UNIVERSITY OF CAPE COAST

STABILITY OF A COMPLETELY DELAYED DYNAMIC EQUATION ON TIME SCALE WITH VARIABLE DELAYS

BY

ANASS ABU AHMAD

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DECLARATION

Candidate's Declaration

I hereby declare that this thesis is the result of my own original research and that no part of it has been presented for another degree in this university or elsewhere.

Candidate's Signature .... Date ....

Name: Anass Abu Ahmad

Supervisor's Declaration

I hereby declare that the preparation and presentation of the thesis were supervised in accordance with the guidelines on supervision of thesis laid down by the University of Cape Coast.

Supervisor's Signature .... Date ....

Name: Prof. Ernest Yankson

ABSTRACT

This thesis is about the stability of completely delayed dynamic equations on time scale.

Fixed point theory is used to study the stability properties of the zero solution of dynamic equations on time scales. In particular, the Banach fixed point theorem is used in this thesis.The dynamic equation is inverted into an equivalent integral dynamic equation and a suitable define mapping is defined based on the equivalent integral dynamic equation which is then used to discuss the stability properties of the zero solution of the dynamic equation. Sufficient conditions are obtained for the zero solution of completely delayed dynamic equations to be asymptotically stable.

KEY WORDS

Dynamic equations

Delay equations

Fixed points

Stability

Variable delays

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DEDICATION

To my family

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CHAPTER ONE

INTRODUCTION

This chapter gives an introduction and background to the study. It also briefly presents the problem statement and outlines the objectives of the study. The chapter is concluded with information on the organization of the thesis.

Background to the Study

The subject dynamic equations on time scale is fairly a new and emerging area of mathematics which describes the theories of both differential and difference equations under one domain. It has attracted a lot of attention by researchers in recent times. The concept was introduced in 1988 by the German Mathematician Stephan Hilger in his Ph.D thesis according to Bohner and Peterson (2001), to allow for the unification and extension of differential and difference equations. Many results of problems concerning differential equations carry over quite easily to corresponding results for difference equations , while other results have a completely different structure from their continuous counterparts. Thus the study of dynamic equations on time scale exposes these discrepancies between differential and difference equations and this helps prevent one from proving results twice, once for differential equations and once for difference equations. In dealing with equations provided by this concept the idea is to obtain a result for a dynamic equation where the domain of the unknown function is a so-called time scale which is an arbitrary closed subsets of reals. For instance, If one chooses the time scale to be the set of reals, we have a continuous case and hence the general results obtained yields the same results concerning an ordinary differential equation. On the other hand, if one chooses the time scale to be the set of integers, we have a discrete case and hence the general results obtained are the same results one would obtain concerning a difference equation. However, since there are many other time scales that one may work with besides the set of reals and integers, one has a more general results.

Bohner and Peterson (2001) and Agarwal et al. (2002) together with their collaborators authored books which deal extensively on the concept of time scale caculus. Differential equations are essential tools in scientific modeling of physical problems which is important in almost every sphere of human endeavor from Engineering, Physical Sciences, Medical Sciences, Agricultural Science to Social sciences.

Therefore, the analysis of qualitative properties of solutions of these equations is necessary for applications. It is therefore important to develop new and efficient methods as well as modify and refine known techniques and adjusting them for the analysis of new classes of problems.
Many mathematical models arising from Engineering, Mechanics, Physics and Social sciences usually involve delays in which the derivative or difference of the past history of the unknown functions are involved as well as those of the present state of the system.

There have been a lot of research activity concerning qualitative theory of dynamic equations on time scales. The first person to carry out a major investigation in the line of qualitative theory was Henry Poincare as noted by Boyer and Merzbach (2011): This was the basis of development of the qualitative theory of differential and difference equations which have important applications in diverse areas such as Engineering,Economics,Physical and Biological sciences.

The investigations of qualitative properties of the solutions of dynamic equations on time scales usually seek to find out about the;
(i)Boundedness,
(ii)Periodicity,
(iii)Stability, and

(iv) Positivity of the solutions.

One of the key areas of qualitative analysis of the dynamic equations is studying the stability of solutions which seeks to harmonize the stability of the continuous and discrete equations. Stability theory of dynamic equation on time scales is very important in the study of qualitative properties of solutions of dynamic equations on time scales and this occurs when a small change in initial data results in a small change in behaviour for future time.

Stability analysis of solutions of these dynamic equations on time scales usually require the use of a wide variety of approaches and mathematical tools. The most fundamental concept for studies in stability of dynamic equations was introduced by Lyapunov in the 19th century. One technique which has widely been the main tool used to study the stability of differential equations, difference equations and dynamic equations on time scales is the Lyapunov's Direct Method. This involves the construction of a suitable function called the Lyapunov function which is positive definite and whose derivative is negative definite.Nevertheless, the application of this method to problem of stability in differential equations with delay usually encounters serious difficulties if the delay is unbounded or if the equation has unbounded term. It has been noticed by Burton (2003) and Ardjouni and Djoudi (2011) that some of theses difficulties vanish by using the fixed point technic. Other advantages of fixed point theory over Lyapunov's method is that the conditions of the former are average while those of the latter are pointwise. This thesis basically investigates the stability of a completely delayed dynamic equation on time scales with variable delays. A delayed differential equation is one in which the derivative of the previous history is involved, as well as those of the present state of the system. Similarly, a difference equation is one in which the difference of the previous history are involved , as well as those of the present state of the system.

Statement of problem

The theory of time scale was developed basically to unify continuous and discrete analysis. Yankson (2009) studied the stability conditions of the solutions of the difference equation

$$x^{\triangle}(t) = -\sum_{i=1}^{N} a_i(t)x(t-\tau_i(t)), \quad t \in \mathbb{T}, \tag{1.1}$$

with variable delays. In particular the fixed point theory was used. Adıvar and Raffoul (2009) studied the dynamic equation

$$x^{\triangle}(t) = -a(t)x(\delta(t))\delta^{\triangle}(t)$$

by means of the fixed point theorem and obtained results for the asymptotic stability of the zero solutions of Equation (1.1).
However, the problem solved by Yankson (2009) holds for only the time scale $\mathbb{T} = \mathbb{Z}$ and the results obtained by Adıvar and Raffoul (2009) holds for a single constant delay dynamic equation.

Research Objectives/Research Questions

The objectives of this thesis are to obtain sufficient conditions for the zero solution of the dynamic Equation (1.1) on time scale to be

(i) Stable,

(ii) Asymptotically stable.

Significance of the Study

This study aims at setting out ways to solve hybrid(time scale calculus) problems by means of the fixed point theorem.Thus the results obtained can be applied in modelling of problems that arecontinuous at some point and also dicrete at some point.

Delimitations

This thesis actually looked into the stability properties of dynamic delay equations on time scale with variable delays using the fixed point theorem.It also looked at time scale calculus which study seeks to bring continuous and discrete calculus under one umbrella.

Limitations

The fixed point theorem was used in this study.it required that the dynamic equation is inverted into an equivalent integral dynamic equation on time scale. When a correct integral dynamic equation is not found, then it will lead to wrong conclusions when the fixed point theorem is used.

Organisation of the Study

This thesis is structured as follows:
This Chapter presents the background of the research including its objectives and statement of the problem. In Chapter Two, Relevant literature for our investigation of dynamic equations on time scale is reviewed. In Chapter Three, we give a general overview of time scale calculus by stating some definitions, theorems and lemmas. In the fifth section of the Chapter, we state fixed point theorems as well as provide some details on how fixed point theorems are used to study qualitative properties. In Chapter Four, criteria for the zero solution of the dynamic equations to be stable and asymptotically stable are obtained.

In Chapter Five, the summary and conclusions of the thesis are given.

Chapter Summary

This chapter introduces the thesis, by first giving the motivation for studying the problem contained in this thesis.I then move on to state the related problem, announce the research objectives and the importance of the results both practically and mathematically.I conclude the chapter by describing the structure of this thesis.

CHAPTER TWO

LITERATURE REVIEW

Introduction

This chapter reviews literature in the areas of dynamic equations on time scale. This starts with the ground breaking exploits of Hilger and some related works on stability of dynamic equations. The background concepts on time scales are taken from Bohner and Peterson (2001) and Agarwal et al. (2002).

Dynamic Equations on Time Scale

The study of qualitative behaviour of dynamic equations on time scale has received a lot of attention by researchers in recent years since the work of Stefan Hilger came to light in 1988. Agarwal et al. (2002) in a study of dynamic equations on time scales outlined some conditions of functions on an arbitrary time scale and used it to solve linear dynamic equations of first order. The fundamental method that has widely been used to study the qualitative properties of solutions of dynamic equations is the Lyapunov method. This method involves the construction a positive definite function which is usually represented as $V(t,x)$ and whose delta-derivative $V^{\triangle}(t,x)$ is negative definite. The Lyapunove method has by far been the general method used in studying dynamic equations on time scales. For instance, Hoffacker and Tisdell (2005) made studies on the stability and instability of the first order dynamic equation $x^{\triangle} = f(t,x)$ on time scales. The Lyapunov functions were used to develop an invariance principle regarding the solutions to the above dynamic equation.

