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Stability in delay Volterra difference equations of neutral type

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Abstract

Sufficient conditions for the zero solution of a certain class of neutral Volterra difference equations with variable delays to be asymptotically stable are obtained. The Banach's fixed point theorem is employed in proving our results.

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1. Introduction

The study of the stability of the zero solution of difference equations has gained the attention of many mathematicians lately, see [1], [2], [3], [5], [7], [9], [11] and [12]. In this paper we consider the nonlinear difference equation with variable delays

$$\begin{aligned}\Delta x(n) &= -\sum_{j=1}^{N} a_j(n)x(n-\tau_j(n)) + \sum_{j=1}^{N} \Delta Q_j(n, x(n-\tau_j(n)))\\ &\quad + \sum_{j=1}^{N} \sum_{s=n-\tau_j(n)}^{n-1} k_j(n,s) f_j(s, x(s)), \end{aligned} \tag{1.1}$$

with the initial condition

$$x(n) = \psi(n) \text{ for } n \in [m(n_0), n_0] \cap \mathbf{Z},$$

where $\psi : [m(n_0), n_0] \cap \mathbf{Z} \to \mathbf{R}$ is a bounded sequence and for $n_0 \geq 0$,

$$m_j(n_0) = \inf\{n - \tau_j(n),\ n \geq n_0\},\ m(n_0) = \min\{m_j(n_0),\ 1 \leq j \leq N\}.$$

Here Δ denotes the forward difference operator. That is, $\Delta x(n) = x(n+1) - x(n)$ for any sequence $\{x(n) : n \in \mathbf{Z}^+\}$. We assume throughout this paper that $a_j : \mathbf{Z}^+ \to \mathbf{R}$, $k_j : \mathbf{Z}^+ \times ([m_j(n_0), \infty) \cap \mathbf{Z}) \to \mathbf{R}$, $f_j : \mathbf{Z}^+ \times \mathbf{R} \to \mathbf{R}$, $Q_j : \mathbf{Z}^+ \times \mathbf{R} \to \mathbf{R}$ and $\tau_j : \mathbf{Z}^+ \to \mathbf{Z}^+$, for $j = 1, ..., N$. Special cases of (1.1) have been considered by a number of researchers in recent times.

For instance, Raffoul in [7] considered the equation

$$\Delta x(n) = -a(n)x(n-\tau), \tag{1.2}$$

where τ is a positive constant. The first author in [11], extended the results obtained in [7] to the equation

$$\Delta x(n) = -\sum_{j=1}^{N} a_j(n)x(n-\tau_j(n)). \tag{1.3}$$

The first author also in [12] considered the the following nonlinear Volterra difference equation

$$x(n+1) = a(n)x(n) + c(n)\Delta x(n-\tau(n)) + \sum_{s=n-\tau(n)}^{n-1} k(n,s)q(x(s)). \tag{1.4}$$

Ardjouni and Djoudi in [1] considered the nonlinear Volterra difference equation with variable delays

$$x(n+1) = a(n)x(n-\tau_1(n)) + c(n)\Delta x(n-\tau_2(n)) + \sum_{s=n-\tau_2(n)}^{n-1} k(n,s)q(x(s)). \tag{1.5}$$

Moreover, Ardjouni and Djoudi in [2] considered the difference equations with variable delays

$$\Delta x(n) = -\sum_{j=1}^{N} a_j(n)x(n-\tau_j(n)) + \sum_{j=1}^{N} c_j(n)\Delta x(n-\tau_j(n)). \tag{1.6}$$

Motivated by the above mentioned papers, we obtain in this paper sufficient conditions for the zero solution of (1.1) to be asymptotically stable.

