}(s)$$

Dividing both sides by

$$\prod_{s=t_0}^{t-1} a^{-1}(s)$$

gives

$$\begin{aligned} x(t) &= x(t_0)\prod_{s=t_0}^{t-1} a(s) + \sum_{r=t_0}^{t-1}\Big[b(r)x(r-g(r)) \\ &\quad + c(r)\Delta x(r-g(r))\Big]\prod_{s=t_0}^{r} a^{-1}(s)\prod_{s=t_0}^{t-1} a(s) \\ &= x(t_0)\prod_{s=t_0}^{t-1} a(s) + \sum_{r=t_0}^{t-1} b(r)x(r-g(r))\prod_{s=r+1}^{t-1} a(s) \\ &\quad + \sum_{r=t_0}^{t-1} c(r)\Delta x(r-g(r))\prod_{s=r+1}^{t-1} a(s) \end{aligned} \quad (3.4)$$

But

$$\begin{aligned} \sum_{r=t_0}^{t-1} c(r)\Delta x(r-g(r))\prod_{s=r+1}^{t-1} a(s) &= c(t-1)x(t-g(t)) - c(t_0-1)x(t_0-g(t_0))\prod_{s=t_0}^{t-1} a(s) \\ &\quad - \sum_{r=t_0}^{t-1} x(r-g(r))\Delta\Big[c(r-1)\prod_{s=r}^{t-1} a(s)\Big] \end{aligned}$$

Thus equation (3.4) becomes

$$\begin{aligned}
x(t) &= x(t_0)\prod_{s=t_0}^{t-1}a(s)+\sum_{r=t_0}^{t-1}b(r)x(r-g(r))\prod_{s=r+1}^{t-1}a(s)\\
&\quad +c(t-1)x(t-g(t))-c(t_0-1)x(t_0-g(t_0))\prod_{s=t_0}^{t-1}a(s)\\
&\quad -\sum_{r=t_0}^{t-1}x(r-g(r))\Delta\Big[c(r-1)\prod_{s=r}^{t-1}a(s)\Big]\\
&= \Big[x(t_0)-c(t_0-1)x(t_0-g(t_0))\Big]\prod_{s=t_0}^{t-1}a(s)+c(t-1)x(t-g(t))\\
&\quad +\sum_{r=t_0}^{t-1}\Big[-x(r-g(r))\Delta[c(r-1)\prod_{s=r}^{t-1}a(s)]\\
&\quad +b(r)x(r-g(r))\prod_{s=r+1}^{t-1}a(s)\Big]\\
&= [x(t_0)-c(t_0-1)x(t_0-g(t_0))]\prod_{s=t_0}^{t-1}a(s)+c(t-1)x(t-g(t))\\
&\quad +\sum_{r=t_0}^{t-1}\Big[b(r)-\phi(r)\Big]x(r-g(r))\prod_{s=r+1}^{t-1}a(s),
\end{aligned}$$

where $\phi(r)=c(r)-c(r-1)a(r)$.

This completes the proof of lemma 3.1.

Theorem 3.1. Suppose $a(t)\neq 0$ for $t\geq t_0$ and $a(t)$ satisfies

$$\Big|\prod_{s=t_0}^{t-1}a(s)\Big|\leq M$$

for $M>0$. Also, suppose that there is an $\alpha\in(0,1)$ such that

$$|c(t-1)|+\sum_{r=t_0}^{t-1}|b(r)-\phi(r)|\Big|\prod_{s=r+1}^{t-1}a(s)\Big|\leq\alpha,\ t\geq t_0. \tag{3.5}$$

Then solutions of (3.1) are equi-bounded.

Proof. Let $B_1 > 0$ be given. Choose B_2 such that

$$|1 - c(t_0 - 1)|MB_1 + \alpha B_2 \leq B_2 \tag{3.6}$$

Let ψ be a bounded initial function satisfying $|\psi(t)| \leq B_1$ on $\mathbb{Z}_0$. Define

$$S = \Big\{\varphi : \mathbb{Z} \to \mathbb{R} |\ \varphi(t) = \psi(t) \text{ on } \mathbb{Z}_0 \text{ and } ||\varphi|| \leq B_2\Big\}, \tag{3.7}$$

where

$$||\varphi|| = \max_{t \in \mathbb{Z}} |\varphi(t)|.$$

Then $\Big(S, ||\cdot||\Big)$ is a complete metric space.
Define mapping $P : S \to S$ by

$$\big(P\varphi\big)(t) = \psi(t) \text{ on } \mathbb{Z}_0$$

and

$$\begin{aligned}\big(P\varphi\big)(t) &= \Big[\psi(t_0) - c(t_0 - 1)\psi(t_0 - g(t_0))\Big] \prod_{s=t_0}^{t-1} a(s) + c(t-1)\varphi(t - g(t)) \\ &\quad + \sum_{r=t_0}^{t-1} \Big[b(r) - \phi(r)\Big]\varphi(r - g(r)) \prod_{s=r+1}^{t-1} a(s),\ t \geq t_0. \end{aligned} \tag{3.8}$$

