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Positive Periodic Solutions in Neutral Delay Difference Equations

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Abstract

We use Krasnoselskii's fixed point theorem to obtain sufficient conditions for the existence of a positive periodic solution of the neutral delay difference equation

$$x(n+1) = a(n)x(n) + c\Delta x(n-\tau) + g(n, x(n-\tau)).$$

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1 Introduction

In this paper we use Krasnoselskii's fixed point theorem to study the existence of positive periodic solutions of a certain type of difference equation with delay which appear in ecological models. The existence of positive periodic solutions of functional differential equations has gained the attention of many researchers in recent times. We are mainly motivated by the work of the first author [5] and the references therein on neutral differential equations.

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Let τ be a nonnegative integer and consider the neutral delay difference equation

$$x(n+1) = a(n)x(n) + c\Delta x(n-\tau) + g(n, x(n-\tau)), \tag{1.1}$$

where g is continuous in x. The operator Δ is defined as $\Delta x(n) = x(n+1) - x(n)$. In this paper, we denote by $\mathbb{E}$ the shift operator defined as $\mathbb{E}x(n) = x(n+1)$. Also, the product of $x(n)$ from $n = a$ to $n = b$ is denoted by $\prod_{n=a}^{b} x(n)$. For more on the calculus of difference equation we refer the reader to [1] and [2]. In the continuous case, equations in the form of 1.1 have applications in food-limited populations, see biological [5] and the references therein. In [3], the first author considered a more complicated form of 1.1 and analyzed the existence of periodic solutions. On the other hand, the second author studied the boundedness of solutions and the stability of the zero solution. In [4], using cone theory, the first author obtained sufficient conditions that guaranteed the existence of multiple positive periodic solutions for the nonlinear delay difference equation

$$x(n+1) = a(n)x(n) \pm \lambda h(n) f(x(n-\tau(n))).$$

2 Preliminaries

We begin this section by introducing some notations. Let P_T be the set of all real T-periodic sequences, where T is an integer with $T \geq 1$. Then P_T is a Banach space when it is endowed with the maximum norm

$$\|x\| = \max_{n \in [0, T-1]} |x(n)|.$$

It is natural to ask for the periodicity condition

$$a(n+T) = a(T), \quad g(n+T, \cdot) = g(n, \cdot), \tag{2.1}$$

to hold for all $n \in \mathbb{Z}$. In addition to (2.1), we assume that

$$0 < a(n) < 1. \tag{2.2}$$

Let

$$G(n, u) = \frac{\prod_{s=u+1}^{n+T-1} a(s)}{1 - \prod_{s=n}^{n+T-1} a(s)}, \quad u \in [n, n+T-1]. \tag{2.3}$$

Note that the denominator in $G(n, u)$ is not zero since $0 < a(n) < 1$ for $n \in [0, T-1]$. Also, let

$$m := \min\{G(n, u) : n \geq 0, u \leq T\} = G(n, n) > 0, \tag{2.4}$$

$$M := \max\{G(n, u) : n \geq 0, u \leq T\} = G(n, n+T-1) = G(0, T-1) > 0. \tag{2.5}$$

Lemma 2.1. *Suppose* (2.1) *and* (2.2) *hold. If* $x(n) \in P_T$*, then* $x(n)$ *is a solution of* (1.1) *if and only if*

$$x(n) = cx(n-\tau) + \sum_{u=n}^{n+T-1} G(n,u)\Big[g(u, x(u-\tau)) - c(1-a(u))x(u-\tau)\Big], \quad (2.6)$$

where $G(n,u)$ *is defined by* (2.3).

Proof. Rewrite (1.1) as

$$\Delta\left[x(n)\prod_{s=0}^{n-1} a^{-1}(s)\right] = [c\Delta x(n-\tau) + g(n, x(n-\tau))]\prod_{s=0}^{n} a^{-1}(s). \quad (2.7)$$

Summing (2.7) from n to $n+T-1$, we obtain

$$\sum_{u=n}^{n+T-1} \Delta\Big[x(u)\prod_{s=0}^{u-1} a^{-1}(s)\Big] = \sum_{u=n}^{n+T-1} [c\Delta x(u-\tau) + g(u, x(u-\tau))]\prod_{s=0}^{u} a^{-1}(s),$$

i.e.,

$$\begin{aligned} x(n+T)\prod_{s=0}^{n+T-1} a^{-1}(s) - x(n)\prod_{s=0}^{n-1} a^{-1}(s) \\ = \sum_{u=n}^{n+T-1} [c\Delta x(u-\tau) + g(u, x(u-\tau))]\prod_{s=0}^{u} a^{-1}(s). \end{aligned}$$

Since $x(n+T) = x(n)$, we obtain

$$\begin{aligned} x(n)\left[\prod_{s=0}^{n+T-1} a^{-1}(s) - \prod_{s=0}^{n-1} a^{-1}(s)\right] \\ = \sum_{u=n}^{n+T-1} [c\Delta x(u-\tau) + g(u, x(u-\tau))]\prod_{s=0}^{u} a^{-1}(s). \end{aligned}$$