2. Stability

Let $n_0 \in \mathbf{Z} \cap [0,\infty)$, be fixed. We let $D(n_0)$ be the set of bounded sequences $\psi : [m(n_0), n_0] \cap \mathbf{Z} \to \mathbf{R}$, with the norm $|\psi|_0 = \max\{|\psi(n)| : n \in [m(n_0), n_0] \cap \mathbf{Z}\}$. Also, let $(\mathbf{B}, ||.||)$ be the Banach space of bounded sequences $x : [m(n_0), \infty) \cap \mathbf{Z} \to \mathbf{R}$ with the maximum norm $||.||$.
In this paper we assume that for j=1,...,N,

$$|Q_j(n,x) - Q_j(n,y)| \leq L_1||x-y||, \tag{2.1}$$

and

$$|f_j(n,x) - f_j(n,y)| \leq L_2||x-y|| \tag{2.2}$$

for some positive constants L_1 and L_2. Also, for j=1,...,N,

$$f_j(n,0)=0, \quad Q_j(n,0)=0, \tag{2.3}$$

and

$$n-\tau_j(n)\to\infty \text{ as } n\to\infty. \tag{2.4}$$

Lemma 2.1 Let $h_j : [m(n_0),\infty)\cap \mathbf{Z} \to \mathbf{R}$ be an arbitrary sequence, for $j = 1,...,N$. Suppose that $H(n) = 1-\sum_{j=1}^{N} h_j(n) \neq 0$, for all $n \in [n_o,\infty)\cap \mathbf{Z}$. Then x is a solution of equation (1.1) if and only if

$$\begin{aligned} x(n) \;=\; & \Big[x(n_0)-\sum_{j=1}^{N} Q_j(n_0, x(n_0-\tau_j(n_0))) - \sum_{j=1}^{N}\sum_{s=n_0-\tau_j(n_0)}^{n_0-1} h_j(s)x(s)\Big] \\ & \prod_{u=n_0}^{n-1} H(u) \\ & +\sum_{j=1}^{N} Q_j(n, x(n-\tau_j(n))) + \sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1} h_j(s)x(s) \\ & +\sum_{s=n_0}^{n-1}\Big[\sum_{j=1}^{N}\{h_j(n-\tau_j(n))-a_j(n)\}x(n-\tau_j(n)) \\ & -[1-H(s)]\sum_{j=1}^{N} Q_j(s, x(s-\tau_j(s))) - [1-H(s)] \\ & \sum_{j=1}^{N}\sum_{r=s-\tau_j(s)}^{s-1} h_j(r)x(r) \\ & +\sum_{j=1}^{N}\sum_{u=s-\tau_j(s)}^{s-1} k_j(s,u)f_j(u,x(u))\Big]\prod_{u=s+1}^{n-1} H(u). \end{aligned} \tag{2.5}$$

Proof. Rewrite (1.1) as

$$\Delta x(n) \;=\; -\sum_{j=1}^{N} h_j(n)x(n) + \Delta_n \sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1} h_j(s)x(s)$$

$$+\sum_{j=1}^{N}\{h_j(n-\tau_j(n))-a_j(n)\}x(n-\tau_j(n))$$
$$+\sum_{j=1}^{N}\Delta Q_j(n,x(n-\tau_j(n)))$$
$$+\sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1}k_j(n,s)f_j(s,x(s)),$$

where Δ_n denotes the difference taken with respect to n.

The above equation is equivalent to

$$\begin{aligned} x(n+1) &= H(n)x(n)+\Delta_n\sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1}h_j(s)x(s) \\ &\quad+\sum_{j=1}^{N}\{h_j(n-\tau_j(n))-a_j(n)\}x(n-\tau_j(n)) \\ &\quad+\sum_{j=1}^{N}\Delta Q_j(n,x(n-\tau_j(n))) \\ &\quad+\sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1}k_j(n,s)f_j(s,x(s)). \end{aligned} \tag{2.6}$$

Rewrite equation (2.6) as

$$\begin{aligned} \Delta_n\Big[\prod_{u=n_0}^{n-1}H(u)^{-1}x(u)\Big] &= \Big[\Delta_n\sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1}h_j(s)x(s) \\ &\quad+\sum_{j=1}^{N}\{h_j(n-\tau_j(n))-a_j(n)\}x(n-\tau_j(n)) \\ &\quad+\sum_{j=1}^{N}\Delta Q_j(n,x(n-\tau_j(n))) \end{aligned} \tag{2.7}$$

$$+\sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1}k_j(n,s)f_j(s,x(s))\Big]\prod_{u=n_0}^{n}H(u)^{-1}.$$