We first show that P maps from S to S. By (3.6)

$$\begin{aligned}|(P\varphi)(t)| &\leq |1 - c(t_0 - 1)|MB_1 + \alpha B_2 \\ &\leq B_2 \qquad \text{for } t \geq t_0\end{aligned}$$

Thus P maps from S into itself. We next show that P is a contraction under the supremum norm. Let $\zeta, \eta \in S$. Then

$$\begin{aligned}|(P\zeta)(t) - (P\eta)(t)| &\leq \Big(|c(t-1)| + \sum_{r=t_0}^{t-1} |b(r) - \phi(r)| \Big| \prod_{s=r+1}^{t-1} a(s)\Big|\Big) ||\zeta - \eta|| \\ &\leq \alpha||\zeta - \alpha||.\end{aligned}$$

This shows that P is a contraction. Thus, by the contraction mapping principle, P has a unique fixed point in S which solves (3.1). This proves that solutions of (3.1) are equi-bounded.

Theorem 3.2. Assume that the hypotheses of Theorem 3.1 hold. Then the zero solution of (3.1) is Liapunov stable.

Proof. Let $\epsilon > 0$ be given. Choose $\delta > 0$ such that

$$|1 - c(t_0 - 1)|M\delta + \alpha\epsilon \leq \epsilon. \tag{3.9}$$

Let ψ be a bounded initial function satisfying $|\psi(t)| \leq \delta, t \in \mathbb{Z}_0$. Define the complete metric space S by

$$S = \Big\{\varphi : \mathbb{Z} \to \mathbb{R} |\ \varphi(t) = \psi(t) \text{ on } \mathbb{Z}_0 \text{ and } ||\varphi|| \leq \epsilon\Big\}. \tag{3.10}$$

Now consider the map $P : S \to S$ defined by (3.8). It follows from the proof of Theorem 3.1 that P is a contraction map and for any $\varphi \in S, ||P\varphi|| \leq \epsilon$.

This proves that the zero solution of (3.1) is Liapunov stable.

Theorem 3.3. Assume that the hypotheses of Theorem 3.1 hold. Also assume that

$$\prod_{s=t_0}^{t-1} a(s) \to 0 \text{ as } t \to \infty, \tag{3.11}$$

$$t - g(t) \to \infty \text{ as } t \to \infty. \tag{3.12}$$

Then the zero solution of (3.1) is asymptotically stable.

Proof. We have already proved that the zero solution of (3.1) is Liapunov stable. Choose $r(t_0)$ to be the δ of the Liapunov stability of the zero solution. Let $\psi(t)$ be any initial discrete function satisfying $|\psi(t)| \leq r(t_0)$. Define

$$S^* = \Big\{\varphi : \mathbb{Z} \to \mathbb{R} |\ \varphi(t) = \psi(t) \text{ on } \mathbb{Z}_0,\ ||\varphi|| \leq \epsilon \text{ and } \varphi(t) \to 0 \text{ as } t \to \infty\Big\}. \tag{3.13}$$

Define $P : S^* \to S^*$ by (3.8). From the proof of Theorem 3.1, the map P is a contraction and it maps from S^* into itself.
We next show that $(P\varphi)(t)$ goes to zero as t goes to infinity.

The first term on the right of (3.8) goes to zero because of condition (3.11). The second term on the right side of (3.8) goes to zero because of condition (3.12) and the fact that $\varphi \in S^*$.

