UNIVERSITY OF CAPE COAST

HOMOGENIZATION OF PARABOLIC PARTIAL DIFFERENTIAL EQUATION USING THE MULTIPLE-SCALE EXPANSION METHOD

VICTOR ZANKONI

2020

UNIVERSITY OF CAPE COAST

HOMOGENIZATION OF PARABOLIC PARTIAL DIFFERENTIAL EQUATION USING THE MULTIPLE-SCALE EXPANSION METHOD

BY

VICTOR ZANKONI

Thesis submitted to the Department of Mathematics of the School of Physical Sciences, College of Agriculture and Natural Sciences, University of Cape Coast, in partial fulfillment of the requirements for the award of Master of Philosophy degree in Mathematics

JULY 2020

DECLARATION

Candidate's Declaration

I hereby declare that this thesis is the result of my own original research and that no part of it has been presented for another degree in this university or elsewhere.

Candidate's Signature .... Date ....

Name: Victor Zankoni

Supervisor's Declaration

I hereby declare that the preparation and presentation of the thesis were supervised in accordance with the guidelines on supervision of thesis laid down by the University of Cape Coast.

Principal Supervisor's Signature .... Date ....

Name: Prof. Emmanuel Kwame Essel

ABSTRACT

In this thesis, we have homogenization of parabolic partial differential equation using the multiple scale expansion method as its central axis. It consists of two introductory chapters into the theory of homogenization, a section is devoted to preliminary concepts and ideas needed to understand the core content of this work. We also highlighted on how the multiple scale expansion technique can be used in homogenizing elliptic partial differential equations. Finally, homogenization of parabolic partial differential equation using the multiple scale expansion method which is the focal point of this work was investigated and the results presented. The rapidly oscillating coefficient of the parabolic partial differential equation is replaced by a constant known as the homogenized coefficient.

KEY WORDS

Composite Materials

Homogenization

Homogenization of Parabolic Partial Differential Equation

Macroscopic Scale

Microscopic Scale

Multiple-Scale Expansion Method

ACKNOWLEDGMENTS

I am most grateful to the almighty God for his grace that is sufficient for me. Also I am very grateful to my father Lawrence Zida Zankoni for his generosity and undying love for me. I would also like to express my sincere gratitude to my mentor and supervisor, Prof. Emmanuel Kwame Essel of the Department of Mathematics, for his professional guidance, advice, encouragement and the goodwill with which he guided this work. I am really very grateful. Special thanks go to Prof. Ernest Yankson and Dr. Albert Lanor Sackitey also of the Department of Mathematics for their constant encouragement and being there as sources of inspiration for me.

I would also like to thank both the teaching and non-teaching staff of the Department of Mathematics at the University of Cape Coast for putting at my disposal the necessary facilities needed for my work. Their marvelous hospitality and support has made my stay here a memorable one.

To Master Faith Shadow Zotor, my best friend, I am sincerely grateful for your unflinching love and support, I owe you a lot. God bless you.

Finally, much thanks go to my siblings Emmanuel, David, Mary, Mathias, Eugenia, Francis, Nathaniel, Benjamin and the entire Zankoni family for their prayers and support.

DEDICATION

To my dad Mr. Lawrence Zida Zankoni and my mum Madam Gerthar Daboni

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF ABBREVIATIONS

PDE	Partial Differential Equation
CFRP	Carbon-Fiber Reinforced Polymer Differential Equation
BVP	Boundary Value Problem

CHAPTER ONE

INTRODUCTION

The theory and applications of the homogenization method sometimes referred to as "Upscaling" or "Zooming Out" is the primary focus of this thesis. Homogenization theory which dates back to the late sixties was very much rapidly developed during the last two decades and is now established as a distinct area within mathematics.(Jimenez, 2009)

Homogenization theory has important applications in the mathematical analysis of different mechanical and physical phenomena-in perforated media, dispersed media, porous media, composites and similar situations. The main task is to determine effective or homogenized properties such as magnetic permeability, flow in porous media, heat transfer etc. of highly heterogeneous multiphase materials. A common feature for such problems is that governing equations involve rapidly oscillating functions due to the heterogeneity of the underlying material. This rapid movements of the functions render a direct numerical treatment of such problems extremely difficult if not impossible. As a consequence of this, one has to perform some type of asymptotic analysis or averaging, which leads to the concept of homogenization.(Bystrom, 2002).

