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ON THE EXISTENCE OF POSITIVE PERIODIC SOLUTIONS FOR TOTALLY NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS OF THE SECOND-ORDER WITH FUNCTIONAL DELAY

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Abstract. We prove that the totally nonlinear second-order neutral differential equation

$$\frac{d^2}{dt^2}x(t) + p(t)\frac{d}{dt}x(t) + q(t)h(x(t)) = \frac{d}{dt}c(t, x(t-\tau(t))) + f(t, \rho(x(t)), g(x(t-\tau(t))))$$

has positive periodic solutions by employing the Krasnoselskii-Burton hybrid fixed point theorem.

Keywords: Krasnoselskii, neutral, positive periodic solution.

Mathematics Subject Classification: 34K20, 45J05, 45D05.

1. INTRODUCTION

The study of existence of positive periodic solutions of neutral differential equations has gained the attention of many researchers in recent times, see [4,6,12,13,16,18,20].

We prove the existence of positive periodic solutions for the totally nonlinear second-order neutral differential equation of the form

$$\begin{aligned} &\frac{d^2}{dt^2}x(t) + p(t)\frac{d}{dt}x(t) + q(t)h(x(t)) = \\ &= \frac{d}{dt}c(t, x(t-\tau(t))) + f(t, \rho(x(t)), g(x(t-\tau(t)))), \end{aligned} \tag{1.1}$$

where p and q are positive continuous real-valued functions. The functions $f : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $c : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, and $h : \mathbb{R} \to \mathbb{R}$ are continuous in their respective arguments.

This work is mainly motivated by the papers [1, 2, 21] and particularly the work of Yankson in [20], in which the existence of positive periodic solutions of the second order neutral delay differential equation

$$\frac{d^2}{dt^2}x(t) + p(t)\frac{d}{dt}x(t) + q(t)x(t) = c\frac{d}{dt}x(t-\tau(t)) + f(t, \rho(x(t)), g(x(t-\tau(t)))) \quad (1.2)$$

is proved. The neutral term $\frac{d}{dt}c(t, x(t-\tau(t)))$ in (1.1) produces non-linearity in the derivative term $\frac{d}{dt}x(t-\tau(t))$, whereas the neutral term $\frac{d}{dt}x(t-\tau(t))$ in (1.2) enters linearly. Also, $h(x(t))$ in (1.1) is equal to $x(t)$ in (1.2), thus making (1.1) totally nonlinear. In view of the above differences between (1.1) and (1.2), our analysis is different from that in [20]. We refer to [5, 7–11, 14, 15], and [19] for results on some qualitative properties of neutral functional differential equations.

The rest of the paper is organized as follows. In Section 2, we provide some preliminary results needed in later sections. We also give the Green's function of (1.1), and provide without proof a statement of the Krasnoselskii-Burton hybrid fixed point theorem. Our main results are presented in Section 3.

2. PRELIMINARIES

For $T > 0$, let P_T be the set of continuous scalar functions x that are periodic in t, with period T. Then $(P_T, \|\cdot\|)$ is a Banach space with the supremum norm

$$\|x\| = \sup_{t\in\mathbb{R}} |x(t)| = \sup_{t\in[0,T]} |x(t)|.$$

In this paper we make the following assumptions:

$$p(t+T) = p(t), \quad q(t+T) = q(t), \quad \tau(t+T) = \tau(t), \quad (2.1)$$

with τ being a scalar function, continuous, and $\tau(t) \geq \tau^* > 0$. Also, we assume that

$$\int_0^T p(s)ds > 0, \quad \int_0^T q(s)ds > 0. \quad (2.2)$$

We also assume that $f(t, \rho, g)$ and $c(t, x)$ are periodic in t with period T, that is,

$$f(t+T, \rho, g) = f(t, \rho, g), \quad c(t+T, x) = c(t, x). \quad (2.3)$$

The following result is found in [13] and will be used to obtain the Green's function for (1.1).

