UNIVERSITY OF CAPE COAST

SOLUTION OF INVERSE EIGENVALUE PROBLEM OF SINGULAR HERMITIAN MATRICES OF RANK GREATER THAN OR EQUAL TO FOUR

BY

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DECLARATION

Candidate's Declaration

I hereby declare that this thesis work is the result of my own original research and that no part of it has been presented for another degree in this university or elsewhere.

Candidate's Signature ……………………………………….. Date ………………

Name: ……………………………………………………………………….

Supervisor's Declaration

I hereby declare that the preparation and presentation of the thesis work were supervised in accordance with the guidelines on supervision of thesis work laid down by the university of Cape Coast.

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ABSTRACT

This work deals with a modification of an algorithm that solves a special Structured Inverse Eigenvalue Problems (SIEP). The problem we consider is the Structured Hermitian Inverse Eigenvalue Problem (SHIEP) where the researcher's purpose is to find the solution of inverse eigenvalue problem of singular Hermitian matrix of rank greater than or equal to four. We modified an algorithm to generate singular symmetric and Hermitian matrices for rank greater than or equal to 4 that meet both the spectral and structural constraint as a solution for the inverse eigenvalue problem. Finally, we proved that given the spectrum and the scalars $k_i = 1,2,\ldots,n-4$, the inverse eigenvalue problem for an $n \times n$ singular symmetric and Hermitian matrices of rank 4 are solvable.

KEY WORDS

Hermitian matrices

Inverse eigenvalue problem

Parameters

Rank

Singular matrices

Symmetric matrices

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DEDICATION

To all loved ones especially students and teachers learning and pushing the boundary of mathematics higher.

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LIST OF ABBREVIATION

AIEP

IEP

LSIEP

MIEP

MVIEP

PDIEP

PIEP

SIEP

CHAPTER ONE

INTRODUCTION

This chapter deals with the background of study, statement of problem, purpose of study, scope of the study, significance of the study, delimitation of the study, limitation of the study, and organization of the rest of the study.

Background to the Study

An inverse problem in science is the process of calculating from a set of observations the causal factors that produced them. It is called an inverse problem because it starts with the effects and then calculates the causes. Inverse problems are some of the most important mathematical problems in science and mathematics because they tell us about parameters that we cannot directly observe. Thus given a system which has be transformed into matrices, we can calculate the solution or technically the sets of eigenvalues known as the spectrum. However, it is also plausible to solve for the inverse problem that is given a particular spectrum (eigenvalues) we can deduce the corresponding system that generated the spectrum. The Weyl laws by Weyl(1911) provides the most easily understood answer to a popular research done in this area by Kac(1966) who proposed the title 'can one hear the shape of a drum'. Inverse problems have wide application in system identification, optics, radar, acoustics, communication theory, signal processing, medical imaging, computer vision, geophysics, oceanography, astronomy, remote sensing, machine learning, nondestructive testing, and many other fields.

In line with inverse problem, an inverse eigenvalue problem entails the reconstruction of a matrix from its eigenvalues. It has been of interest not only to algebraists but also to numerical analysts, control theorists, statisticians and engineers. Most research effort have been directed at solving the inverse eigenvalue problem for nonsingular symmetric matrices (Chu & Golub, 2005; Gladwell, 2004; Deakin & Luke, 1992; Chu, 1995). Recently, however, the case of singular symmetric matrices of arbitrary order and rank has been virtually solved provided linear dependency relations are specified (Gyamfi, Oduro, & Aidoo, 2013; Aidoo, Gyamfi, Ackora-Prah, & Oduro, 2013). The focus of this research will be on inverse eigenvalue problem of certain entry-wise non vanishing singular Hermitian matrices which has received little attention in terms of research. Inverse eigenvalue problem can be seen in a lot of fields which include control design, system identification, exploration and remote sensing, antenna array processing, circuit theory, structure analysis, geophysics and many other fields

Depending on the application, inverse eigenvalue problems maybe described in several different forms. Translated into mathematics, it is often necessary in order that the inverse eigenvalue problem be meaningful to restrict the construction to special classes of matrices, especially to those with specified structures. In this research we focus on the construction of singular symmetric Hermitian matrices from a given spectral data. The solution to an inverse eigenvalue problem therefore should satisfy two constraints, the spectral constraint referring to the prescribed spectral data and the structural constraint referring to the desirable structure.

Eigenvalues are a special set of scalars associated with a linear system of equations (i.e., a matrix equation) that are sometimes also known as characteristic roots, characteristic values (Hoffman and Kunze 1971), proper values, or latent roots (Marcus and Minc, 1988). The determination of the eigenvalues and eigenvectors of a system is extremely important in physics and engineering, where it is equivalent to matrix diagonalization and arises in such common applications as stability analysis, the physics of rotating bodies, and small oscillations of vibrating systems, to name only a few. Each eigenvalue is paired with a corresponding so-called eigenvector. Given a square matrix A,a scalar λ is an eigenvalue of A if and only if there is a nonzero vector v, called an eigenvector, such that $Av = \lambda v$ or equivalently $(\lambda I - A)v = 0$where I is the identity matrix. Since v must be non-zero, it implies that the eigenvalues of A are the roots of $\det(\lambda I - A) = 0$, which is a polynomial in λ. Consider $\mathrm{A}X = 0$ where $X \neq 0$ is required, then A is singular that is $|A| = 0$. Now suppose $\mathrm{A} = B - \lambda I$, then the homogenous linear equation $(\mathrm{A}X = 0)$ is said to be an eigenvalue problem, where admissible values of λ are referred to as eigenvalues and the corresponding solutions X_λ are called eigenvectors belonging to the eigenvalues.

In this research work, notations, concept and some theorems associated with linear systems of equations transformed into matrices such as singular and nonsingular matrices, rank of matrices, transpose of matrix, conjugate transpose, trace of matrix, and consistency conditions for systems of linear equation are presented as background concept to help fully appreciate the research work.

A rectangular grid or array of numbers, symbols, and expressions arranged in rows and columns is known as a matrix. We often write $A := (a_{i,j})_{m\times n}$ to define an $m \times n$ matrix A with each entry in the matrix $A[i,j]$ called a_{ij} for all $1 \le i \le m$ and $1 \le j \le n$. However, the convention that the indices i and j start at 1 is not universal: some programming languages start at zero, in which case we have $0 \le i \le m - 1$ and $0 \le j \le n - 1$. The following are types of matrices: see for example (Strang, 1980 and Kreyszig, 1999)

1. Column vector; $x \in \mathbb{C}^{m\times 1}$
2. Row vector; $y \in \mathbb{C}^{1\times m}$
3. Square matrix; $A \in \mathbb{C}^{n\times n}$
4. Hermitian matrix; $A = \bar{A}^T$
5. Anti Hermitian matrix $A = -\bar{A}^T$

We also, for the purpose of this research, note the following three basic operations associated with matrices.

1. Elementary row or column operations on matrices where rows (columns) are be interchanged
2. Addition; where matrices of same dimensions or same space R^n are added
3. Scalar multiplication; where if A is matrix with scalar k then we have kA, $k \in \mathbb{C}\ or\ \mathbb{R}$.

A symmetric matrix is a square matrix that is equal to its transpose. Because equal matrices have equal dimensions, only square matrices can be symmetric.

The entries of a symmetric matrix are symmetric with respect to the main diagonal. Suppose $A = (a_{ij})$ is an $n \times n$ square matrix then matrix A is called symmetric if $A = A^T$ for all $(a_{ij}) \in \mathbb{R}$. Eigenvectors corresponding to distinct eigenvalues are orthogonal and A^T and A have the same eigenvalues and eigenvectors. The relationship between eigenvalues, eigenvectors and symmetric matrices are established in the following two theorems;

Theorem 1.1. All eigenvalues of a real symmetric matrix are real.

Theorem 1.2. Eigenvectors of distinct eigenvalues of a symmetric real matrix are orthogonal.

If an eigenvalue λ has multiplicity m then we can always find a set of m orthonormal eigenvectors for λ. We conclude that by normalizing the eigenvectors of A, we get an orthonormal set of vectors $u_1, u_2, \dots, u_n$. Thus, if $A \in \mathbb{R}^{n\times n}$ is a symmetric matrix then it has an orthonormal set of eigenvectors $u_1, u_2, \dots, u_n$ corresponding to (not necessarily distinct) eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, then we have the spectral decomposition: $Q^T AQ = \Lambda$ where $Q = \{u_1, u_2, \dots, u_n\}$ is an orthogonal matrix with $Q^{-1} = Q^T$ and $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ is diagonal.

A Hermitian matrix (or self-adjoint matrix) is a complex square matrix that is equal to its own conjugate transpose. Hermitian matrices can be understood as the complex extension of real symmetric matrices. Let $\text{A} = (a_{ij})$ be an $n \times n$ square matrix then matrix A is called Hermitian if $A = (\bar{A})^T$ or $a_{ij} = \bar{a}_{ji}$ for $(a_{ij}) \in \mathbb{C}$ except $(i = j) \in \mathbb{R}$. The conjugate transpose of a matrix A, denoted by A^*, is by taking the complex conjugate of its transpose: $A^* = (\bar{A})^T = \overline{A^T}$. The eigenvalues of $A^*$are the complex conjugates of the eigenvalues of A. However, their eigenvectors are not related. A matrix with only real entries, $a_{ij} \in \mathbb{R}$ for all indexes i and j, is Hermitian if and only if it is a symmetric matrix with respect to its main diagonal entries. The sum of any two Hermitian matrices is Hermitian, but, the product of two Hermitian matrices will only be Hermitian if they commute, thus if for example A and B are two Hermitian matrices then their product will be Hermitian if $AB = BA$. Also, the inverse of an invertible Hermitian matrix is Hermitian. Hermitian matrices are normal matrices. A matrix A is said to be normal if $A\bar{A}^T = \bar{A}^T A$. In other words, a matrix is normal if it has a complete orthonormal set of eigenvectors. Hermitian matrices also obeys the concept of finite-dimensional spectral theorem. Thus, any Hermitian matrix can be diagonalized by a unitary matrix, and that the resulting diagonal matrix has only real entries. This means that all eigenvalues of a Hermitian matrix are real, and eigenvectors with distinct eigenvalues are orthogonal. Examples of real and complex Hermitian matrices are;

$$A = \begin{bmatrix} 6 & 5 & 3 \\ 5 & 4 & 1 \\ 3 & 1 & 7 \end{bmatrix} \text{ and } B = \begin{bmatrix} 7 & 2-3i & 1 \\ 2+3i & 4 & 5i \\ 1 & -5i & 6 \end{bmatrix}$$

Theorem 1.3. Hermitian Matrices have Orthogonal Eigenvectors

Theorem 1.4. The characteristic roots of a Hermitian matrix are real and the characteristic roots of a real symmetric matrix are real.

The trace of a square matrix A is the sum of entries in the main diagonal that is

$$tr(A) = \sum_{i=1}^{n} a_{ij} \ , i = j$$

The secular determinant of an $n \times n$ matrix A is the determinant of $A - \lambda I$. If we put the secular determinant equal to zero we obtain the secular equation or characteristic equation of A.

$$\Delta(\lambda) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

The secular determinant is a polynomial in λ:

$$\Delta(\lambda) = (-\lambda)^n + p_1(-\lambda)^{n-1} + p_2(-\lambda)^{n-2} + \cdots + p_{n-1}(-\lambda) + p_n = 0$$

The coefficient p_1 of $(-\lambda)^{n-1}$ is equal to the trace of A and p_n is the determinant of A. If the field F is algebraically closed such as the field of complex numbers then the fundamental theorem of algebra states that the secular equation has exactly n roots $\lambda_i, i = 1,2, \ldots, n$ the eigenvalues of A and the following factorization holds

$$\Delta(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \ldots (\lambda_n - \lambda)$$

Theorem 1.5. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of a matrix A.Then

$$Tr(A) = \sum_{i=1}^{n} \lambda_i$$

Theorem 1.6. The trace of a matrix is invariant under a similarity transformation

$Tr(B^{-1}AB) = Tr(A)$

Theorem 1.7. Let S be a symmetric matrix, $S^T = S$, and A be an anti-symmetric matrix, $A^T = -A$. Then $Tr(SA) = Tr(AS) = 0$

Definition: A square matrix is A is idempotent if $A^2 = A$ and nilpotent if $A^m = 0$ for some integer greater than 1.

Theorem 1.8. If A is idempotent then rank$(A) = tr(A)$

A square matrix is non-singular if and only if its determinant is nonzero (Lipschutz 1991, p. 45). In other words an $n \times n$ matrix A is called non–singular or invertible if there exists an $n \times n$ matrix B such that $AB = I_n = BA$. Any matrix B with the above property is called an inverse of A. If A does not have an inverse, A is called singular. That is, a matrix is singular if and only its determinant is 0.

Theorem 1.9.: If the coefficient matrix A of a system of n equations in n unknowns is non–singular, then the system $AX = B$ has the unique solution $X = A^{-1}B$.

Theorem 1.1.0. If A is an $n \times n$ non–singular matrix, then the homogeneous system $AX = 0$ has only the trivial solution $X = 0$. Hence if the system $AX = 0$ has a non–trivial solution, A is singular.

The rank of a matrix is the order of the largest square matrix whose determinant is not zero or equivalently, rank of a matrix is the maximum number of linearly independent column vectors in the matrix or the maximum number of linearly independent row vectors in the matrix. The rank of matrix simply measures the non-singularity of a matrix. In this work we denote the rank of a matrix by rank(A). If the rank of an $n \times n$ matrix is smaller than n, the determinant will be zero. The rank of a matrix is computed for both square and non-square matrices. There are various methods for computing the rank of a matrix, this research uses determinant of a matrix and minors of matrices, other method include the Gauss elimination to reduce to echelon form.

Properties of rank: (Banerjee, Sudipto, Roy, Anindya, 2014)

Given the linear map $f(x) = Ax$, where A an $m \times n$ matrix, then the following properties hold;

1. The rank of an $m \times n$ matrix is a nonnegative integer and cannot be greater than m or n. That is $\text{rank}(A) \leq \min(m, n)$. A matrix that has rank $\min(m, n)$ is said to have full rank; otherwise, the matrix is rank deficient.
2. Only a zero matrix has zero rank
3. f is injective (one to one) if and only if A has rank $= n$ (full column rank)
4. f is surjective (onto) if and only if A has rank $= m$ (full row rank)
5. If A is a square matrix $(m = n)$, then A is invertible if and only if A has rank n (full rank or determinant is nonzero)
6. If A is a matrix over the complex numbers and $\bar{A}$ denote complex conjugate of A and A^* the conjugate transpose of A (i.e., the adjoint of A), then $\text{rank}(A) = \text{rank}(\bar{A}) = \text{rank}(A^T) = \text{rank}(A^*) = \text{rank}(A^T A) = \text{rank}(AA^T)$

One useful application of calculating the rank of a matrix is the computation of the number of solutions of a system of linear equations. The solution is unique if and only if the rank equals the number of variables. For instance in control theory, the rank of a matrix can be used to determine whether a linear system is controllable, or observable. In the field of communication complexity, the rank of the communication matrix of a function gives bounds on the amount of communication needed for two parties to compute the function.

A linear system is consistent if and only if its coefficient matrix has the same rank as does its augmented matrix otherwise it is inconsistent. A system of linear equations can have one solution, an infinite number of solutions, or no solution. Systems of equations can be classified by the number of solutions.