Homogenization is thus a mathematical technique used to determine the average properties of composite materials. The combination of materials with different chemical and physical properties produce a material which may partly possess essential properties of their constituents, the resulting material is referred to as a composite material and the study of such materials is an important aspect in material science. The problems modeled on such materials lead to homogenization problems. Some examples of composite materials are plastics, ceramics, fiberglass, cement, concretes, superconducting multifilamentary composites, Carbon-Fiber Reinforced Polymer (CFRP) which is a light and extremely strong composite materials made up of carbon fibers and polymer resins. This material is mainly used whenever there is a need for high strength to weight ratio and rigidity e.g. automotive and aerospace engineering and some modern bicycles and motorcycles. Geophysics, fluid mechanics, electricity, material science and engineering are fields where researchers and scientists study physical properties such as linear elasticity, thermal conductivity, electrical conductivity, transport of fluids in porous media etc. of composite (heterogeneous) media.

In recent times, composite materials are widely used by industries because they have better properties as compared to their individual constituents, this makes researchers in recent years to study extensively the use of composite materials in place of individual materials. Due to the increased usage in composite materials there is a huge need for studying and understanding the behavior of these materials mathematically. Intuitively, the size of the heterogeneities in composite materials are smaller as compared to the global dimension of the composite material. As a result of this, a composite material can be represented with two scales, the microscopic scale which represents the rapid oscillations between the heterogeneities in the composite material also known as the local scale and the macroscopic or global scale which represents the global behavior of the material. Additionally, since the mathematical equations describing the physical properties of these materials display high oscillating coefficients which in turn reflect in the solution, applying a direct numerical computation to these equations to predict the bulk/overall behavior of these materials becomes highly non-trivial and extremely difficult. To overcome this challenge, the theory of Homogenization presents us with an asymptotic analysis of the problem which yields an auxiliary equation with constant coefficients to replace the original equation with highly oscillating coefficients for easy analysis.

Research Objectives

Materials having varied length scales can be modeled using partial differential equations (PDEs) by reducing the material to a body covering a smooth domain R^n. However, it is difficult and very expensive to determine the properties of the highly heterogeneous body since the coefficients at the local or microscopic level are rapidly oscillating functions. A problem of great theoretical or practical interest is that of analyzing the average properties of composite materials. Examples of such properties include linear elasticity, transport of fluids in porous media, electrical properties, thermal properties etc. In many instances such as steady heat conduction in a composite material, it is reasonable to assume that these properties do not depend on time as such they can be modeled by using elliptic partial differential equations of the form:

$$\begin{cases} -\frac{d}{dx}\left(a(x)\frac{du}{dx}\right) = f(x), \quad \text{for} \quad \text{x} \in \Omega \\ \quad u = 0, \quad \text{on} \quad \partial\Omega \end{cases}$$

where Ω is the periodic domain, $f \in L^2(\Omega)$ and $a \in L^\infty(\Omega)$.
This assumption upon a cursory look may appear restrictive but it has numerous practical applications.

We would like to extend ideas by considering time dependent properties of composite materials. Since time dependent properties of materials can be modelled by parabolic partial differential equations, we task ourselves to homogenize such PDEs by using the multiple scale expansion method which is based on the assumption that the solution u^ε to problem (1.1) is of the form.

$$u^\varepsilon(x,t) = \varepsilon^0 u_0\left(x, \frac{x}{\varepsilon}, t\right) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}, t\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}, t\right) + \cdots = \sum_{i=0}^{\infty} \varepsilon^i u_i\left(x, \frac{x}{\epsilon}, t\right)$$

where x is the slow variable and $y = \frac{x}{\varepsilon}$ is the fast variable.

In this thesis, the homogenization of a parabolic partial differential equation problem with periodic coefficient, which is finding time-dependent average properties or characteristics of composite materials is studied.

$$\begin{cases} c^\varepsilon(x)\frac{\partial u_\varepsilon(x,t)}{\partial t} - \frac{d}{dx}\left(a^\varepsilon(x)\frac{du_\varepsilon(x,t)}{dx}\right) = f(x,t), \quad \text{in} \quad \Omega \times (0,T) \\ u_\varepsilon(x,t) = 0 \quad \text{on} \quad \partial\Omega \times (0,T) \\ u_\varepsilon(0,T) = u_0(x) \quad \text{in} \quad \Omega \end{cases} \tag{1.1}$$

where ε is a small parameter representing the heterogeneities.
In other words, a limit equation known as the homogenized or effective equation which describes the global behaviors of the heterogeneities is obtained. The approach is to study the corresponding sequence of the weak formulation of (1.1) using the multiple scale expansion technique.