Now we show that the last term

$$|\sum_{r=t_0}^{t-1}[b(r)-\phi(r)]\varphi(r-g(r))\prod_{s=r+1}^{t-1}a(s)|$$

on the right of (3.8) goes to zero as $t\to\infty$.
Let $\varphi\in S^*$ then $|\varphi(t-g(t))|\le\epsilon$. Also, since $\varphi(t-g(t))\to 0$ as $t-g(t)\to\infty$, there exists a $t_1>0$ such that for $t>t_1$, $|\varphi(t-g(t))|<\epsilon_1$ for $\epsilon_1>0$. Due to condition (3.11) there exists a $t_2>t_1$ such that for $t>t_2$ implies that

$$\Big|\prod_{s=t_1}^{t}a(s)\Big|<\frac{\epsilon_1}{\alpha\epsilon}.$$

Thus for $t>t_2$, we have

$$\begin{aligned}
&\Big|\sum_{r=t_0}^{t-1}\Big[b(r)-\phi(r)\Big]\varphi(r-g(r))\prod_{s=r+1}^{t-1}a(s)\Big|\le\sum_{r=t_0}^{t-1}\Big|[b(r)-\phi(r)]\varphi(r-g(r))\prod_{s=r+1}^{t-1}a(s)\Big|\\
\le\ &\sum_{r=t_0}^{t_1-1}\Big|[b(r)-\phi(r)]\varphi(r-g(r))\prod_{s=r+1}^{t-1}a(s)\Big|\\
&+\sum_{r=t_1}^{t-1}\Big|[b(r)-\phi(r)]\varphi(r-g(r))\prod_{s=r+1}^{t-1}a(s)\Big|\\
\le\ &\epsilon\sum_{r=t_0}^{t_1-1}\Big|[b(r)-\phi(r)]\prod_{s=r+1}^{t-1}a(s)\Big|+\epsilon_1\alpha\\
\le\ &\epsilon\sum_{r=t_0}^{t_1-1}\Big|[b(r)-\phi(r)]\prod_{s=r+1}^{t_1-1}a(s)\prod_{s=t_1}^{t-1}a(s)\Big|+\epsilon_1\alpha\\
\le\ &\epsilon\Big|\prod_{s=t_1}^{t-1}a(s)\Big|\sum_{r=t_0}^{t_1-1}\Big|[b(r)-\phi(r)]\prod_{s=r+1}^{t_1-1}a(s)\Big|+\epsilon_1\alpha\\
\le\ &\epsilon\alpha|\prod_{s=t_1}^{t-1}a(s)|+\epsilon_1\alpha\\
\le\ &\epsilon_1+\epsilon_1\alpha.
\end{aligned}$$

Thus showing that the last term of (3.8) goes to zero as t goes to infinity. Therefore $(P\varphi)(t)\to 0$ as $t\to\infty$.
By the contraction mapping principle, P has a unique fixed point that solves (3.1) and goes to zero as t goes to infinity. Therefore, the zero solution of (3.1) is asymptotically stable.

Example 3.1. Consider the linear difference equation

$$x(t+1) = \frac{1}{1+t}x(t) + \frac{2^{t+1}}{8(1+t)!}x(t-2) + \frac{2^{t+1}}{8(1+t)!}\Delta x(t-2),\ t \geq 0 \quad (3.14)$$

Equi-boundedness

For this example we let $t_0 = 0$ thus $\mathbb{Z}_0 = [-2, 0]$.
Let $B_1 > 0$ be given and $\psi(t) : \mathbb{Z}_0 \to \mathbb{R}$ be a given initial function with $|\psi(t)| \leq B_1$. Choose B_2 satisfying

$$\frac{7B_1}{8} \leq \frac{B_2}{2}. \quad (3.15)$$

Let S be the set definded by (3.7).
In our example

$$c(r) = \frac{2^{r+1}}{8(1+r)!}.$$

Therefore, $\phi(r)$ of (3.8) yields

$$\phi(r) = c(r) - c(r-1)a(r)$$
$$= \frac{2^{r+1}}{8(1+r)!} - \frac{2^r}{8(r)!}\frac{1}{(1+r)}$$

Thus $b(r) - \phi(r)$ of (3.8) also yields

$$b(r) - \phi(r) = \frac{2^{r+1}}{8(1+r)!} - \frac{2^{r+1}}{8(1+r)!} + \frac{2^r}{8(r+1)!}$$

$$= \frac{2^r}{8(r+1)!} \quad (3.16)$$

Now using (3.16) in (3.8) we define $\big(P\varphi\big)(t) = \psi(t)$ on $\mathbb{Z}_0$
and for $t \geq 0$

$$\big(P\varphi\big)(t) = [\psi(0) - \frac{1}{8}\psi(-2)]\prod_{s=0}^{t-1}\frac{1}{1+s} + \frac{2^t}{8t!}\varphi(t-2)$$
$$+ \sum_{r=0}^{t-1}\frac{2^r}{8(1+r)!}\varphi(r-2)\prod_{s=r+1}^{t-1}\frac{1}{1+s} \quad (3.17)$$