Lemma 2.1. *Suppose that* (2.1) *and* (2.2) *hold and*

$$\frac{R_1[\exp(\int_0^T p(u)du) - 1]}{Q_1 T} \geq 1, \quad (2.4)$$

where

$$R_1 = \max_{t\in[0,T]} \left| \int_t^{t+T} \frac{\exp(\int_t^s p(u)du)}{\exp(\int_0^T p(u)du) - 1} q(s)ds \right|,$$

$$Q_1 = \left(1 + \exp\left(\int_0^T p(u)du\right)\right)^2 R_1^2.$$

Then there are continuous and T-periodic functions a and b such that $b(t) > 0$, $\int_0^T a(u)du > 0$, and

$$a(t) + b(t) = p(t), \quad \frac{d}{dt}b(t) + a(t)b(t) = q(t), \quad \textit{for } t \in \mathbb{R}.$$

The following result displays the Green's function, $G(t,s)$, which is found in [19] and is used in the inversion of (1.1).

Lemma 2.2. *Suppose the conditions of Lemma 2.1 hold and $\phi \in P_T$. Then the equation*

$$\frac{d^2}{dt^2}x(t) + p(t)\frac{d}{dt}x(t) + q(t)x(t) = \phi(t)$$

has a T-periodic solution. Moreover, the periodic solution can be expressed as

$$x(t) = \int_t^{t+T} G(t,s)\phi(s)ds,$$

where

$$G(t,s) = \frac{\int_t^s \exp[\int_t^u b(v)dv + \int_u^s a(v)dv]du + \int_s^{t+T} \exp[\int_t^u b(v)dv + \int_u^{s+T} a(v)dv]du}{[\exp\left(\int_0^T a(u)du\right) - 1][\exp\left(\int_0^T b(u)du\right) - 1]}.$$

The next result which is found in [19] contains properties of the Green's function, $G(t,s)$, needed in the inversion of (1.1) and in later sections.

Corollary 2.3. *Green's function G satisfies the following properties*

$$G(t,t+T) = G(t,t), \quad G(t+T,s+T) = G(t,s),$$

$$\frac{\partial}{\partial s}G(t,s) = a(s)G(t,s) - \frac{\exp\left(\int_t^s b(v)dv\right)}{\exp\left(\int_0^T b(v)dv\right) - 1},$$

$$\frac{\partial}{\partial t}G(t,s) = -b(t)G(t,s) + \frac{\exp\left(\int_t^s a(v)dv\right)}{\exp\left(\int_0^T a(v)dv\right) - 1}.$$

We next state and prove the following lemma which will play an essential role in obtaining our results.

Lemma 2.4. *Suppose* (2.1)–(2.4) *hold. If* $x \in P_T$*, then* x *is a solution of* (1.1) *if and only if*

$$\begin{aligned} x(t) = &\int_t^{t+T} G(t,s)q(s)[x(s) - h(x(s))]ds + \\ &+ \int_t^{t+T} \Big[c(s, x(s-\tau(s)))[E(t,s) - a(s)G(t,s)] + \\ &\qquad + G(t,s)f(s, \rho(x(s)), g(x(s-\tau(s))))\Big]ds, \end{aligned} \tag{2.5}$$

where

$$E(t,s) = \frac{\exp(\int_t^s b(v)dv)}{\exp(\int_0^T b(v)dv) - 1}. \tag{2.6}$$

Proof. Let $x \in P_T$ be a solution of (1.1). Rewrite (1.1) as

$$\begin{aligned} \frac{d^2}{dt^2}x(t) + p(t)\frac{d}{dt}x(t) + q(t)x(t) = q(t)[x(t) - h(x(t))] + \frac{d}{dt}c(t, x(t-\tau(t))) + \\ + \ f(t, \rho(x(t)), g(x(t-\tau(t)))). \end{aligned}$$