1. If a system has at least one solution, it is said to be consistent.
    a. If a consistent system has exactly one solution, it is independent. Thus, a system with the same number of equations and unknowns
    b. If a consistent system has an infinite number of solutions, it is dependent. When you graph the equations, both equations represent the same line.
2. If a system has no solution, it is said to be inconsistent. The graphs of the lines do not intersect, so the graphs are parallel and there is no solution. In other words, a system with more equations than unknowns has no solution.

Statement of the Problem

This research seeks a systematic way to solve the inverse eigenvalue problem of singular symmetric and Hermitian matrices of rank greater than or equal to four. We define a map between a space of eigenvalues and the space of the corresponding singular Hermitian matrices. Thus every symmetric matrix has a reflectional symmetry in the main diagonal therefore for the singular matrices there would be the need to reduce the number of independent matrix elements from $\frac{n(n+1)}{2}$ to the extent that there is an invertible (isomorphism) map between such matrix elements and the non zero eigenvalues. Specifically, the research problem can be stated as follows;

1. Given non-zero scalars $\{\lambda_1, \lambda_2, \dots, \lambda_r\} \subset F, F = \mathbb{R}, r < n$ and some parameters, we generate a matrix $A_{(n,r)}$ such that $\text{diag}(A) = \{\lambda_1, \lambda_2, \dots, \lambda_r\}$,$\det(A) \neq 0$ for $n = r$ and $A^T = A$.

2. Given non-zero scalars $\{\lambda_1, \lambda_2, \dots, \lambda_r\} \subset F, F = \mathbb{R}, r < n$ and some parameters, we generate a matrix $A_{(n,r)}$such that $\text{diag}(A) = \{\lambda_1, \lambda_2, \dots, \lambda_r\}$, $\det(A) \neq 0$ for $n = r$ and $A^{\bar{T}} = A$.

Purpose of Study

The purpose of this study is to construct singular symmetric and Hermitian matrices with rank greater than or equal to four that maintain certain specific structure as well as satisfy a given spectral property. For IEP, two fundamental questions arise that is the theoretic issue on solvability and the practical issue on computability. On solvability, we seek to determine a necessary and a sufficient condition under which an IEP has a solution.

Theorem 1.11. A necessary and sufficient condition for the existence of a matrix with eigenvalues $\lambda_1, \dots, \lambda_n$ and main diagonal elements $a_1, \dots, a_n$ is that

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} \lambda_i$$

Naturally, the next level of question is to find any connection between the main diagonal entries and the singular values of a general matrix, as was posed by (Mirsky, 1964). Such a relationship was discovered independently by (Sing, 1976) and (Thompson, 1977). Similar to the notion of majorization, it turns out that the necessary and sufficient conditions for the existence of a matrix with prescribed main diagonal entries and prescribed singular values also involve a set of inequalities which we state as follows.

Theorem 1.12. (Sing-Thompson theorem). Let $d, s \in \mathbb{R}^n$ be two vectors with entries arranged in the order $s_1 \geq s_2 \ldots \geq s_n$ and $|d_1| \geq |d_2| \geq \cdots \geq |d_n|$respectively. Then there exists a real matrix $A \in \mathbb{R}^{n\times n}$ with singular values s and main diagonal entries d (possibly different order) if and only if

$$\sum_{i=1}^{n} |d_i| \leq \sum_{i=1}^{n} s_i$$

for all $i = 1,2, \ldots, n$ and

$$\left(\sum_{i=1}^{n-1} |d_i|\right) - |d_n| \leq \left(\sum_{i=1}^{n-1} s_i\right) - s_n$$

The main concern in computability, on the other hand, has been to develop a procedure by which, knowing a priori that the given spectral data are feasible, a matrix can be constructed numerically. In this work, we would review previous results obtained by (Annor, Gyamfi, & Boadi, 2016) in respect of the inverse eigenvalue problem for singular Hermitian matrices of rank 2 and 3. Thus our specific objectives will include;

- Construct singular symmetric matrices of rank greater than or equal to four from their eigenvalues.
- Construct singular Hermitian matrices of rank greater than or equal to four from their eigenvalues.

Significance of the Study

The current state of detail research done in the area of inverse eigenvalue problem for singular Hermitian matrices to the best of the researcher's knowledge and findings is up to rank 3, so an extension to rank greater than or equal to four will add up to the academic knowledge.

Delimitation

In determining the solutions of inverse eigenvalue problem for singular symmetric Hermitian matrices of rank greater than or equal to four, the study could have considered inverse eigenvalue problem for nonsingular symmetric matrices and skew Hermitian (anti-Hermitian). However, for the sake of this study, emphasis was laid on inverse eigenvalue problem for singular symmetric Hermitian matrices of rank greater than or equal to four.

Limitation

The research work covered solutions of inverse eigenvalue problem on singular symmetric Hermitian matrices of rank greater than or equal to four, thus the research work on inverse eigenvalue problem is limited to singular symmetric

Hermitian matrices of rank greater than or equal to four. Also, in this research we will restrict ourselves to mappings within the same space, such as $\boldsymbol{T}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\boldsymbol{T}: \mathbb{C}^n \rightarrow \mathbb{C}^n$ then $\boldsymbol{T}$ will be associated with a square $n \times n$ matrix.

Definition of Terms

Hermitian matrices: Complex matrices that are equal to their conjugate transpose.

Inverse eigenvalue problem: The reconstruction of a matrix from its eigenvalue.

Parameters: The important physical quantities that modulate an inverse problem.

Rank: The order of the largest square matrix with non-zero determinants.

Singular matrices: Matrices whose determinants are zero.

Symmetric matrices: Matrices equal to their transpose.

Organization of the Study

The study basically is made up of five chapters of which Chapter one has already been discussed. The rest of the Chapters have been outlined as follows; Chapter two is on review of related literature on solutions of inverse eigenvalue problem. Chapter three deals with research methodology and approach. Main results of the study is presented in the fourth Chapter. The fifth Chapter covers the summary, conclusions and recommendation of the study.

CHAPTER TWO

LITERATURE REVIEW

Introduction

This chapter seeks to review the work of some authors on the solutions of inverse eigenvalue problem of singular Hermitian matrices of specified rank up to three. The researcher first and foremost as a matter of recognizing the fact that some but limited literatures exist on the research topic decided to look at the various forms of the inverse eigenvalue problems so far solved on symmetric matrices. However, despite a critical look at the research on inverse eigenvalue problem, note must be taken that inverse eigenvalue problems are characterized based on the mathematical attributes exhibited by the inverse eigenvalue problem. A physical process is often described by a mathematical model of which the parameters represent important physical quantities. An important step in the construction of a mathematical model for engineering applications is to verify the model by comparing the predicted behavior of the model with experimental results and then to update the model to more accurately represent the physical process. An inverse eigenvalue problem amounts to one such modeling process in which quantities are represented in terms of matrices whereas the comparison is based upon the spectral information and the update is governed by the underlying structure constraint. Studies on inverse eigenvalue problems have been intensive, ranging from engineering application to algebraic theorization. Yet the results are scattered even within the same field of discipline.

In this research, inverse eigenvalue problems are discussed based on their characteristics such parameterized, addictive, multiplicative, partially described and structured. According to Gladwell (1986) as cited by (Chu & Golub, 2005) Figure 1 indicates the overlapping mathematical attribute of the forms of inverse eigenvalue problem thus Multi-Variate Inverse Eigenvalue Problem (MVIEP), Least Square Inverse Eigenvalue Problem(LSIEP), Parameterized Inverse Eigenvalue Problem (PIEP), Structured Inverse Eigenvalue Problem (SIEP), Partially Described Inverse Eigenvalue Problem (PDIEP), Additive Inverse Eigenvalue Problem (AIEP) and Multiplicative Inverse Eigenvalue Problem (MIEP).

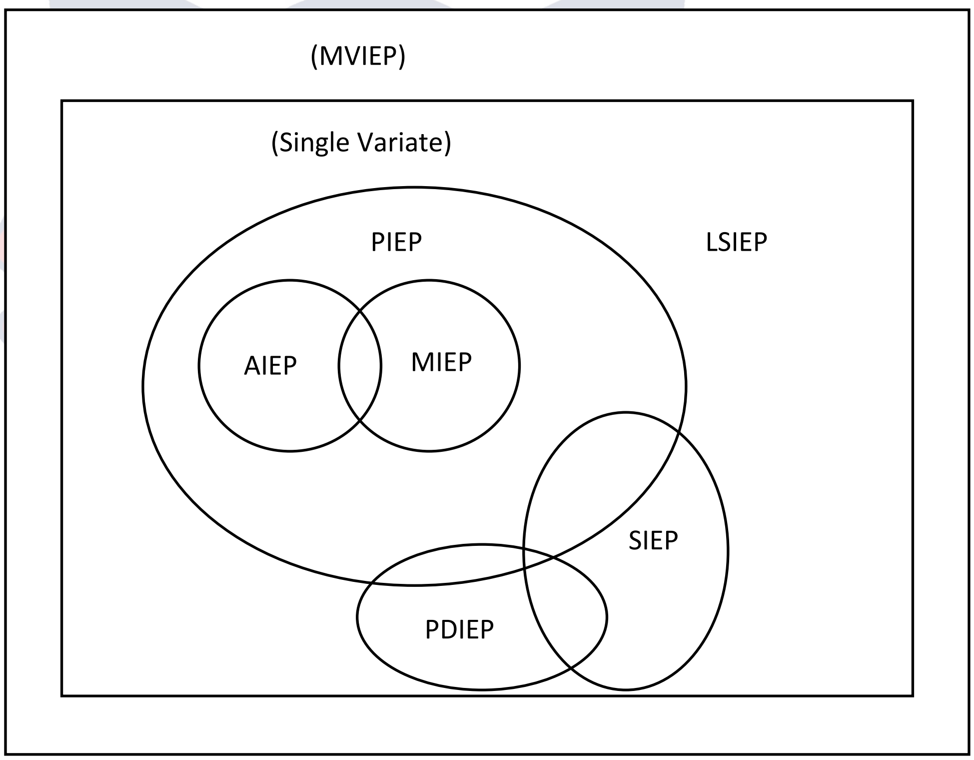

Figure 1: A Venn diagram showing the overlapping mathematical attribute of the forms of inverse eigenvalue problem, Chu and Golub, (2005).

The MVIEP which deals with having knowledge on the partition pattern as well as the spectral information to help determine whether a sample matrix can be constructed is an unexplored territory because most of the studies have been for the single variate inverse eigenvalue problem which deals with spectral information to help construct a sample matrix. Our research work is largely a single variate inverse eigenvalue problem but before we discuss in detail our research work, we will discuss the various structures or forms of the single variate inverse eigenvalue problem and what each of them seeks to achieve. In the PIEP, which is sub divided into two that is AIEP and MIEP, emphasis is on the way that parameters modulate a problem. In SIEP, our attention will be on the structure that a solution matrix is supposed to maintain, in LSIEP the focus is on the best solution that exists only in the sense of least squares approximation and finally the PDIEP arises when there are simply no reasonable tools available to evaluate the entire spectral information due to, for instance, the complexity or the size of the physical system.

Parameterized Inverse Eigenvalue Problem

In general, given a family of matrices $A(c) \in M$ with $c = \{c_1, \dots c_m\} \in \mathrm{F}^m$, where F is a scalar field of real or complex numbers and scalars $\{\lambda_1, \dots, \lambda_n\} \subset \mathrm{F}$, find a parameter c such that the spectrum of the family of matrices that is $\sigma\big(A(c)\big) = \{\lambda_1, \dots, \lambda_n\}$. Note that the number m of parameters in c may be different from n. Depending upon how the family of matrices $A(c)$ is specifically defined in terms of c, the PIEP can appear and be solved very differently. Inverse eigenvalue problems in the above PIEP format arise frequently in discrete modeling (Gladwell,1984; Hald, 1972; Osborne, 1971) and factor analysis (Harman, 1967) both references cited in the work of (Chu & Golub, 2005).We now discuss the two main forms of PIEP that is the AIEP and MIEP.

Addictive inverse eigenvalue problem (AIEP)

Given a matrix $A \in M$, scalars$\{\lambda_1, \dots, \lambda_n\} \subset$ F, where F is a scalar field of real numbers and a class of matrices N, find a matrix $X \in N$such that $\sigma(A+X) = \{\lambda_1, \dots, \lambda_n\}$. Thus, AIEP is a special case of the PIEP with $A(X) = A + X$ and X playing the role of c and A perturbed by the addition of a specially structured matrix X in order to match the eigenvalues. Another form of AIEP, $M = H(n)$, F $= \mathbb{R}, N = D_R(n)$ (Downing and Householder, 1956) as cited by (Chu & Golub, 2005) represent a special case of another form of the AIEP that is $M = S(n)$, F $= \mathbb{R}, N = D_R(n)$, with M being a Jacobi matrix is of particular interest because the discretization of the boundary value problem, for example,

$$-u''(x) + p(x)u(x) = \lambda u(x),$$

$$u(0) = u(\pi) = 0,$$

by the central difference formula with uniform mesh $h = \frac{\pi}{n+1}$ naturally leads to the eigenvalue problem in tridiagonal structure,

$$\left(\frac{1}{h^2}[M + X]\right)u = \lambda u$$

where X is a diagonal matrix representing the discretization of $p(x)$. Thus, $M = S(n)$, F $= \mathbb{R}, N = D_R(n)$ may be interpreted as a discrete analog of the inverse Strum-Liouville problem, a classical subject where the potential $p(x)$ is unknown but the spectrum is given to be used to determine the potential $p(x)$ so that the system possesses a prescribed spectrum. Another form of the AIEP with the structure $M = R(n)$, we find $X \in N$ where $\sigma(M + X)$ lies in a certain fixed region, say the right-half, of the complex plane. This form of the AIEP can be found in the field of control or algorithm design where stability is at stake. In such a problem it is more practically critical to have eigenvalues located in a certain region than at a certain points. Also, related to this form AIEP is the nearest unstable matrix problem (Byers, 1988) as cited by (Chu & Golub). The problem concerns the distance from a given matrix, stable in the sense that all its eigenvalues have negative real part, to the nearest matrix with one eigenvalue on the imaginary axis. Also related is the communality problem in factor analysis (Harman, 1967) as cited by (Chu & Golub) and the educational testing problem (Chu and Wright, 1995; Fletcher, 1985). The former concerns finding a diagonal matrix D so that the sum $A + D$ in which A is a given real symmetric matrix with zero diagonal entries has as many zero eigenvalues as possible. The latter concerns finding a positive diagonal matrix D so that the difference $A - D$ in which A is a given real symmetric positive definite matrix remains positive semi-definite while the trace (D) is maximized.

There is a rich literature on both the theoretic and the numerical aspects for the AIEP, notably we have the following main result addressing the existence question for the AIEP with the form$M = C(n)$, $\mathrm{F} = \mathbb{C}, N = D_R(n)$ by (Friedland, 1977) as cited by (Chu & Golub, 2005), which is stated without proof;

Theorem 2.1. For any specified $\{\lambda_1, \dots, \lambda_n\}$,the AIEP with the form $M = C(n)$, $F = \mathbb{C}, N = D_R(n)$ is solvable.

The number of solutions is finite and does not exceed $n!$. Moreover, for almost all $\{\lambda_1, \dots, \lambda_n\}$, there are exactly $n!$ solutions. However, the existence question for the following forms of AIEP with structure $M = S(n)$, $F = \mathbb{R}, N = D_R(n)$or $M = R(n)$, $F = \mathbb{R}, N = D_R(n)$ has yet to be settled.