Thesis Outline

The first chapter of this thesis presents a brief introduction to the theory of homogenization coupled with the main objective of the thesis. The basic concept of periodic homogenization and a brief introduction to the theory of homogenization are presented in Chapter two. Also, theorems, definitions and some important results which are used throughout the thesis are discussed here. The necessary function spaces are also defined. Chapter three gives an introduction to the method of asymptotic expansion also known as the multiple scale expansion method. We show how the effective or homogenized equation is computed from an elliptic partial differential equation. In Chapter four which is the focal chapter of this thesis, we illustrate using the multiple scale expansion method to homogenize a parabolic partial differential equation. i.e. we find the homogenized equation and the effective coefficients of the parabolic partial differential equation. Finally, Chapter five gives a brief summary of the entire thesis and conclusions.

CHAPTER TWO

LITERATURE REVIEW

Introduction

The growth of continuum mechanics and theoretical physics over the last few decades has triggered the question of justifying the global or microscopic view of physical phenomena by zooming out or "upscaling" the implied microscopic rules for particle attraction at the atomic level through the phenomena at the immediate "mesoscopic" level. This motivation has led to an extensive global programme of study and research which is still far from being complete as of now. Giving a crude but a more universally applicable approximation is the objective of this extensive activity, in other words, it aims at coming out with techniques to averaging out to some extent properties of materials at one level with the goal of obtaining a less detailed but almost equally exact description of the material properties. In this chapter we look at some mathematical techniques that can be employed in homogenizing partial differential equations. Homogenization is a technical word that aims at giving a proper description of materials that are composed of several constituents, intimately mixed together(Alouges, 2017). Indeed, when one considers a mixture of materials, e.g. a composite, it is expected that the new material will benefit from properties that each of its constituent only partly possess. The applications of such materials are numerous. Foam and wool are very classically used for thermic and acoustic insulation. Composed of fibers in the air or bubbles of air inside a rubber matrix, they only partly reproduce the behavior of their constituents. Other examples are given by the so-called "spring magnets" which are composed of hard and soft magnets mixed together, porous media which are a solid matrix with micro-channels in which a fluid may flow or multilayer materials. All these properties can be modeled by using partial differential equations. Numerically, analyzing partial differential equations with rapidly oscillating coefficients (highly heterogeneous domain) do not give classical solution but instead a sequence of solutions due to the heterogeneous nature of the material involved, this situation makes it very expensive and extremely difficult if not impossible to find the solution of such Partial Differential Equations by using a direct numerical approach. As a result, we seek to replace these PDEs with rapidly oscillating coefficients with an auxiliary one of similar characteristic but with a stable coefficient for easy analysis, this is the main idea in Homogenization Theory. Homogenization is therefore a mathematical technique used to analyze partial differential equations i.e. elliptic, hyperbolic, parabolic, etc. with rapidly varying or oscillating coefficients, equations in perforated domains, equations with rough random coefficients and many other objects with practical and theoretical interests. Similarly, homogenization can be said to be a branch of the theory of partial differential equations which provides the mathematical basis for describing the effective properties of materials with inhomogeneous microstructures. Due to the numerous applications of homogenization theory which include but not limited to predicting properties of composite materials even before they are engineered and macroscopic modeling of microscopic systems, mathematicians have come out with novel ideas and techniques as far as the theory of homogenization is concerned. Some examples of these techniques are: G-Convergence Method, H-Convergence Method, the Two-Scale Convergence Method and the Multiple Scale Expansion Method. These approaches make use of certain assumptions and theorems such as the coerciveness and bounded assumption, the periodicity theorem, the Eberlein-Smuljan theorem, the Lax Milgram theorem and the Riesz representation theorem for reflexive spaces and so forth to ensure that solutions of prospective PDEs are unique. The two scale convergence method for studying boundary value problems with periodic rapidly oscillating coefficients was first introduced by Nguetseng(1989) which was additionally developed by Allaire(1992). Again Nguetseng(2003) further developed the two-scale convergence method to tackle problems beyond periodic settings under the name Σ-convergence. Also, the G-Convergence method was introduced by Spagnolo(1967) which deals with convergence of solutions to symmetric problems with periodic and non-periodic coefficients, this technique was later extended to the H-Convergence method by Tartar and Murat(1977) to non-symmetric problems.