From Lemma 2.2 we have

$$\begin{aligned} x(t) = &\int_t^{t+T} G(t,s)q(s)[x(s) - h(x(s))]ds + \\ &+ \int_t^{t+T} G(t,s)\Big[\frac{\partial}{\partial s}c(s, x(s-\tau(s))) + f(s, \rho(x(s)), g(x(s-\tau(s))))\Big]ds. \end{aligned} \tag{2.7}$$

Integrating by parts, we have

$$\begin{aligned} \int_t^{t+T} G(t,s)\frac{\partial}{\partial s}c(s, x(s-\tau(s)))ds &= -\int_t^{t+T} \Big[\frac{\partial}{\partial s}G(t,s)\Big]c(s, x(s-\tau(s)))ds = \\ &= \int_t^{t+T} c(s, x(s-\tau(s)))[E(t,s) - a(s)G(t,s)]ds, \end{aligned} \tag{2.8}$$

where E is given by (2.6). Then substituting (2.8) in (2.7) completes the proof. ☐

The next result contains minimum and maximum values for obtaining bounds for the functions $G(t,s)$ and $E(t,s)$ and its proof is found in [19].

Lemma 2.5. *Let* $A = \int_0^T p(u)du$*,* $B = T^2 \exp\left(\frac{1}{T}\int_0^T \ln(q(u))du\right)$*. If*

$$A^2 \geq 4B, \tag{2.9}$$

then

$$\min\left\{\int_0^T a(u)du, \int_0^T b(u)du\right\} \geq \frac{1}{2}(A - \sqrt{A^2 - 4B}) := l,$$

$$\max\left\{\int_0^T a(u)du, \int_0^T b(u)du\right\} \leq \frac{1}{2}(A + \sqrt{A^2 - 4B}) := m.$$

The bounds for the functions $G(t,s)$ and $E(t,s)$ are given in the following result which is found in [19].

Corollary 2.6. *Functions G and E satisfy*

$$\frac{T}{(e^m - 1)^2} \leq G(t,s) \leq \frac{T \exp\left(\int_0^T p(u)du\right)}{(e^l - 1)^2}, \quad |E(t,s)| \leq \frac{e^m}{e^l - 1}.$$

To simplify notation, we introduce the constants

$$\beta = \frac{e^m}{e^l - 1}, \quad \alpha = \frac{T \exp\left(\int_0^T p(u)du\right)}{(e^l - 1)^2}, \quad \gamma = \frac{T}{(e^m - 1)^2}. \tag{2.10}$$

Next we provide definitions for large contraction and equicontinuity. We then state the Krasnoselskii-Burton hybrid fixed point theorem which constitutes a basis for our main result. We refer to [17] for Krasnoselskii's fixed point theorem.

Definition 2.7. Let $(\mathbb{M}, d)$ be a metric space and $B : \mathbb{M} \to \mathbb{M}$. B is said to be a large contraction if $\psi, \varphi \in \mathbb{M}$, with $\psi \neq \varphi$ then $d(B\varphi, B\psi) < d(\varphi, \psi)$ and if for all $\epsilon > 0$ there exists $\delta < 1$ such that

$$[\psi, \varphi \in \mathbb{M}, d(\varphi, \psi) \geq \epsilon] \Rightarrow d(B\varphi, B\psi) \leq \delta d(\varphi, \psi).$$

Definition 2.8. Let U be an interval on $\mathbb{R}$ and let $\{f_n\}$ be a sequence of functions with $f_n : U \to \mathbb{R}^d$. Denote by $|\cdot|$ any norm on $\mathbb{R}^d$. Then $\{f_n\}$ is equicontinuous if for any $\epsilon > 0$ there exist $\delta > 0$ such that $t_1, t_2 \in U$ and $|t_1 - t_2| < \delta$ imply $|f_n(t_1) - f_n(t_2)| < \epsilon$ for all n.

Theorem 2.9 ([3]). *Let $\mathbb{M}$ be a closed bounded convex non-empty subset of a Banach space $(S, \|\cdot\|)$. Suppose that A, B map $\mathbb{M}$ into $\mathbb{M}$ and that:*

(i) *for all $x, y \in \mathbb{M} \Rightarrow Ax + By \in \mathbb{M}$,*
(ii) *A is continuous and AM is contained in a compact subset of M,*
(iii) *B is a large contraction.*

Then there is a $z \in \mathbb{M}$ with $z = Az + Bz$.