Multiplicative inverse eigenvalue problem (MIEP)

Given a matrix $A \in M$, scalars$\{\lambda_1, \dots, \lambda_n\} \subset F$, and a class of matrices N, find a matrix $X \in N$such that $\sigma(XA) = \{\lambda_1, \dots, \lambda_n\}$. The MIEP takes the form of $A(X) = XA$ with X representing the parameter c in the general PIEP. MIEP is obtained by pre-multiply a given matrix A by a specially structured matrix X to reposition or to precondition the distribution of its eigenvalues. The MIEP can arise from engineering application, (Chu and Golub, 2005 & Yamamoto, 1990). For example, consider the vibration of particles on a string sketched in the Figure below,

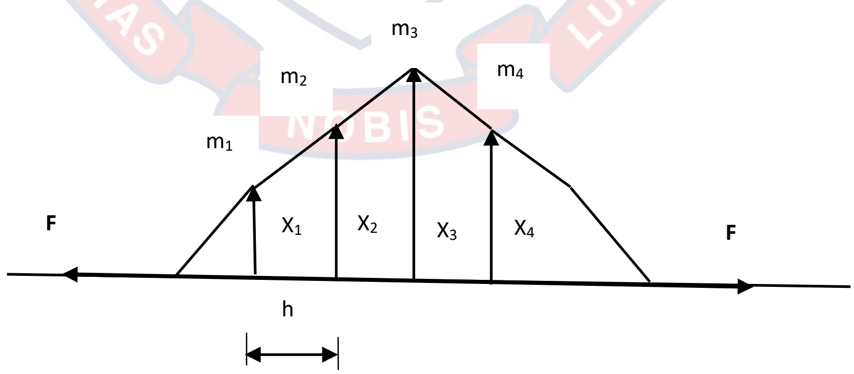

Figure 2: A graph showing the vibration of particles on a string, Chu and Golub, (2005)

where four particles, each with mass $m_i, i = 1,2 \dots 4$, are uniformly spaced with distance h and are vibrating vertically subject to the horizontal tension F. Then the equation of motion is given by (Zhou et al.,1991):

$$m_1 \frac{d^2 x_1}{dt^2} = -F \frac{x_1}{h} + F \frac{x_2 - x_1}{h}$$

$$m_2 \frac{d^2 x_2}{dt^2} = -F \frac{x_2 - x_1}{h} + F \frac{x_3 - x_2}{h}$$

$$m_3 \frac{d^2 x_3}{dt^2} = -F \frac{x_3 - x_2}{h} + F \frac{x_4 - x_3}{h}$$

$$m_4 \frac{d^2 x_4}{dt^2} = -F \frac{x_4 - x_3}{h} - F \frac{x_4}{h}$$

Which can be summarized as the system

$$\frac{d^2 x}{dt^2} = -DAx \qquad (2.1)$$

where A is a tridiagonal real symmetric matrix, $x = [x_1, x_2, \dots, x_n]^T$ and

$D = diag(d_1, d_2, \dots, d_n)$ with

$d_i = \frac{F}{m_i h}$.

To solve (2.1)we consider the eigenvalue problem,

$$DAx = \lambda x$$

where λ is the square of the natural frequency of the system. The inverse problem then amounts to calculating the mass $m_i, i = 1,2,, \dots ,4$ so that the resulting system vibrates at a prescribed natural frequency. Similarly, a discretization of the boundary value problem

$$-u''(x) = \lambda p(x)u(x)$$

yields the eigenvalue problem

$$Au = \lambda Xu, \tag{2.2}$$

where X is a positive diagonal matrix representing $p(x)$. Thus, an MIEP is to determine the density function $p(x) > 0$ from the prescribed spectrum. A conservative, n degrees of freedom mass-spring system with mass matrix X and stiffness matrix A also ends with the formulation (2.2). Since the physical realizability of the stiffness matrix A usually is more complex than the mass matrix X, a practical way of ensuring the overall physical realizability in engineering design is to determine A from static constraints and then to find a positive diagonal matrix X so that some desired natural frequencies are achieved.

In the complex context, as cited by Chu and Golub (2005), (Downing & Householder, 1956) proposed another form of the MIEP with the structure; suppose a matrix $A \in H_n$ thus all Hermitian matrices and scalars $\{\lambda_1, \dots, \lambda_n\} \subset \mathbb{R}$, find a matrix $X \in D_R(n)$ such that $\sigma(X^{-1}AX^{-1}) = \{\lambda_1, \dots, \lambda_n\}$. Also, although in practice one does not need a preconditioner that exactly repositions the eigenvalues, an understanding of the MIEP might shed some insights into the design of a good preconditioner. In line this as cited by Chu and Golub, Friedland (1977) proved using degree theory theorem 2.2 where his results on the solution suggests that in the complex context a matrix can be perfectly conditioned by a diagonal matrix.

Theorem 2.2. If all principal minors of a matrix are distinct from zero, then this structure of MIEP in the form, given a matrix $M \in C(n)$, $F = \mathbb{C}, M \in D_R(n)$, is solvable for arbitrary $\{\lambda_1, \ldots, \lambda_n\}$ and there exist at most $n!$ distinct solutions.

Partially Described Inverse Eigenvalue Problem (PDIEP)

This form of inverse eigenvalue problem arise when not all the set of eigenvalues are known due to the complexity or the size of the physical system one cannot find any reasonable analytical tools to evaluate the entire spectral information. It takes the form; given vectors $\{v^{(1)}, \ldots, v^{(k)}\} \subset F^n$ and scalars $\{\lambda_1, \ldots, \lambda_k\} \subset F$ where $1 \leq k < n$, find a matrix $X \in N$ such that $Xv^{(i)} = \lambda_i v^{(i)}$ for $i = 1, \ldots, k$. For example, consider the dynamical system

$$M\frac{d^2}{dt^2}v + C\frac{d}{dt}v + Kv = 0, \quad (2.3)$$

where M, C and K are symmetric and in addition M is positive definite, that arises in a wide range of applications. Upon separation of variables, the system naturally leads to the quadratic λ −matrix problem:

$$P(\lambda)x = 0$$

with the quadratic pencil

$$P(\lambda) = M\lambda^2 + C\lambda + K. \quad (2.4)$$

Suppose now a state feedback forcing function of the form

$$u(t) = b\left(f^T\frac{d}{dt}v(t) + g^T v(t)\right),$$

where, $b, f, g \in \mathbb{R}^n$ are constant vectors, is applied to the system. The resulting closed loop system leads to the λ −matrix problem with pencil

$$Q(\lambda) = M\lambda^2 + (C - bf^T)\lambda + (K - bg^T).$$

The goal of this feedback control $u(t)$ according to Chu and Golub (2005) is to relocate those bad eigenvalues in the quadratic pencil (2.4) that either are unstable or lead to large vibration phenomena in the dynamic system (2.3) while maintaining those good eigenvalues. Thus, modifying the behaviour of a dynamical system by the feedback control $u(t)$ result in a partial pole assignment. Similar to the above PDIEP is also another form of PDIEP which takes the form; given matrices M, C, K, its associated eigenvalues $\{\lambda_1, \dots, \lambda_{2n}\}$ of the quadratic pencil $P(\lambda) = M\lambda^2 + C\lambda + K$, a fixed vector $b \in \mathbb{R}^n$, and m complex numbers $\{\mu_1, \dots, \mu_m\}, m \leq n,$ find $f, g \in \mathbb{C}^n$ such that the spectrum of the closed loop pencil $Q(\lambda) = M\lambda^2 + (C - bf^T)\lambda + (K - bg^T)$ has spectrum $\{\mu_1, \dots, \mu_m, \lambda_{m+1}, \dots, \lambda_{2n}\}$. Hochstadt, (1967) through Theorem 2.3 provides solution for this type of PDIEP.

Theorem 2.3. Let the eigenvector matrix and eigenvalue matrix of $P(\lambda) = M\lambda^2 + C\lambda + K$ be partitioned into $X = [X_1, X_2]$ and $\Lambda = diag(\Lambda_1, \Lambda_2)$, respectively, where $X_1 \in \mathbb{C}^{n\times m}, X_2 \in \mathbb{C}^{n\times(2n-m)}, \Lambda_1 \in D_n(m)$ and $\Lambda_2 \in D_n(2n - m)$. Define $\beta = [\beta_1, \dots, \beta_m]^T \in \mathbb{C}^m$ by

$$\beta_1 \coloneqq \frac{1}{b^T x_j} \frac{\mu_j - \lambda_j}{\lambda_j} \prod_{i=1, i\neq j}^{m} \frac{\mu_i - \lambda_j}{\lambda_i - \lambda_j}$$

Then the pair of vectors

$$f \coloneqq MX_1\Lambda_1\beta$$

$$g \coloneqq -KX_1\beta$$

solve the partially described inverse eigenvalue problem.

Structured Inverse Eigenvalue Problem (SIEP)

The most focused inverse eigenvalue problems are the structured problem where a matrix with a specified structure as well as a designated spectrum is sought after. For the purpose of this research, more attention is focused on the SIEP, due the similarities with this structure of the single variate inverse eigenvalue problem. A generic structured inverse eigenvalue problem may be stated as follows. Given scalars $\{\lambda_1, \dots, \lambda_n\} \in \mathrm{F}$, find $X \in N$ which consists of specially structured matrices such that $\sigma(X) = \{\lambda_1, \dots, \lambda_n\}$. Where N denotes subset of square matrices. By demanding X to belong to N, where a structure is defined, the structured inverse eigenvalue problem is required to meet both the spectral constraint and the structural constraint. The structural constraint usually is imposed due to the realizability of the underlying physical system. In literature, many variations to the structured inverse eigenvalue problem (SIEP) have been studied and for the sake of difference we choose to label the SIEP numerically to differentiate them. Thus,

1. $\mathrm{F} = \mathbb{R}$ and $N = \{\text{All Toeplitz matrices in S(n)}\}$ (Delsarte and Genin, 1984; Friedland, 1992; Landau, 1994; Trench, 1996).
2. $\mathrm{F} = \mathbb{R}$ and $N = \{\text{All per} - \text{symmetric Jacobi matrices in S(n)}\}$ (Boley and Golub, 1984; Hochstadt, 1967).
3. $\mathrm{F} = \mathbb{R}$ and $N = \{\text{All nonnegative matrices in S(n)}\}$ (Chu and Driessel, 1991)
4. $\mathrm{F} = \mathbb{R}$ and $N = \{\text{All nonnegative matrices in R(n)}\}$

5. $\mathrm{F} = \mathbb{C}$ and $N = \{\text{All row} - \text{stochastic matrices in } \mathrm{R(n)}\}$ (Chu and Guo, 1996)

Among the variations of the SIEP, which has gained extensive research in literature is also the inverse eigenvalue problem for Jacobi and periodic Jacobi matrices. For example, given scalars $\{\lambda_1, \ldots, \lambda_n\}$ and $\{\mu_1, \ldots, \mu_{n-1}\} \subset \mathbb{R}$ that satisfy the interlacing property $\lambda_i \leq \mu_1 \leq \lambda_{i+1}$ for $i = 1, \ldots, n-1$, find a Jacobi matrix J so that $\sigma(J) = \{\lambda_1, \ldots, \lambda_n\}$ and $\sigma(\check{J}) = \{\mu_1, \ldots, \mu_{n-1}\}$ where $\check{J}$ is the leading $(n-1) \times (n-1)$ principal submatrix of J(Boley and Golub, 1987; Gladwell, 1986) as cited by (Chu & Golub, 2005). One of the methods for constructing such symmetric Jacobi matrices with two sets of eigenvalues is as follows;

Given a Jacobi matrix H which is a real symmetric tridiagonal matrix for the form:

$$H = \begin{bmatrix} a_1 & b_1 & \cdots & 0 \\ b_1 & a_2 & \cdots & 0 \\ 0 & \cdots & \cdots & b_{n-1} \\ 0 & \cdots & b_{n-1} & a_n \end{bmatrix}$$

with $b_i > 0$. We form arbitrary symmetric matrix $H = \begin{bmatrix} a_{11} & \hat{b}^t \\ \hat{b} & K \end{bmatrix}$ whose eigenvalues are $\{\lambda_i\}_1^n$ and whose lower principal sub matrix K has eigenvalues $\{\mu_i\}_1^{n-1}$. These eigenvalues are distinct and satisfy the following interlacing property $\lambda_i \leq \mu_1 \leq \lambda_{i+1}$ for $= 1, \ldots, n-1$. We compute the first row eigenvectors of the matrix H using the relation

$$q_{1i^2} = \frac{\prod_{j=1}^{n-1}(\mu_i - \lambda_i)}{\prod_{j=1, j\neq i}^{n}(\mu_i - \lambda_i)}$$

All the eigenvectors are normalized to have norm equal to 1 and a_{11} is obtained from the relation$\sum_1^n \lambda_i - \sum_1^{n-1} \mu_i$ (Er-Xiong, 2003). Lanczos algorithm is then applied to construct the tridiagonal matrix from the arbitrary symmetric tridiagonal matrix H.This work deals with a modification of an algorithm that solve a special Structured Inverse Eigenvalue Problems (SIEP). The problem we consider is the structured Hermitian Inverse Eigenvalue Problem (SHIEP) where the researcher's purpose is to find the solution of inverse eigenvalue problem of singular Hermitian matrix of rank greater than or equal to four. According to Gladwell (1986) as cited by Chu and Golub, the Jacobi matrices enjoys an interesting physical interpretation in vibrations. Thus, it may be regarded as identifying the spring configurations of an undamped system from its spectrum and the spectrum of the constrained system where the last mass is restricted to have no motion. When the damper comes into the system, the question becomes an inverse eigenvalue problem for symmetric quadratic pencil which is of the form;

Given two sets of distinct eigenvalues $\{\lambda_1, \dots, \lambda_{2n}\}$ and $\{\mu_1, \dots, \mu_{2n-2}\} \in \mathbb{C}$, find tridiagonal symmetric matrices M and N such that the determinant $\det(Q(\lambda))$ of the $\lambda - matrix Q(\lambda) = \lambda^2 I + \lambda M + N$ has zeros precisely $\{\lambda_1, \dots, \lambda_{2n}\}$ and $\det(\tilde{Q}(\lambda))$ has zeros precisely $\{\mu_1, \dots, \mu_{2n-2}\}$ where $\tilde{Q}(\lambda)$ is obtained by deleting the last row and the last column of $Q(\lambda)$ (Ram and Elhay, 1996).

The inverse eigenvalue problem for the periodic Jacobi matrix has the generic form, thus, given scalars $\{\lambda_1, \dots, \lambda_n\}$ and $\{\mu_1, \dots, \mu_{n-1}\} \subset \mathbb{R}$ satisfying $\lambda_i \le \mu_1 \le \lambda_{i+1}$ for $i = 1, \dots, n-1$, and a positive number β, find a periodic Jacobi matrix J_n of the form

$$J_n = \begin{bmatrix} a_1 & b_1 & & \cdots & & b_n \\ b_1 & a_2 & b_2 & & & 0 \\ 0 & b_2 & a_3 & & & 0 \\ \vdots & & & \ddots & & \\ & & & & a_{n-1} & b_{n-1} \\ b_n & 0 & \cdots & & b_{n-1} & a_n \end{bmatrix}$$

and also form the following matrices from J_n

$$J^+ = \begin{bmatrix} a_1 & (b^+)^t \\ b^+ & K \end{bmatrix}$$

and

$$J^- = \begin{bmatrix} a_1 & (b^-)^t \\ b^- & K \end{bmatrix}$$

so that $\sigma(J_n) = \{\lambda_1, \dots, \lambda_n\}$ and $\sigma(J_{n-1}) = \{\lambda_1, \dots, \lambda_n\}$ where K is a Jocobi matrix given by J_{n-1} which is also the leading $(n-1) \times (n-1)$ principal submatrix of J_n, and $\prod_1^n b_i = \beta$ (Boley and Golub, 1987; Boley and Golub, 1984; Ferguson, 1980). This inverse problem normally arises in inverse scattering theory problems. The periodic Jacobi matrix is a tridiagonal matrix with real entries. The eigenvalues are therefore real and their corresponding eigenvectors are orthonormal. This tridiagonal matrix is constructed with two sets of eigenvalues, the eigenvalues of the main matrix and the eigenvalues of the leading principal sub matrix and a set of scalars. The solution of the periodic Jacobi matrix is not unique and the number of solution is at most 2^{n-m-1}, where m is the number of common eigenvalues of the main matrix and the leading principal submatrix. The eigenvalue of J and $\check{J}$ satisfy an interlacing property which shows that they have common eigenvalues. The following theorem therefore provides the necessary and sufficient conditions for the two matrices to have common eigenvalues, (Xu and Jiang, 2006). We denote the first component of s_i as s_{1i} and the last one as $s_{n-1,i}$.