From its inception till now loads of work has been done in homogenizing elliptic, parabolic and hyperbolic partial differential equations as well as finding the average properties (eg. thermal properties, flow in porous media, electrical properties, linear elasticity etc.) of composite materials (materials having two or more components with different physical properties, some examples of composite materials are cellulose fiber, concrete, wood, textile, aluminum,plastic and so forth) where time is of no essence, that is analyzing time-independent partial differential equations.

Bensoussan,Boccardo and Murat(1986) published their work on Homogenization of elliptic equations among others which primarily focused on analyzing elliptic PDEs with principal part not in divergence form and Hamiltonian with quadratic growth. Bourgeat and Pankratov(1996) also contributed their quota to the field of Homogenization by analyzing semi linear parabolic equations in domains with spherical traps. Again Efendiev and Pankov(2004) in their work used numerical approach in homogenizing nonlinear random parabolic operators where they proposed and analyzed a numerical homogenization procedure that is applicable to heterogeneities of general nature.

A thesis by Lobkova(2017) used the method of two-scale convergence to homogenize parabolic partial differential equations and most recently, Sackitey(2019) also highlighted on how the multiple scale expansion method can be used to find the average properties of composite materials where time is of no essence. In all these, very little has been done in analyzing time dependent partial differential equations by using multiple scale expansion method i.e. finding average properties that are time dependent of composite materials. By way of contributing to this field, I seek to extend previous knowledge by Sackitey(2019) to analyzing parabolic Partial Differential Equations using the multiple scale expansion method. In other words I aim at using the multiple scale expansion method to find the average properties of composite materials where the contribution of time is taken into consideration. Properties of composite materials where time is of essence can be modeled by using parabolic PDE of the form:

$$\begin{cases} c^{\varepsilon}(x)\frac{\partial u_{\varepsilon}(x,t)}{\partial t} - \frac{d}{dx}\left(a^{\varepsilon}(x)\frac{du_{\varepsilon}(x,t)}{dx}\right) = f(x,t), \quad \text{in} \quad \Omega\times(0,T) \\ u_{\varepsilon}(x,t) = 0 \quad \text{on} \quad \partial\Omega\times(0,T) \\ u_{\varepsilon}(x,0) = u_0(x) \quad \text{in} \quad \Omega \end{cases} \tag{2.1}$$

where ε is a small positive number converging to zero, $u_{\varepsilon}(x,t)$ is assumed to be positive and belongs to the space $L^{\infty}(\Omega)$, $a_{\varepsilon}(x)$ satisfies the boundedness condition $0 < \alpha < a_{\varepsilon}(x) < \beta < \infty$ a.e. on R and $f \in L_2(\Omega)$. The multiple scale expansion method finds the asymptotic behavior of $u_{\varepsilon}(x,t)$ by making use of the assumption that u_{ε} has a two scale expansion of the form:

$u^{\varepsilon}(x,t) = \varepsilon^0 u_0\left(x,\frac{x}{\varepsilon},t\right) + \varepsilon u_1\left(x,\frac{x}{\varepsilon},t\right) + \varepsilon^2 u_2\left(x,\frac{x}{\varepsilon},t\right) + \cdots = \sum_{i=0}^{\infty}\varepsilon^i u_i\left(x,\frac{x}{\varepsilon},t\right)$
and by making use of lemma 7, equation (2.1) with rapidly oscillating coefficient can be replaced with equation (2.2) the homogenized equation below

$$\begin{cases} c^{*}\frac{\partial u_{\varepsilon}(x,t)}{\partial t} - \frac{d}{dx}\left(a^{*}\frac{du_{\varepsilon}(x,t)}{dx}\right) = f(x,t), \quad \text{in} \quad \Omega\times(0,T) \\ u_{\varepsilon}(x,t) = 0 \quad \text{on} \quad \partial\Omega\times(0,T) \\ u_{\varepsilon}(x,0) = u_0(x) \quad \text{in} \quad \Omega \end{cases} \tag{2.2}$$

where a^* and c^* are constants.

