DIFFERENTIAL AND DELAY DIFFERENTIAL APPROACH IN THE ANALYSIS OF STABLE STATE EQUILIBRIUM PRICES USING CHARACTERISTIC EQUATION TECHNIQUES

THE DIFFERENTIAL AND DELAY DIFFERENTIAL APPROACH IN THE ANALYSIS OF STABLE STATE EQUILIBRIUM PRICES USING CHARACTERISTIC EQUATION TECHNIQUES

[1]MARTIN ANOKYE, [2]HENRY AMANKWAH, [3]EMMANUEL KWAME ESSEL, [4]IRENE KAFUI AMPONSAH

[1]Department of Mathematics and Statistics, University of Cape Coast, Cape Coast, Ghana
[2,3,4]Department of Mathematics and Statistics, University of Cape Coast, Cape Coast, Ghana
E-mail: [1]martin.anokye@ucc.edu.gh

Abstract - The study compares the stability states of price adjustment differential models with and without delay parameter using roots of characteristic equations. The states of stability of both models were simulated using their particular solutions with inputs from same source. It was found that irrespective of initial prices set for the commodity, the current price for the differential models will always have the propensity to move monotonically to the equilibrium price defined for the system. On the other hand, the current price for the delay- differential models tends to oscillate and move away from the initial prices due to the delay associated with the supply, then with time decreases and turns towards to the defined system equilibrium prices.

It was deduced that whilst the current prices in the delay-differential models are not predictable at the initial stages due to the time delay parameters associated with them, its counterpart without the delay are predictable, since differential models converge monotonically to the equilibrium price points defined for the system. Since most economic and natural phenomena are associated with delays, it is recommended that such systems are modeled using delay-differential equations to reflect realities of the phenomena.

Keywords - Stability Analysis, Delay Differential Equations, Differential Equations and Equilibrium Prices.

I. INTRODUCTION

Price stability of basic foodstuffs has important role in the macro economy and welfare of consumers. Thus price stabilization has significant benefits for the consumer and great impact on the macro economy of the country [1].

Seasonal food crops are associated with time delay in between planting and harvesting periods and therefore according to Timmer[1],seasonal price stabilization between post-harvest lows and pre-harvest highs is very important.

In nature many other processes are time delay inherent yet such systems are usually modeled and described in terms of ordinary differential equations (ODE). The reason is connected to the fact that ODE modeled systems are simpler and easier to analyze than the delay differential equations (DDE) modeled systems [2].Thus it is well known that the behaviour of solutions to DDE can be much more complicated than the behaviour of solutions to its counterpart system with delay equal to zero[3]-[7].

In other words the oscillatory behavior of a delay differential equation and the associated differential equation without delay are not always the same [1].In both theory and applications,the crucial difference is that a delay differential equation is infinite dimensional, considered source of instability and italso requiresa bit more mathematical expediency as compared to their ordinary differential equations counterparts[7],[9].

Characteristic roots (roots of the characteristic equation) usual provide qualitative information about variables whose evolutions are described by dynamic equations. Generally, if a differential equation is parameterized on time, the variable's evolution will be stable if and only if the real part of each root is negative. Also persistent fluctuations occur if there is at least one pair of complex roots [10].

It is observed that not in all situations one can approximate time delay to zero and model the system using differential approach [6]. For instance in prices of commodities that take a unit of time to plant and harvest, the dynamics of price during harvest and planting seasons are not the same. Therefore this paper intends to use characteristic equation techniques to solve price adjustment system modeled in both differential and delay-differential so that qualitative behaviour difference between the solutions of DDE and DE could be established.

This method is being applied to the study because the characteristic roots though meant for solution of higher order differential equations, will offer both models equal platform to qualitatively provide information about the behaviour of price whose evolution is described by the price adjustment equation.

There is no doubt that the outcomes of the study would enhance the understanding of the qualitative behavior and solutions of delay differential and differential equations so that real-life systems that are time parameterized could be modeled using appropriate mathematical approach.

II. METHODS

This paper presents the dynamics of stability conditions of both differential and delay-differential equations applied as price adjustment functions. Therefore, the two mathematical methods are used to model time dependent processes in economics, specifically, demand and supply of commodities and then qualitatively examine their stability behaviour using characteristic equation techniques.

2.1 Price Adjustment Differential Model without Delay

Given demand and supply functions of price with time equation respectively of the form

$$D_{p(t)} = \alpha + \beta P_{(t)}; \quad \beta < 0 \qquad (1)$$

$$S_{p(t)} = \lambda + \delta P_{(t)}; \quad \delta > 0 \ (2)$$

where α and λ are intercepts, β is the change in demand and δ is the change in supply.