Also, Adivar and Raffoul (2011) used the Lyapunov method to study the stability of the dynamic equation

$$x^{\triangle} = a(t)x(t) + b(t)x(\delta_-(h,t))\delta_-^{\triangle}(h,t). \tag{2.1}$$

In the same way, Akin-Bohner and Raffoul (2006) used nonnegative definite Lyapunov functionals and proved theorems for the boundedness of functional dynamic equations on time scales.
Raffoul (2006) cosidered the equation

$$\triangle x(n) = a(n)x(n-\tau), \tag{2.2}$$

and obtained sufficient conditions for the zero solution for the equation to be asymptotically stable. By using matrix-valued functions in dynamic equations on time scales, Bohner and Martynyuk (2007) obtained sufficient conditions ensuring stability of dynamic equations on time scales. This matrix-valued functions makes it possible to construct heterogenous Lyapunov functions(ie functions with both continuous and discrete components).
However the construction of an appropriate lyapunov function usually pose a challenge and also there is a problem with the type of conditions which are imposed on the dynamic equation.
Burton and Furumochi (2002) noticed in their study that a number of dificulties encountered when the Lyapunov method is used to study stability of solutions are overcome or disappear when the fixed point theory is used instead.
Islam and Yankson (2005) dealt with the stability and boundedness of the zero solution of the nonlinear difference equation

$$x(t+1) = a(t)x(t) + c(t)\triangle x(t-g(t)) + q(x(t), x(t-g(t))), \tag{2.3}$$

The authors used the fixed point theorem and established asymptotic Stability of the zero solution of the equation.

Yankson (2009) also considered the difference equation

$$\triangle x(n) = -a(n)x(n - \tau(n)) \tag{2.4}$$

and its generalization

$$\triangle x(n) = -\sum_{i=1}^{N} a(n)x(n - \tau_j(n)), \tag{2.5}$$

with variable delays. The author obtained sufficient conditions for the difference equation to be asymptotically stable using the fixed point theory. In the same way Kaufmann and Raffoul (2007) also dealt with the nonlinear dynamic equation on time scale

$$x^{\triangle}(t) = -a(t)x^{\sigma}(t) + (Q(t, x(t), x(t - g(t))))^{\triangle} + Gt, x(t), x(t - g(t)), \tag{2.6}$$

The contraction mapping principle was used to establish asymptotic stability of the zero solution of the dynamic equation provided $Q(t, 0, 0) = G(t, 0, 0) = 0$.

Adıvar and Raffoul (2009) studied the stability and periodicity of the dynamic delay equation on time scale and obtained by means of the fixed point theorem sufficient conditions for the stability and periodicity of the dynamic equation on time scale of the form

$$x^{\triangle}(t) = -a(t)x(\delta(t))\delta^{\triangle}(t). \tag{2.7}$$

Chapter Summary

In this chapter,some literature on stability of dynamic equations is reviewed.

CHAPTER THREE

TIME SCALE CALCULUS AND METHODOLOGY

Introduction

This chapter captures the methods and tools utilized in this study. These methods are theoretical and analytical in nature and limited to time scales calculus. In this chapter we outline central concepts and definitions of the time scale calculus initiated by Hilger in 1988 under the supervision of Bernd Aulbach. Attention is given to the concepts such as continuity, Rd-Continuity, differentiability which are relevant in the analysis of continuous and discrete systems.

The time scale calculus

The following defnitions and theorems, as well as a general introduction to the theory, can be found in the text by Bohner and Peterson (2001).

Definition 1

A time scale is an arbitrary nonempty closed subset of the real.

Thus

$$\mathbb{R}, \mathbb{Z}, \mathbb{N}, \mathbb{N}_{\o},$$

that is, the real numbers, the integers, the natural numbers, and the non-negative integers are examples of time scales.

Definition 2

Let T be a time scale. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma(t) : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := inf\{s \in \mathbb{T} : s > t\}$$

while the backward jump operator $\rho(t) : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := sup\{s \in \mathbb{T} : s < t\}.$$

In the case that $s \in \mathbb{T} : s > t$ is empty, put $\inf \phi = \sup \mathbb{T}$ (that is, $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t). Similarly, if $s \in \mathbb{T} : s < t$ is empty, we put $\sup \phi = \inf \mathbb{T}$ (that is, $\rho(t) = t$ if $\mathbb{T}$ has a minimum t). If $f : \mathbb{T} \to \mathbb{R}$ is a function, we define the function $f^{\sigma} : \mathbb{T} \to \mathbb{R}$ by $f^{\sigma}(t) = f(\sigma(t))$ for all $t \in \mathbb{T}$. Points are classified as follows: If $\sigma(t) > t$, we say t is right-scattered, while if $\rho(t) < t$ we say t is left-scattered. Also, if $t < supT$ and $\sigma(t) = t$, then t is said to be right-dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. Points that are right-scattered and left-scattered at the same time are called isolated, and points that are both right and left dense are called dense.

The graininess function, $\mu : \mathbb{T} \to [0; \infty)$, is defined by $\mu(t) := \sigma(t) - t$.

Example 1

Consider the examples $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.

(i) If $\mathbb{T} = \mathbb{R}$, then we have for any $t \in \mathbb{R}$

$$\sigma(t) = \inf\{s \in \mathbb{R} : s > t\} = \inf\,(t, \infty)\,,$$

and similarly $\rho(t) = t$. Hence every point $t \in \mathbb{R}$ is dense. The graininess function μ turns out to be

$$\mu(t) \equiv 0 \;\; for\ all \;\; t \in \mathbb{T}$$

(ii) If $\mathbb{T} = \mathbb{Z}$ then we have $t \in \mathbb{Z}$

$$\sigma(t) = \inf\{s \in \mathbb{Z} : s > t\} = \inf\{t+1, t+2, t+3, ...\} = t + 1,$$

and similarly $\rho(t) = t - 1$. Hence every point $t \in \mathbb{Z}$ is isolated. The graininess function μ in this case is

$$\mu(t) \equiv 1 \ \ for\ all \ \ t \in \mathbb{T}.$$

Differentiation

Definition 3

Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we define $f^{\triangle}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighbourhood $\mathbb{U}$ of t (that is, $\mathbb{U} = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$| [f(\sigma(t) - f(s))] - f^{\triangle}(t)[\sigma(t) - s] | \leq \varepsilon[\sigma(t) - s]$$

We call $f^{\triangle}(t)$ the delta (or Hilger) derivative of f at t.
Moreover, we say that f is delta (or Hilger) differentiable (or in short: differentiable) on $\mathbb{T}^k$ provided $f^{\triangle}(t)$ exists for all $t \in \mathbb{T}^k$. The function $f^{\triangle} : \mathbb{T}^k \to \mathbb{R}$ is then called the (delta) derivative of f on $\mathbb{T}^k$.

Theorem 1. *The delta derivative is well defined.*

Proof. Let $t \in \mathbb{T}^k$ and $f_i^{\triangle}(t), i = 1, 2,$ be such that

$$| f(\sigma(t)) - f(s) - f_1^{\triangle}(t)(\sigma(t) - s) | \leq \frac{\epsilon}{2} | \sigma(t) - s |,$$

and

$$| f(\sigma(t)) - f(s) - f_2^{\triangle}(t)(\sigma(t) - s) | \leq \frac{\epsilon}{2} | \sigma(t) - s |,$$

for all $\epsilon > 0$ and all s belonging to a neighborhood U of t, $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0, s \neq \sigma(t)$. Hence

$$
\begin{aligned}
&\mid f_1^{\triangle}(t) - f_2^{\triangle}(t) \mid \\
&= \left| f_1^{\triangle}(t) - \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} + \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} - f_2^{\triangle}(t) \right| \\
&\leq \left| f_1^{\triangle}(t) - \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} \right| + \left| \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} - f_2^{\triangle}(t) \right| \\
&= \frac{\mid f(\sigma(t)) - f(s) - f_1^{\triangle}(t)(\sigma(t) - s) \mid}{\mid \sigma(t) - s \mid} + \frac{\mid f(\sigma(t)) - f(s) - f_1^{\triangle}(t)(\sigma(t) - s) \mid}{\mid \sigma(t) - s \mid} \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}
$$

Since $\epsilon > 0$ was chosen arbitrarily, we conclude that

$$f_1^{\triangle}(t) = f_2^{\triangle}(t),$$

which completes the proof.

Theorem 2. *Assume $f : \mathbb{T} \to \mathbb{R}$ are functions and let $t \in \mathbb{T}^k$. Then we have the following:*

(i) If f is differentiabble at t, then f is continuous at t.

(ii) If f is continuous at t and t is right-scattered, then f is differentiable at t with

$$f^{\triangle}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(iii) If t is right-dense, then f is differentiable at

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s},$$

exists as a finite number. In this case,

$$f^{\triangle}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

(iv) If f is differentiable at t, then $f(\sigma(t)) = f(t) + \mu(t)f^{\triangle}(t)$.

Proof. Part (i). Assume that f is differentiable at t. Let $\varepsilon \in (0,1)$. Define $\varepsilon^* = \varepsilon[1+ \mid f^{\triangle}(t) \mid +2\mu]^{-1}$. Then, $\varepsilon^* \in (0,1)$. By Definition 3 there exists a neighbourhood U of t such that

$$\mid [f(\sigma(t) - f(s))] - f^{\triangle}(t)[\sigma(t) - s] \mid \leq \varepsilon^*[\sigma(t) - s]$$

for all $s \in U$. Therefore we have for all $s \in U \cap (t - \varepsilon^*, t + \varepsilon)$

$$\begin{aligned}
&\mid f(t) - f(s)\mid \\
&=\mid \{f(\sigma(t)) - f(s) - f^{\triangle}(t)[\sigma(t) - s]\} \\
&- \{f(\sigma(t)) - f(t) - \mu(t)f^{\triangle}(t)\} + (t - s)f^{\triangle}(t) \mid \\
&\leq \varepsilon^* \mid \sigma(t) - s \mid +\varepsilon^*\mu(t)+ \mid (t - s) \mid\mid f^{\triangle}(t) \mid \\
&\leq \varepsilon^*\mu(t)+ \mid t - s \mid +\mu(t)+ \mid f^{\triangle}(t) \mid] \\
&= \varepsilon.
\end{aligned}$$

It follows that f is continuous at t.