Summing (2.7) from n_0 to $n-1$ we obtain

$$\begin{aligned}\sum_{s=n_0}^{n-1}\Delta_s\Big[\prod_{u=n_0}^{s-1}H(u)^{-1}x(s)\Big] &= \sum_{s=n_0}^{n-1}\Big[\Delta_s\sum_{j=1}^{N}\sum_{r=s-\tau_j(s)}^{s-1}h_j(r)x(r)\\ &\quad+\sum_{j=1}^{N}\{h_j(n-\tau_j(n))-a_j(n)\}x(n-\tau_j(n))\\ &\quad+\sum_{j=1}^{N}\Delta Q_j(n,x(n-\tau_j(n)))\\ &\quad+\sum_{j=1}^{N}\sum_{u=s-\tau_j(s)}^{s-1}k_j(s,u)f_j(u,x(u))\Big]\prod_{u=n_0}^{s}H(u)^{-1}.\end{aligned}$$

Consequently, we have

$$\begin{aligned}&\prod_{u=n_0}^{n-1}H(u)^{-1}x(n)-\prod_{u=n_0}^{n_0-1}H(u)^{-1}x(n_0)\\ &=\sum_{s=n_0}^{n-1}\Big[\Delta_s\sum_{j=1}^{N}\sum_{r=s-\tau_j(s)}^{s-1}h_j(r)x(r)\\ &+\sum_{j=1}^{N}\{h_j(n-\tau_j(n))-a_j(n)\}x(n-\tau_j(n))\\ &+\sum_{j=1}^{N}\Delta Q_j(n,x(n-\tau_j(n)))\\ &+\sum_{j=1}^{N}\sum_{u=s-\tau_j(s)}^{s-1}k_j(s,u)f_j(u,x(u))\Big]\prod_{u=n_0}^{s}H(u)^{-1}.\end{aligned}\tag{2.8}$$

Dividing both sides of (2.8) by $\prod_{u=n_0}^{n-1}H(u)^{-1}$ we obtain

$$\begin{aligned}x(n) &= x(n_0)\prod_{u=n_0}^{n-1}H(u)+\sum_{s=n_0}^{n-1}\Big[\Delta_s\sum_{j=1}^{N}\sum_{r=s-\tau_j(s)}^{s-1}h_j(r)x(r)\\ &\quad+\sum_{j=1}^{N}\{h_j(n-\tau_j(n))-a_j(n)\}x(n-\tau_j(n))\\ &\quad+\sum_{j=1}^{N}\Delta Q_j(n,x(n-\tau_j(n)))\end{aligned}$$

$$+\sum_{j=1}^{N}\sum_{u=s-\tau_j(s)}^{s-1}k_j(s,u)f_j(u,x(u))\Big]\prod_{u=s+1}^{n-1}H(u). \tag{2.9}$$

Using the summation by parts formula, we obtain

$$\sum_{s=n_0}^{n-1}\prod_{u=s+1}^{n-1}H(u)\Big[\sum_{j=1}^{N}\Delta Q_j(n,x(n-\tau_j(n)))\Big]$$
$$=\sum_{j=1}^{N}Q_j(n,x(n-\tau_j(n)))-\sum_{j=1}^{N}Q_j(n_0,x(n_0-\tau_j(n_0)))\prod_{u=n_0}^{n-1}H(u)$$
$$-\sum_{s=n_0}^{n-1}\sum_{j=1}^{N}Q_j(s,x(s-\tau_j(s)))[1-H(s)]\prod_{u=s+1}^{n-1}H(u), \tag{2.10}$$

and

$$\sum_{s=n_0}^{n-1}\prod_{u=s+1}^{n-1}H(u)\Big[\Delta_s\sum_{j=1}^{N}\sum_{r=s-\tau_j(s)}^{s-1}h_j(r)x(r)\Big]$$
$$=\sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1}h_j(s)x(s)-\prod_{u=n_0}^{n-1}H(u)\sum_{j=1}^{N}\sum_{s=n_0-\tau_j(n_0)}^{n_0-1}h_j(s)x(s)$$
$$-\sum_{s=n_0}^{n-1}[1-H(s)]\prod_{u=s+1}^{n-1}H(u)\sum_{j=1}^{N}\sum_{r=s-\tau_j(s)}^{s-1}h_j(r)x(r). \tag{2.11}$$

Substituting (2.10) and (2.11) into (2.9) gives the desired results.