For some non-negative constant K and a positive constant L we define the set

$$\mathbb{D} = \{\varphi \in P_T : K \leq \varphi \leq L\},$$

which is a closed convex and bounded subset of the Banach space P_T. In addition we assume that there exists non-negative constants σ, c^* and μ such that

$$\sigma < E(t,s) \text{ for all } (t,s) \in [0,T] \times [0,T], \tag{2.11}$$

$$c^* \le c(t, t-\tau(t)), \tag{2.12}$$

$$\|c(t,x)\| \le \mu, \tag{2.13}$$

$$\beta\mu T < L, \quad c^*\sigma T < K, \tag{2.14}$$

and for all $s \in \mathbb{R}$, $\phi, \varphi \in \mathbb{D}$

$$\frac{K - c^*\sigma T}{\gamma T} \le q(s)[\phi(s) - h(\phi(s))] + f(s, \rho(\varphi), g(\varphi)) - a(s)c(s,\varphi) \le \frac{L - \beta\mu T}{\alpha T}. \tag{2.15}$$

To apply Theorem 2.9 we define $\mathcal{A} : \mathbb{D} \to \mathbb{D}$ and $\mathcal{B} : \mathbb{D} \to \mathbb{D}$, respectively, by

$$\begin{aligned}(\mathcal{A}\varphi)(t) = \int_t^{t+T} \Big[&c(s, \varphi(s-\tau(s)))[E(t,s) - a(s)G(t,s)] + \\ &+ G(t,s) f(s, \rho(\varphi(s)), g(\varphi(s-\tau(s))))\Big] ds,\end{aligned} \tag{2.16}$$

and

$$(\mathcal{B}\varphi)(t) = \int_t^{t+T} G(t,s)q(s)[\varphi(s) - h(\varphi(s))]ds. \tag{2.17}$$

We end this section by making the following assumptions on the function $h : \mathbb{R} \to \mathbb{R}$. Let U represent the closed interval $[K, L]$. Then:

(H1) h is continuous and differentiable on U,
(H2) h is strictly increasing on U,
(H3) $\sup_{s \in U} h'(s) \le 1$.

3. EXISTENCE OF POSITIVE PERIODIC SOLUTIONS

In this section we present our main result. In order to establish our main result we first prove the following lemmas.

We begin this section by stating the following result which is found in [1], and is required for proving that $\mathcal{B}$ is a large contraction.

Lemma 3.1. *Let $h : \mathbb{R} \to \mathbb{R}$ be a function satisfying* (H1)–(H3) *and $\digamma = \varphi(s) - h(\varphi(s))$. Then $\digamma : \mathbb{D} \to \mathbb{D}$ is a large contraction on the set $\mathbb{D}$.*

The next result gives a relationship between the mappings $\digamma$ and $\mathcal{B}$ in the sense of a large contraction.

Lemma 3.2. *Suppose that condition* (2.15) *holds, then* $\mathcal{B}\varphi \in \mathbb{D}$ *for all* $\varphi \in \mathbb{D}$. *Moreover, if* $\digamma$ *is a large contraction on* $\mathbb{D}$, *and*

$$\alpha\|q\|T \leq 1, \tag{3.1}$$

then so is the mapping $\mathcal{B}$.

Proof. It is easy to check that $(\mathbb{B}\varphi)(t+T) = (\mathbb{B}\varphi)(t)$. Observe that if $c(t,x) = 0$ then in view of (2.12) and (2.13) we have that $c^* = \mu = 0$. Therefore, condition (2.15) with $f(t,\rho,g) = 0$ becomes

$$\frac{K}{\gamma T} \leq q(s)[\varphi(s) - h(\varphi(s))] \leq \frac{L}{\alpha T}. \tag{3.2}$$

Now, let $\varphi \in \mathbb{D}$, then

$$(\mathcal{B}\varphi)(t) \leq \alpha T\Big(\frac{L}{\alpha T}\Big) = L.$$

On the other hand,

$$(\mathcal{B}\varphi)(t) \geq \gamma T\Big(\frac{K}{\gamma T}\Big) = K.$$

Thus showing that $\mathcal{B}\varphi \in \mathbb{D}$ for all $\varphi \in \mathbb{D}$.