Theorem 2.4. For $j \in \{1, 2, \ldots, n-1\}$, μ_i is an eigenvalue of J_n if and only if

$$b_n s_{1j} + b_{n-1} s_{n-1,j} = 0 \tag{2.5}$$

$b_n, n = 1,2, \ldots, n$ are the eigenvalues of the matrix J_{n-1} which are represented by the single scalar quantity β and s_n are the components of the matrix.

Proof. .

Let $y^t = (b_n, 0, 0, \ldots, b_{n-1}) \in R^{n-1}$, then

$$J_n = \begin{bmatrix} J_{n-1} & y \\ y^t & a_n \end{bmatrix}$$

$$\det(\lambda I - J_n) = det \begin{bmatrix} \lambda I - J_{n-1} & -y \\ 0 & \lambda - a_n - y^t(\lambda I - J_{n-1})^{-1} y \end{bmatrix}$$

$$= (\lambda I - J_{n-1})(\lambda - a_n - y^t(\lambda I - J_{n-1})^{-1} y)$$

$$= \prod_{i=1}^{n-1} (\lambda - \mu_i)(\lambda - a_n - y^t(\lambda I - J_{n-1})^{-1} y)$$

$(\lambda I - J_{n-1})^{-1}$ can be expressed as $(\lambda I - J_{n-1})^{-1} = \sum_{i=1}^{n-1} \frac{1}{\lambda - \mu_i} s_i s_i^t$, where μ_i are the eigenvalues of K and s_i are the components of the matrix J_{n-1} and s_i^t is the transpose of s_i. Therefore,

$$y^t(\lambda I - J_{n-1})^{-1} y = \sum_{i=1}^{n-1} \frac{(s_i^t y)^2}{\lambda - \mu_i} = \sum_{i=1}^{n-1} \frac{(b_n s_{1i} + b_{n-1} s_{n-1,i})^2}{\lambda - \mu_i}$$

and

$$det(\lambda I - J_n) = \prod_{i=1}^{n-1} (\lambda - \mu_i)(\lambda - a_n - \sum_{i=1}^{n-1} \frac{(b_n s_{1i} + b_{n-1} s_{n-1,i})^2}{\lambda - \mu_i} \quad (2.6)$$

From equation (2.6) we know that

$$det(\mu_i I - J_n) = -\prod_{i=1, i \neq j}^{n-1} (\mu_i - \mu_i) x (b_n s_{1i} + b_{n-1} s_{n-1,i})^2$$

This implies that μ_j is an eigenvalue of J_n if and only if equation (2.3) is valid. Now, on the other hand, if the two matrices J_n and J_{n-1} have no common eigenvalues, that is, the eigenvalues are distinct, then the following theorem holds.

Theorem 2.5. If

$$b_n s_{1i} + b_{n-1} s_{n-1,j}, \text{ where } j = 1,2, \dots, n-1 \quad (2.7)$$

then the eigenvalues of the matrix J_n are equal to n roots of the following equation:

$$F(\lambda) = \lambda - a_n - \sum_{i=1}^{n-1} \frac{\left(b_n s_{1i} + b_{n-1} s_{n-1,j}\right)^2}{\lambda - \mu_i} = 0$$

and μ_i strictly separate λ_i as follows:

$$\lambda_1 < \mu_1 < \lambda_2 < \cdots < \lambda_{n-1} < \mu_{n-1} < \lambda_n \tag{2.8}$$

Proof. (Jiang, 2003).

Applying theorem (2.5), we can conclude that, under condition (2.7), for $i = 1,2,\dots,n-1, \mu_i$ are the eigenvalues of J_n. Combining this with (2.6), we know $det(\lambda I - J_n) = 0$, is equivalent to equation (2.9).

As $\left(b_n s_{1i} + b_{n-1}s_{n-1,j}\right)^2 > 0, i = 1,2,\dots,n-1$ for a sufficiently small positive number,

$$F(\mu_i - \varepsilon) > 0 \text{ and } F(\mu_i + \varepsilon) < 0, i = 1,2,\dots,n-1,$$

$$F(-\infty) < 0 \text{ and } F(+\infty) > 0. \text{ Hence (2.6) holds.}$$

Suppose some of the eigenvalues of J_{n-1} are the eigenvalues of J_n. The following theorem therefore gives the relationship between the rest of the eigenvalues of J_n that do not belong to J_{n-1}.

Theorem 2.6. Let $N = \{1,2,\dots,n-1\}$. If there is a set $N_1 = \{i_1, i_2, \dots, i_m\} \subset N$ such that $b_n s_{1j} + b_{n-1}s_{n-1,j} = 0, j \in N_1$ and $b_n s_{1j} + b_{n-1}s_{n-1,j} \neq 0, j \in \frac{N}{N_1}$ then $\mu_1, \mu_2, \dots, \mu_m$ are the eigenvalues of J_n, and the rest of the eigenvalues of J_n are given as $n-m$ roots of the equation

$$F(\lambda) = \lambda - a_n - \sum_{i=1, i\notin N_1}^{n-1} \frac{\left(b_n s_{1i} + b_{n-1}s_{n-1,i}\right)^2}{\lambda - \mu_i} = 0 \tag{2.9}$$

From theorem 2.5., we deduce that except for $\mu_1, \mu_2, \dots, \mu_m$, the rest of the eigenvaluesof J_n are $n-m$ roots of $F(\lambda) = 0$. The necessary and sufficient conditions for an inverse eigenvalue problem for periodic Jacobi to be solvable are related to the following two theorems. If J_n and J_{n-1} have distinct eigenvalue then the periodic Jacobi problem is solvable provided the theorem below holds (Jiang, 2003).

Theorem 2.7. If all the elements in the two sets $\lambda = \{\lambda_j\}_{j=1}^{n}$ and $\mu = \{\mu_j\}_{j=1}^{n-1}$ are distinct, then the periodic Jacobi inverse eigenvalue problem (PJI) is solvable if and only if

$$\prod_{i=1}^{n}|\lambda_i - \mu_i| \geq 4\beta(-1)^{n-j+1}, j = 1,2,\ldots,n-1 \tag{2.10}$$

Furthermore, uniqueness of solution is not guaranteed and there are at most 2^{n-1} different solutions. Finally, we state without proof, (see for example Jiang, 2003) for proof. The following theorem establishes the fact that if J_n and J_{n-1} have common eigenvalues then the inequality 2.10 holds.

Theorem 2.8. If two sets $\lambda = \{\lambda_j\}_{j=1}^{n}$ and $\mu = \{\mu_j\}_{j=1}^{n-1}$ have common elements, and the number of common eigenvalues is m, then the periodic Jacobi inverse eigenvalue problem (PJIEP) is solvable if and only if (2.10) is valid. Furthermore, if the problem PJIEP is solvable, there are at most 2^{n-m-1} different solutions.

The algorithm for constructing periodic Jacobi matrix is as follows as cited by (Gyamfi,2012). (See for example Boley and Golub, 1987).

Algorithm:

1. Two sets of eigenvalues $\{\lambda_i^{+}\}_{i=1}^{n}, \{\mu^{i}\}_{i=1}^{n-1}$ and the single scalar β
2. Compute the first row of Q, the eigenvectors of J^{+} using

$$q_{1j}^{2} = \frac{\prod_{k=1}^{n}(\mu_k - \lambda_j^{+})}{\prod_{k=1,k\neq j}^{n}(\lambda_k^{+} - \lambda_j^{+})}, j = 1,2,\ldots,n$$

3. Compute b+ and b− using the equations below;

$$(b_k^+)^2 = -\frac{\prod_{k=1}^{n}(\lambda_j^+ - \mu_k)}{\prod_{j=1,j\neq k}^{n-1}(\mu_j - \mu_k)},$$

$$(b_k^-)^2 = -\frac{\prod_{k=1}^{n}(\lambda_j^- - \mu_k)}{\prod_{j=1,j\neq k}^{n-1}(\mu_j - \mu_k)}, k = 1,2,\dots,n-1$$

4. Compute the eigenvectors of K using, $P_{n-1} = \frac{b^+ - b^-}{2bn}$
5. Compute the last row of $Q, z_n = r_n = [q_{n1}, \dots, q_{nn}]$ using

$$q_{n,k} = -q_{1k}\sum_{j=1}^{n-1}\frac{P_{n-1,j}b_j^+}{(\mu_k - \lambda_k^+)}$$

6. Using the initial values z_1 and z_n and $\Lambda^+ = Q^t J^+ Q$, apply Lanczos algorithm to generate the tridiagonal matrix.

Another important matrix structure that arises frequently from applications is banded matrices. A symmetric banded matrix with bandwidth $2r+1$ contains $\sum_{k=n-r}^{n} k$ entries. For example, suppose $\left\{\lambda_1^{(k)}, \dots, \lambda_k^{(k)}\right\}, k = n-r, \dots, n$ is a scalar and satisfying the interlacing property $\lambda_i^{(k)} \leq \lambda_i^{(k-1)} \leq \lambda_{i+1}^{(k)}$ for $i = 1, \dots, k-1$ and $k = n-r+1, \dots, n$, construct a symmetric banded matrix A with bandwidth $2r+1$ such that each leading $k \times k$ principal submatrix of A has spectrum precisely $\left\{\lambda_1^{(k)}, \dots, \lambda_k^{(k)}\right\}$ (Biegler-König, 1981; Boley and Golub, 1977; Mattis and Hochstadt, 1981; Ram, 1995). There are several structural constraint that can be imposed on inverse eigenvalue problem. For example, the structure could be a block Jacobi matrix (Zhu, Jackson and Chan, 1993), unitary Hessenberg matrix (Ammar and He, 1995) and many others. Similar to the PIEP, is the parameterized inverse singular value problem (PISVP). The significant difference between the

PIEP and the PISVP is that the matrices involved in the PISVP can be rectangular. Since eigenvalues of the symmetric matrix, for example, $\begin{bmatrix} 0 & A(c) \\ A(c)^T & 0 \end{bmatrix}$ where $A(c)$ is a family of matrices, are plus and minus of singular values of the matrix $A(c)$, the PISVP can be solved by conversion to a special parameterized SIEP. In fact, each of the inverse problems discussed so far for eigenvalues have a counterpart problem for singular values and just like many of the inverse eigenvalue problems, the existence question for the inverse singular value problem remains open.

In structured inverse eigenvalue problem, some entries are specified beforehand, sometimes certain submatrix is specified (Deift and Nanda, 1984; Silva, 1987) or sometimes the characteristic polynomial is prescribed (Silva, 1987). An example that plays an important role in majorization (Arnold, 1987) is of the form; given two sets of real values $\{a_1, \dots, a_n\}$ and $\{\lambda_1, \dots, \lambda_n\}$, construct a Hermitian matrix H with diagonal $\{a_1, \dots, a_n\}$ such that $\sigma(H) = \{\lambda_1, \dots, \lambda_n\}$. The existence question for the SHIEP (Structured Hermitian Inverse eigenvalue problem) can be completely settled by the Schur-Horn theorem (Horn and Johnson, 1991). Thus, it has been observed that the main diagonal entries and the eigenvalues of any Hermitian matrix enjoy an interesting relationship. This relationship is completely characterized by what is now known as the Schur-Horn theorem.

Theorem 2.9. (Schur-Horn)

1. Let H be a Hermitian matrix. Let $\lambda = [\lambda_i] \in \mathbb{R}^n$ and $a = [a_i] \in \mathbb{R}^n$ denote the vectors of eigenvalues and diagonal entries of H, respectively. If the entries are arranged in increasing order $a_{j1} \leq \cdots \leq a_{jn}, \lambda_{m1} \leq \cdots \leq \lambda_{mn}$, then

$$\sum_{i=1}^{k} a_{ji} \geq \sum_{i=1}^{k} \lambda_{mi} \tag{2.11}$$

   for all $k = 1,2, \dots, n$ with equality for $k = n$

2. Given any $a, \lambda \in \mathbb{R}^n$ satisfying (2.11), there exists a Hermitian matrix H with eigenvalues λ and diagonal entries a

The notion of (2.11) is also known as a majorizing λ, which has arisen as the precise relationship between two sets of numbers in many areas of disciplines, including matrix theory and statistics. The theorem asserts that $\{a_1, \dots, a_n\}$ majorizes $\{\lambda_1, \dots, \lambda_n\}$ if and only if there exists a Hermitian matrix H with eigenvalues $\{\lambda_1, \dots, \lambda_n\}$ and diagonal entries $\{a_1, \dots, a_n\}$. The second part of the Schur-Horn theorem gives rise to an interesting inverse eigenvalue problem, namely, to construct a Hermitian matrix with the prescribed eigenvalues and diagonal entries. Numerical methods for such a construction were first proposed in (Chu, 1995). Later an efficient recursive method was discussed in (Zha and Zhang, 1995).

Now we discuss two algorithms which have been proposed for constructing Hermitian matrices with diagonal and spectral properties. In what follows, we use M_n to denote the set of complex $n \times n$ matrices and $M_{d,n}$ to denote the set of complex $d \times n$ matrices.

The Bendel–Mickey algorithm produces random (Hermitian) correlation matrices with given spectrum (Bendel and Mickey, 1978). Suppose that $A \in M_n$ is a Hermitian matrix with $Tr\ A = n$. If A does not have a unit diagonal, we can locate two diagonal elements so that $A_{jj} < 1 < A_{kk}$; otherwise, the trace condition would be violated. It is then possible to construct a real rotation Q in the $jk - plane$ for which $(Q^*AQ)_{jj} = 1$. The transformation $A \longrightarrow Q^*AQ$ preserves the conjugate symmetry and the spectrum of A, but it reduces the number of non-unit diagonal entries by at least one. Therefore, at most $(n-1)$ rotations are required before the resulting matrix has a unit diagonal. If the output matrix is Z, it follows that $[Z] \succcurlyeq [A]$. Indeed, $[Z]$ is the unique $\succcurlyeq -$maximal element in every chain that contains$[A]$.

The Chan–Li algorithm, on the other hand, was developed as a constructive proof of the Schur–Horn theorem (Chan and Li, 1983). Suppose that $a \succcurlyeq \lambda$. The Chan–Li algorithm begins with the diagonal matrix $\Lambda \stackrel{\text{def}}{=} diag\lambda$. Then it applies a sequence of $(n-1)$ cleverly chosen (real) plane rotations to generate a real, symmetric matrix A with the same eigenvalues as Λ but with diagonal entries listed by a. Once again, the output and input satisfy the relationship $[A] \succcurlyeq [\Lambda]$. While the Bendel–Mickey algorithm starts from any element of a chain and moves to the top; the Chan–Li algorithm starts at the bottom of a chain and moves upward. The Bendel–Mickey algorithm is a surjective map from the set of Hermitian matrices with spectrum λ onto the set of correlation matrices with spectrum λ. If the initial matrix is chosen uniformly at random (which may be accomplished with standard techniques (Stewart, 1980)), the result may be construed as a random correlation matrix. The distribution of the output, however, is unknown (Holmes, 1991). On the other hand, due to the special form of the initial matrix and the rigid choice of rotations, the Chan–Li algorithm cannot construct very many distinct matrices with a specified diagonal.