But

$$\begin{aligned} \sum_{u=n}^{n+T-1} c\Delta x(u-\tau)\prod_{s=0}^{u} a^{-1}(s) &= c\sum_{u=n}^{n+T-1} \mathbb{E}\Big[\prod_{s=0}^{u-1} a^{-1}(s)\Big]\Delta x(u-\tau) \\ &= cx(n-\tau)\Big[\prod_{s=0}^{n+T-1} a^{-1}(s) - \prod_{s=0}^{n-1} a^{-1}(s)\Big] - c\sum_{u=n}^{n+T-1} x(u-\tau)\Delta\Big[\prod_{s=0}^{u-1} a^{-1}(s)\Big] \\ &= cx(n-\tau)\left[\prod_{s=0}^{n+T-1} a^{-1}(s) - \prod_{s=0}^{n-1} a^{-1}(s)\right] \\ &\quad - \sum_{u=n}^{n+T-1} x(u-\tau)c[1-a(u)]\prod_{s=0}^{u} a^{-1}(s). \end{aligned}$$

Thus (2.7) becomes

$$x(n)\left[\prod_{s=0}^{n+T-1} a^{-1}(s) - \prod_{s=0}^{n-1} a^{-1}(s)\right] = cx(n-\tau)\left[\prod_{s=0}^{n+T-1} a^{-1}(s) - \prod_{s=0}^{n-1} a^{-1}(s)\right]$$
$$-\sum_{u=n}^{n+T-1} x(u-\tau)c\,[1-a(u)]\prod_{s=0}^{u} a^{-1}(s) + \prod_{s=0}^{u} a^{-1}(s)\sum_{u=n}^{n+T-1} g(u, x(u-\tau)).$$

Dividing both sides of the above equation by $\displaystyle\prod_{s=0}^{n+T-1} a^{-1}(s) - \prod_{s=0}^{n-1} a^{-1}(s)$ completes the proof. □

Now for $n \leq 0$, Equation (1.1) is equivalent to

$$\Delta\Big[x(n)\prod_{s=n-1}^{0} a^{-1}(s)\Big] = [c\Delta x(n-\tau) + g(n, x(n-\tau))]\prod_{s=n+1}^{0} a^{-1}(s).$$

Summing the above expression from n to $n+T-1$, we obtain (1.1) by a similar argument.

We next state Krasnoselskii's theorem in the following lemma.

Lemma 2.2 (Krasnoselskii). *Let $\mathbb{M}$ be a closed convex nonempty subset of a Banach space $(\mathbb{B}, \|\cdot\|)$. Suppose that C and B map $\mathbb{M}$ into $\mathbb{B}$ such that*

(i) $x, y \in \mathbb{M}$ *implies* $Cx + By \in \mathbb{M}$*;*

(ii) C *is continuous and* $C\mathbb{M}$ *is contained in a compact set;*

(iii) B *is a contraction mapping.*

Then there exists $z \in \mathbb{M}$ *with* $z = Cz + Bz$*.*

3 Main Results

In this section we obtain the existence of positive periodic solution of (1.1). For some nonnegative constant L and a positive constant K we define the set

$$\mathbb{M} = \{\phi \in P_T : \ L \leq \phi \leq K\}, \tag{3.1}$$

which is a closed convex and bounded subset of the Banach space P_T. In addition we assume that for all $u \in \mathbb{Z}$ and $\rho \in \mathbb{M}$,

$$\frac{(1-c)L}{mT} \leq g(u,\rho) - c[1-a(u)]\rho \leq \frac{(1-c)K}{MT}, \tag{3.2}$$

where m and M are defined by (2.4) and (2.5), respectively. We will treat separately the cases $0 \leq c < 1$ and $-1 < c \leq 0$. Thus, for our first theorem we assume

$$0 \leq c < 1. \tag{3.3}$$

To apply the theorem stated in Lemma 2.2, we will need to construct two mappings; one is contraction and the other is compact. In view of this we define the map $B : \mathbb{M} \to P_T$ by

$$(B\varphi)(n) = cx(n - \tau).$$

In a similar way we define the map $C : \mathbb{M} \to P_T$ by

$$(C\varphi)(n) = \sum_{u=n}^{n+T-1} G(n,u) \left[g(u, x(u-\tau)) - c(1-a(u))x(u-\tau) \right].$$

It is clear from condition (3.3) that B defines a contraction map under the supremum norm.

Lemma 3.1. *If* (2.1), (2.2), (3.2) *and* (3.3) *hold, then the operator C is completely continuous on* $\mathbb{M}$.