If $\digamma$ is a large contraction on $\mathbb{D}$, then for $x, y \in \mathbb{D}$, with $x \neq y$, we have $\|\digamma x - \digamma y\| \leq \|x - y\|$. Thus,

$$|\mathcal{B}x(t) - \mathcal{B}y(t)| \leq \alpha\|q\|T\|x - y\| \leq \|x - y\|.$$

Thus,

$$\|Bx - By\| \leq \|x - y\|.$$

Also, let $\epsilon \in (0,1)$. Then for the δ of the proof of Theorem 3.4 in [1], we have that $\|\digamma x - \digamma y\| \leq \delta\|x - y\|$. Thus,

$$\|\mathcal{B}x - \mathcal{B}y\| \leq \delta\alpha\|q\|T\|x - y\| \leq \delta\|x - y\|.$$

The proof is complete. □

Lemma 3.3. *Suppose that conditions* (2.1)–(2.3), *and* (2.11)–(2.15) *hold. Then the image of* $\mathcal{A}$ *is contained in a compact set and* $\mathcal{A}$ *is continuous.*

Proof. Let $\mathcal{A}$ be defined by (2.16). It is easy to see that $(\mathcal{A}\varphi)(t+T) = (\mathcal{A}\varphi)(t)$. Using Corollary 2.3, and conditions (2.13), (2.15) we obtain that for $t \in [0,T]$ and for $\varphi \in \mathbb{D}$

$$(\mathcal{A}\varphi)(t) \leq \mu\beta T + \alpha T\Big(\frac{L - \beta\mu T}{\alpha T}\Big) \leq L.$$

Also, in view of Corollary 2.3, and conditions (2.11), (2.12), (2.13) we have that

$$(\mathcal{A}\varphi)(t) \geq c^*\sigma T + \gamma T\Big(\frac{K - c^*\sigma T}{\gamma T}\Big) \geq K.$$

Thus $\mathcal{A}\varphi \in \mathbb{D}$ for all $\varphi \in \mathbb{D}$.

Moreover, Corollary 2.3 and conditions (2.13), (2.15) give

$$\begin{aligned}|(\mathcal{A}\varphi)(t)| \leq & \Big|\int_t^{t+T} c(s,\varphi(s-\tau(s)))E(t,s)ds\Big| + \\ & + \Big|\int_t^{t+T} G(t,s)\Big[f(s,h(\varphi(s)),g(\varphi(s-\tau(s)))) - a(s)c(s,\varphi(s-\tau(s)))\Big]ds\Big| \leq \\ & \leq \mu\beta T + \alpha T\Big(\frac{L-\beta\mu T}{\alpha T}\Big) \leq L.\end{aligned}$$

Thus from the estimation of $|(\mathcal{A}\varphi)(t)|$ we have that

$$\|\mathcal{A}\varphi\| \leq L.$$

This shows that $\mathcal{A}(\mathbb{D})$ is uniformly bounded. We next show that $\mathcal{A}(\mathbb{D})$ is equicontinuous by first computing $\frac{d}{dt}(\mathcal{A}\varphi_n(t))$. We obtain by taking the derivative in (2.14) that