Concept of Homogenization

In analyzing a homogeneous material the assumption that the length scale of the heterogeneities or the oscillation is very small ($\varepsilon << 1$) as compared to the global dimension of the domain under investigation is made. Due to this, the physical characteristics of the domain can be modeled by using partial differential equations with rapidly oscillating coefficients or appropriate differential equations with complex structures. Example being in non-periodic domains or perforated domains. The rapidly oscillating coefficients makes it extremely difficult to use a direct numerical approach or technique to solve the resulting boundary value problems.

By holding the assumption that the microscopic scale much less than the scale of the global dimension of the heterogeneous material then the global or macroscopic description of the material can be given. In such situations, the material usually emits stable characteristic such as thermal conductivity, electrical conductivity, flow in porous media, elasticity etc., which may to a large extent differ from its properties or characteristics on a microscopic scale.

In order to mathematically analyze and describe a heterogeneous material, it is assumed that its properties on the microscopic scale is dependent on a minute parameter ε converging to zero which is the length scale of the microscopic structure(Lobkova,2017).

The effective model of the phenomenon under discussion is arrived at by an asymptotic analysis of the boundary value problem as the small parameter ε converges to zero i.e. as $\varepsilon \to 0$ or equivalently as the global length scale(L) approaches infinity i.e. as $L \to \infty$ hence the name “zooming out” or “upscaling”.

Now the limit of the solution to the original problem satisfies an auxiliary differential equation with a much more stable coefficient as compared to the original differential equation.

Illustrative Example

Consider the following model problem of conductivity of a composite material occupying a domain $\Omega \in R^n$ where $n \in [1, 2]$, $A^\varepsilon(x)$ is a heat conductivity matrix that is periodic with period $(0, 1)^N$, the source term is denoted by f and (u^ε) is an unknown function that represent the temperature in Ω and equals zero on the surface $\partial\Omega$ of the body.

$$\begin{cases} -\nabla(A^\varepsilon \nabla u^\varepsilon) = f \quad \text{in} \quad \Omega \\ u^\varepsilon = 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

Suppose the length scale of the heterogeneities is small as compared to the global dimension of the domain, then the domain occupied by the material can be divided into periodic cells with period ε which represents a small positive number converging to zero $\varepsilon \to 0$ as shown in Figure 1.

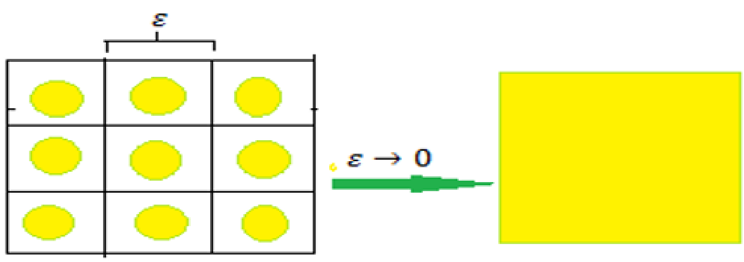

Figure 1: Homogenizing the mixture

Now by selecting one of the cells and "upscaling" it to a unit cell, and denoting it by Y=[0,1] as seen in Figure 2.

Clearly it can be noticed that our model will depend on two scales, the first scale $x \in \Omega$ which is the slower between the two shows the position of a point in the entire domain under investigation it is also known as the macroscopic scale

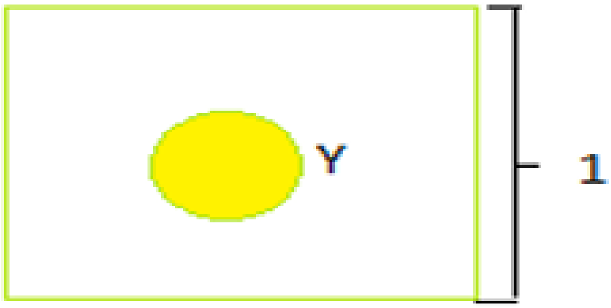

Figure 2: The representative cell upscaled to a unit cell

or the global scale. The second scale denoted by $y = \frac{x}{\varepsilon} \in Y$ is referred to as the microscopic scale which shows the rapid oscillations between the individual cells, it is also known as local scale.