The assumption is that the demand and supply of the commodity are dependent on only the price set for the commodity in a single market where there are no substitutes. Also, from the Walrasian assumption, the rate of change of price involving the (1) and (2) is given by

$$P'_{(t)} = \gamma(D_{P_{(t)}} - S_{P_{(t)}}). \qquad (3)$$

Substituting (1) and (2) into (3) gives

$$P'_{(t)} = \gamma(\alpha - \lambda + (\beta - \delta)P_{(t)}). \qquad (4)$$

Equation (4) is simplified if the following notations are adopted: let $-\rho = \gamma(\beta - \delta) > 0$ and $p_e = \dfrac{\alpha - \lambda}{\delta - \beta} > 0$, then dividing through (4) by $(\delta - \beta)$, we have

$$P'_{(t)} = -\rho(P_{(t)} - P_e), \ (5)$$

where P_e is the equilibrium price. At $P_t = P_e$ there is no price change. If $Z_{(t)} = P_{(t)} - P_e$ denotes the deviation from equilibrium then $Z'_{(t)} = P'_{(t)}$ and (5) is reduced to the form

$$Z'_{(t)} = -\rho Z_{(t)}, \ (6)$$

Which could be solved by method of characteristics if $Z_{(t)} = Ce^{\varphi t}$ where C is a constant and φ is the root of the (6). This implies that

$$\varphi Ce^{\varphi t} = -\rho Ce^{\varphi t} \ (7)$$

and thus,

$$(\varphi + \rho)Ce^{\varphi t} = 0 \Rightarrow \varphi = -\rho.$$

Using the notation under (5), the general solution in terms of price is given by

$$P_{(t)} = P_e + Ce^{-\rho t}. \ (8)$$

The particular solution at time $t = 0$, with initial condition $P_{(0)} = P_0$ is also obtained as

$$P_{(t)} = P_e + (P_0 - P_e)e^{-\rho t} \ (9)$$

This means that irrespective of the P_0 set for the (9), as $t \to \infty$, $e^{\infty} \to 0$ and so $P_{(t)} = P_e$ after a long period of time[11]..

2.2 Price Adjustment Differential Model with Delay

If (2) is considered to respond to price change with certain time-delay then supply functions of price with time-delay equation is given in the form

$$S_{p(t)} = \lambda + \delta P_{(t-\tau)}; \quad \delta > 0, \ (10)$$

where τ is the time needed for change in supply, in response to a unit change in price of the commodity. Then from (3), the rate of change is mathematically expressed as follows

$$P'_{(t)} = \gamma(\alpha - \lambda + \beta P_{(t)} - \delta P_{(t-\tau)}). \ (11)$$

If $q = \gamma(\alpha - \lambda)$ with $\alpha > \lambda$, $r = -\gamma\beta$ and $s = \gamma\delta$ then

$$P'_{(t)} = q - rP_{(t)} - sP_{(t-\tau)}. \ (12)$$

Using the transformation $Z_{(t)} = r(P_{(t)} - P_e)$, where P_e is the equilibrium price (same as used) in (4), then diving (12) through by $(\delta - \beta)$ gives

$$P'_{(t)} + r^2(P_{(t)} - P_e) + sr(P_{(t-\tau)} - P_e) = 0$$

and

$$Z'_{(t)} + rZ_{(t)} + sZ_{(t-\tau)} = 0. \ (13)$$

Equation (13) could be solved using characteristic techniques such that if $Z_{(t)} = Ce^{\mu t}$ then

$$\mu Ce^{\mu t} + rCe^{\mu t} + sCe^{\mu(t-\tau)} = 0. \ (14)$$

This implies that

$$(\mu + r)Ce^{\mu t} = -sCe^{\mu(t-\tau)}$$

and

$$(\mu + r) = -se^{-\mu\tau}.$$

By setting $m = \mu + r$, we have

$$me^{(m-r)\tau} = -s.$$

By multiplying through by $e^{r\tau}$, we obtain

$$me^{m\tau} = -se^{r\tau} \quad (15)$$

Since s, r and τ are provided, the right hand side of the (15) could be represented by σ so that

$$F_{(m)} = me^{m\tau} + \sigma. \qquad (16)$$

It is obvious that no real and positive values of m satisfy this transcendental equation, however, for some range of σ there are complex roots with positive real parts. In this case, the solution would provide oscillations that increase exponentially [11].