To see that P defines a contraction mapping, we let $\zeta, \eta \in S$. Then

$$\begin{aligned}
|(P\eta)(t) - (P\zeta)(t)| &\leq \frac{2^t}{8t!}||\eta - \zeta|| + \sum_{r=0}^{t-1} \frac{2^r}{8(1+r)!} \prod_{s=r+1}^{t-1} \frac{1}{1+s}||\eta - \zeta|| \\
&= \frac{2^t}{8t!}||\eta - \zeta|| + \sum_{r=0}^{t-1} \frac{2^r}{8(1+r)!} \frac{1}{(r+2)(r+3)...(t)}||\eta - \zeta|| \\
&= \frac{2^t}{8t!}||\eta - \zeta|| + \sum_{r=0}^{t-1} \frac{2^r}{8t!}||\eta - \zeta|| \\
&= \frac{2^t}{8t!}||\eta - \zeta|| + \frac{1}{8t!}(2^t - 1)||\eta - \zeta|| \\
&\leq \frac{2^t}{8t!}||\eta - \zeta|| + \frac{2^t}{8t!}||\eta - \zeta|| \\
&\leq \frac{1}{2}||\eta - \zeta||.
\end{aligned}$$

We now show that P maps from S into itself. Let $\varphi \in S$. Then

$$\begin{aligned}
\Big|(P\varphi)(t)\Big| &= \Big|[\psi(0) - \frac{1}{8}\psi(-2)] \prod_{s=0}^{t-1} \frac{1}{1+s} + \frac{2^t}{8t!}\varphi(t-2) \\
&\quad + \sum_{r=0}^{t-1} \frac{2^r}{8(1+r)!}\varphi(r-2) \prod_{s=r+1}^{t-1} \frac{1}{1+s}\Big| \\
&\leq \frac{7}{8}B_1 + \frac{2}{8}B_2 + \sum_{r=0}^{t-1} \frac{2^r}{8(1+r)!} \prod_{s=r+1}^{t-1} \frac{1}{1+s}B_2 \\
&= \frac{7}{8}B_1 + \frac{2}{8}B_2 + \sum_{r=0}^{t-1} \frac{2^r}{8(1+r)!} \frac{1}{(r+2)(r+3)...(t)}B_2 \\
&\leq \frac{7}{8}B_1 + \frac{2B_2}{8} + \frac{2B_2}{8} \\
&\leq \frac{7}{8}B_1 + \frac{B_2}{2} \\
&\leq B_2
\end{aligned}$$

by (3.15).

Therefore solutions of (3.14) are equi-bounded at $t_0 = 0$.

Stability

Let $\epsilon > 0$ be given. Choose $\delta > 0$ satisfying

$$\frac{7\delta}{8} \leq \frac{\epsilon}{2}. \tag{3.18}$$

Let $\psi(t) : \mathbb{Z}_0 \to \mathbb{R}$ be a given initial function with $|\psi(t)| \leq \delta$. Let S be the set defined by (3.10).
Define the map $P : S \to S$ by (3.17). Then P is a contraction map and for any $\varphi \in S$, $||P\varphi|| \leq \epsilon$. Therefore the zero solution of (3.14) is Liapunov stable at $t_0 = 0$.

4. Boundedness and Stability of Nonlinear Difference Equation Employing Fixed Point Theory

We continue our study of boundedness and stability using fixed point theory by considering the nonlinear difference equation with delay

$$x(t+1) = a(t)x(t) + c(t)\Delta x(t-g(t)) + q(x(t), x(t-g(t))) \tag{4.1}$$

where a, c and g are defined as before. Here we assume that, $q(0,0) = 0$ and q is locally Lipschitz in x and y. That is, there is a $K > 0$ so that if $|x|, |y|, |z|$ and $|w| \leq K$ then

$$|q(x,y) - q(z,w)| \leq L|x-z| + E|y-w|$$

for some positive constants L and E.
Note that

$$\begin{aligned} |q(x,y)| &= |q(x,y) - q(0,0) + q(0,0)| \\ &\leq |q(x,y) - q(0,0)| + |q(0,0)| \\ &\leq L|x| + E|y|. \end{aligned}$$

Lemma 4.1. Suppose that $a(t) \neq 0$ for all $t \in \mathbb{Z}$. Then x is a solution of equation (4.1) if and only if

$$x(t) = \Big[x(t_0) - c(t_0-1)x(t_0-g(t_0))\Big] \prod_{s=t_0}^{t-1} a(s) + c(t-1)x(t-g(t))$$
$$+ \sum_{r=t_0}^{t-1} \Big[- x(r-g(r))\Phi(r) + q(x(r), x(r-g(r)))\Big] \prod_{s=r+1}^{t-1} a(s), \; t \geq t_0$$

where $\Phi(r) = c(r) - c(r-1)a(r)$.