Let $a(y)$ be a matrix-valued function representing the variations in the thermal conductivity in the "upscale" periodic cell. Now by substituting $y = \frac{x}{\varepsilon} \in Y$, we arrive at a rapidly oscillating function $a\left(\frac{x}{\varepsilon}\right)$ with period ε at any point x$\in \Omega$, describing the thermal conductivity in the material. Suppose that the material is placed in a medium with zero temperature and a heat source given by a function f is introduced, then the boundary value problem takes the form:

$$\begin{cases} -\frac{d}{dx}\left(a\left(\frac{x}{\varepsilon}\right)\frac{du_\varepsilon(x)}{dx}\right) = f \quad \text{in} \quad \Omega \\ u_\varepsilon = 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{2.3}$$

for small values of ε, by using numerical approach, it is very difficult to solve (2.3).

Additionally, as ε gets smaller and approaches zero, the heterogeneities of the material also becomes smaller thereby assuming the appearance of a homogeneous material. To this end, it is evident that the material will assume a homogeneous status on a macroscopic scale. A natural question to ask is as ε approaches zero, can we determine a limit equation similar to (2.3) and satisfied by the limit u_0?

That is problems of the form

$$\begin{cases} -\frac{du_0}{dx}\left(\bar{a}\frac{du_0}{dx}\right) = f \quad \text{in} \quad \Omega \\ u_0 = 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{2.4}$$

Equation 2.4 is known as the effective or homogenized boundary value problem and $\bar{a}$ is the homogenized thermal conductivity matrix. Many natural questions arise:

1. Convergence to a limit. Is there a limit u_0 of u_ε, as $\varepsilon \to 0$? In which sense should we understand the convergence (i.e., in which norm, which topology etc.)? What is the convergence rate?

2. Characterization of the limiting process. What kind of equation does the limit satisfy? Suppose that the limit equation is of the following form:

$$\begin{cases} -\text{div}\left(\bar{a}\nabla u_0\right) = f \quad \text{in} \quad \Omega \\ u_0 = 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{2.5}$$

Is (2.5) of the same type as the original equation (2.3)? Are the effective coefficients of the limit equation constant? How can we compute the effective coefficients?

3. Properties of the limiting equation. How do the properties of the solution u_0 of the limiting equation compare with those of u_ε? Is u_0 a good approximation of u_ε?

4. Numerical homogenization. Can we design and implement efficient numerical algorithms for problems of the form (2.3), based on the method of homogenization?

Finding answers to all these questions is the main aim of mathematical homogenization. These questions are very important in applications, since if one can give positive answers to these problems, then the limit coefficients as it is well known from engineers and physicists are good approximations of the global characteristics of the composite material when regarded as a homogeneous one.

Moreover replacing the problem by the limit one, allows us to make easy numerical analysis and computations. There are several problems in engineering and physics that can be solved by using the theory of homogenization. Examples are flow in porous media, composite materials, etc. it also has applications in mathematical finance and atmospheric turbulence. A common feature of all these problems is that the phenomena occur at various length scales and time, as a result, the partial differential equations used to model these physical phenomena have solutions with very complicated multi-structure. This renders the numerical simulation and analysis of such problems extremely difficult.

Hilbert Space $L_2(a, b)$

The space $L_2(a, b)$ is a linear space, i.e. the elements in this space are additive and homogeneous, the elements of which are square integrable functions (in the Lebesgue sense) in the bounded interval $[a, b]$.

Thus the integrals,

$$\int_a^b u(x)dx \qquad \text{and} \qquad \int_a^b u^2(x)dx \tag{2.6}$$

exists and are finite. In this space, the following definition holds

1. Scalar product

$$(u, v) = \int_a^b uvdx \tag{2.7}$$

2. The norm

$$\|u\| = \sqrt{(u, u)} \tag{2.8}$$

3. The Distance or metric

$$d(u, v) = \|u - v\| \tag{2.9}$$

Remark 1

From the theory of Lesbegue integral we know that

i. If the functions $u(x)$ and $v(x)$ are square integrable in the domain Ω, then the function

$$a_1 u(x) + a_2 v(x)$$

where a_1 and a_2 are arbitrary real constants, is also square integrable in Ω

ii. If the functions $u(x)$ and $v(x)$ are square integrable in the domain Ω, then the integral

$$\int_\Omega u(x)v(x)dx$$

is convergent with the following rules being valid

$$\int_\Omega u(x)v(x)dx = \int_\Omega v(x)u(x)dx$$

$$\int_\Omega [a_1 u_1(x) + a_2 u_2(x)]v(x)dx = a_1 \int_\Omega u_1(x)v(x)dx + a_2 \int_\Omega u_2(x)v(x)dx$$

Property (i) shows that the set of functions square integrable in the considered domain Ω, constitutes a linear set.