Also, if $m = x \pm iy$, and $me^{m\tau} + \sigma = 0$, then

$$(x + iy)e^{(x+iy)\tau} + \sigma = 0$$

For which

$(x+iy)e^{iy\tau} = -\sigma e^{-x\tau}$

or

$(x+iy)(\cos(y\tau)+i\sin(y\tau)) = -\sigma e^{-x\tau}$.

Then the following equations are obtained:

$x\cos(y\tau) - y\sin(y\tau) = -\sigma e^{-x\tau}$ (17)

and

$y\cos(y\tau) + x\sin(y\tau) = 0$ (18)

From (18), $x = -y\cot(y\tau), \quad y \neq 0$

We notice that

$$\lim_{y\to 0} - ycot(y\tau) = \lim_{y\to 0}\frac{-y\tau\cos(y\tau)}{\tau\sin(y\tau)}$$
$$= \lim_{y\to 0}\left(\frac{-\cos(y\tau)}{\tau}\right)\lim_{y\to 0}\left(\frac{y\tau}{\sin(y\tau)}\right)$$
$$= \frac{-1}{\tau}.$$

Now, if $x \to \frac{-1}{\tau}, as, y \to 0$ and (17) and (18) are satisfied by $(x,y) = \left(\frac{-1}{\tau}, 0\right)$ then $\sigma = \frac{1}{\tau e}$.

Therefore, a particular solution of (13) in terms of price at $t = \tau = 0$ is given by

$P_{(t)} = P_e + \frac{1}{2}(P_0 - P_e)e^{-\mu t} + \frac{1}{2}(P_0 - P_e)e^{-\mu(t-\tau)}$,

(19)

where $\mu = (\frac{1+\tau r}{\tau})$ and (19) will also move to equilibrium irrespective of the initial price of the commodity. However, the movement would be controlled by the delay parameter in the system [12].

III. RESULTS AND DISCUSSION

The results of the paper are now presented using same parameter values forboth the differential and the delay differential price adjustment models (9) and (19) respectively, to simulate the equilibrium prices when the systems are in their stable states using MatLab.

3.1 Equilibrium Price Set between the Initial Prices

The equilibrium prices are fixed at GHS 0.60 and GHS 0.65, in between the initial prices of GHS1.70 and GHS 0.20 respectively. The price adjustment models are numerically run for both delay differential and differential systems and we had the following results.

Delay-Differential Models:

The time-delays are set at 1:00 and 0.20 at different occasions and then defined the equilibrium prices at stated earlier. The delay differential model was run numerically and following two graphical results obtained.

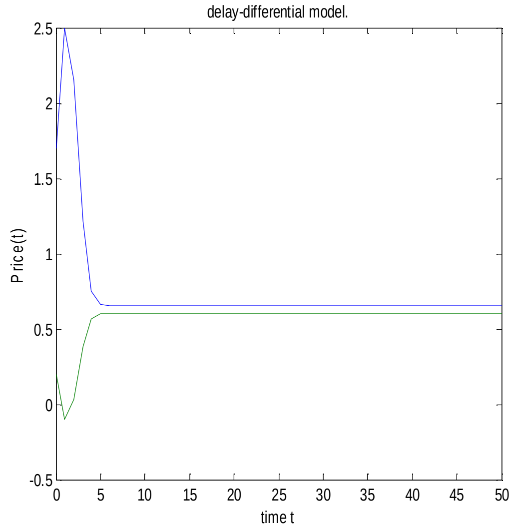

Figure 1: For delay of 1.00 and System Defined Equilibria of GHS 0.60and GHS 0.65.

In Figure 1, when the initial prices were fixed at GHS 0.20, GHS 1.70 and the equilibrium also at GHS 0.60 GHS 0.65 respectively, the current price started rising far above the initial prices due to the delay (1.00) incorporated in the model,then at point in time began decreasingto settle at equilibria defined above.

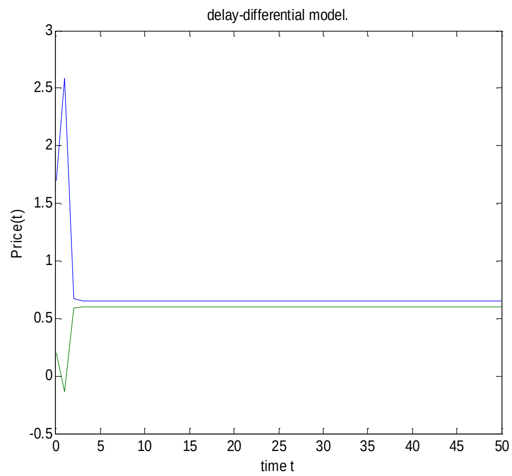

Figure 2: For a delay of 0.20 and System Defined Equilibria of GHS 0.60 and GHS 0.65.