Theorem 2.1 Suppose (2.1), (2.2), (2.3) and (2.4) hold and let $h_j : [m(n_0),\infty)\cap\mathbf{Z}\to\mathbf{R}$ be an arbitrary sequence, for $j=1,...,N$, such that $H(n)=1-\sum_{j=1}^{N}h_j(n)\neq 0$, for all $n\in[n_o,\infty)\cap\mathbf{Z}$. Suppose further that there exist a constant $\alpha\in(0,1)$ such that

$$NL_1+\sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1}|h_j(s)|$$
$$+\sum_{s=n_0}^{n-1}\Big[\sum_{j=1}^{N}|h_j(n-\tau_j(n))-a_j(n)|$$

$$+|1-H(s)|NL_1+|1-H(s)|\sum_{j=1}^{N}\sum_{r=s-\tau_j(s)}^{s-1}|h_j(r)|$$

$$+L_2\sum_{j=1}^{N}\sum_{u=s-\tau_j(s)}^{s-1}|k_j(s,u)|\Big]\Big|\prod_{u=s+1}^{n-1}H(u)\Big|\leq\alpha. \tag{2.12}$$

Moreover, assume that there exist a positive constant G such that

$$\Big|\prod_{u=n_0}^{n-1}H(u)\Big|\leq G, \tag{2.13}$$

and

$$\prod_{u=n_0}^{n-1}H(u)\to 0 \text{ as } n\to\infty. \tag{2.14}$$

Then the zero solution of (1.1) is asymptotically stable.

Proof. Let $\epsilon>0$ be given. Choose $\delta>0$ such that

$$\delta G[1+\alpha]+\epsilon\alpha\leq\epsilon.$$

Let $\psi\in D(n_0)$ such that $|\psi(n)|\leq\delta$ and define

$$S=\{\varphi\in B:\ \varphi(n)=\psi(n)\ if\ n\in[m(n_0),n_0]\cap\mathbf{Z},$$
$$||\varphi||\leq\epsilon \text{ and } \varphi(n)\to 0 \text{ as } n\to\infty\}.$$

Then $(S,||.||)$ is a complete metric space where, $||.||$ is the maximum norm.

Define the mapping $P:S\to S$ by

$$(P\varphi)(n)=\psi(n) \text{ for } n\in[m(n_0),n_0]\cap\mathbf{Z},$$

and

$$(P\varphi)(n)=\Big[\psi(n_0)-\sum_{j=1}^{N}Q_j(n_0,\psi(n_0-\tau_j(n_0)))-\sum_{j=1}^{N}\sum_{s=n_0-\tau_j(n_0)}^{n_0-1}h_j(s)\psi(s)\Big]$$
$$\prod_{u=n_0}^{n-1}H(u)$$

$$
\begin{aligned}
&+\sum_{j=1}^{N} Q_j(n,\varphi(n-\tau_j(n))) + \sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1} h_j(s)\varphi(s) \\
&+\sum_{s=n_0}^{n-1}\Big[\sum_{j=1}^{N}\{h_j(n-\tau_j(n)) - a_j(n)\}\varphi(n-\tau_j(n)) \\
&-[1-H(s)]\sum_{j=1}^{N} Q_j(s,\varphi(s-\tau_j(s))) - [1-H(s)]\sum_{j=1}^{N}\sum_{r=s-\tau_j(s)}^{s-1} h_j(r)\varphi(r) \\
&+\sum_{j=1}^{N}\sum_{u=s-\tau_j(s)}^{s-1} k_j(s,u) f_j(u,\varphi(u))\Big]\prod_{u=s+1}^{n-1} H(u),\ n \ge n_0.
\end{aligned}
$$