Proof. For $n \in [0, T-1]$ and for $\varphi \in \mathbb{M}$, we have by (3.2) that

$$\begin{aligned} |(C\varphi)(n)| &\leq \left| \sum_{u=n}^{n+T-1} G(n,u)[g(u, x(u-\tau)) - c(1-a(u))x(u-\tau)] \right| \\ &\leq TM \frac{(1-c)K}{MT} = (1-c)K. \end{aligned}$$

From the estimation of $|C\varphi(n)|$ it follows that

$$\|C\varphi\| \leq (1-c)K \leq K.$$

This shows that $C(\mathbb{M})$ is uniformly bounded. Due to the continuity of all terms, we have that C is continuous. Next, we show that A maps bounded subsets into compact sets. Let J be given, $S = \{\varphi \in P_T : \|\varphi\| \leq J\}$ and $Q = \{(C\varphi)(t) : \varphi \in S\}$. Then S is a subset of $\mathbb{R}^T$ which is closed and bounded and thus compact. As C is continuous in φ, it maps compact sets into compact sets. Therefore $Q = C(S)$ is compact. This completes the proof. $\square$

Theorem 3.2. *Suppose that* (2.1), (2.2), (3.2) *and* (3.3) *hold. Then equation* (1.1) *has a positive periodic solution z satisfying $L \leq \|z\| \leq K$.*

Proof. Let $\varphi, \psi \in \mathbb{M}$. Then, by (3.2), we have that

$$\begin{aligned}
&(A\varphi)(n) + (C\psi)(n) \\
&\quad = c\varphi(n-\tau) + \sum_{u=n}^{n+T-1} G(n,u)\Big[g(u,\psi(u-\tau)) - c(1-a(u))\psi(u-\tau)\Big] \\
&\quad \leq cK + M\sum_{u=n}^{n+T-1}\Big[g(u,\psi(u-\tau)) - c(1-a(u))\psi(u-\tau)\Big] \\
&\quad \leq cK + MT\frac{(1-c)K}{MT} = K.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&(A\varphi)(n) + (C\psi)(n) \\
&\quad = c\varphi(n-\tau) + \sum_{u=n}^{n+T-1} G(n,u)\Big[g(u,\psi(u-\tau)) - c(1-a(u))\psi(u-\tau)\Big] \\
&\quad \geq cL + m\sum_{u=n}^{n+T-1}\Big[g(u,\psi(u-\tau)) - c(1-a(u))\psi(u-\tau)\Big] \\
&\quad \geq cL + mT\frac{(1-c)L}{mT} = L.
\end{aligned}$$

This shows that $A\varphi + C\psi \in \mathbb{M}$. All the hypotheses of the theorem stated in Lemma 2.2 are satisfied and therefore equation (1.1) has a positive periodic solution, say z, residing in $\mathbb{M}$. This completes the proof. □

For the next theorem we substitute conditions (3.2) and (3.3) with

$$-1 < c \leq 0 \tag{3.4}$$

and for all $u \in \mathbb{R}$ and $\rho \in \mathbb{M}$

$$\frac{L-cK}{mT} \leq g(u,\rho) - c[1-a(u)]\rho \leq \frac{K-cL}{MT}, \tag{3.5}$$

where M and m are defined by (2.4) and (2.5), respectively.

Theorem 3.3. *If* (2.1), (2.2), (3.2), (3.4) *and* (3.5) *hold, then Equation* (1.1) *has a positive periodic solution* z *satisfying* $L \leq \|z\| \leq K$.

Proof. The proof follows along the lines of Theorem 3.2, and hence we omit it. □

4 Example

The neutral difference equation

$$x(n+1) = \frac{1}{8}x(n) + \frac{1}{10}\Delta x(n-4) + \frac{1}{x^2(n-4)+100} + \frac{7}{80}x(n-4) + \frac{1}{20} \tag{4.1}$$

has a positive periodic solution x of period 4 satisfying $\frac{1}{18428} \le \|\phi\| \le 2$. To see this, we have

$$g(u,\rho) = \frac{1}{\rho^2+100} + \frac{7}{80}\rho + \frac{1}{20},$$
$$a(n) = \frac{1}{8},\ c = \frac{1}{10},\ \text{ and } T = 4.$$

A simple calculation yields $M = \frac{4096}{4095}$ and $m = \frac{8}{4095}$. Let $K = 2$, $L = \frac{1}{18428}$, and define the set

$$M = \left\{ \phi \in P_4 :\ \frac{1}{18428} \le \phi \le 2 \right\}.$$

Then for $\rho \in \left[\frac{1}{18428}, 2\right]$ we have

$$\begin{aligned} g(u,\rho) - c[1-a(u)]\rho &= \frac{1}{\rho^2+100} + \frac{7}{80}\rho + \frac{1}{20} - \frac{1}{10}\left[1-\frac{1}{8}\right]\rho \\ &\le \frac{1}{100} + \frac{5}{20} = 0.26 < \frac{K(1-c)}{MT}. \end{aligned}$$

On the other hand,

$$\begin{aligned} g(u,\rho) - c[1-a(u)]\rho &= \frac{1}{\rho^2+100} + \frac{7}{80}\rho + \frac{1}{20} - \frac{1}{10}\left[1-\frac{1}{8}\right]\rho \\ &> \frac{7}{80}\rho + \frac{1}{20} - \frac{7}{80}\rho \\ &> \frac{7}{20}\frac{1}{18428} + \frac{1}{20} - \frac{7}{80}2 > \frac{L(1-c)}{mT}. \end{aligned}$$

By Theorem 3.2, equation (4.1) has a positive periodic solution x with period 4 such that $\frac{1}{18428} \le \|x\| \le 2$.

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