Remark. The details of the inversion of equation (4.1) is similar to that of equation (3.1) discussed in the previous section and so we omit the proof of lemma 4.1.

Theorem 4.1. Suppose $a(t) \neq 0$ for $t \geq t_0$ and a satisfies

$$\Big|\prod_{s=t_0}^{t-1} a(s)\Big| \leq M$$

for $M > 0$. Also, suppose that there is an $\alpha \in (0,1)$ such that

$$\big|c(t-1)\big| + \sum_{r=t_0}^{t-1}\Big[|\Phi(r)| + (L+E)\Big]\Big|\prod_{s=r+1}^{t-1} a(s)\Big| \leq \alpha,\ t \geq t_0. \tag{4.2}$$

Then solutions of (4.1) are equi-bounded.

Proof. Let $B_1 > 0$ be given. Choose B_2 such that

$$|1 - c(t_0-1)|MB_1 + \alpha B_2 \leq B_2. \tag{4.3}$$

Let $\psi : \mathbb{Z}_0 \to \mathbb{R}$ be an initial function satisfying $|\psi(t)| \leq B_1, t \in \mathbb{Z}_0$. Let S be the set defined by (3.7).
Define mapping $P : S \to S$ by

$$(P\varphi)(t) = \psi(t) \text{ on } \mathbb{Z}_0,$$

and

$$\begin{aligned}(P\varphi)(t) \quad &= \quad \Big[\psi(t_0) - c(t_0-1)\psi(t_0 - g(t_0))\Big]\prod_{s=t_0}^{t-1} a(s) + c(t-1)\varphi(t-g(t)) \quad (4.4)\\ &+ \sum_{r=t_0}^{t-1}\Big[-\varphi(r-g(r))\Phi(r) + q(\varphi(r), \varphi(r-g(r)))\Big]\prod_{s=r+1}^{t-1} a(s),\ t \geq t_0\end{aligned}$$

We first show that P maps from S into itself.

$$\begin{aligned}\big|(P\varphi)(t)\big| &\leq \big|B_1 - c(t_0-1)B_1\big|M + \big|c(t-1)\big|B_2 \\ &\quad + \sum_{r=t_0}^{t-1}\Big[|\Phi(r)|B_2 + L|\varphi(r)| + E|\varphi(r-g(r))|\Big]\Big|\prod_{s=r+1}^{t-1}a(s)\Big| \\ &\leq \big|(1-c(t_0-1))\big|MB_1 + |c(t-1)|B_2 + \sum_{r=t_0}^{t-1}\Big[|\Phi(r)| + L + E\Big]\Big|\prod_{s=r+1}^{t-1}a(s)\Big|B_2 \\ &\leq \big|(1-c(t_0-1))\big|MB_1 + \Big(|c(t-1)| + \sum_{r=t_0}^{t-1}\Big[|\Phi(r)| + L + E\Big]\Big|\prod_{s=r+1}^{t-1}a(s)\Big|\Big)B_2 \\ &\leq |(1-c(t_0-1))|MB_1 + \alpha B_2 \\ &\leq B_2\end{aligned}$$

by (4.3). Thus showing that P maps from S into itself.

We next show that P is a contraction map. For $\zeta, \eta \in S$, we get

$$\begin{aligned}|(P\zeta)(t) - (P\eta)(t)| &\leq \Big(|c(t-1)| + \sum_{r=t_0}^{t-1}\Big[|\Phi(r)| + L + E\Big]\Big|\prod_{s=r+1}^{t-1}a(s)\Big|\Big)||\zeta-\eta|| \\ &\leq \alpha||\zeta-\eta||\end{aligned}$$

Thus, by the contraction mapping principle, P has a unique fixed point in S which solves (4.1). This proves that solutions of (4.1) are equi-bounded.

Theorem 4.2. Assume that the hypotheses of Theorem 4.1 hold. Then the zero solution of (4.1) is Liapunov stable.