Part (ii). Assume f is continuous at t and t is right-scattered. By continuity

$$\lim_{s\to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} = \frac{f(\sigma(t)) - f(t)}{\mu(t)}$$

Hence, given $\varepsilon > 0$,there is a neighborhood U of t such that

$$\left| \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} - \frac{f(\sigma(t)) - f(t)}{\mu(t)} \right| < \varepsilon$$

for all $s \in U$. It follows that

$$\left| [f(\sigma(t)) - f(s)] - \frac{f(\sigma(t)) - f(t)}{\mu(t)} \right| \leq \varepsilon \mid \sigma(t) - s \mid$$

for all $s \in U$. Hence we get the desired result

$$f^{\triangle}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

Part (iii). Assume f is differentiable at t and t is right-dense. Let $\varepsilon > 0$ be given. Since f is differentiable at t, there is a neighbourhood U of t such that

$$\mid [f(\sigma(t)) - f(s)] - f^{\triangle}(t)[\sigma(t) - s]\varepsilon \mid \sigma(t) - s \mid \quad for\ all\ s \in U$$

Since $\sigma(t) = t$ we have that

$$\mid [f(t) - f(s)] - f^{\triangle}(t)(t - s) \mid \leq \varepsilon \mid t - s \mid \quad for\ all\ s \in U.$$

It follows that

$$\left| \frac{f(t) - f(s)}{t - s} - f^{\triangle}(t) \right| < \varepsilon$$

for all $s \in U$, $s \neq t$. Therefore we get the desired result

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

Part (iv). If $\sigma(t) = t$, then $\mu(t) = 0$ and we have that

$$f(\sigma(t)) = f(t) = f(t) + \mu(t) f^{\triangle}(t).$$

On the other hand if $\sigma(t) > t$, then by (ii)

$$\begin{aligned} f(\sigma(t)) &= f(t) + \mu(t).\frac{f(\sigma(t)) - f(t)}{\mu(t)} \\ &= f(t) + \mu(t) f^{\triangle}(t), \end{aligned}$$

and the proof of part (iv) is complete. Next we would like to find the derivatives of sums, products, and quotients of differentiable functions. This is possible according to the following theorem.

Theorem 3. *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$. Then:*

(i) The sum $f + g : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(f+g)^{\triangle}(t) = f^{\triangle}(t) + g^{\triangle}(t).$$

(ii) For any constant $\alpha, \alpha f : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(\alpha f)^{\triangle}(t) = \alpha f^{\triangle}(t).$$

(iii) The product $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$\begin{aligned}(fg)^{\triangle}(t) &= f^{\triangle}(t)g(t) + f(\sigma(t))g^{\triangle}(t)\\ &= f(t)g^{\triangle}(t) + f^{\triangle}(t)g(\sigma(t)).\end{aligned}$$

(iv) If $f(t)f(\sigma(t)) \neq 0$, then $\frac{1}{f}$ is differentiable at t with

$$\left(\frac{1}{f}\right)^{\triangle}(t) = -\frac{f^{\triangle}(t)}{f(t)f(\sigma(t))}.$$

(v) If $g(t)g(\sigma(t)) \neq 0$, then fg is differentiable at t and

$$\left(\frac{f}{g}\right)^{\triangle}(t) = \frac{f^{\triangle}(t)g(t) - f(t)g^{\triangle}(t)}{g(t)g(\sigma(t))}.$$

Proof. Assume that f and g are delta differentiable at $t \in \mathbb{T}^k$.

Part (i). Let $\varepsilon > 0$. Then there exist neighbourhoods U_1 and U_2 of t with

$$\mid f(\sigma(t)) - f(s) - f^{\triangle}(t)(\sigma(t) - s) \mid \leq \frac{\epsilon}{2} \mid \sigma(t) - s \mid, \quad \textit{for all } s \in U_1$$

and

$$| g(\sigma(t)) - g(s) - g^{\triangle}(t)(\sigma(t) - s) | \leq \frac{\epsilon}{2} | \sigma(t) - s |, \quad for\ all\ s \in U_2$$

Let $U = U_1 \cap U_2$. Then we have for all $s \in U$

$$\begin{aligned}
&| (f+g)(\sigma(t)) - (f+g)(s) - [f^{\triangle}(t) + g^{\triangle}(t)](\sigma(t) - s)| \\
&= | f(\sigma(t)) - f(s) - f^{\triangle}(t)(\sigma(t) - s) + g(\sigma(t)) - g(s) - g^{\triangle}(t)(\sigma(t) - s) | \\
&\leq | f(\sigma(t)) - f(s) - f^{\triangle}(t)(\sigma(t) - s) | + | g(\sigma(t)) - g(s) - g^{\triangle}(t)(\sigma(t) - s) | \\
&\leq \frac{\varepsilon}{2} | \sigma(t) - s | + \frac{\varepsilon}{2} | \sigma(t) - s | \\
&= \varepsilon | \sigma(t) - s |
\end{aligned}$$

Therefore $f + g$ is differentiable at t and $(f+g)^{\triangle} = f^{\triangle} + g^{\triangle}$ holds at t.

Part (iii). Let $\varepsilon \in (0, 1)$. Define $\varepsilon^* = \varepsilon[1 + | f(t) | + |g(\sigma(t)) | + | g^{\triangle}(t) |]^{-1}$. Then $\varepsilon^* \in (0, 1)$ and hence there exist neighbourhoods U_1, U_2, and U_3 of t such that

$$| [f(\sigma(t) - f(s))] - f^{\triangle}(t)[\sigma(t) - s] | \leq \varepsilon^*[\sigma(t) - s]$$

for all $s \in U_1$ and

$$| [f(\sigma(t) - f(s))] - f^{\triangle}(t)[\sigma(t) - s] | \leq \varepsilon^*[\sigma(t) - s]$$

for all $s \in U_2$ and

$$| f(t) - f(s) | \leq \varepsilon^*$$

for all $s \in U_3$. Put $U = U_1 \cap U_2 \cap U_3$ and let $s \in U$. Then

$$| (fg)(\sigma(t)) - (fg(s) - [f^{\triangle}(t)g(\sigma(t)) + f(t)g^{\triangle}(t)](\sigma(t) - s) |$$

$$=| [f(\sigma(t)) - f(s) - f^{\triangle}(t)(\sigma(t) - s)]g(\sigma(t)) + [g(\sigma(t)) - g(s) - g^{\triangle}(t)(\sigma(t) - s)]f(t) + [g$$

$$\times (\sigma(t) - s)][f(s) - f(t)] + (\sigma(t) - s)g\triangle(t)[f(s) - f(t)]|$$

$$\leq \varepsilon^* \mid \sigma(t) - s \mid\mid g(\sigma(t)) \mid + \varepsilon^* \mid \sigma(t) - s \mid\mid f(t) \mid + \varepsilon^* \varepsilon^* \mid \sigma(t) - s \mid$$

$$+ \varepsilon^* \mid \sigma(t) - s \mid\mid g^{\triangle}(t) \mid$$

$$= \varepsilon^* \mid \sigma(t) - s \mid [\mid g(\sigma(t)) \mid + \mid f(t) \mid + \varepsilon^* + \mid g^{\triangle}(t) \mid]$$

$$\leq \varepsilon^* \mid \sigma(t) - s \mid [1 + \mid f(t) \mid + \mid g(\sigma(t)) \mid + \mid g^{\triangle}(t) \mid]$$

$$= \varepsilon \mid \sigma(t) - s \mid .$$

Thus $(fg)^{\triangle} = f^{\triangle} g^{\sigma} + fg^{\triangle}$ holds at t.

Theorem 4. *Let $fg : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable. Then $fog : \mathbb{T} \to \mathbb{R}$ is delta differentiable on $\mathbb{T}^k$ and the formula*

$$(fog)^{\triangle}(t) = \left\{ \int_0^1 f'(g(t) + h\mu(t)g^{\triangle}(t))dh \right\} g^{\triangle}(t)$$

holds for $t \in \mathbb{T}^k$

Integration

Definition 4

A function $f : \mathbb{R} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in T and its left-sided limits exist (finite) at all left dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{R} \to \mathbb{R}$ will be denoted in this thesis by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$. The set of functions $f : \mathbb{R} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by

$$C'_{rd} = C'_{rd}(\mathbb{T}) = C'_{rd}(\mathbb{T}, \mathbb{R}).$$

Theorem 5. *(i) If f is continuous, then f is rd-continuous.*

ii) If f is rd-continuous then f is regulated.

(iii) The jump operator σ is rd-continuous.

(iv) If f is regulated or rd-continuous, then so is f^{σ}.

(v) Assume f is continuous. If $g : \mathbb{T} \to \mathbb{R}$ is regulated or rd-continuous, then fog has that property too.

Theorem 6. *(Existence of Pre-Antiderivatives) Let f be regulated. Then there exists a function F which is pre-differentiable with region of differentiation D such that $F^{\triangle}(t) = f(t)$ holds for all $t \in D$.*

Definition 5

Assume $f : \mathbb{T} \to \mathbb{R}$ is a regulated function. Any function F as in Theorem 6 is called a pre-antiderivative of f . We define the indefinite integral of a regulated function f by

$$\int f(t)\triangle = F(t) + C,$$

where C is an arbitrary constant and F is a pre-antiderivative of f . We define the Cauchy integral by

$$\int_r^s f(t)\triangle t = F(s) - F(r)$$

for all $r, s \in \mathbb{T}$.

A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided

$$F^{\triangle}(t) = f(t)$$

holds for all $t \in \mathbb{T}^k$

Definition 6

If $a \in \mathbb{T}$, $\sup \mathbb{T} = \infty$, and f is rd- continuous on $[0, \infty)$, then we define the improper integral by

$$\int_a^{\infty} f(t)\triangle t := \lim_{b\to\infty}\int_a^b f(t)\triangle t$$

provided this limit exists, and we say that the improper integral converges in this case. If this limit does not exist, then we say that the improper integral diverges.