Property (ii) makes it possible to define the inner products of two functions on this set.

Definition 3

A function is said to be normed in the $L_2(a, b)$ space, if its norm is equal to unity.

Convergence of L_2 space

A sequence of functions $\{v_1, v_2, v_3, \ldots\}$ is said to converge in the space $L_2(a,b)$ or in the mean on the interval $[a,b]$ and to have the limit v, if

$$\lim_{n\to\infty} d(v, v_n) = 0 \tag{2.10}$$

we write

$$\lim_{n\to\infty} v_n = v \text{ in } L_2(a,b)$$

According to equations (2.15)-(2.17) the limit (2.18) can be written as:

$$\begin{aligned}
\lim_{n\to\infty} |v - v_n| &= 0 \\
\lim_{n\to\infty} (v - v_n, v - v_n)^{\frac{1}{2}} &= 0 \\
\lim_{n\to\infty} \left(\int_a^b (v - v_n)(v - v_n)dx \right)^{\frac{1}{2}} &= 0
\end{aligned}$$

$$\lim_{n\to\infty} \int_a^b (v - v_n)^2 dx = 0 \tag{2.11}$$

In an example, the sequence of functions

$$v_n(x) = x^n \quad n = 1, 2, \ldots \tag{2.12}$$

converges in the space $L_2(0,1)$ (in the mean on the interval [0,1] to the zero function $v = 0$ because according to (2.19))

$$\lim_{n\to\infty} ||v - v_n|| = \lim_{n\to\infty} \left(\int_a^b (v - v_n)^2 dx \right) = \lim_{n\to\infty} \left(\int_0^1 (0 - x_n)^2 dx \right)$$

$$= \lim_{n\to\infty} \left(\int_0^1 (x_n)^2 dx \right) = \lim_{n\to\infty} \left[\frac{1}{2n+1} x^{2n+1} \right]_0^1 = \lim_{n\to\infty} \frac{1}{2n+1}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}}{2 + \frac{1}{n}} = 0.$$

thus

$$\lim_{n\to\infty} ||v - v_n|| = 0$$

and so $v_n(x) = x^n$ converges to zero as $n \to \infty$

Remark 2

1. The sequence (2.20) converges pointwise in the interval [0,1] to the function

$$v(x) = \begin{cases} 0 & \text{if} \quad x \in [0,1) \\ 1 & \text{if} \quad x = 1 \end{cases}$$

since for every $x \in [0,1)$, we have

$$\lim_{n\to\infty} x^n = 0$$

and for x=1

$$\lim_{n\to\infty} 1^n = 1$$

2. The space of $C[0,1]$ of continuous functions is not extensive enough to accommodate the L_2-metric. The space is completed by adding the limits of all non-convergent Cauchy sequences to form $L_2[0,1]$. The $L_2(\Omega)$ space contains discontinuous functions that are square integrable over the domain Ω.
3. $||f||_{L_p(a,b)} = \left(\int_a^b |f|^p dx\right)^{\frac{1}{p}}$ represents the area under the curve of $f(x)$ and between a and b.
4. $||f||_{L_\infty} = \max|f(x)|$ for $a \leq x \leq b$ i.e. $L_\infty = \max L_p$. As p increases the area under the curve depends on the maximum of the function.

Definition 4 (Strong convergence)

A sequence (v_n) in a normed space V is said to be strongly convergent or convergent in a norm if there is a $v \in V$ such that

$$\lim_{n\to\infty} \|v - v_n\| = 0 \text{ or } \|v - v_n\| \leq \varepsilon \quad \forall n \in \mathbb{N}$$

This is written as $\lim_{n\to\infty} v_n = v$ or simply $v_n \to v$. v is called the strong limit of (v_n), and we say that (v_n) converges strongly to v.

Equivalent Function

Two functions for which $d(u, v) = 0$ i.e. $\|u - v\| = 0$ or $\int_a^b (u - v)^2 dx = 0$ hold, are called equivalent in the space $L_2(a, b)$; we write $u = v$ in $L_2(a, b)$.

1. All mutually equivalent functions are considered as one element of the space $L_2(a, b)$.

2. Equivalent functions may differ in the interval $[a, b)$ only on a set of measure zero, for example at infinite number of points.