In the same light of using algorithms to produce specified matrices, Gyamfi (2012) proposed an algorithm that generates singular Hermitian matrices of rank one when the eigenvalues and some parameters are given. In his work he proposed the theorem on singular symmetricand singular Hermitian matrices of rank one. Thus;

Theorem 2.10. Given the spectrum and row multipliers $k_{i,} i = 1,..,n-1$, the inverse eigenvalue problem for a $n \times n$ singular symmetric matrix of rank one is solvable. See proof of theorem in (Gyamfi, 2012).

Theorem 2.11. Given the spectrum and row multipliers $k_{i,} i = 1,..,n-1$, the inverse eigenvalue problem for a $n \times n$ singular Hermitian matrix of rank one is solvable. See proof of theorem in (Gyamfi, 2012).

Annor et al (2016), in their work also proved theorems on solution of inverse eigenvalue problem for singular Hermitian matrices of rank two and three.

Theorem 2.12. Given the spectrum and row multipliers $k_{i,} i = 1,..,n-1$, the inverse eigenvalue problem for a $n \times n$ singular Hermitian matrix of rank two is solvable. See proof of theorem in (Annor, et al, 2016).

Theorem 2.13. Given the spectrum and row multipliers $k_{i,} i = 1, .., n-1$, the inverse eigenvalue problem for a $n \times n$ singular Hermitian matrix of rank three is solvable. See proof of theorem in (Annor, et al, 2016).

Chapter summary

In conclusion, a number of various types of inverse eigenvalue problem have been reviewed and discussed, most importantly related to solvability, existence of numerical methods and in some case sensitivity analysis. Some of the inverse problems were solvable and others still open for solution. Some algorithms for the construction of Hermitian matrices such as the Bendel–Mickey algorithm and the Chan–Li algorithm but the algorithms were somehow challenging. Therefore, this research, presents a step by step algorithm in chapter four to generate singular symmetric and Hermitian matrices of rank greater or equal to four.

CHAPTER THREE

METHODOLOGY

Introduction

This chapter describes in some detail, the method and procedure that is used in obtaining singular symmetric and Hermitian matrices given the dimension and rank of the matrices. Specifically, the research concentrate on the method employed to obtain the solution of inverse eigenvalue problem of singular symmetric and Hermitian matrices. Based on solvability lemma, an algorithm by Gyamfi (2012) and Annor et al. (2016) is modified to construct $n \times n$ singular symmetric matrices and then an extension is made to construct $n \times n$ singular Hermitian matrices of rank greater than or equal to four. First a method for generating singular symmetric and Hermitian matrices is discussed and then the modified algorithm for the solution of the inverse eigenvalue problem for both singular symmetric and Hermitian matrices.

Method for generating singular symmetric and Hermitian matrices

This research uses a 2×2 matrix as a base for the development of the method for generating subsequent singular symmetric and Hermitian matrices of given dimension and rank. Let $A_{(n,r)}$ denote a matrix A with $n \times n$ dimension and rank r.

Suppose A denote a 2×2 matrix with the following entries

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Our aim is to make A both singular and symmetric. For symmetric matrix, it implies $A = A^T$ where A^T denote transpose. Thus,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$$

which implies $a_{12} = a_{21}$.

Now for singularity,det $A = 0$. Which implies

$$detA = a_{11}a_{22} - a_{12}a_{21} = 0 \Longrightarrow a_{11}a_{22} = a_{12}a_{21}$$

Now writing matrix A in terms of a_{11} thus by making the column or row a scalar multiple of each other thus linear dependency of rows, we let $a_{12} = a_{21} = ka_{11}$ where k is a parameter.

$$\Rightarrow a_{11}a_{22} = ka_{11}ka_{11}$$

$$a_{22} = k^2a_{11}$$

Hence, for a 2×2 singular symmetric matrix we have

$$A = \begin{pmatrix} a_{11} & ka_{11} \\ ka_{11} & k^2a_{11} \end{pmatrix} \tag{3.0}$$

From (3.0), the number of scalars is determined by $n - r$, thus for $A_{(2,1)}$ with $rank(r) = 1$, we have $n - r = 2 - 1 = 1 \Longrightarrow scalar = k$. Based on (3.0) we formulate the following steps to generated singular symmetric matrices given any dimension (square) and rank.

Step 1: Form the entries of the symmetric square matrix based on the dimension. Thus,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} = A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} & \cdots & a_{n1} \\ a_{12} & a_{22} & a_{32} & \cdots & a_{n2} \\ a_{13} & a_{23} & a_{33} & \cdots & a_{n3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & a_{3n} & \cdots & a_{nn} \end{bmatrix}$$

Step 2: Determine the number of scalar multipliers, $n - r$

Step 3: Depending on the number of scalars, we use based on equation (3.0) the following

$$\begin{array}{ll} \text{Number of scalars} & \\ 0 & \\ 1 & \\ 2 & \\ 3 & \\ 4 & \\ \vdots & \\ N & \end{array} \quad a_{11}\left[\begin{array}{llllll} 1 & & & & & \\ 1 & k_1 & & & & \\ 1 & k_1 & k_1k_2 & & & \\ 1 & k_1 & k_1k_2 & k_1k_2k_3 & & \\ 1 & k_1 & k_1k_2 & k_1k_2k_3 & k_1k_2k_3k_4 & \\ \vdots & & & & & \\ 1 & k_1 & k_1k_2 & k_1k_2k_3\ldots & k_2k_3k_4 \ldots k_n & \end{array}\right. \qquad (3.1)$$

entries which represent the first, second, third, fourth up the desired entry in either the first column (C_1) or row (R_1) of the matrix. The entries in (3.1) replaces, based on the number of scalars, the entries of the first column or row in the respective order of the symmetric matrix. Details are presented in the table below;

Table 1: Summary of entries to replace symmetric entries given the dimension, rank and number of scalars of a matrix:

Dimension of matrix	Rank (r)	Number of scalars $n-r$	Entries to replace symmetric entries in the first column or row
2×2	1	1	$a_{11}, k_1 a_{11}$
3×3	1	2	$a_{11}, k_1 a_{11}, k_1 k_2 a_{11}$
4×4	1	3	$a_{11}, k_1 a_{11}, k_1 k_2 a_{11}, k_1 k_2 k_3 a_{11}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n \times n$	1	$n-1$	$a_{11}, k_1 a_{11}, k_1 k_2 a_{11}, \dots, k_1 k_2 k_3 \dots k_{n-1} a_{11}$
3×3	2	1	$a_{11}, k_1 a_{11}, a_{13}$ Note here only two entries in the first column or row can be replaced hence a_{13} will not be replaced
4×4	3	1	$a_{11}, k_1 a_{11}, a_{13}, a_{14}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n \times n$	$r > 1$	$n-r$	$a_{11}, k_1 a_{11}, \dots, k_1 k_2 \dots k_{n-r} a_{11}, a_{13}, a_{14}, \dots, a_{1n}$

Step 4a: Use row (R) or column (C) dependency of the form

$$C_2 = k_1 C_1$$

$$C_3 = k_2 C_2$$

$$\vdots$$

$$C_n = k_{n-1} C_{n-1}$$

to generate second, third up to the specified row or column number. The subsequent specified number of columns or rows depends strictly on the number of the scalars.

Step 4b: In the case of Hermitian matrix with complex entries, we go through step 1 to 3 then we use the same row (R) or column (C) dependency, however, we multiply the columns or rows by the conjugate of the scalars. This is done because whenever we turn column vectors into row vectors and vice versa, we have to conjugate to preserve the meaning of inner products which determine the geometry of complex space. Thus;

$$C_2 = \bar{k}_1 C_1$$

$$C_3 = \bar{k}_2 C_2$$

$$\vdots$$

$$C_n = \bar{k}_{n-1} C_{n-1}$$

is used to generate second, third up to the specified row or column number. Again the subsequent specified number of columns or rows depends strictly on the number of the scalars.

Modified algorithm to generate solution of inverse eigenvalue problem of singular symmetric and Hermitian matrices

We suppose, that given the non-zero scalars $\{\lambda_1, \lambda_2, \dots, \lambda_r\} \subset F, F = \mathbb{R}$ of the matrix and some parameters generated by difference between the dimension of the square matrix and the rank greater than or equal to four, thus $n - r = m$, resulting to $k_1, k_2, \dots, k_{n-r}$, we can generate the system.

From theorem 1.5., we established that for the real case

$$Tr(A_{(n,r)}) = \lambda_1 + \lambda_2 + \cdots + \lambda_r$$

$$= diag(a_{11}[1 + k_1^2 + k_1^2 k_2^2 + \cdots + k_1^2 \dots k_{n-r}^2] + \sum_{i=m+2}^{n} a_{ij}, i = j)$$

Similarly, in the complex case we have

$$Tr(A_{(n,r)}) = \lambda_1 + \cdots + \lambda_r$$

$$= diag(a_{11}[1 + |k|_1^2 + |k|_1^2 |k|_2^2 + \cdots + |k|_1^2 \dots |k|_{n-r}^2]$$

$$+ \sum_{i=m+2}^{n} a_{ij}, i = j)$$

We then generate in corresponding terms to the entries of the main diagonal sequence of the products of the eigenvalues, thus

$$\lambda_1 \lambda_i, i = 2, \dots, r, \tag{3.3}$$

$$\lambda_2 \lambda_i, i = 3, \dots, r \tag{3.4}$$

$$\lambda_3 \lambda_i, \dots, \lambda_{i-1} \lambda_i, i = 4, \dots, r. \tag{3.5}$$

$$\lambda_1 \lambda_2 \lambda_i, \dots, \lambda_{i-2} \lambda_{i-1} \lambda_i, i = 3, \dots, r \tag{3.6}$$

$$\lambda_1 \lambda_3 \lambda_i, \dots, \lambda_{i-3} \lambda_{i-1} \lambda_i, i = 4, \dots, r \tag{3.7}$$

$$\lambda_2 \lambda_3 \lambda_i, \dots, \lambda_{i-2} \lambda_{i-1} \lambda_i, i = 4, \dots, r \tag{3.8}$$

$$\lambda_1 \lambda_2 \lambda_3 \lambda_i, \dots, \lambda_1 \lambda_2 \lambda_3 \lambda_{i-1}, i = 5, \dots, r \tag{3.9}$$

Finally, we evaluate the sum of (3.3), (3.4) and (3.5) and also the sum of (3.6), (3.7) and (3.8) to establish the characteristic polynomial of the matrix. Which is then solved to establish a relation between the eigenvalues and entries of the main diagonal of the matrix. This relation together with other free variables is then used to generate the system to form singular Hermitian matrices.

CHAPTER 4

RESULTS AND DISCUSSION

Introduction

The purpose of this study is to find solution of inverse eigenvalue problem of singular Hermitian matrix of rank greater or equal to four. Thus our specific objectives were to;

- Construct singular symmetric matrices of rank greater than or equal to four from their eigenvalues.
- Construct singular Hermitian matrices of rank greater than or equal to four from their eigenvalues.

In this chapter, findings from the study are presented and discussed in relation to the research problem below;

1. Given non-zero scalars $\{\lambda_1, \lambda_2, \dots, \lambda_r\} \subset F, F = \mathbb{R},, r < n$ and some parameters, we generate a matrix $A_{(n,r)}$ such that $diag(A) = \{\lambda_1, \lambda_2, \dots, \lambda_r\}$, $\det(A) = 0$ for $n \neq r$ and $A^T = A$.
2. Given non-zero scalars $\{\lambda_1, \lambda_2, \dots, \lambda_r\} \subset F, F = \mathbb{R}, r < n$ and some parameters, we generate a matrix $A_{(n,r)}$ such that $diag(A) = \{\lambda_1, \lambda_2, \dots, \lambda_r\}$, $\det(A) = 0$ for $n \neq r$ and $A^{\bar{T}} = A$.

Generating singular symmetric matrix

To answer the research problem above, we start with a 5×5 matrix with rank 4. We first generate a singular symmetric matrix with dimension 5×5 and rank 4. Suppose

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix} = A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} & a_{41} & a_{51} \\ a_{12} & a_{22} & a_{32} & a_{42} & a_{52} \\ a_{13} & a_{23} & a_{33} & a_{43} & a_{53} \\ a_{14} & a_{24} & a_{34} & a_{44} & a_{54} \\ a_{15} & a_{25} & a_{35} & a_{45} & a_{55} \end{bmatrix}$$

Next, number of scalars equal 1 which implies we replace a_{12} with ka_{11}. Then

$$C_2 = kC_1$$

which ends our column operation, but the entries are symmetric about the main diagonal. Hence;

$$A_{(5,4)} = \begin{bmatrix} a_{11} & ka_{11} & a_{13} & a_{14} & a_{15} \\ ka_{11} & k^2a_{11} & ka_{13} & ka_{14} & ka_{15} \\ a_{13} & ka_{13} & a_{33} & a_{34} & a_{35} \\ a_{14} & ka_{14} & a_{34} & a_{44} & a_{45} \\ a_{15} & ka_{15} & a_{35} & a_{45} & a_{55} \end{bmatrix} \quad (4.1)$$

Secondly, for $A_{(5,4)}$ in (4.1) we have

$$\lambda_1 \neq 0 , \lambda_2 \neq 0 , \lambda_3 \neq 0, \lambda_4 \neq 0 \text{ and } \lambda_5 = 0.$$

such that,

$$trA_{(5,4)} = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = a_{11}[1 + k^2] + a_{33} + a_{44} + a_{55}$$

But

$$\lambda_2 = a_{11}[1 + k^2]a_{33}$$

$$\lambda_1\lambda_3 = a_{11}[1 + k^2]a_{44}$$

$$\lambda_1\lambda_4 = a_{11}[1 + k^2]a_{55}$$

$$\lambda_2\lambda_3 = a_{33}a_{44}$$

$$\lambda_2\lambda_4 = a_{33}a_{55}$$

$$\lambda_3\lambda_4 = a_{44}a_{55}$$

$$\lambda_1\lambda_2\lambda_3 = a_{11}[1+k^2]a_{33}a_{44}$$

$$\lambda_1\lambda_2\lambda_4 = a_{11}[1+k^2]a_{33}a_{55}$$

$$\lambda_1\lambda_3\lambda_4 = a_{11}[1+k^2]a_{44}a_{55}$$

$$\lambda_2\lambda_3\lambda_4 = a_{33}a_{44}a_{55}$$

$$\lambda_1\lambda_2\lambda_3\lambda_4 = a_{11}[1+k^2]a_{33}a_{44}a_{55} \Longrightarrow a_{33}a_{44}a_{55} = \frac{\lambda_1\lambda_2\lambda_3\lambda_4}{a_{11}[1+k^2]}$$

Now,

$$\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4$$
$$= a_{11}[1+k^2]a_{33}a_{44} + a_{11}[1+k^2]a_{33}a_{55}$$
$$+ a_{11}[1+k^2]a_{44}a_{55} + a_{33}a_{44}a_{55}$$