Theorem 7. *(Chain Rule). Assume that $\nu : \mathbb{R} \to \mathbb{R}$ is strictly increasing and $\mathbb{T} = \nu(\mathbb{T})$ is a time scale. Let $\omega : \mathbb{T} \to \mathbb{R}$. If $\nu^{\triangle}(t)$ and $\omega^{\triangle}(\nu(t))$ exist for $t \in \mathbb{T}^k$, then $(\omega\ o\ \nu)^{\triangle} = (\omega^{\triangle}\ o\ \nu)\nu^{\triangle}$.*

Definition 7

A function $F : \mathbb{R} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{R} \to \mathbb{R}$ provided $F^{\triangle}(t) = f(t)$ *for all* $t \in \mathbb{T}^k$. Then we define the Cauchy integral by

$$\int_a^b f(t)\triangle t = f(b) - f(a), \quad \textit{for all } t \in \mathbb{T}.$$

Theorem 8. *Every rd-continuous function has an antiderivative. In particular, if $t_0 \in \mathbb{T}$, then F defined by*

$$F(t) := \int_{t_0}^{t} f(\tau)\triangle\tau \ \textit{for } t \in \mathbb{T}$$

is an antiderivative of f.

Theorem 9. *(Substitution) Assume $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is an rd-continuous function and ν is differentiable with rd-continuous derivative, then for $a, b \in \mathbb{T}$,*

$$\int_a^b f(t)\nu^{\triangle}(t) = \int_{\nu(a)}^{\nu(b)} (f\ o\ \nu^{-1})(s)\triangle s$$

The following theorem provides useful properties of delta integrals.

Theorem 10. *If $a, b, c \in \mathbb{T}, \alpha \in \mathbb{R}$ and $f, g \in C_{rd}$ then*

(i) $\int_a^b [f(t) + g(t)]\triangle t = \int_a^b f(t)\triangle t + \int_a^b g(t)\triangle t$;

(ii) $\int_a^b (\alpha f)(t)\triangle t = \alpha \int_a^b f(t)\triangle t$;

(iii) $\int_a^b f(t)\triangle t = -\int_b^a f(t)\triangle t$;

(iv) $\int_a^b f(t)\triangle t = \int_a^c f(t)\triangle t + \int_c^b f(t)\triangle t$;

(v) $\int_a^b f(\sigma(t))g^{\triangle}(t)\triangle t = fg(b) - fg(a) - \int_a^b f^{\triangle}(t)g(t)\triangle t$;

(vi) $\int_a^b f(t)g^{\triangle}(t)\triangle t = fg(b) - fg(a) - \int_a^b f^{\triangle}(t)g(\sigma(t))\triangle t$;

(vii) $if \mid f(t) \mid \leq g(t)$ *on* $[a, b)$, *then* $\mid \int_a^b f(t)\triangle t \mid \leq \int_a^b g(t)\triangle t$;

(viii) $f(t) \geq 0$ *for all* $a \leq t < b$ *then* $\int_a^b f(t)\triangle t \geq 0$;

(ix) $\int_a^a f(t)\triangle t = 0$.

Definition 8

A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathbb{R}$ while the set $\mathbb{R}^+$ is given by $\Re^+ = \Big\{ f \in \mathbb{R} : \mu(t)f(t) > 0 \; for \; all \; t \in \mathbb{T} \Big\}$.

Definition 9

Let $p \in \mathbb{R}$ and $\mu(t) \neq 0$ for all $t \in \mathbb{T}$. The exponential function on $\mathbb{T}$ is defined by

$$e_p(t, s) = exp\Big(\int_s^t \frac{1}{\mu(z)} \log(1 + \mu(z)p(z))\triangle z \Big),$$

It is well known that if $p \in \mathbb{R}^+$, then $e_p(t, s) > 0$ for all $t \in \mathbb{T}$. Also, the exponential function $y(t) = e_p(t, s)$ is the solution to the initial value problem $y^{\triangle} = p(t)y, y(s) = 1$. Other properties of the exponential function are given in the following lemma.

Lemma 1

Let $p, q \in \mathbb{R}$. Then

(i) $e_0(t, s) \equiv 1$ and $e_p(t, t) \equiv 1$;

(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t)e_p(t, s)$;

(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$ where, $\ominus p(t) = -\frac{p(t)}{(1+\mu(t)p(t))}$;

(iv) $e_p(t,s) = \frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$;

(v) $e_p(t,s)e_p(s,r) = e_p(s,t)$;

(vi) $\left(\frac{1}{e_p(.,s)}\right)^{\triangle} = -\frac{p(t)}{e_p^{\sigma}(.,s)}$

Next we consider the first order non-homogeneous linear equation

$$y^{\triangle} = p(t)y + f(t)$$

and the corresponding homogeneous equation $y^{\triangle} = p(t)y$ on a time scale $\mathbb{T}$.

Definition 10

If $p \in \Re$ and $f : \mathbb{T} \rightarrow \mathbb{R}$ is rd-continuous, then the dynamic equation $y^{\triangle}(t) = p(t)y(t) + f(t)$ is called regressive.

Theorem 11. *(Variation of Constants). Suppose $p \in \Re$ and $f \in C_{rd}$. Let $t_0 \in \mathbb{T}$ is fixed in $\mathbb{T}$ and $y(t_0) = y_0 \in \mathbb{R}$, then the unique solution to the first order dynamic equation on $\mathbb{T}$*

$$y^{\triangle}(t) = p(t)y(t) + f(t)\ y(t_0) = y_0;$$

exists and is given by

$$y(t) = y_0 e_p(t,t_0) + \int_{t_0}^{t} e_p(t,\sigma(\tau))f(\tau)\triangle\tau$$

Definition 11

The zero solution of a dynamic equation on time scale $\mathbb{T}$ is said to be stable if, for every $t_0 \in \mathbb{T}$ and for every $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that, for any solution $x(t, t_0, x_0)$ of the dynamic equation, the inequality $|| x_0 || < \delta$ implies $|| x || < \varepsilon$,for all $t \geq t_0$, $t_0 \in \mathbb{T}$.

Definition 12

The zero solution of a dynamic equation on time scale $\mathbb{T}$ is said to be asymptotically stable if it is stable and for every $t_0 \in \mathbb{T}$, there exist a $\delta = \delta(t_0) > 0$ such that the inequality $|| x_0 || < \delta$ implies

$$\lim_{t\to\infty} | x(t) | = 0.$$

.

Definition 13

The function $f : \mathbb{T} \to \mathbb{R}$ is said to be continuous at $t_0 \in \mathbb{T}$ for all $\varepsilon > 0$, if there exists a neighbourhood $N_\varepsilon(t_0)$ such that $| f(t) - f(t_0) | < \varepsilon$ for all $N_\varepsilon(t_0)$

The next Lemma is a result obtained by Adıvar and Raffoul (2009) which will be used to establish the proof in the next chapter.

Lemma 2 (Adıvar and Raffoul (2009))

Let $f : \mathbb{T} \to \mathbb{R}$ *be an rd − continuous function, Then*

$$\int_{\delta(t)}^{t} f(s)\triangle s = f(t) - f(\delta(t))\delta^{\triangle}(t).$$

Proof. Case 1. Let $\sigma(\delta(t)) \neq t$. Then $\sigma(\delta(t)) < t$. Thus, there exists a constant $\tau_0 \in [\delta(t), t) \cap \mathbb{T}$ such that $\sigma(\delta(t)) = \tau_0$. The result is immediate from

$$\int_{\delta(t)}^{t} f(s)\triangle t = \int_{\delta(t)}^{\tau_0} f(s)\triangle t + \int_{\tau_0}^{t} f(s)\triangle s$$

Case 2. Let $\sigma(\delta(t)) = t$. Hence, we arrive at

$$
\begin{aligned}
[\int_{\delta(t)}^{t} f(s)\triangle s]^{\triangle} &= [\mu(\delta(t)f(\delta(t)))]^{\triangle} \\
&= [(\sigma(\delta(t)) - \delta(t)f(\delta(t))]^{\triangle} \\
&= (1 - \delta(t))f(\delta(t)) + (\sigma(t) - \delta(\sigma(t)))[f(\delta(t))]^{\triangle} \\
&= f(\delta(t) - \delta^{\triangle} f(\delta(t) + \mu(t)[f(\delta(t))]^{\triangle} \\
&= f(\delta(t)) - \delta^{\triangle} f(\delta(t)) + f(\delta(t)) - f(\delta(t)) \\
&= f(t) - \delta^{\triangle} f(\delta(t))
\end{aligned}
$$

This completes the proof.

Research Approach

The fixed point theorem is the main approach that will be used in this study. Thus we give appropriate definitions and theorems that will be relevant in this thesis. A fixed point of a function is an element of the function's domain that is mapped to itself by the function. A set of fixed points is sometimes called a fixed set. That is to say, c is a fixed point of the function $f(x)$ if and only if $f(c) = c$.

Many different kinds of problems can be solved by means of fixed point theory. Generally, to solve a problem with fixed point theory is to find:

(a) a set S consisting of points which would be acceptable solutions;

(b) a mapping $P : \mathbb{S} \rightarrow \mathbb{S}$ with the property that a fixed point solves the problem;

(c) a fixed point theorem stating that this mapping on this set will have a fixed point.

Definition 14

A pair (S, ρ) is a metric space if S is a set and $\rho : S \times S \rightarrow [0, \infty)$ such that when y, z, and u are in S then

(a) $\rho(y,z) \geq 0, \rho(y,y) = 0$ and $\rho(y,z) = 0$ implies $y = z$,

(b) $\rho(y,z) = \rho(z,y)$, and

(c) $\rho(y,z) \leq \rho(y,u) + \rho(u,z)$.

Definition 15

A fixed point of a function $\mathbb{T} : \mathbb{X} \to \mathbb{X}$ is a point $x \in \mathbb{X}$ such that $Tx = x$.