Proof. Let $\epsilon > 0$ be given. Choose $\delta > 0$ such that

$$|1 - c(t_0-1)|M\delta + \alpha\epsilon \leq \epsilon. \tag{4.5}$$

Let $\psi : \mathbb{Z}_0 \to \mathbb{R}$ be an initial discrete function satisfying $|\psi(t)| \leq \delta, t \in \mathbb{Z}_0$. Let S be the set defined by (3.10).
Define the map $P : S \to S$ by (4.4). It follows from the proof of Theorem 4.1 that P is a contraction map and for any $\psi \in S$, $||P\varphi|| \leq \epsilon$.
This proves that the zero solution of (4.1) is Liapunov stable.

Theorem 4.3 Assume that the hypotheses of Theorem 4.1 hold. Also assume that

$$\prod_{s=t_0}^{t-1}a(s) \to 0 \text{ as } t \to \infty, \tag{4.6}$$

$$t - g(t) \to \infty \text{ as } t \to \infty. \tag{4.7}$$

Then the zero solution of (4.1) is asymptotically stable.

Proof We have already proved that the zero solution of (4.1) is Liapunov stable. Choose $r(t_0)$ to be the δ of the Liapunov stability of the zero solution. Let ψ be any initial discrete function satisfying $|\psi(t)| \leq r, t \in \mathbb{Z}_0$. Let S^* be the set defined by (3.13).
Define the map $P : S^* \to S^*$ by (4.4). From the proof of Theorem 4.1, the map P is a contraction and it maps S^* into itself.

Next we show that $(P\varphi)(t)$ goes to zero as t goes to infinity.
The first term on the right of (4.4) goes to zero because of condition (4.6). The second term on the right goes to zero because of condition (4.7) and the fact that $\varphi \in S^*$.
Finally we show that the last term

$$\sum_{r=t_0}^{t-1} \Big[-\Phi(r)\varphi(r-g(r)) + q(\varphi(r), \varphi(r-g(r)))\Big] \prod_{s=r+1}^{t-1} a(s)$$

on the right of (4.4) goes to zero as $t \to \infty$.
Let $\varphi \in S^*$ then $|\varphi(t-g(t))| \leq \epsilon$. Also, since $\varphi(t-g(t)) \to 0$ as $t-g(t) \to \infty$, there exists a $t_1 > 0$ such that for $t > t_1$, $|\varphi(t-g(t))| < \epsilon_1$ for $\epsilon_1 > 0$. Due to condition (4.6) there exists a $t_2 > t_1$ such that for $t > t_2$ implies that

$$\Big|\prod_{s=t_1}^{t} a(s)\Big| < \frac{\epsilon_1}{\alpha\epsilon}.$$

Thus for $t > t_2$, we have

$$
\begin{aligned}
&\Big|\sum_{r=t_0}^{t-1}\big[-\varphi(r-g(r))\Phi(r)+q(\varphi(r),\varphi(r-g(r)))\big]\prod_{s=r+1}^{t-1}a(s)\Big| \\
\leq\ & \sum_{r=t_0}^{t-1}\Big|\big[-\varphi(r-g(r))\Phi(r)+q(\varphi(r),\varphi(r-g(r)))\big]\prod_{s=r+1}^{t-1}a(s)\Big| \\
\leq\ & \sum_{r=t_0}^{t_1-1}\Big|\big[\varphi(r-g(r))\Phi(r)+L\varphi(r)+E\varphi(r-g(r))\big]\prod_{s=r+1}^{t-1}a(s)\Big| \\
& +\sum_{r=t_1}^{t-1}\Big|\big[\varphi(r-g(r))\Phi(r)+L\varphi(r)+E\varphi(r-g(r))\big]\prod_{s=r+1}^{t-1}a(s)\Big| \\
\leq\ & \epsilon\sum_{r=t_0}^{t_1-1}\Big[|\Phi(r)|+L+E\Big]\Big|\prod_{s=r+1}^{t-1}a(s)\Big|+\epsilon_1\alpha \\
\leq\ & \epsilon\sum_{r=t_0}^{t_1-1}\Big[|\Phi(r)|+L+E\Big]\Big|\prod_{s=r+1}^{t_1-1}a(s)\prod_{s=t_1}^{t-1}a(s)\Big|+\epsilon_1\alpha \\
\leq\ & \epsilon\alpha\Big|\prod_{s=t_1}^{t-1}a(s)\Big|+\epsilon_1\alpha \\
\leq\ & \epsilon_1+\epsilon_1\alpha.
\end{aligned}
$$

Thus showing that $(P\varphi)(t) \to 0$ as $t \to \infty$. Therefore the zero solution of (4.1) is asymptotically stable.

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