Which gives,

$$\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4 = a_{11}[1+k^2]$$
$$[a_{33}a_{44} + a_{33}a_{55} + a_{44}a_{55}] + a_{33}a_{44}a_{55}$$

$$\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4$$
$$= a_{11}[1+k^2][a_{33}a_{44} + a_{33}a_{55} + a_{44}a_{55}] + \frac{\lambda_1\lambda_2\lambda_3\lambda_4}{a_{11}[1+k^2]}$$

$$a_{33}a_{44} + a_{33}a_{55} + a_{44}a_{55}$$
$$= \frac{a_{11}[1+k^2][\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4] - \lambda_1\lambda_2\lambda_3\lambda_4}{{a_{11}}^2[1+k^2]^2} \quad (4.2)$$

Also,

$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4$$
$$= a_{11}[1+k^2][a_{33} + a_{44} + a_{55}] + a_{33}a_{44} + a_{33}a_{55} + a_{44}a_{55}$$

$$a_{11}[1+k^2][a_{33}+a_{44}+a_{55}]$$
$$=\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_3\lambda_4-[a_{33}a_{44}+a_{33}a_{55}$$
$$+a_{44}a_{55}]$$

From (4.2)

$$a_{11}[1+k^2][a_{33}+a_{44}+a_{55}]=\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_3+\lambda_2\lambda_4+$$
$$\lambda_3\lambda_4-\frac{a_{11}[1+k^2][\lambda_1\lambda_2\lambda_3+\lambda_1\lambda_2\lambda_4+\lambda_1\lambda_3\lambda_4+\lambda_2\lambda_3\lambda_4]-\lambda_1\lambda_2\lambda_3\lambda_4}{{a_{11}}^2[1+k^2]^2}$$

$$a_{33}+a_{44}+a_{55}=$$
$$\frac{{a_{11}}^2[1+k^2]^2[\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_3\lambda_4]-a_{11}[1+k^2][\lambda_1\lambda_2\lambda_3+\lambda_1\lambda_2\lambda_4+\lambda_1\lambda_3\lambda_4+\lambda_2\lambda_3\lambda_4]+\lambda_1\lambda_2\lambda_3\lambda_4}{{a_{11}}^3[1+k^2]^3} \tag{4.3}$$

But

$$\lambda_1+\lambda_2+\lambda_3+\lambda_4=a_{11}[1+k^2]+a_{33}+a_{44}+a_{55}$$

Substituting (4.3) we have,

$$\lambda_1+\lambda_2+\lambda_3+\lambda_4$$
$$=a_{11}[1+k^2]$$
$$+\frac{{a_{11}}^2[1+k^2]^2[\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_3\lambda_4]-a_{11}[1+k^2][\lambda_1\lambda_2\lambda_3+\lambda_1\lambda_2\lambda_4+\lambda_1\lambda_3\lambda_4+\lambda_2\lambda_3\lambda_4]+\lambda_1\lambda_2\lambda_3\lambda_4}{{a_{11}}^3[1+k^2]^3}$$

Hence,

$${a_{11}}^4[1+k^2]^4-[\lambda_1+\lambda_2+\lambda_3+\lambda_4]{a_{11}}^3[1+k^2]^3$$
$$+[\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_3\lambda_4]{a_{11}}^2[1+k^2]^2$$
$$-[\lambda_1\lambda_2\lambda_3+\lambda_1\lambda_2\lambda_4+\lambda_1\lambda_3\lambda_4+\lambda_2\lambda_3\lambda_4]a_{11}[1+k^2]+\lambda_1\lambda_2\lambda_3\lambda_4$$
$$=0$$

Solving the quartic equation, we obtain

$$a_{11} = \frac{\lambda_1}{1+k^2}, \lambda_2 = a_{33}, \lambda_3 = a_{44}, and\ \lambda_4 = a_{55}$$

Also the free variables are; $a_{13}, a_{14}, a_{15}, a_{35}, a_{34}\ and\ a_{45}$

Numerical example 1.0

Given that

$\lambda_1 = 10, \lambda_2 = 6, \lambda_3 = 7, \lambda_4 = 11, k = 2, a_{13} = 8, a_{14} = -4, a_{15} = 5, a_{35} = 14, a_{34} = 6, a_{45} = 17$. We are to generate a 5×5 singular symmetric matrix with rank equal to four. Now from data given we have;

$$\Rightarrow a_{11} = \frac{10}{1+(2)^2} = 2$$

Hence,

$$A_{(5,4)} = \begin{bmatrix} 2 & 4 & 8 & -4 & 5 \\ 4 & 8 & 16 & -8 & 10 \\ 8 & 16 & 6 & 6 & 14 \\ -4 & -8 & 6 & 7 & 17 \\ 5 & 10 & 14 & 17 & 11 \end{bmatrix}$$

Extension to singular Hermitian matrix

We generate a Hermitian singular symmetric matrix with dimension 5×5 and $rank = 4$. Suppose

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix} = A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} & a_{41} & a_{51} \\ a_{12} & a_{22} & a_{32} & a_{42} & a_{52} \\ a_{13} & a_{23} & a_{33} & a_{43} & a_{53} \\ a_{14} & a_{24} & a_{34} & a_{44} & a_{54} \\ a_{15} & a_{25} & a_{35} & a_{45} & a_{55} \end{bmatrix}$$

then we perform the column operation,

$$C_2 = \bar{k}C_1$$

but in this case the entries are conjugated about the main diagonal to form the symmetry of the matrix. Hence,

$$A_{(5,4)} = \begin{bmatrix} a_{11} & \bar{k}a_{11} & \bar{a}_{13} & \bar{a}_{14} & \bar{a}_{15} \\ ka_{11} & |k|^2 a_{11} & k\bar{a}_{13} & k\bar{a}_{14} & k\bar{a}_{15} \\ a_{13} & \bar{k}a_{13} & a_{33} & \bar{a}_{34} & \bar{a}_{35} \\ a_{14} & \bar{k}a_{14} & a_{34} & a_{44} & \bar{a}_{45} \\ a_{15} & \bar{k}a_{15} & a_{35} & a_{45} & a_{55} \end{bmatrix} \quad (4.4)$$

From (4.4), we have

$$trA_{(5,4)} = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = a_{11}[1 + |k|^2] + a_{33} + a_{44} + a_{55}$$

Using similar procedure above, we obtain the quartic polynomial below;

$$a_{11}{}^4[1 + |k|^2]^4 - [\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4]a_{11}{}^3[1 + |k|^2]^3$$
$$+ [\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4]a_{11}{}^2[1 + |k|^2]^2$$
$$- [\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4]a_{11}[1 + |k|^2]$$
$$+ \lambda_1\lambda_2\lambda_3\lambda_4 = 0$$

Hence factoring the quartic equation we obtain;

$$a_{11} = \frac{\lambda_1}{1 + |k|^2}, \lambda_2 = a_{33}, \lambda_3 = a_{44}, and\ \lambda_4 = a_{55}$$

with$a_{13}, a_{14}, a_{15}, a_{35}, a_{34}$ and a_{45} as free variables

Numerical example 2.0

Given that $\lambda_1 = 10\ \lambda_2 = 4, \lambda_3 = 2, \lambda_4 = 6, k = 2i, a_{13} = 5i, a_{14} = 2 + i, a_{15} = 3i, a_{35} = i, a_{34} = 4i, a_{45} = -2i$. We are to generate a 5×5 singular Hermitian matrix with rank equal to four. Now from data given we have;

$$\Rightarrow a_{11} = \frac{10}{1 + (2)^2} = 2$$

Hence,

$$A_{(5,4)} = \begin{bmatrix} 2 & -4i & -5i & 2-i & -3i \\ 4i & 8 & 10 & 2+4i & 6 \\ 5i & 10 & 4 & -4i & -i \\ 2+i & 2-4i & 4i & 2 & 2i \\ 3i & 6 & i & -2i & 6 \end{bmatrix}$$

which can be proved by theorem 2.9 (Schur-Horn) for $A_{(5,4)}$ existence as a Hermitian matrix.

We also generate a 6×6 singular symmetric matrix with $rank = 4$.

$$A_{(6,4)} = \begin{bmatrix} a_{11} & k_1 a_{11} & k_1 k_2 a_{11} & a_{14} & a_{15} & a_{16} \\ k_1 a_{11} & k_1^2 a_{11} & k_1^2 k_2 a_{11} & k_1 a_{14} & k_1 a_{15} & k_1 a_{16} \\ k_1 k_2 a_{11} & k_1^2 k_2 a_{11} & k_1^2 k_2^2 a_{11} & k_1 k_2 a_{14} & k_1 k_2 a_{15} & k_1 k_2 a_{16} \\ a_{14} & k_1 a_{14} & k_1 k_2 a_{14} & a_{44} & a_{45} & a_{46} \\ a_{15} & k_1 a_{15} & k_1 k_2 a_{15} & a_{45} & a_{55} & a_{56} \\ a_{16} & k_1 a_{16} & k_1 k_2 a_{16} & a_{46} & a_{56} & a_{66} \end{bmatrix} \qquad (4.5)$$

From (4.5), $\lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 \neq 0, \lambda_4 \neq 0, \lambda_5 = 0, \lambda_6 = 0$ and

$$trA = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = a_{11}[1 + k_1^2 + k_1^2 k_2^2] + a_{44} + a_{55} + a_{66}$$

Using similar procedure, we obtain

$$a_{11}{}^4[1 + k_1^2 + k_1^2 k_2^2]^4 - [\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4]a_{11}{}^3[1 + k_1^2 + k_1^2 k_2^2]^3$$
$$+ [\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4$$
$$+ \lambda_3\lambda_4]a_{11}{}^2[1 + k_1^2 + k_1^2 k_2^2]^2$$
$$- [\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4]a_{11}[1 + k_1^2 + k_1^2 k_2^2]$$
$$+ \lambda_1\lambda_2\lambda_3\lambda_4 = 0$$

Solving the quartic equation, we have

$$a_{11} = \frac{\lambda_1}{1 + k_1^2 + k_1^2 k_2^2}\ , \lambda_2 = a_{44}\ , \lambda_3 = a_{55}\ , and\ \lambda_4 = a_{66}$$

The free variables are; $a_{14}, a_{15}, a_{16}, a_{45}, a_{46}\ and\ a_{56}$

Numerical example 3.0

Given that

$\lambda_1 = 4, \lambda_2 = 2, \lambda_3 = 7, \lambda_4 = 8, k_1 = 3, k_2 = 2, a_{14} = 3, a_{15} = 9, a_{16} = 10, a_{45} = 5, a_{46} = 2, a_{56} = 1$. We are to generate a 6×6 singular symmetric matrix with rank equal to four. Now from similar process;

$$\Rightarrow a_{11} = \frac{4}{1 + 9 + (9)(4)} = \frac{2}{23}$$

Therefore,

$$A_{(6,4)} = \begin{bmatrix} \frac{2}{23} & \frac{6}{23} & \frac{12}{23} & 3 & 9 & 10 \\ \frac{6}{23} & \frac{18}{23} & \frac{36}{23} & 9 & 27 & 30 \\ \frac{12}{23} & \frac{36}{23} & \frac{72}{23} & 18 & 54 & 60 \\ 3 & 9 & 18 & 2 & 5 & 2 \\ 9 & 27 & 54 & 5 & 7 & 1 \\ 10 & 30 & 60 & 2 & 1 & 8 \end{bmatrix}$$

We again extend the same process to a 6×6 singular Hermitian matrix with $rank = 4$

$$A_{(6,4)} = \begin{bmatrix} a_{11} & \overline{k_1} a_{11} & \overline{k_1}\,\overline{k_2} a_{11} & \bar{a}_{14} & \bar{a}_{15} & \bar{a}_{16} \\ k_1 a_{11} & |k_1|^2 a_{11} & |k_1|^2 \bar{k}_2 a_{11} & k_1 \bar{a}_{14} & k_1 \bar{a}_{15} & k_1 \bar{a}_{16} \\ k_1 k_2 a_{11} & |k_1|^2 k_2 a_{11} & |k_1|^2 |k_2|^2 a_{11} & k_1 k_2 \bar{a}_{14} & k_1 k_2 \bar{a}_{15} & k_1 k_2 \bar{a}_{16} \\ a_{14} & \bar{k}_1 a_{14} & \bar{k}_1 \bar{k}_2 a_{14} & a_{44} & \bar{a}_{45} & \bar{a}_{46} \\ a_{15} & \bar{k}_1 a_{15} & \bar{k}_1 \bar{k}_2 a_{15} & a_{45} & a_{55} & \bar{a}_{56} \\ a_{16} & \bar{k}_1 a_{16} & \bar{k}_1 \bar{k}_2 a_{16} & a_{46} & a_{56} & a_{66} \end{bmatrix} \qquad (4.6)$$

From (4.6),

$$trA = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = a_{11}[1 + |k_1|^2 + |k_1|^2|k_2|^2] + a_{44} + a_{55} + a_{66}$$

Using similar procedure, we obtain

$$\begin{aligned} & {a_{11}}^4[1 + |k_1|^2 + |k_1|^2|k_2|^2]^4 \\ & \quad - [\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4]{a_{11}}^3[1 + |k_1|^2 + |k_1|^2|k_2|^2]^3 \\ & \quad + [\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 \\ & \quad + \lambda_3\lambda_4]{a_{11}}^2[1 + |k_1|^2 + |k_1|^2|k_2|^2]^2 \\ & \quad - [\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4]a_{11}[1 + |k_1|^2 \\ & \quad + |k_1|^2|k_2|^2] + \lambda_1\lambda_2\lambda_3\lambda_4 = 0 \end{aligned}$$

Solving the quartic equation, we have

$$a_{11} = \frac{\lambda_1}{1 + |k_1|^2 + |k_1|^2|k_2|^2}, \lambda_2 = a_{44}, \lambda_3 = a_{55}, and\ \lambda_4 = a_{66}$$

The free variables are; $a_{14}, a_{15}, a_{16}, a_{45}, a_{46}$ and a_{56}

Numerical example 4.0

Given that $\lambda_1 = 2, \lambda_2 = 5, \lambda_3 = 1, \lambda_4 = 5, k_1 = 3i, k_2 = 2i, a_{14} = 5i, a_{15} = 1 + i, a_{16} = 2 + i, a_{45} = -i, a_{46} = 4i, a_{56} = 3 + 2i$. We are to generate a 6×6 singular Hermitian matrix with rank equal to four. Now from data given we have;

$$\Rightarrow a_{11} = \frac{2}{1+9+(9)(4)} = \frac{1}{23}$$

Hence,

$$A_{(6,4)} = \begin{bmatrix} \frac{1}{23} & -\frac{3}{23}i & -\frac{6}{23} & -5i & 1-i & 2-i \\ \frac{3}{23}i & \frac{9}{23} & -\frac{18}{23}i & 15 & 3+3i & 3+6i \\ -\frac{6}{23} & \frac{18}{23}i & \frac{36}{23} & 30i & -6+6i & -12+6i \\ 5i & 15 & -30i & 5 & i & -4i \\ 1+i & 3-3i & -3-6i & -i & 1 & 3-2i \\ 2+i & 3-6i & -12-6i & 4i & 3+2i & 5 \end{bmatrix}$$

Our next attention is on generating a singular Hermitian matrix with $n \times n$ dimension and $rank = 5$. From the process outlined in chapter 3 of this work, suppose $A_{(6,5)}$ a singular Hermitian matrix, then

$$A_{(6,5)} = \begin{bmatrix} a_{11} & \bar{k}a_{11} & \bar{a}_{13} & \bar{a}_{14} & \bar{a}_{15} & \bar{a}_{16} \\ ka_{11} & |k|^2 a_{11} & k\bar{a}_{13} & k\bar{a}_{14} & k\bar{a}_{15} & k\bar{a}_{16} \\ a_{13} & \bar{k}a_{13} & a_{33} & \bar{a}_{34} & \bar{a}_{35} & \bar{a}_{36} \\ a_{14} & \bar{k}a_{14} & a_{34} & a_{44} & \bar{a}_{45} & \bar{a}_{46} \\ a_{15} & \bar{k}a_{15} & a_{35} & a_{45} & a_{55} & \bar{a}_{56} \\ a_{16} & \bar{k}a_{16} & a_{36} & a_{46} & a_{56} & a_{66} \end{bmatrix} \quad (4.7)$$