(2.15)

Clearly, $P\varphi$ is continuous. We first show that $P : S \to S$. Using (2.15) we obtain

$$
\begin{aligned}
|(P\varphi)(n)| \le\ & \delta G[1+\alpha] + \Big\{NL_1 + \sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1} |h_j(s)| \\
& + \sum_{s=n_0}^{n-1}\Big[\sum_{j=1}^{N} |h_j(n-\tau_j(n)) - a_j(n)| \\
& + |1-H(s)|NL_1 + |1-H(s)|\sum_{j=1}^{N}\sum_{r=s-\tau_j(s)}^{s-1} |h_j(r)| \\
& + L_2\sum_{j=1}^{N}\sum_{u=s-\tau_j(s)}^{s-1} |k_j(s,u)|\Big]\Big|\prod_{u=s+1}^{n-1} H(u)\Big|\Big\}||\varphi|| \\
\le\ & \delta G[1+\alpha] + \alpha\epsilon \\
\le\ & \epsilon.
\end{aligned}
$$

We next show that $(P\varphi)(n) \to 0$ as $n \to \infty$. The first term on the right hand side of (2.15) goes to zero in view of condition (2.14). Since $\varphi(n) \to 0$ and $n - \tau_j(n) \to \infty$ as $n \to \infty$, we have that $Q_j(n, \varphi(n - \tau_j(n))) \to Q_j(n, 0) = 0$ as $n \to \infty$ for $j = 1, ..., N$. Thus showing that the second term on the right hand side of (2.15) goes to zero as $n \to \infty$.

Let $\varphi \in S$ be fixed. The fact that $\varphi(n) \to 0$ and $n - \tau_j(n) \to \infty$ as $n \to \infty$, implies that, given $\epsilon_1 > 0$ there exists $N_1 > n - \tau_j(n)$ for $j = 1, ..., N$ such that $|\varphi(s)| \leq \epsilon_1$ for $s \geq N_1$. Thus

$$\begin{aligned}\Big|\sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1} h_j(s)\varphi(s)\Big| &\leq \epsilon_1 \sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1} |h_j(s)| \\ &\leq \alpha\epsilon_1 < \epsilon_1.\end{aligned}$$

Thus showing that the third term on the right hand side of (2.15) goes to zero as $n \to \infty$. We next show that the last term on the right hand side of (2.15) goes to zero as $n \to \infty$. Since $\varphi(n) \to 0$ and $n - \tau_j(n) \to \infty$ as $n \to \infty$, for each $\epsilon_2 > 0$, there exists $N_2 > n_0$ such that $s \geq N_2$ implies $|\varphi(s - \tau_j(s))| < \epsilon_2$ for $j = 1, ..., N$. Thus for $n \geq N_2$, the last term on the right hand side of (2.15) satisfies

$$\begin{aligned}&\Big|\sum_{s=n_0}^{n-1}\Big[\sum_{j=1}^{N}\{h_j(n-\tau_j(n)) - a_j(n)\}\varphi(n-\tau_j(n)) \\ &-[1-H(s)]\sum_{j=1}^{N} Q_j(s,\varphi(s-\tau_j(s))) - [1-H(s)]\sum_{j=1}^{N}\sum_{r=s-\tau_j(s)}^{s-1} h_j(r)\varphi(r) \\ &+\sum_{j=1}^{N}\sum_{u=s-\tau_j(s)}^{s-1} k_j(s,u)f_j(u,\varphi(u))\Big]\prod_{u=s+1}^{n-1} H(u)\Big| \\ &\leq \sum_{s=n_0}^{N_2-1}\Big[\sum_{j=1}^{N}|h_j(s-\tau_j(s)) - a_j(s)||\varphi(s-\tau_j(s))| \\ &+|1-H(s)|L_1\sum_{j=1}^{N}|\varphi(s-\tau_j(s))| + |1-H(s)|\sum_{j=1}^{N}\sum_{r=s-\tau_j(s)}^{s-1}|h_j(r)||\varphi(r)| \\ &+L_2\sum_{j=1}^{N}\sum_{u=s-\tau_j(s)}^{s-1}|k_j(s,u)||\varphi(u)|\Big]\Big|\prod_{u=s+1}^{n-1} H(u)\Big| \\ &+\sum_{s=N_2}^{n-1}\Big[\sum_{j=1}^{N}|h_j(s-\tau_j(s)) - a_j(s)||\varphi(s-\tau_j(s))| \\ &\\ &+|1-H(s)|L_1\sum_{j=1}^{N}|\varphi(s-\tau_j(s))| + |1-H(s)|\sum_{j=1}^{N}\sum_{r=s-\tau_j(s)}^{s-1}|h_j(r)||\varphi(r)|\end{aligned}$$