Definition 16

A vector space $(V, +, .)$ is a normed vector space if for each $x, y \in V$ there is a nonnegative real number $\| x \|$ called the norm of x,such that

(a) $\| x \| = 0$ if and only if $x = 0$,

(b) $\| \alpha x \| = | \alpha | \| x \|$ for each $\alpha \in \mathbb{R}$,

(c) $\| x + y \| \leq \| x \| + \| y \|$.

Definition 17

A banach space is a complete normed space.

Theorem 12. *Let (X, d) be a nonempty complete metric space and $T : X \to X$ is a contraction mapping, if there exist a constant α with $0 \leq \alpha < 1$ such that $d(T(x), T(y)) \leq \alpha d(x, y)$ for all $x, y \in X$, then T has a unique fixed point x such that $T(x) = x$.*

Chapter Summary

This chapter is firstly concerned about providing some definitions, theorems and lemmas together with some proofs. We also gave a general overview of the fixed point theory which will be our main tool for analyzing our results.

CHAPTER FOUR

RESULTS AND DISCUSSION

In this chapter we state and prove the main results of the thesis.The dynamic equation is inverted into an equivalent integral dynamic equation and a mapping is defined based on the integral equation. The stability results of the dynamic equation considered in this thesis are then proven.

Let $\mathbb{T}$ be a time scale which is unbounded above and below with $0 \in \mathbb{T}$. Consider the dynamic equation on a time scale,

$$x^{\triangle}(t) = -\sum_{i=1}^{N} a_i(t)x(t-\tau_i(t)), \quad t \in \mathbb{T},$$

where, $a_i : \mathbb{T}^+ \to \mathbb{R}$ and $\tau_i : \mathbb{T}^+ \to \mathbb{T}^+$ are continuous with $t - \tau(t) \to \infty$ as $t \to \infty$. For each t_0, we define $m_i(t_0) = \inf\{s - \tau_i(s) : \ s \geq t_0\}, m(t_0) = \min\{m_i(t_0) \ : 1 \leq i \leq N\}$. Let $D(t_0)$ be the set of bounded Δ-differentiable functions $\psi : [m(t_0), t_0]_{\mathbb{T}} \to \mathbb{R}$ with the supremum norm. Also, let $C_{rd} = C_{rd}(\mathbb{T}, \mathbb{R})$ be the space of all rd-continuous functions from $\mathbb{T} \to \mathbb{R}$.

In studying the stability properties of dynamic equation using a fixed point technique, it usually involves the construction of a suitable fixed point mapping and this can be a difficult task. So, to construct our mapping, we begin by transforming Equation (1.1) into a more manageable and equivalent equation that possesses the same basic structure and properties as the dynamic equation and then define a fixed point mapping. This is therefore done in the next lemma.

Lemma 3

Suppose that $g_i(t)$ is the inverse of $\delta_i(t)$ for $i = 1, ..., N$, then Equation

(1.1) is equivalent to the equation

$$x^{\triangle}(t) = \sum_{i=1}^{N} b_i(t)x(t) + \sum_{i=1}^{N} b_i(\delta_i(t)x(\delta_i(t))\tau_i^{\triangle}(t) - \left(\sum_{i=1}^{N} \int_{\delta_i(t)}^{t} b_i(g_i(s))x(s)\triangle s\right)^{\triangle} \tag{4.1}$$

where $b_i(t) = -a_i(g_i(t))$ and $\delta_i(t) = t - \tau_i(t)$.

Proof. Differentiating the integral term in Equation (4.1) using the results by Adıvar and Raffoul (2009) in Lemma 2, we obtain

$$\begin{aligned}\left(\sum_{i=1}^{N} \int_{\delta_i(t)}^{t} b_i(s)x(s)\triangle s\right)^{\triangle} &= \sum_{i=1}^{N} b_i(t)x(t) - \sum_{i=1}^{N} b_i(\delta_i(t))x(\delta_i(t))\delta_i^{\triangle}(t) \\ &= \sum_{i=1}^{N} b_i(t)x(t) - \sum_{i=1}^{N} b_i(\delta_i(t))x(\delta_i(t)) \\ &\quad \times \left(1 - \tau_i^{\triangle}(t)\right) \\ &= \sum_{i=1}^{N} b_i(t)x(t) - \sum_{i=1}^{N} b_i(\delta_i(t)x(\delta_i(t)) \\ &\quad + \sum_{i=1}^{N} b_i(\delta_i(t))x(\delta_i(t))\tau_i^{\triangle}(t) \end{aligned} \tag{4.2}$$

Substituting Equation (4.2) into Equation (4.1), we arrive at

$$\begin{aligned} x^{\triangle}(t) &= \sum_{i=1}^{N} b_i(t)x(t) + \sum_{i=1}^{N} b_i(\delta_i(t))x(\delta_i(t))\tau_i^{\triangle}(t) - \sum_{i=1}^{N} b_i(t)x(t) \\ &\quad + \sum_{i=1}^{N} b_i(\delta_i(t)x(\delta_i(t)) - \sum_{i=1}^{N} b_i(\delta_i(t))x(\delta_i(t))\tau_i^{\triangle}(t) \\ &= \sum_{i=1}^{N} b_i(\delta_i(t))x(\delta_i(t)) \\ &= -\sum_{i=1}^{N} a_i(g_i(\delta_i(t)))x(\delta_i(t)) \end{aligned}$$

But

$$g_i(\delta_i(t)) = t.$$

Thus,

$$x^{\triangle}(t) = -\sum_{i=1}^{N} a_i(t)x(\delta_i(t))$$

This shows that Equation (4.1) is equivalent to Equation (1.1).

The fixed point theorem requires that we define a mapping based on the integral equation which is equivalent to the dynamic equation and thus obtain the equivalent integral equation in the next Lemma.

Lemma 4

The function $x(t)$ is a solution a solution of Equation 1.1 if and only if

$$\begin{aligned} x(t) &= x(t_0)e_p(t,t_0) + \int_{t_0}^{t}\sum_{i=1}^{N} b_i(\delta_i(u)x(\delta_i(u))\tau_i^{\triangle}(u)e_p(t,u)\triangle u \\ &\quad - \sum_{i=1}^{N}\int_{\delta_i(t)}^{t} b_i(s)x(s)\triangle s + e_p(t,t_0)\sum_{i=1}^{N}\int_{\delta_i(t_0)}^{t_0} a_i(g_i(s))x(s)\triangle s \\ &\quad - \int_{t_0}^{t}\sum_{i=1}^{N} b_i(s)e_p(t,u)\left(\sum_{i=1}^{N}\int_{\delta_i(u)}^{u} b_i(s)x(s)\triangle s\right)\triangle u \end{aligned} \tag{4.3}$$

Proof. Rewrite Equation (4.1) as

$$x^{\triangle}(t) - \sum_{i=1}^{N} b_i(t)x(t) = \sum_{i=1}^{N} b_i(\delta_i(t)x(\delta_i(t))\tau_i^{\triangle}(t) - \left(\sum_{i=1}^{N}\int_{\delta_i(t)}^{t} b_i(s)x(s)\triangle s\right)^{\triangle} \tag{4.4}$$

Multiplying both sides of Equation (4.4) by $e_{\ominus p}(t,t_0)$ where

$$p(t) = \sum_{i=1}^{N} b_i(t)$$

we obtain

$$\begin{aligned} &(x(t)e_{\ominus p}(t,t_0))^{\triangle} \\ &= \left[\sum_{i=1}^{N} b_i(\delta_i(t)x(\delta_i(t))\tau_i^{\triangle}(t) - \left(\sum_{i=1}^{N}\int_{\delta_i(t)}^{t} b_i(s)x(s)\triangle s\right)^{\triangle}\right] e_{\ominus p}(t,t_0) \end{aligned}$$

$$= \sum_{i=1}^{N} b_i(\delta_i(t)x(\delta_i(t))\tau_i^{\triangle}(t)e_{\ominus p}(t,t_0) - \left(\sum_{i=1}^{N}\int_{\delta_i(t)}^{t} b_i(s)x(s)\triangle s\right)^{\triangle} e_{\ominus p}(t,t_0) \tag{4.5}$$

Integrating both sides of the Equation (4.5) from t_0 to t gives

$$\begin{aligned}\int_{t_0}^{t}(x(u)e_{\ominus p}(u,t_0))^{\triangle}\triangle u &= \int_{t_0}^{t}\sum_{i=1}^{N} b_i(\delta_i(u)x(\delta_i(u))\tau_i^{\triangle}(u)e_{\ominus p}(u,t_0)\triangle u \\ &- \int_{t_0}^{t}\left(\sum_{i=1}^{N}\int_{\delta_i(u)}^{u} b_i(s)x(s)\triangle s\right)^{\triangle} e_{\ominus p}(u,t_0))\triangle u\end{aligned}$$

This implies that

$$\begin{aligned}x(t)e_{\ominus p}(t,t_0) =& x(t_o)e_{\ominus p}(t_0,t_0) + \int_{t_0}^{t}\sum_{i=1}^{N} b_i(\delta_i(u)x(\delta_i(u))\tau_i^{\triangle}(u)e_{\ominus p}(u,t_0)\triangle u \\ &- \int_{t_0}^{t}\left(\sum_{i=1}^{N}\int_{\delta_i(u)}^{u} b_i(s)x(s)\triangle s\right)^{\triangle} e_{\ominus p}(u,t_0))\triangle u \end{aligned} \tag{4.6}$$