$$\begin{aligned}\frac{d}{dt}(\mathcal{A}\varphi)_n(t) = & \frac{\exp\big(\int_t^{t+T} b(v)dv - 1\big)}{\exp\big(\int_0^T b(v)dv\big) - 1} c(t,\varphi_n(t-\tau(t))) + \\ & + \int_t^{t+T} c(s,\varphi_n(s-\tau(s)))\Big[-b(t)E(t,s) - a(s)\Big(-b(t)G(t,s) + \\ & \qquad + \frac{\exp(\int_t^s a(v)dv)}{\exp(\int_0^T a(v)dv) - 1}\Big)\Big]ds + \\ & + \int_t^{t+T}\Big(-b(t)G(t,s) + \frac{\exp(\int_t^s a(v)dv)}{\exp(\int_0^T a(v)dv) - 1}\Big) \times \\ & \qquad \times f(s,h(\varphi_n(s)),g(\varphi_n(s-\tau(s))))ds =\end{aligned}$$

$$= \frac{\exp\big(\int_t^{t+T} b(v)dv - 1\big)}{\exp\big(\int_0^T b(v)dv\big) - 1} c(t, \varphi_n(t-\tau(t)))+$$

$$+ \int_t^{t+T} c(s, \varphi_n(s-\tau(s)))\Big[-b(t)E(t,s) - a(s)\frac{\exp(\int_t^s a(v))dv}{\exp(\int_0^T a(v)dv) - 1}\Big]ds+$$

$$+ \int_t^{t+T} \frac{\exp(\int_t^s a(v)dv)}{\exp(\int_0^T a(v)dv) - 1} f(s, h(\varphi_n(s)), g(\varphi_n(s-\tau(s))))ds+$$

$$+ \int_t^{t+T} -b(t)G(t,s)\Big[f(s, h(\varphi_n(s)), g(\varphi_n(s-\tau(s))))-$$

$$- a(s)c(s, \varphi_n(s-\tau(s)))\Big]ds.$$

Consequently, by invoking (2.10), (2.13) and (2.15) we obtain

$$\Big|\frac{d}{dt}(\mathcal{A}\varphi)(t)\Big| \le \beta\mu + T\mu[\|b\|\beta + \|a\|\beta] + T\beta\Big(\frac{L-\beta\mu T}{\alpha T}\Big) + \|b\|\alpha T\Big(\frac{L-\beta\mu T}{\alpha T}\Big) \le M,$$

for some positive constant M. Hence $(\mathcal{A}\varphi)$ is equicontinuous. Then by the Ascoli-Arzelà theorem we obtain that $\mathcal{A}$ is a compact map. Due to the continuity of all the terms in (2.16), we have that $\mathcal{A}$ is continuous. This completes the proof. □

Theorem 3.4. *Let α, β and γ be given by* (2.10). *Suppose that conditions* (2.1)–(2.4), (2.11)–(2.15) *hold, then equation* (1.1) *has a positive periodic solution z satisfying $K \le z \le L$.*

Proof. Let $\varphi, \psi \in \mathbb{D}$. Using (2.16) and (2.17) we obtain

$$(\mathcal{B}\psi)(t) + (\mathcal{A}\varphi)(t) =$$

$$= \int_t^{t+T} G(t,s)q(s)[\psi(s) - h(\psi(s))]ds + \int_t^{t+T} \Big[c(s, \varphi(s-\tau(s)))[E(t,s) - a(s)G(t,s)]+$$

$$+ G(t,s)f(s, h(\varphi(s)), g(\varphi(s-\tau(s))))\Big]ds =$$

$$= \int_t^{t+T} c(s, \varphi(s-\tau(s)))E(t,s)ds+$$

$$+ \int_t^{t+T} G(t,s)\Big(q(s)[\psi(s) - h(\psi(s))]+$$

$$+ f(s, h(\varphi(s)), g(\varphi(s-\tau(s)))) - a(s)c(s, \varphi(s-\tau(s)))\Big)ds \le$$

$$\le \beta\mu T + \alpha T\Big(\frac{L-\beta\mu T}{\alpha T}\Big) = L.$$

On the other hand,

$$\begin{aligned}(\mathcal{B}\psi)(t)+(\mathcal{A}\varphi)(t) &= \int\limits_t^{t+T} c(s,\varphi(s-\tau(s)))E(t,s)ds+\\ &+\int\limits_t^{t+T} G(t,s)\Big(q(s)[\psi(s)-h(\psi(s))]+\\ &\qquad + f(s,h(\varphi(s)),g(\varphi(s-\tau(s))))-\\ &\qquad - a(s)c(s,\varphi(s-\tau(s)))\Big)ds \geq\\ &\geq c^*\sigma T+\gamma T\Big(\frac{K-c^*\sigma T)}{\gamma T}\Big)=K.\end{aligned}$$