The eigenvalues for (4.7) will exhibit the following;

$$\lambda_1 \neq 0\ ,\ \lambda_2 \neq 0\ ,\ \lambda_3 \neq 0,\ \lambda_4 \neq 0,\ \lambda_5 \neq 0 \ \text{ and } \lambda_6 = 0$$

such that

$$tr(A_{(6,5)}) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = a_{11}[1+|k|^2] + a_{33} + a_{44} + a_{55} + a_{66}$$

Then from similar process the characteristic polynomial will be

$$a_{11}{}^5[1+|k|^2]^5 - [\lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5]a_{11}{}^4[1+|k|^2]^4$$
$$+[\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_1\lambda_5+\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_2\lambda_5+\lambda_3\lambda_4$$
$$+\lambda_3\lambda_5+\lambda_4\lambda_5]a_{11}{}^3[1+|k|^2]^3$$
$$-[\lambda_1\lambda_2\lambda_3+\lambda_1\lambda_2\lambda_4+\lambda_1\lambda_2\lambda_5+\lambda_1\lambda_3\lambda_4+\lambda_1\lambda_3\lambda_5+\lambda_2\lambda_3\lambda_4$$
$$+\lambda_2\lambda_3\lambda_5+\lambda_3\lambda_4\lambda_5]a_{11}{}^2[1+|k|^2]^2+[\lambda_1\lambda_2\lambda_3\lambda_4+\lambda_1\lambda_2\lambda_3\lambda_5$$
$$+\lambda_2\lambda_3\lambda_4\lambda_5]a_{11}[1+|k|^2]+\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5=0$$

Solving the quintic equation we have

$$a_{11}=\frac{\lambda_1}{1+|k|^2}, \lambda_2=a_{33}\lambda_3=a_{44}, \lambda_4= a_{55}, and\ \lambda_5=a_{66}$$

The free variables for $A_{(6,5)}$ are; $a_{13}, a_{14}, a_{15}, a_{16}, a_{34}, a_{35}, a_{36}, a_{45}, a_{46}$ $and\ a_{56}$.

In a Similar process the characteristics polynomial for a matrix with an $n\times n$ dimension and $rank(r)$ of a singular Hermitian matrix, thus$A_{(n,r)}$ will exhibit the following forms. Suppose q represent the number of scalars, then

$$n-r=q \Longrightarrow r=n-q$$

Now, if $q=1$

$A_{(n,n-1)}$:

$$a_{11}{}^{n-1}[1+|k|^2]^{n-1}-[\lambda_1+\lambda_2+\cdots+\lambda_n]a_{11}{}^{n-2}[1+|k|^2]^{n-2}$$
$$+[\lambda_1\lambda_2+\cdots+\lambda_1\lambda_n+\lambda_2\lambda_3+\cdots+\lambda_2\lambda_n+\lambda_3\lambda_4+\cdots+\lambda_3\lambda_n$$
$$+\cdots+\lambda_{n-1}\lambda_n]a_{11}{}^{n-3}[1+|k|^2]^{n-3}$$
$$-[\lambda_1\lambda_2\lambda_3+\cdots+\lambda_1\lambda_2\lambda_n+\lambda_1\lambda_3\lambda_4+\ldots+\lambda_1\lambda_3\lambda_n+\lambda_2\lambda_3\lambda_4$$
$$+\cdots\lambda_2\lambda_3\lambda_n]a_{11}{}^{n-4}[1+|k|^2]^{n-4}+\cdots+\lambda_1\lambda_2\lambda_3\ldots\lambda_n=0$$

Then,

$$\lambda_1=a_{11}[1+|k|^2], \lambda_2=a_{33}\lambda_3=a_{44}, \ldots\ and\ \lambda_r=a_{(n+1)(n+1)}$$

If $q = 2$

$A_{(n,n-2)}$:

$$a_{11}{}^{n-2}[1 + |k_1|^2 + |k_1|^2|k_2|^2]^{n-2}$$
$$- [\lambda_1 + \lambda_2 + \cdots + \lambda_n]a_{11}{}^{n-3}[1 + |k_1|^2 + |k_1|^2|k_2|^2]^{n-3}$$
$$+ [\lambda_1\lambda_2 + \cdots + \lambda_1\lambda_n + \lambda_2\lambda_3 + \cdots + \lambda_2\lambda_n + \lambda_3\lambda_4 + \cdots + \lambda_3\lambda_n$$
$$+ \cdots + \lambda_{n-1}\lambda_n]a_{11}{}^{n-4}[1 + |k_1|^2 + |k_1|^2|k_2|^2]^{n-4}$$
$$- [\lambda_1\lambda_2\lambda_3 + \cdots + \lambda_1\lambda_2\lambda_n + \lambda_1\lambda_3\lambda_4 + \ldots + \lambda_1\lambda_3\lambda_n + \lambda_2\lambda_3\lambda_4$$
$$+ \cdots \lambda_2\lambda_3\lambda_n]a_{11}{}^{n-5}[1 + |k_1|^2 + |k_1|^2|k_2|^2]^{n-5} + \cdots$$
$$+ \lambda_1\lambda_2\lambda_3 \ldots \lambda_n = 0$$

Then,

$$\lambda_1 = a_{11}[1 + |k|^2 + |k_1|^2|k_2|^2], \lambda_2 = a_{33}\lambda_3 = a_{44}, \ldots \; and \; \lambda_r = a_{(n+1)(n+1)}$$

If $q = 3$

$A_{(n,n-3)}$:

$$a_{11}{}^{n-3}[1 + |k_1|^2 + |k_1|^2|k_2|^2 + |k_1|^2|k_2|^2|k_3|^2]^{n-3}$$
$$- [\lambda_1 + \lambda_2 + \cdots$$
$$+ \lambda_n]a_{11}{}^{n-4}[1 + |k_1|^2 + |k_1|^2|k_2|^2 + |k_1|^2|k_2|^2|k_3|^2]^{n-4}$$
$$+ [\lambda_1\lambda_2 + \cdots + \lambda_1\lambda_n + \lambda_2\lambda_3 + \cdots + \lambda_2\lambda_n + \lambda_3\lambda_4 + \cdots + \lambda_3\lambda_n$$
$$+ \cdots$$
$$+ \lambda_{n-1}\lambda_n]a_{11}{}^{n-5}[1 + |k_1|^2 + |k_1|^2|k_2|^2 + |k_1|^2|k_2|^2|k_3|^2]^{n-5}$$
$$- [\lambda_1\lambda_2\lambda_3 + \cdots + \lambda_1\lambda_2\lambda_n + \lambda_1\lambda_3\lambda_4 + \ldots + \lambda_1\lambda_3\lambda_n + \lambda_2\lambda_3\lambda_4$$
$$+ \cdots \lambda_2\lambda_3\lambda_n]a_{11}{}^{n-6}[1 + |k_1|^2 + |k_1|^2|k_2|^2 + |k_1|^2|k_2|^2|k_3|^2]^{n-6}$$
$$+ \cdots + \lambda_1\lambda_2\lambda_3 \ldots \lambda_n = 0$$

Likewise, the eigenvalues will be;

$$\lambda_1 = a_{11}[1 + |k|^2 + |k_1|^2|k_2|^2 + |k_1|^2|k_2|^2|k_3|^2], \lambda_2 = a_{33}\lambda_3 = a_{44}, \dots \ and \ \lambda_r = a_{(n+1)(n+1)}$$

In general, by induction if the number of scalar $= q$, q is a positive integer, then

$$n - q = r$$

Hence,

$$A_{(n,r)}:$$

$$a_{11}{}^r[1 + |k_1|^2 + |k_1|^2|k_2|^2 + |k_1|^2|k_2|^2|k_3|^2 + \cdots + |k_1|^2|k_2|^2|k_3|^2 \dots |k_q|^2]^r - [\lambda_1 + \lambda_2 + \cdots + \lambda_n]a_{11}{}^{r-1}\left[1 + |k_1|^2 + |k_1|^2|k_2|^2 + |k_1|^2|k_2|^2|k_3|^2 + \cdots + |k_1|^2|k_2|^2|k_3|^2 \dots |k_q|^2\right]^{r-1} + [\lambda_1\lambda_2 + \cdots + \lambda_1\lambda_n + \lambda_2\lambda_3 + \cdots + \lambda_2\lambda_n + \lambda_3\lambda_4 + \cdots + \lambda_3\lambda_n + \cdots + \lambda_{n-1}\lambda_n]a_{11}{}^{r-2}\left[1 + |k_1|^2 + |k_1|^2|k_2|^2 + |k_1|^2|k_2|^2|k_3|^2 + \cdots + |k_1|^2|k_2|^2|k_3|^2 \dots |k_q|^2\right]^{r-2} - [\lambda_1\lambda_2\lambda_3 + \cdots + \lambda_1\lambda_2\lambda_n + \lambda_1\lambda_3\lambda_4 + \dots + \lambda_1\lambda_3\lambda_n + \lambda_2\lambda_3\lambda_4 + \cdots \lambda_2\lambda_3\lambda_n]a_{11}{}^{r-3}\left[1 + |k_1|^2 + |k_1|^2|k_2|^2 + |k_1|^2|k_2|^2|k_3|^2 + \cdots + |k_1|^2|k_2|^2|k_3|^2 \dots |k_q|^2\right]^{r-3} + \cdots + \lambda_1\lambda_2\lambda_3 \dots \lambda_n = 0$$

Where,

$$\lambda_1 = a_{11}[1 + |k|^2 + |k_1|^2|k_2|^2 + |k_1|^2|k_2|^2|k_3|^2 + \cdots$$
$$+ |k_1|^2|k_2|^2|k_3|^2 \dots |k_q|^2], \lambda_2 = a_{33}\lambda_3 = a_{44}, \dots \ and \ \lambda_r$$
$$= a_{(n+1)(n+1)}$$

The above method for singular symmetric and singular Hermitian matrices is then generalize in the following two theorems

1. We generalize for an $n \times n$ matrix of rank four
2. We generalize for an $n \times n$ matrix of rank r, where $4 \leq r < n$

Theorem 4.1.

Given the spectrum and the number of scalars$k_i = 1,2, \dots, n - 4$, the inverse eigenvalue problem for an $n \times n$ singular symmetric matrix of rank four is solvable.See similar proof by (Gyamfi, 2012)

Proof: Given the spectrum $\Lambda_n = \{\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n\}$, since the rank of $\Lambda_4 = 4$, it follows that

$$\lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 \neq 0, \lambda_4 \neq 0 \text{ and } \lambda_i = 0, \text{ for } i = 5, 6, \dots, n.$$

But the number of scalars $= n - r$, where $r = 4$ implying that

$$\text{number of scalars} = n - 4.$$

Therefore we replace either the column or row with

$$a_{11}\ [1 \quad k_1 k_1 k_2 k_1 k_2 k_3 \dots k_{n-4}]$$

and perform the column operation until all the scalars are used to generate subsequent columns and finally since the entries are symmetric about the main diagonal we obtain

$A_{(n,4)} =$

$$\begin{bmatrix} a_{11} & k_1a_{11} & k_1k_2a_{11} & \cdots & k_1k_2...k_{n-4}a_{11} & a_{1(n-2)} & a_{1(n-1)} & a_{1n} \\ k_1a_{11} & k_1^2a_{11} & k_1^2k_2a_{11} & \cdots & k_1^2k_2...k_{n-4}a_{11} & k_1a_{1(n-2)} & k_1a_{1(n-1)} & k_1a_{1n} \\ k_1k_2a_{11} & k_1^2k_2a_{11} & k_1^2k_2^2a_{11} & \cdots & k_1^2k_2^2...k_{n-4}a_{11} & k_1k_2a_{1(n-2)} & k_1k_2a_{1(n-1)} & k_1k_2a_{1n} \\ \vdots & \vdots & \vdots & \ddots & & \cdots & \cdots & \vdots \\ k_1k_2...k_{n-4}a_{11} & k_1^2k_2...k_{n-4}a_{11} & k_1^2k_2^2...k_{n-4}a_{11} & \cdots & k_1^2k_2^2...k_{n-4}^2a_{11} & k_1k_2...k_{n-4}a_{1(n-2)} & \cdots & k_1k_2...k_{n-4}a_{1n} \\ a_{1(n-2)} & k_1a_{1(n-2)} & k_1k_2a_{1(n-2)} & \cdots & k_1k_2...k_{n-4}a_{1(n-2)} & a_{(n-2)(n-2)} & a_{(n-2)n} & a_{(n-1)n} \\ a_{1(n-1)} & k_1a_{1(n-1)} & k_1k_2a_{1(n-1)} & \cdots & \vdots & a_{(n-2)n} & a_{(n-1)(n-1)} & a_{(n-1)n} \\ a_{1n} & k_1a_{1n} & k_1k_2a_{1n} & \cdots & k_1k_2...k_{n-4}a_{1n} & a_{(n-1)n} & a_{(n-1)n} & a_{nn} \end{bmatrix}$$

Where,

$$tr\left(A_{(n,4)}\right) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$$

$$= a_{11}[1 + k_1^2 + k_1^2k_2^2 + \cdots + k_1^2k_2^2 \times \ldots \times k_{n-4}^2] + a_{(n-2)(n-2)}$$

$$+ a_{(n-1)(n-1)} + a_{nn}$$

Which gives us the quartic equation

$$a_{11}^4[1 + k_1^2 + k_1^2k_2^2 + \cdots + k_1^2k_2^2 \times \ldots \times k_{n-4}^2]^4 - [\sum_{i=1}^{4} \lambda_i]a_{11}^3[1 + k_1^2 + k_1^2k_2^2$$

$$+ \cdots + k_1^2k_2^2 \times \ldots \times k_{n-4}^2]^3 + [\sum_{i=1}^{3} \lambda_1\lambda_{i+1} + \sum_{i=1}^{2} \lambda_2\lambda_{i+2}$$

$$+ \sum_{i=1}^{1} \lambda_3\lambda_{i+3}]a_{11}^2[1 + k_1^2 + k_1^2k_2^2 + \cdots + k_1^2k_2^2 \times \ldots \times k_{n-4}^2]^2$$

$$- \left[\sum_{i=1}^{2} \lambda_1\lambda_2\lambda_{i+2} + \sum_{i=1}^{1} \lambda_1\lambda_3\lambda_{i+3} + \sum_{i=1}^{1} \lambda_2\lambda_3\lambda_{i+3}\right] a_{11}[1 + k_1^2$$

$$+ k_1^2k_2^2 + \cdots + k_1^2k_2^2 \times \ldots \times k_{n-4}^2] + \left[\prod_{i=1}^{4} \lambda_i\right] = 0$$

Now solving the above quartic equation we obtain

$$a_{11} = \frac{\lambda_1}{1 + k_1^2 + k_1^2 k_2^2 + \cdots + k_1^2 k_2^2 \times \ldots \times k_{n-4}^2}, \lambda_2 = a_{(n-2)(n-2)}, \lambda_3$$

$$= a_{(n-1)(n-1)}, \ \ \lambda_4 = a_{nn}$$

Hence solvable.