Using integration by parts formula from Theorem 10 of the chapter three on the last term of Equation (4.6), gives

$$\begin{aligned}&\int_{t_0}^{t}\left(\sum_{i=1}^{N}\int_{\delta_i(u)}^{u} b_i(s)x(s)\triangle s\right)^{\triangle} e_{\ominus p}(u,t_0))\triangle u \\ &= e_{\ominus p}(u,t_0)(\sum_{i=1}^{N}\int_{\delta_i(u)}^{u} b_i(u))x(u)\triangle u)\ |_{t_0}^{t} \\ &- \int_{t_0}^{t} e_{\ominus p}^{\triangle}(u,t_0)\left(\sum_{i=1}^{N}\int_{\delta_i(u)}^{u} b_i(s)x(s)\triangle s\right)\triangle u \\ &= e_{\ominus p}(t,t_0)\sum_{i=1}^{N}\int_{\delta_i(t)}^{t} b_i(s)x(s)\triangle s - e_{\ominus p}(t_0,t_0)\sum_{i=1}^{N}\int_{\delta_i(t_0)}^{t_0} b_i(s)x(s)\triangle s \\ &- \int_{t_0}^{t} e_{\ominus p}^{\triangle}(u,t_0)\left(\sum_{i=1}^{N}\int_{\delta_i(u)}^{u} b_i(s)x(s)\triangle s\right)\triangle u\end{aligned} \tag{4.7}$$

Substituting Equation (4.7) into Equation (4.6), we obtain

$$\begin{aligned}
&x(t)e_{\ominus p}(t,t_0) - x(t_0)e_{\ominus p}(t_0,t_0)\\
&= \int_{t_0}^{t}\sum_{i=1}^{N} b_i(\delta_i(s)x(\delta_i(s))\tau_i^{\triangle}(t)e_{\ominus p}(s,t_0)\triangle s - \ e_{\ominus p}(t,t_0)\sum_{i=1}^{N}\int_{\delta_i(t)}^{t} b_i(s)x(s)\triangle s\\
&+\sum_{i=1}^{N}\int_{\delta_i(t_0)}^{t_0} b_i(s)x(s)\triangle s + \int_{t_0}^{t} e_{\ominus p}^{\triangle}(u,t_0)\left(\sum_{i=1}^{N}\int_{\delta_i(u)}^{u} b_i(s)x(s)\triangle s\right)\triangle u
\end{aligned}$$

Simplifying the above equation, gives,

$$\begin{aligned}
x(t)e_{\ominus p}(t,t_0) &= x(t_0) + \int_{t_0}^{t}\sum_{i=1}^{N} b_i(\delta_i(s)x(\delta_i(s))\tau_i^{\triangle}(s)e_{\ominus p}(s,t_0)\triangle s\\
&\quad - e_{\ominus p}(t,t_0)\sum_{i=1}^{N}\int_{\delta_i(t)}^{t} b_i(s)x(s)\triangle s + \sum_{i=1}^{N}\int_{\delta_i(t_0)}^{t_0} b_i(s)x(s)\triangle s\\
&\quad + \int_{t_0}^{t} \ominus p e_{\ominus p}(t,u)\left(\sum_{i=1}^{N}\int_{\delta_i(u)}^{u} b_i(s)x(s)\triangle s\right)\triangle u \qquad (4.8)
\end{aligned}$$

Dividing through Equation (4.8) by

$$e_{\ominus p}(t,t_0),$$

gives

$$\begin{aligned}
x(t) &= x(t_0)e_p(t,t_0) + \int_{t_0}^{t}\sum_{i=1}^{N} b_i(\delta_i(u)x(\delta_i(u))\tau_i^{\triangle}(u)e_p(t,u)\triangle u\\
&\quad - \sum_{i=1}^{N}\int_{\delta_i(t)}^{t} b_i(s)x(s)\triangle s + e_p(t,t_0)\sum_{i=1}^{N}\int_{\delta_i(t_0)}^{t_0} a_i(g_i(s))x(s)\triangle s\\
&\quad - \int_{t_0}^{t}\sum_{i=1}^{N} b_i(s)e_p(t,u)\left(\sum_{i=1}^{N}\int_{\delta_i(u)}^{u} b_i(s)x(s)\triangle s\right)\triangle u,
\end{aligned}$$

which completes the proof of Lemma 3.

A dynamic equation can be asymptotically stable provided it is first of all stable. In the next theorem we state sufficient conditions for the dynamic equation considered to be stable.

Theorem 13. *Suppose the inverse function $g_i(t)$ of $\delta_i(t)$ exist for $i = 1, ..., N$ and assume there is a constant $\alpha \in (0,1)$ such that*

$$\int_{t_0}^{t} \sum_{i=1}^{N} |b_i(\delta_i(u))||\tau_i^{\Delta}(u)|e_p(t,u)\Delta u + \sum_{i=1}^{N} \int_{\delta_i(t)}^{t} | b_i(u) | \triangle u$$
$$+ \int_{t_0}^{t} \sum_{i=1}^{N} | b_i(u) | \Big(\sum_{i=1}^{N} \int_{\delta_i(u)}^{u} | b_i(s) | \triangle s \Big) e_p(t,u)\triangle u \leq \alpha. \quad (4.9)$$

Then the zero solution of Equation (1.1) is stable. **Proof**. Let $\varepsilon > 0$ be given. Choose $\delta > 0$ such that

$$(1+\alpha)\delta + \alpha\varepsilon \leq \varepsilon. \quad (4.10)$$

Let $\psi \in D(t_0)$ such that $| \psi(t) |\leq \delta$.

Define $S = \{\varphi \in C_{rd}\ \varphi(t) = \psi(t)$ if $t \in [m(t_0), t_0]_{\mathbb{T}}, \| \varphi \|\leq \epsilon\}$. Then $(S, \| \ . \ \|)$ is a complete metric space where, $\| \ . \ \|$ is the supremum norm. Define the mapping $Q : S \to S$ by $(Q\varphi)(t) = \psi(t)$ for $t \in [m(t_0), t_0]_{\mathbb{T}}$ and

$$(Q\varphi) = \psi(t_0) + \sum_{i=1}^{N} \int_{\delta_i(t_0)}^{t_0} a_i(g_i(s))\psi(s)\triangle s \Big) e_p(t,t_0) + \int_{t_0}^{t} \sum_{i=1}^{N} b_i(\delta_i(u)$$
$$\times \varphi(\delta_i(u))\tau_i^{\triangle}(u)e_p(t,u)\triangle u - \sum_{i=1}^{N} \int_{\delta_i(t)}^{t} b_i(s)\varphi(s)\triangle s - \int_{t_0}^{t} \sum_{i=1}^{N} b_i(u)e_p(t,u)$$
$$\times \Big(\sum_{i=1}^{N} \int_{\delta_i(u)}^{u} b_i(s)\varphi(s)\triangle s \Big) \triangle u, t \geq t_0. \quad (4.11)$$

We first show that Q maps from S to S. From Eq. (4.10) we have

$$|(Q\varphi)(t)|$$
$$= \Big| \psi(t_o)e_p(t,t_0) + \int_{t_0}^{t} \sum_{i=1}^{N} b_i(\delta_i(u)\psi(\delta_i(u))\tau_i^{\triangle}(u)e_p(t,u)\triangle u$$
$$- \sum_{i=1}^{N} \int_{\delta_i(t)}^{t} b_i(s)\varphi(s)\triangle s + e_p(t,t_0)(\sum_{i=1}^{N} \int_{\delta_i(t_0)}^{t_0} a_i(g_i(s))\psi(s)\triangle s$$
$$- \int_{t_0}^{t} pe_p(t,u) \left(\sum_{i=1}^{N} \int_{\delta_i(u)}^{u} b_i(s)\varphi(s)\triangle s \right) \triangle u \Big|$$

$$
\begin{aligned}
&\leq \Big| \Big(\psi(t_0) + (\sum_{i=1}^{N} \int_{\delta_i(t_0)}^{t_0} a_i(g_i(s))\psi(s)\triangle s \Big) e_p(t,t_0) \Big| \\
&\quad + \Big| \int_{t_0}^{t} \sum_{i=1}^{N} b_i(\delta_i(u)\psi(\delta_i(u))\tau_i^{\triangle}(u) e_p(t,u)\triangle u \Big| + \Big| \Big\{ \sum_{i=1}^{N} \int_{\delta_i(t)}^{t} \Big| b_i(s) \Big| \triangle s \\
&\quad + \int_{t_0}^{t} \Big| p(s) \mid e_p(t,u) \sum_{i=1}^{N} \int_{\delta_i(u)}^{u} \mid b_i(s) \mid \triangle s \triangle u \Big\} \Big| \\
&\leq \delta + \alpha\delta + \Big\{ \sum_{i=1}^{N} \int_{\delta_i(t)}^{t} \mid b_i(s) \mid\mid \varphi(s) \mid \triangle s + \int_{t_0}^{t} \mid p(s) \mid e_p(t,u) \\
&\quad \times \Big(\sum_{i=1}^{N} \int_{\delta_i(u)}^{u} \mid b_i(s) \mid\mid \varphi(s) \mid \triangle s \Big) \triangle u \Big\} \\
&\leq (1+\alpha)\delta + \Big\{ \sum_{i=1}^{N} \int_{\delta_i(t)}^{t} \mid b_i(s) \mid \triangle s + \int_{t_0}^{t} \mid p(s) \mid e_p(t,u) \\
&\quad \times \Big(\sum_{i=1}^{N} \int_{\delta_i(u)}^{u} \mid b_i(s) \mid \triangle s \Big) \triangle u \Big\} \mid\mid \varphi \mid\mid \\
&\leq (1+\alpha)\delta + \alpha \mid\mid \varphi \mid\mid \\
&\leq (1+\alpha)\delta + \alpha\varepsilon \\
&\leq \varepsilon
\end{aligned}
$$

This therefore shows that Q maps from S into itself.