This shows that $\mathcal{B}\psi+\mathcal{A}\varphi\in\mathbb{D}$. Thus all the hypotheses of Theorem 2.9 are satisfied and therefore equation (1.1) has a periodic solution in $\mathbb{D}$. This completes the proof. □

Remark 3.5. The problem considered in this paper can be extended to a system of totally nonlinear neutral functional delay equations.

Finally, we provide an example to illustrate our results.

4. EXAMPLE

Let $a(t)=\frac{2}{\pi}\cos^2(t)$ and $b(t)=\frac{2}{\pi}$. Then $p(t)=\frac{2}{\pi}\cos^2(t)+\frac{2}{\pi}$ and $q(t)=\frac{4}{\pi^2}\cos^2(t)$. Thus, the neutral second order differential equation

$$\begin{aligned}&\frac{d^2}{dt^2}x(t)+p(t)\frac{d}{dt}x(t)+q(t)x(t)=\\ &=\frac{1}{100\,000}\frac{d}{dt}\Big(\sin^2(t)x(t-\pi)\Big)+\frac{0.0006\cos^2(t)}{x^2(t-\pi)+1}+\frac{1}{10\,000}\end{aligned} \tag{4.1}$$

has a positive π periodic solution x satisfying $0\leq\|x\|\leq 10$. To see this, we have

$$f(u,\rho)=\frac{0.0006\cos^2(u)}{\rho^2+1}+\frac{1}{10\,000},\quad c(u,\rho)=\frac{1}{100\,000}\Big(\sin^2(u)\rho\Big),\quad \text{and } T=\pi.$$

A simple calculation yields $l=1, m=2, \beta=4.3, \mu=\frac{1}{200}$ and $\alpha=66.931\,71$.

Let $K = 0$, and $L = 10$ and define the set $\mathcal{D} = \{\phi \in P_\pi : 0 \leq \|\phi\| \leq 10\}$. Then for $\rho \in [0, 10]$ we have

$$\begin{aligned}
&q(u)[\phi(u) - h(\phi(u))] + f(u, h(\varphi), g(\varphi)) - a(u)c(u, \varphi) = \\
&= [-\sin(u) + \sin(u)(\cos(u) + 1)](\rho - \rho) + \frac{0.0006\cos^2(u)}{\rho^2 + 1} + \\
&\quad + \frac{1}{10000} - \frac{1}{100000}\sin^3(u)\rho \leq \\
&\leq 0.0006 + \frac{1}{10000} + \frac{10}{100000} = 0.0008 \leq \\
&\leq \frac{L - \beta\mu T}{\alpha T} = 0.047.
\end{aligned}$$

On the other hand, with $c^* = 0$ we have

$$\begin{aligned}
&q(u)[\phi(u) - h(\phi(u))] + f(u, h(\varphi), g(\varphi)) - a(u)c(u, \varphi) = \\
&= [-\sin(u) + \sin(u)(\cos(u) + 1)](\rho - \rho) + \frac{0.0006\cos^2(u)}{\rho^2 + 1} + \\
&\quad + \frac{1}{10000} - \frac{1}{100000}\sin^3(u)\rho \geq \\
&\geq \frac{0.0006}{101} + \frac{1}{10000} - \frac{10}{100000} = 0.000005941 \geq \\
&\geq \frac{K - c^*\sigma T}{\gamma T} = 0.
\end{aligned}$$

By Theorem 3.4, equation (4.1) has a positive π periodic solution x such that $0 \leq \|x\| \leq 10$.

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