$$+ L_2\sum_{j=1}^{N}\sum_{u=s-\tau_j(s)}^{s-1}|k_j(s,u)||\varphi(u)|\Big]\Big|\prod_{u=s+1}^{n-1}H(u)\Big|$$

$$\leq \max_{\sigma\geq m(n_0)}\varphi(\sigma)\sum_{s=n_0}^{N_2-1}\Big[\sum_{j=1}^{N}|h_j(s-\tau_j(s))-a_j(s)|$$

$$+|1-H(s)|L_1N+|1-H(s)|\sum_{j=1}^{N}\sum_{r=s-\tau_j(s)}^{s-1}|h_j(r)|$$

$$+ L_2\sum_{j=1}^{N}\sum_{u=s-\tau_j(s)}^{s-1}|k_j(s,u)|\Big]\Big|\prod_{u=s+1}^{n-1}H(u)\Big|$$

$$+ \epsilon_2\sum_{s=N_2}^{n-1}\Big[\sum_{j=1}^{N}|h_j(s-\tau_j(s))-a_j(s)|$$

$$+|1-H(s)|L_1N+|1-H(s)|\sum_{j=1}^{N}\sum_{r=s-\tau_j(s)}^{s-1}|h_j(r)|$$

$$+ L_2\sum_{j=1}^{N}\sum_{u=s-\tau_j(s)}^{s-1}|k_j(s,u)|\Big]\Big|\prod_{u=s+1}^{n-1}H(u)\Big|$$

$$\leq \epsilon_2+\epsilon_2\alpha < 2\epsilon_2.$$

Thus showing that the last term on the right hand side of (2.15) goes to zero as $n\to\infty$. Therefore, $(P\varphi)\to 0$ as $n\to\infty$. It therefore follows that P maps S into S.

We finally show that P is a contraction. Let $\varphi,\eta\in S$. Then

$$\begin{aligned}|(P\varphi)(n)-(P\eta)(n)| \leq\ & \Big\{NL_1+\sum_{j=1}^{N}\sum_{s=n-\tau_j(n)}^{n-1}|h_j(s)|\\ &+\sum_{s=n_0}^{n-1}\Big[\sum_{j=1}^{N}|h_j(n-\tau_j(n))-a_j(n)|\\ &+|1-H(s)|NL_1+|1-H(s)|\sum_{j=1}^{N}\sum_{r=s-\tau_j(s)}^{s-1}|h_j(r)|\end{aligned}$$

$$+ L_2\sum_{j=1}^{N}\sum_{u=s-\tau_j(s)}^{s-1}|k_j(s,u)|\Big]\Big|\prod_{u=s+1}^{n-1}H(u)\Big|\Big\}||\varphi-\eta||$$

$$\leq \alpha||\varphi-\eta||.$$

This shows that P is a contraction. Therefore, by the contraction mapping principle, P has a unique fixed point in S which solves (1.1) and for any $\varphi \in S$, $||P\varphi|| \leq \epsilon$. This shows that the zero solution of (1.1) is stable. Moreover, $(P\varphi) \to 0$ as $n \to \infty$, showing that the zero solution of (1.1) is asymptotically stable. This completes the proof.

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