We next show that Q is continuous. Let $\varphi, \eta \in S$. Given $\varepsilon > 0$, Choose $\delta = \dfrac{\varepsilon}{\alpha}$ such that $\mid \varphi - \eta \mid < \delta$.Then,

$$
\begin{aligned}
&\mid\mid (Q\varphi) - (Q\eta) \mid\mid \\
=& \Big| \Big(\psi(t_0) + \sum_{i=1}^{N} \int_{\delta_i(t_0)}^{t_0} a_i(g_i(s))\psi(s)\triangle s \Big) e_p(t,t_0) \\
&+ \int_{t_0}^{t} \sum_{i=1}^{N} b_i(\delta_i(u))\varphi(\delta_i(u))\tau_i^{\triangle}(u) e_p(t,u)\triangle u - \sum_{i=1}^{N} \int_{\delta_i(t)}^{t} b_i(s)\varphi(s)\triangle s \\
&- \int_{t_0}^{t} \Big(\sum_{i=1}^{N} b_i(u) \Big) e_p(t,u) \Big(\sum_{i=1}^{N} \int_{\delta_i(u)}^{u} b_i(s)\varphi(s)\triangle s \Big) \triangle u \Big) \\
&- \Big(\psi(t_0) + \sum_{i=1}^{N} \int_{\delta_i(t_0)}^{t_0} a_i(g_i(s))\psi(s)\triangle s \Big) e_p(t,t_0) \\
&+ \int_{t_0}^{t} \sum_{i=1}^{N} b_i(\delta_i(u))\eta(\delta_i(u))\tau_i^{\triangle}(u) e_p(t,u)\triangle u - \sum_{i=1}^{N} \int_{\delta_i(t)}^{t} b_i(s)\eta(s)\triangle s
\end{aligned}
$$

$$
\begin{aligned}
& - \int_{t_0}^{t} \sum_{i=1}^{N} b_i(u) e_p(t,u) \Big(\sum_{i=1}^{N} \int_{\delta_i(u)}^{u} b_i(s)\eta(s)\triangle s \Big) \triangle u \Big| \\
=& \Big| \int_{t_0}^{t} \sum_{i=1}^{N} b_i(\delta_i(u))\varphi(\delta_i(u))\tau_i^{\triangle}(u) e_p(t,u)\triangle u - \int_{t_0}^{t} \sum_{i=1}^{N} b_i(\delta_i(u))\eta(\delta_i(u))\tau_i^{\triangle}(u) \\
& \times e_p(t,u)\triangle u + \sum_{i=1}^{N} \int_{\delta_i(t)}^{t} b_i(s)\varphi(s)\triangle s - \sum_{i=1}^{N} \int_{\delta_i(t)}^{t} b_i(s)\eta(s)\triangle s \\
& + \int_{t_0}^{t} \sum_{i=1}^{N} b_i(u) e_p(t,u) \Big(\sum_{i=1}^{N} \int_{\delta_i(u)}^{u} b_i(s)\varphi(s)\triangle s \Big) \triangle u \Big) - \int_{t_0}^{t} \sum_{i=1}^{N} b_i(u) e_p(t,u) \\
& \times \Big(\sum_{i=1}^{N} \int_{\delta_i(u)}^{u} b_i(s)\eta(s)\triangle s \Big) \triangle u \Big| \\
\leq & \Big| \int_{t_0}^{t} \sum_{i=1}^{N} b_i(\delta_i(u))\varphi(\delta_i(u))\tau_i^{\triangle}(u) e_p(t,u)\triangle u - \int_{t_0}^{t} \sum_{i=1}^{N} b_i(\delta_i(u))\eta(\delta_i(u))\tau_i^{\triangle}(u) \\
& \times e_p(t,u)\triangle u \Big| + \Big| \sum_{i=1}^{N} \int_{\delta_i(t)}^{t} b_i(s)\varphi(s)\triangle s - \sum_{i=1}^{N} \int_{\delta_i(t)}^{t} b_i(s)\eta(s)\triangle s \Big| \\
& + \Big| \int_{t_0}^{t} \sum_{i=1}^{N} b_i(u) e_p(t,u) \Big(\sum_{i=1}^{N} \int_{\delta_i(u)}^{u} b_i(s)\varphi(s)\triangle s \Big) \triangle u \Big) - \int_{t_0}^{t} \sum_{i=1}^{N} b_i(u) e_p(t,u) \\
& \times \Big(\sum_{i=1}^{N} \int_{\delta_i(u)}^{u} b_i(s)\eta(s)\triangle s \Big) \triangle u \Big| \\
\leq & \int_{t_0}^{t} \sum_{i=1}^{N} \mid b_i(\delta_i(u) \mid\mid \tau_i^{\triangle}(u) \mid e_p(t,u)\triangle \mid \varphi(\delta_i(u)) - \eta(\delta_i(u)) \mid \\
& + \sum_{i=1}^{N} \int_{\delta_i(t)}^{t} \mid b_i(s) \mid \triangle s \mid \varphi(s) - \eta(s) \mid + \int_{t_0}^{t} \mid \sum_{i=1}^{N} b_i(u) \mid e_p(t,u) \\
& \times \sum_{i=1}^{N} \int_{\delta_i(u)}^{u} \mid b_i(s) \mid \triangle s \triangle u \mid \varphi(s) - \eta(s) \mid \\
\leq & \int_{t_0}^{t} \sum_{i=1}^{N} \mid b_i(\delta_i(u)) \mid\mid \tau_i^{\triangle}(u) \mid e_p(t,u)\triangle u \mid\mid \varphi - \eta \mid\mid + (\sum_{i=1}^{N} \int_{\delta_i(t)}^{t} \mid b_i(s) \mid \triangle s) \mid \\
& \times \mid \varphi - \eta \mid\mid + \int_{t_0}^{t} \sum_{i=1}^{N} \mid b_i(u) \mid e_p(t,u) (\sum_{i=1}^{N} \int_{\delta_i(u)}^{u} \mid b_i(s) \mid \triangle s)\triangle u) \mid\mid \varphi - \eta \mid\mid . \\
\leq & \Big\{ \int_{t_0}^{t} \sum_{i=1}^{N} \mid b_i(\delta_i(u) \mid\mid \tau_i^{\triangle}(u) \mid e_p(t,u)\triangle u + \sum_{i=1}^{N} \int_{\delta_i(t)}^{t} \mid b_i(s) \mid \triangle s \\
& + \int_{t_0}^{t} \mid \sum_{i=1}^{N} b_i(u) \mid e_p(t,u) \sum_{i=1}^{N} \int_{\delta_i(u)}^{u} \mid b_i(s) \mid \triangle s \triangle u \Big\} \mid\mid \varphi - \eta \mid\mid . \\
\leq & \alpha \mid\mid \varphi - \eta \mid\mid \\
\leq & \varepsilon
\end{aligned}
$$

This therefore shows that $Q\varphi$ is continuous. We next show that Q is a contraction under the supremum norm. Let $\varphi, \phi \in S$. Then,

$$
\begin{aligned}
& \mid (Q\varphi)(t) - (Q\phi)(t) \mid \\
=& \Big| \Big(\psi(t_0)e_p(t,t_0) + \int_{t_0}^{t} \sum_{i=1}^{N} b_i(\delta_i(u))\varphi(\delta_i(u))\tau_i^{\triangle}(u)e_p(t,u)\triangle u \\
& - \sum_{i=1}^{N} \int_{\delta_i(t)}^{t} b_i(s)\varphi(s)\triangle s + e_p(t,t_0)(\sum_{i=1}^{N} \int_{\delta_i(t_0)}^{t_0} a_i(g_i(s))\psi(s)\triangle s \\
& - \int_{t_0}^{t} \sum_{i=1}^{N} b_i(u)e_p(t,u)(\sum_{i=1}^{N} \int_{\delta_i(u)}^{u} b_i(s)\varphi(s)\triangle s)\triangle u \Big) \\
& - \Big(\psi(t_0)e_p(t,t_0) + \int_{t_0}^{t} \sum_{i=1}^{N} b_i(\delta_i(u))\phi(\delta_i(u))\tau_i^{\triangle}(u)e_p(t,u)\triangle u \\
& - \sum_{i=1}^{N} \int_{\delta_i(t)}^{t} b_i(s)\phi(s)\triangle s + e_p(t,t_0)(\sum_{i=1}^{N} \int_{\delta_i(t_0)}^{t_0} a_i(g_i(s))\psi(s)\triangle s \\
& - \int_{t_0}^{t} \sum_{i=1}^{N} b_i(u)e_p(t,u)(\sum_{i=1}^{N} \int_{\delta_i(u)}^{u} b_i(s)\phi(s)\triangle s)\triangle u \Big) \Big|. \\
=& \Big| \int_{t_0}^{t} \sum_{i=1}^{N} b_i(\delta_i(u)\varphi(\delta_i(u))\tau_i^{\triangle}(u)e_p(t,u)\triangle u + \sum_{i=1}^{N} \int_{\delta_i(t)}^{t} b_i(s)\varphi(s)\triangle s \\
& - \int_{t_0}^{t} \sum_{i=1}^{N} b_i(u)e_p(t,u)(\sum_{i=1}^{N} \int_{\delta_i(u)}^{u} b_i(s)\varphi(s)\triangle s\triangle u \\
& + \int_{t_0}^{t} \sum_{i=1}^{N} b_i(\delta_i(u)\phi(\delta_i(u))\tau_i^{\triangle}(u)e_p(t,u)\triangle u - \sum_{i=1}^{N} \int_{\delta_i(t)}^{t} b_i(s)\phi(s)\triangle s \\
& - \int_{t_0}^{t} \sum_{i=1}^{N} b_i(u)e_p(t,u)(\sum_{i=1}^{N} \int_{\delta_i(u)}^{u} b_i(s)\phi(s)\triangle s)\triangle u \Big|. \\
\leq& \Big| \int_{t_0}^{t} \sum_{i=1}^{N} b_i(\delta_i(u))\varphi(\delta_i(u))\tau_i^{\triangle}(u)e_p(t,u)\triangle u - \int_{t_0}^{t} \sum_{i=1}^{N} b_i(\delta_i(u))\varphi(\delta_i(u))\tau_i^{\triangle}(u) \\
& \times e_p(t,u)\triangle u \Big| + \Big| \sum_{i=1}^{N} \int_{\delta_i(t)}^{t} b_i(s)\varphi(s)\triangle s - \sum_{i=1}^{N} \int_{\delta_i(t)}^{t} b_i(s)\varphi(s)\triangle s \Big| \\
& + \Big| \int_{t_0}^{t} \sum_{i=1}^{N} b_i(u)e_p(t,u)\Big(\sum_{i=1}^{N} \int_{\delta_i(u)}^{u} b_i(s)\varphi(s)\triangle s \Big)\triangle u \Big) - \int_{t_0}^{t} \sum_{i=1}^{N} b_i(u)e_p(t,u) \\
& \times \Big(\sum_{i=1}^{N} \int_{\delta_i(u)}^{u} b_i(s)\varphi(s)\triangle s \Big)\triangle u \Big| \\
\leq& \int_{t_0}^{t} \sum_{i=1}^{N} \Big| b_i(\delta_i(u)) \Big| \Big| \tau_i^{\triangle}(u) \Big| e_p(t,u)\triangle u \Big| \varphi(\delta_i(u)) - \phi(\delta_i(u)) \Big|
\end{aligned}
$$