From Figure 2, the current price initiated upward movement due to the time delay value of 0.02, then suddenly turned, decreasing towards equilibrium prices of GHS 0.65andGHS0.60.

Differential Models:

The differential model was also run numerically using the system defined equilibrium prices of GHS 0.60 and GHS 0.65 and we obtained results as appeared in Figure 3.

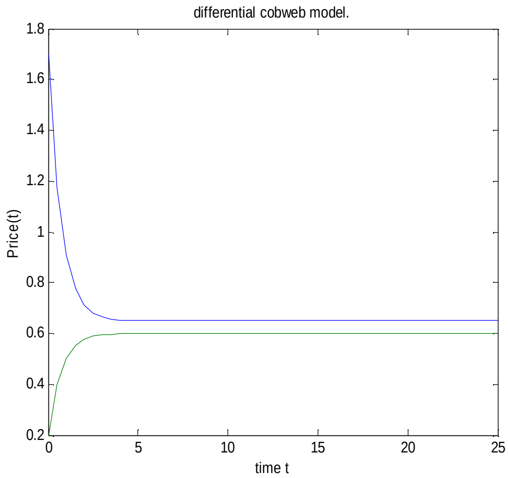

Figure 3: System Defined Equilibria of GHS 0.60 and GHS0.65.

It is observed in Figure 3, that the current price moved monotonically to the system defined equilibrium prices of GHS0.65 and GHS0.60 from the initial prices of GHS 1.70 and GHS 0.20 respectively.

3.2 Equilibrium Price Set beyond the Initial Prices

The equilibrium price is now fixed above the initial prices at GHS 23.0, whilst the initial prices of the commodity maintained at GHS 0.20 and GHS 1.70. The system was run numerically and the following graphical representations obtained for both delay differential and differential models.

Delay-Differential Models:

The parameter values are run for the delay-differential price adjustment model with differential time-delay values but at same system defined equilibrium prices GHS 2.30 and we had the resultsas shown in Figures 4 and 5.

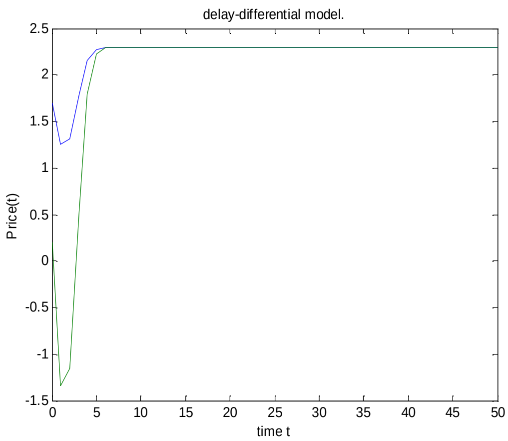

Figure 4: For a delay of 1.00 and System Defined Equilibrium of GHS 2.30.

In Figure 4, when only one equilibrium price was set at GHS 2.30,above the initial prices of GHS 0.20 and GHS 1.70, the oscillatory behavoiur of the delay price function did not change because of the time-delay value of 1.00. At different initial prices, the current price moved upwards and then with time settled at equilibrium price of GHS 2.30 defined for the system.

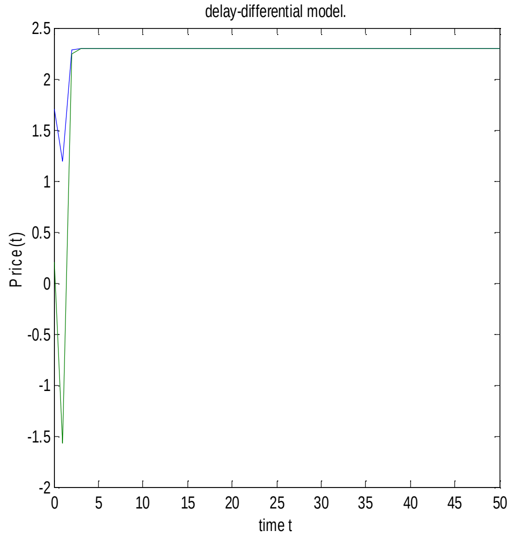

Figure 5: For a delay of 0.2 and System Defined Equilibrium of GHS 2.30.