The free variables are;

1. For 5×5 singular symmetric matrix we get one scalar that is k, hence the free variables $a_{13}, a_{14}, a_{15}, a_{35}, a_{34}$ *and* a_{45}
2. For 6×6 singular symmetric matrix we get two scalars that is $k_1 k_2$, hence the free variables $a_{14}, a_{15}, a_{16}, a_{45}, a_{46}$ *and* a_{56}
3. In general for $n \times n$ singular symmetric matrix we get $n - 4$ scalars, hence the free variables $a_{1(n-2)}, a_{1(n-1)}, a_{1n}, a_{(n-2)(n-1)}, a_{(n-2)n}$ *and* $a_{(n-1)n}$

Theorem 4.2.

Given the spectrum and the scalars $k_i = 1,2, \ldots, n - 4$, the inverse eigenvalue problem for an $n \times n$ singular Hermitian matrix of rank four is solvable

Proof; Given the spectrum $\Lambda_n = \{\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n\}$, since the rank of $\Lambda_4 = 4$, it follows that

$\lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 \neq 0, \lambda_4 \neq 0$ and $\lambda_i = 0$, for $i = 5, 6, \ldots, n$.

But the number of scalars $= n - r$, where $r = 4$ implying that

$$\text{number of scalars} = n - 4.$$

Therefore we replace either the column or row with

$$a_{11} \; [1 \quad k_1 k_1 k_2 k_1 k_2 k_3 \ldots k_{n-4}]$$

and perform the column operation until all the scalars are used to generate subsequent columns and finally since the entries are symmetric about the main diagonal but in the complex plane we conjugate the entries so that we obtain

$A_{(n,4)} =$

$$\begin{bmatrix} a_{11} & \bar{k}_1 a_{11} & \bar{k}_1\bar{k}_2 a_{11} & \cdots & \bar{k}_1\bar{k}_2\ldots\bar{k}_{n-4}a_{11} & \bar{a}_{1(n-2)} & \bar{a}_{1(n-1)} & \bar{a}_{1n} \\ k_1 a_{11} & |k_1|^2 a_{11} & |k_1|^2\bar{k}_2 a_{11} & \cdots & |k_1|^2\bar{k}_2\ldots\bar{k}_{n-4}a_{11} & k_1\bar{a}_{1(n-2)} & k_1\bar{a}_{1(n-1)} & k_1\bar{a}_{1n} \\ k_1k_2 a_{11} & |k_1|^2 k_2 a_{11} & |k_1|^2|k_2|^2 a_{11} & \cdots & |k_1|^2|k_2|^2\ldots\bar{k}_{n-4}a_{11} & k_1k_2\bar{a}_{1(n-2)} & k_1k_2\bar{a}_{1(n-1)} & k_1k_2\bar{a}_{1n} \\ \vdots & \vdots & \vdots & \ddots & & \cdots & \cdots & \vdots \\ k_1k_2\ldots k_{n-4}a_{11} & |k_1|^2 k_2\ldots k_{n-4}a_{11} & |k_1|^2|k_1|^2\ldots k_{n-4}a_{11} & \cdots & |k_1|^2|k_2|^2\ldots|k_{n-4}|^2 a_{11} & k_1k_2\ldots k_{n-4}\bar{a}_{1(n-2)} & \cdots & k_1k_2\ldots k_{n-4}\bar{a}_{1n} \\ a_{1(n-2)} & \bar{k}_1 a_{1(n-2)} & \bar{k}_1\bar{k}_2 a_{1(n-2)} & \cdots & \bar{k}_1\bar{k}_2\ldots\bar{k}_{n-4}a_{1(n-2)} & a_{(n-2)(n-2)} & \bar{a}_{(n-2)n} & \bar{a}_{(n-1)n} \\ a_{1(n-1)} & \bar{k}_1 a_{1(n-1)} & \bar{k}_1\bar{k}_2 a_{1(n-1)} & \cdots & \vdots & a_{(n-2)n} & a_{(n-1)(n-1)} & \bar{a}_{(n-1)n} \\ a_{1n} & \bar{k}_1 a_{1n} & \bar{k}_1\bar{k}_2 a_{1n} & \cdots & \bar{k}_1\bar{k}_2\ldots\bar{k}_{n-4}a_{1n} & a_{(n-1)n} & a_{(n-1)n} & a_{nn} \end{bmatrix}$$

Where

$$tr\left(A_{(n,4)}\right) = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$$

$$= a_{11}[1 + |k_1|^2 + |k_1|^2|k_2|^2 + \cdots + |k_1|^2|k_2|^2 \times \ldots \times |k_{n-4}|^2]$$

$$+ a_{(n-2)(n-2)} + a_{(n-1)(n-1)} + a_{nn}$$

Which gives us the quartic equation

$$a_{11}^4[1 + |k_1|^2 + |k_1|^2|k_2|^2 + \cdots + |k_1|^2|k_2|^2 \times \ldots \times |k_{n-4}|^2]^4 -$$
$$[\textstyle\sum_{i=1}^4 \lambda_i]a_{11}^3[1 + |k_1|^2 + |k_1|^2|k_2|^2 + \cdots + |k_1|^2|k_2|^2 \times \ldots \times |k_{n-4}|^2]^3 +$$
$$[\textstyle\sum_{i=1}^3 \lambda_1\lambda_{i+1} + \sum_{i=1}^2 \lambda_2\lambda_{i+2} + \sum_{i=1}^1 \lambda_3\lambda_{i+3}]a_{11}^2[1 + |k_1|^2 + |k_1|^2|k_2|^2 + \cdots +$$
$$|k_1|^2|k_2|^2 \times \ldots \times |k_{n-4}|^2]^2 - [\textstyle\sum_{i=1}^2 \lambda_1\lambda_2\lambda_{i+2} + \sum_{i=1}^1 \lambda_1\lambda_3\lambda_{i+3} +$$
$$\textstyle\sum_{i=1}^1 \lambda_2\lambda_3\lambda_{i+3}]a_{11}[1 + |k_1|^2 + |k_1|^2|k_2|^2 + \cdots + |k_1|^2|k_2|^2 \times \ldots \times |k_{n-4}|^2] +$$
$$[\textstyle\prod_{i=1}^4 \lambda_i] = 0$$

Hence proved with the following eigenvalues

$$a_{11} = \frac{\lambda_1}{1 + |k_1|^2 + |k_1|^2|k_2|^2 + \cdots + |k_1|^2|k_2|^2 \times \ldots \times |k_{n-4}|^2},$$

$$\lambda_2 = a_{(n-2)(n-2)}, \lambda_3 = a_{(n-1)(n-1)}, \quad \lambda_4 = a_{nn}$$

The free variables are;

1. For 5×5 singular symmetric matrix we obtain one scalar that is k, hence the free variables $a_{13}, a_{14}, a_{15}, a_{35}, a_{34}$ *and* a_{45}
2. For 6×6 singular symmetric matrix we obtain two scalars that is $k_1 k_2$, hence the free variables $a_{14}, a_{15}, a_{16}, a_{45}, a_{46}$ *and* a_{56}
3. In general for $n \times n$ singular symmetric matrix we obtain $n - 4$ scalars, hence the free variables $a_{1(n-2)}, a_{1(n-1)}, a_{1n}, a_{(n-2)(n-1)}, a_{(n-2)n}$ *and* $a_{(n-1)n}$

Theorem 4.3.

The inverse eigenvalue problem for an $n \times n$ singular symmetric matrix of rank r is solvable provided that $n - r$ arbitrary parameters are prescribed.(Gyamfi, 2012)

Proof: Let

$A_{(n,r)} =$

$$\begin{bmatrix}
a_{11} & k_1 a_{11} & k_1 k_2 a_{11} & \cdots & k_1 k_2 \ldots k_{n-r} a_{11} & a_{1(n-2)} & a_{1(n-1)} & a_{1n} \\
k_1 a_{11} & k_1^2 a_{11} & k_1^2 k_2 a_{11} & \cdots & k_1^2 k_2 \ldots k_{n-r} a_{11} & k_1 a_{1(n-2)} & k_1 a_{1(n-1)} & k_1 a_{1n} \\
k_1 k_2 a_{11} & k_1^2 k_2 a_{11} & k_1^2 k_2^2 a_{11} & \cdots & k_1^2 k_2^2 \ldots k_{n-r} a_{11} & k_1 k_2 a_{1(n-2)} & k_1 k_2 a_{1(n-1)} & k_1 k_2 a_{1n} \\
\vdots & \vdots & \vdots & \ddots & & \cdots & \cdots & \vdots \\
k_1 k_2 \ldots k_{n-r} a_{11} & k_1^2 k_2 \ldots k_{n-r} a_{11} & k_1^2 k_2^2 \ldots k_{n-r} a_{11} & \cdots & k_1^2 k_2^2 \ldots k_{n-r}^2 a_{11} & k_1 k_2 \ldots k_{n-r} a_{1(n-2)} & \cdots & k_1 k_2 \ldots k_{n-4} a_{1n} \\
a_{1(n-2)} & k_1 a_{1(n-2)} & k_1 k_2 a_{1(n-2)} & \cdots & k_1 k_2 \ldots k_{n-r} a_{1(n-2)} & a_{(n-2)(n-2)} & a_{(n-2)n} & a_{(n-1)n} \\
a_{1(n-1)} & k_1 a_{1(n-1)} & k_1 k_2 a_{1(n-1)} & \cdots & \vdots & a_{(n-2)n} & a_{(n-1)(n-1)} & a_{(n-1)n} \\
a_{1n} & k_1 a_{1n} & k_1 k_2 a_{1n} & \cdots & k_1 k_2 \ldots k_{n-r} a_{1n} & a_{(n-1)n} & a_{(n-1)n} & a_{nn}
\end{bmatrix}$$

where $n \geq 5, \; n > r$ We have,

$$tr\left(A_{(n,r)}\right) = \lambda_1 + \cdots + \lambda_r$$

$$= a_{11}[1 + k_1^2 + k_1^2 k_2^2 + \cdots + k_1^2 k_2^2 \times \ldots \times k_{n-r}^2] + a_{(n-2)(n-2)}$$

$$+ a_{(n-1)(n-1)} + a_{nn}$$

Such that,

$$a_{11} = \frac{\lambda_1}{1 + k_1^2 + k_1^2 k_2^2 + \cdots + k_1^2 k_2^2 \times \ldots \times k_{n-r}^2},$$

and

$a_{ij} = \lambda_2$, where $i = j = n - r + 2$

$\vdots \; \vdots$

$a_{nn} = \lambda_r$,

for all $\lambda_1, \lambda_2, \ldots, \lambda_r \in \mathbb{R}$

Theorem 4.4.

The inverse eigenvalue problem for an $n \times n$ singular Hermitian matrix of rank r is solvable provided that $n - r$ arbitrary parameters are prescribed.

Proof: Let

$A_{(n,r)} =$

$$\begin{bmatrix}
a_{11} & \bar{k}_1 a_{11} & \bar{k}_1 \bar{k}_2 a_{11} & \cdots & \bar{k}_1 \bar{k}_2 \ldots \bar{k}_{n-r} a_{11} & \bar{a}_{1(n-2)} & \bar{a}_{1(n-1)} & \bar{a}_{1n} \\
k_1 a_{11} & |k_1|^2 a_{11} & |k_1|^2 \bar{k}_2 a_{11} & \cdots & |k_1|^2 \bar{k}_2 \ldots \bar{k}_{n-r} a_{11} & k_1 \bar{a}_{1(n-2)} & k_1 \bar{a}_{1(n-1)} & k_1 \bar{a}_{1n} \\
k_1 k_2 a_{11} & |k_1|^2 k_2 a_{11} & |k_1|^2 |k_2|^2 a_{11} & \cdots & |k_1|^2 |k_2|^2 \ldots \bar{k}_{n-r} a_{11} & k_1 k_2 \bar{a}_{1(n-2)} & k_1 k_2 \bar{a}_{1(n-1)} & k_1 k_2 \bar{a}_{1n} \\
\vdots & \vdots & \vdots & \ddots & & \cdots & \cdots & \vdots \\
k_1 k_2 \ldots k_{n-r} a_{11} & |k_1|^2 k_2 \ldots k_{n-r} a_{11} & |k_1|^2 |k_1|^2 \ldots k_{n-r} a_{11} & \cdots & |k_1|^2 |k_2|^2 \ldots |k_{n-r}|^2 a_{11} & k_1 k_2 \ldots k_{n-r} \bar{a}_{1(n-2)} & \cdots & k_1 k_2 \ldots k_{n-r} \bar{a}_{1n} \\
a_{1(n-2)} & \bar{k}_1 a_{1(n-2)} & \bar{k}_1 \bar{k}_2 a_{1(n-2)} & \cdots & \bar{k}_1 \bar{k}_2 \ldots \bar{k}_{n-r} a_{1(n-2)} & a_{(n-2)(n-2)} & \bar{a}_{(n-2)n} & \bar{a}_{(n-1)n} \\
a_{1(n-1)} & \bar{k}_1 a_{1(n-1)} & \bar{k}_1 \bar{k}_2 a_{1(n-1)} & \cdots & \vdots & a_{(n-2)n} & a_{(n-1)(n-1)} & \bar{a}_{(n-1)n} \\
a_{1n} & \bar{k}_1 a_{1n} & \bar{k}_1 \bar{k}_2 a_{1n} & \cdots & \bar{k}_1 \bar{k}_2 \ldots \bar{k}_{n-r} a_{1n} & a_{(n-1)n} & a_{(n-1)n} & a_{nn}
\end{bmatrix}$$

where $n \geq 5, \; n > r$ We have,

$$tr(A_{(n,r)}) = \lambda_1 + \lambda_2 + \lambda_3 + \dots + \lambda_r$$

$$= a_{11}[1 + |k_1|^2 + |k_1|^2|k_2|^2 + \cdots + |k_1|^2|k_2|^2 \times \dots \times |k_{n-r}|^2]$$

$$+ a_{(n-2)(n-2)} + a_{(n-1)(n-1)} + a_{nn}$$

Such that

$$a_{11} = \frac{\lambda_1}{1 + |k_1|^2 + |k_1|^2|k_2|^2 + \cdots + |k_1|^2|k_2|^2 \times \dots \times |k_{n-r}|^2},$$

and

$a_{ij} = \lambda_2$, where $i = j = n - r + 2$

$\vdots \vdots$

$a_{nn} = \lambda_r$,

for all $\lambda_1, \lambda_2, \dots, \lambda_r \in \mathbb{R}$ *and* $k \in \mathbb{R}$ *or* $\mathbb{C}$. Hence Theorem 4.4. is solvable.

CHAPTER 5

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

Summary

The main purpose of the research was to find the solution of inverse eigenvalue problem of singular symmetric and Hermitian matrix of rank greater than or equal to four. Before, this was done there was a review of some existing literature specifically a related work on the singular symmetric and Hermitian matrix for rank 1, 2 and 3. In relation to these literatures (Gyamfi, 2012 & Annor et al, 2016), the researcher modified an algorithm to include rank greater than or equal to four for the solution of inverse eigenvalue problem of singular symmetric and Hermitian matrices where numerical examples were also presented.

Conclusions

1. We modified an algorithm to generate singular symmetric matrices for rank greater than or equal to four.
2. We modified an algorithm to generate singular Hermitian matrices for rank greater than or equal to four.
3. Finally, we proved that given the non zero scalars and the parameters $k_i, i = 1,2, \dots, n-4$, the inverse eigenvalue problem for an $n \times n$ singular symmetric and Hermitian matrices of rank four are solvable.

Recommendations

Based on the findings, we recommend for consideration the following for future research;

1. Generate non-singular symmetric matrix from singular symmetric matrix of order greater than four using some direct iterative method.

2. Consider the application in some engineering field the solution of inverse eigenvalue problem for singular Hermitian matrix with rank greater than or equal to four.

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