$$+\sum_{i=1}^{N}\int_{\delta_i(t)}^{t}\Big|b_i(s)\Big|\triangle s\mid\varphi(s)-\phi(s)\Big| - \int_{t_0}^{t}\sum_{i=1}^{N}\Big|b_i(u)\Big|e_p(t,u)$$
$$\times\sum_{i=1}^{N}\int_{\delta_i(u)}^{u}\Big|b_i(s)\Big|\triangle s)\triangle u\Big|\varphi(s)-\phi(s)\Big|$$
$$\leq\int_{t_0}^{t}\sum_{i=1}^{N}\mid b_i(\delta_i(u)\mid\mid\tau_i^{\triangle}(u)\mid e_p(t,u)\triangle u\mid\mid\varphi-\phi\mid\mid$$
$$+(\sum_{i=1}^{N}\int_{\delta_i(t)}^{t}\mid b_i(s)\mid\triangle s)\mid\mid\varphi-\phi\mid\mid+\int_{t_0}^{t}\sum_{i=1}^{N}\mid b_i(u)\mid e_p(t,u)$$
$$\times(\sum_{i=1}^{N}\int_{\delta_i(u)}^{u}\mid b_i(s)\mid\triangle s)\triangle u\mid\mid\varphi-\phi\mid\mid.$$
$$\leq\Big\{\int_{t_0}^{t}\sum_{i=1}^{N}\mid b_i(\delta_i(u))\mid\mid\tau_i^{\triangle}(u)\mid e_p(t,u)\triangle u+\sum_{i=1}^{N}\int_{\delta_i(t)}^{t}\mid b_i(s)\mid\triangle s$$
$$+\int_{t_0}^{t}\sum_{i=1}^{N}\mid b_i(u)\mid e_p(t,u)(\sum_{i=1}^{N}\int_{\delta_i(u)}^{u}\mid b_i(s)\mid\triangle s)\triangle u\Big\}\mid\mid\varphi-\phi\mid\mid.$$
$$\leq\alpha\mid\mid\varphi-\phi\mid\mid$$

This shows that Q is a contraction. By the contraction mapping principle, Q has a unique fixed point in S which solves Equation (1.1) and for any $\varphi\in S$, $\|Q\varphi\|\leq\varepsilon$.

This proves that the zero solution of Equation (1.1) is stable. In the next theorem we state sufficient conditions for the dynamic equation to be asymptotically stable.

Theorem 14. *Assume the hypothesis of Theorem 13 hold. Assume further that*

$$t-\tau_i(t)\to\infty \;\; as \;\; t\to\infty \; for \; i=1,...,N. \tag{4.12}$$

Then the zero solution of Equation (1.1) is asymptotically stable.

Proof. It has been proved already that the zero solution of Equation (1.1) is stable. Let $\psi\in D(t_o)$ such that $|\psi(t)|\leq\delta$ and define $S^*=\{\varphi\in C_{rd}$: $\varphi(t)=\psi(t)$ if $t\in[m(t_0),t_0]_{\mathbb{T}},\|\varphi\|\leq\varepsilon$ and $||\varphi||\to 0$ as $t\to\infty\}$. Define

$Q : S^\star \to S^\star$ by (4.11). It has been proved that the map Q is a contraction and for every $\varphi \in S^\star, \|Q\varphi\| \leq \varepsilon$. We next show that $(Q\varphi) \to 0$ as $t \to \infty$. The first term on the right hand side of Equation (4.11) given by

$$\Big(\psi(t_0) + \sum_{i=1}^{N} \int_{\delta_i(t_0)}^{t_0} a_i(g_i(s))\psi(s)\triangle s\Big) e_p(t, t_0)$$

goes to zero as $t \to \infty$ since $e_p(t, t_0) \to 0$ $t \to \infty$.

Since $\varphi \in S$, we have $\varphi(t) \to 0$ as $t \to \infty$. From continuity of norm we obtain

$$||\varphi|| \to 0 \text{ as } t \to \infty. \tag{4.13}$$

Hence, we obtain for the third term

$$\mid \sum_{i=1}^{N} \int_{\delta_i(t)}^{t} b_i(s)\varphi(s)\triangle s \mid \;\leq\; || \varphi || \sum_{i=1}^{N} \int_{\delta_i(t)}^{t} \mid b_i(s) \mid \triangle s \to 0 \text{ as } t \to \infty.$$

We now show that the second and the last terms on the right hand side of Equation (4.11) go to zero as $t \to 0$. Since $\varphi(t) \to 0$ and $\delta_i(t) \to \infty$ as $t \to \infty$, for $\varepsilon_1 > 0$, there exist a $T_1 > t_0$ such that $t \geq T_1$ implies $|\varphi(\delta_i(t)) \mid < \varepsilon_1$ for $j = 1, 2, 3, ..., N$. Also, due to the fact that $e_p(t, 0) \to 0$ as $t \to \infty$, there exist $T_2 > T_1$ such that $t > T_2$ implies that

$$e_p(t, T_1) \leq \frac{\epsilon_1}{\alpha\epsilon}$$

Then,

$$\begin{aligned}
&\mid I_3 \mid \\
&\leq \int_{t_0}^{T_1} \Big[\sum_{i=1}^{N} \Big(\Big| b_i(\delta_i(u))\varphi(\delta_i(u))\tau_i^{\triangle}(u) + \int_{t_0}^{t} \sum_{i=1}^{N} b_i(u) \Big(\sum_{i=1}^{N} \int_{\delta_i(u)}^{u} b_i(s)\varphi(s)\triangle s \Big) \Big] \\
&\times e_p(t, u) \triangle u \Big| + \Big| \int_{T_1}^{t} \Big[\sum_{i=1}^{N} \Big(b_i(\delta_i(u))\varphi(\delta_i(u))\tau_i^{\triangle}(u) + \int_{t_0}^{t} \sum_{i=1}^{N} b_i(u)
\end{aligned}$$

$$\times\Big(\sum_{i=1}^{N}\int_{\delta_i(u)}^{u} b_i(s)\varphi(s)\triangle s\Big)\Big]e_p(t,u)\,\triangle u\Big|$$

$$\leq ||\varphi||e_p(t,T_1)\mid\int_{t_0}^{T_1}\Big[\sum_{i=1}^{N}\Big(\mid b_i(\delta_i(u))\mid\mid\tau_i^{\triangle}(u)\mid+\mid\int_{t_0}^{t}\sum_{i=1}^{N}b_i(u)\mid$$

$$\times\Big(\sum_{i=1}^{N}\int_{\delta_i(u)}^{u}\mid b_i(s)\mid\triangle s\Big)\Big]e_p(T_1,u)\,\triangle u\;+\varepsilon_1\int_{T_1}^{t}\Big[\sum_{i=1}^{N}\Big(\mid b_i(\delta_i(u))\mid$$

$$\times\mid\varphi(\delta_i(u))\tau_i^{\triangle}(u)\mid+\mid\int_{t_0}^{t}\sum_{i=1}^{N}b_i(u)\mid\Big(\sum_{i=1}^{N}\int_{\delta_i(u)}^{u}\mid b_i(s)\varphi(s)\mid\triangle s\Big)\Big]e_p(t,u)\,\triangle u$$

$$\leq |e_p(t,T_1)\alpha\varepsilon+\alpha\varepsilon_1$$

$$\leq \varepsilon_1+\alpha\varepsilon_1$$

This yields $I_3 \to 0$ as $t \to \infty$. Hence $(Q\varphi)(t) \to 0$ as $t \to \infty$ and so $P\varphi \in S^{\star}$. Hence all the conditions of the Banach fixed point theorem has been established. Therfore, by the Banach fixed point theorem, the mapping Q has a unique fixed point which solves Equation (1.1) and goes to zero as t goes to infinity. Therefore the zero solution of Equation (4.10) is asymptotically stable.

Chapter Summary

In this chapter,an equivalent integral dynamic equation to the dynamic equation obtained.A mapping is then defined based on the equivalent integral dynamic equation. Sufficient conditions conditions are obtained for the zero solution of the dynamic equation to be stable and asymptotically stable.

CHAPTER FIVE

SUMMARY,CONCLUSIONS AND RECOMENDATIONS

Summary

The stability properties of a completely delayed dynamic equations on time scale is investigated in this research work.The dynamic equation is inverted or transformed into an equivalent integral dynamic equations on time scale. The equivalent integral equation is then used to define a mapping that was used for the discussion of the stability behaviour of the dynamic equation considered. The Banach fixed point theorem is used to prove the asymptotic stability of the zero solution of the dynamic equation on time scale.

Conclusion

Sufficient conditions for the zero solution of a completely delayed dynamic equation to be stable has been established. Moreover,sufficient conditions for the zero solution of the completely delayed dynamic equation on time scale have also been obtained.

Recommendations

For the study of stability properties of dynamic equations with variable delay i recommend that the fixed point theorem be used.

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