In Figure 5, it is also observed that at common defined equilibrium price of GHS 2.30 and time-delay of 0.2, the current price converged to the equilibrium point of GHS2.30, when the system was fixed with initial prices of GHS 0.20 and GHS1.70.

Differential Models:

For common system defined equilibrium price of GHS 2.30, the numerical results of the differential price adjustment model also provided the results as follows.

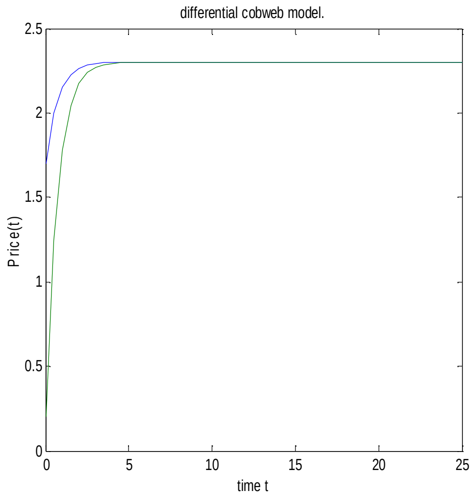

Figure 6: System Defined Equilibria of GHS 2.30.

From Figure 6, it is deduced that whenever the defined equilibrium price point of differential model is set beyond the initial prices, the current price will also move towards it.

CONCLUSION

This paper studies the dynamics of stability states of price adjustment differential equations with and without delay parameter using characteristic equation techniques. The states of stability of the two models were simulated using their particular solutions with inputs from same source.

It was found that irrespective of initial prices set for the commodity, the current price for the differential models will always have the propensity to move monotonically to the equilibrium prices defined for the system.

On the other hand, the current price for the delay-differential models tends to oscillate and move away from the initial price due to the delay associated with the supply, then with time decreases and turns towards to the defined system equilibrium prices.

The current prices in the delay-differential models are not predictable at the initial stages due the time-delay parameters associated with them. However, the current prices of differential models without delay could be predicted as such models do not oscillate and converge monotonically to the equilibrium price points defined to be operated by the system. Since most economic and natural phenomena are associated with delays, it is recommended that such systems are modeled using delay-differential equations to reflect realities of the phenomena.

REFERENCES

[1] Timmer C.Peter (1989). Food price policy-the rationale for government intervention, Butterworth & Co (Publishers) Ltd.
[2] Marek Bodnar (2004).”On the differences and similarities of the first order delay and ordinary differential equations” Journal of Mathematical Analysis and Applications 300,172–188.
[3] Liangliang W. and Jiguo C. (2011). “Estimating Parameters in Delay Differential Equation Models”Journal of Agricultural, Biological, and Environmental Statistics, Accessed 23 April 2018, available www.stat.sfu.ca/~cao/Research/EstimateDDE.pdf
[4] Nelson, P. (2006). “Research Statement”, Accessed 16 March 2018 Available www.math.lsa.umich.edu/~pwn/Research06.pdf.
[5] Driver R.D. (1977), “Ordinary and Delay Differential Equation”, Springer-Verlag, New York.
[6] KuangY. (1993). “Delay Differential Equations With Application in Population Dynamics”, Academic Press, London.
[7] Fridman, E. and Uri, S. (2003). “Special issue on time-delay systems”,Int. J. Robust Nonlinear Control 2003; Vol 13, pg. 791–792 (DOI: 10.1002/rnc.844).
[8] Grace, S. R. and Lalli, B. S. (1985).”Oscillation Theorems for Certain Delay Differential Inequalities” Journal of Mathematical Analysis and Applications, Vol 106, gp.414-426.
[9] Bodnar M. and Piotrowska M. J. (2010). “Delay differential equations: theory and applications”MathematicaApplicanda (MatematykaStosowana),Vol 38, No. 1 pg , 17-56.
[10] Smith, D. E. (1906). "History of Modern Mathematics: Differential Equations". University of South Florida.
[11] Soltes V.,Baculikova, B. and Dzurina J. (2012). “Oscillation in Price Adjustment Models” International Journal of Business and Social Science, Vol 3, No. 15.
[12] Falbo C. E. (1995). “Analytic and Numerical Solutions to the Delay Differential Equation-Revised” Sonoma State University, Accessed 19 April 2018 Available www.mathfile.net/My_San_Luis_Obispo_paper.pdf.

★★★