3. If two functions are equivalent and continuous in $[a, b]$, then they are equal in the whole space i.e., then it holds that $u = v$ in $L_2(a, b) \iff u(x) = v(x) \quad \forall x \in [a, b]$ Especially if u is a constant function in $[a, b]$ then $u = 0$ in $L_2(a, b) \iff u(x) = 0 \quad \forall x \in [a, b]$.

Especially if u is a constant function in [a,b] then $u = 0$ in $L_2(a, b) \iff u(x) = 0 \quad \forall x \in [a, b]$.

Properties in L^p Spaces

1. Let $\varepsilon \in \mathbb{R}$ with $1 \leq p \leq \infty$. The set $L^p(\Omega)$ is a Banach space with the norm

$$\|u\|_{L^p(\Omega)} = \begin{cases} \left(\int_\Omega |u(x)|^p dx\right)^{\frac{1}{p}} & \text{for} \quad p < +\infty \\ \inf\{C, |u| \leq C \quad \text{a.e. on} \quad \Omega \quad \text{for} \quad p = +\infty\} \end{cases}$$

2. If $p = 2$ the space $L^2(\Omega)$ is a Hilbert space for the scalar product

$$(u, v)_{L^2(\Omega)} = \int_\Omega u(x)v(x)dx$$

3. The space $L^p(\Omega)$ is separable for $1 \leq p \leq +\infty$ and is uniformly convex $1 < p < +\infty$

4. The space $L^2(\Omega)$ for $1 < p < +\infty$ is reflexive. Note that the space $L^1(\Omega)$ is not reflexive and also that the space $L^\infty(\Omega)$ is neither reflexive nor separable

Sobolev Space $W^{1,p}(\Omega)$

Definition 5

Let Ω be an open subset of $\mathbb{R}$ and $1 \leq p < +\infty$. The Sobolev space

$$W^{1,P}(\Omega) = \{u \in L_p(\Omega) : Du \in L_p(\Omega; \mathbb{R}^n)\}$$

where $Du = (Du_1, Du_2, \dots, Dn_u) = \left\{ \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n} \right\}$ denotes the first distributional derivative of the function u. We define the norm

$$W^{1.2} = (\Omega) \left(\|u\|^p_{L_2} + \|u\|^p_{L_2} \right)^{\frac{1}{p}}$$

When $p = 2$ we get the $W^{1,2}(\Omega)$ space and it is defined as the set of all $u \in L_2(\Omega)$ such that all the first partial derivatives $\frac{\partial u}{\partial x_i} \in L_2(\Omega)$. The exponent $1, 2$ in $W^{1,2}(\Omega)$ means the function u and its first partial derivative of order 1 are square integrable. Functions belonging to $W^{1,2}(\Omega)$ do not have to be differentiable at every point. For example, it is enough if they are continuous with piecewise continuous partial derivatives in the domain of definition and satisfy the above conditions.

Definition 6

Let $1 \leq p < \infty$. $W_0^{1,2}(\Omega)$ denotes the closure of $C_0^\infty(\Omega) \in W^{1,2}(\Omega)$. $W^{-1,q}(\Omega)$ where $\frac{1}{p} + \frac{1}{q} = 1$ is the dual space of $W_0^{1,2}(\Omega)$.

Definition 7

Let Y be a cell in $\mathbb{R}^n$ (the cell is a rectangle if $n = 2$; a box if $n = 3$). Then

$$W_{per}^{1,2}(Y) = \left\{u \in W^{1,2}(Y) : u \text{ is } Y - \text{periodic}\right\}$$

(We read like this: W-periodic, $1, 2$, of Y is the set of all functions u which is an element of the space $W, 1, 2$ of Y where u is Y-periodic). The statement that the function u is Y-periodic means that u have the same-values on opposite faces of the cell Y. The space

$$W_0^{1,2}(\Omega) = \left\{u \in W^{1,2}(Y) : u \text{ is } 0 \text{ on the boundary } \partial\Omega\right\}$$

(We read it like this: $W, 0, 1, 2$ of Y): It is the set of all functions u which are elements of the space $W^{1,2}(\Omega)$ where u is zero on the boundary $\partial\Omega$. In particular the following holds

$$W_0^{1,2}(\Omega) \subset W_{per}^{1,2}(\Omega) \subset W^{1,2}(\Omega) \subset L_2(Y)$$

The W-Spaces are often called Sobolev Spaces.

Remark 3

If $p = 2$ the following notations hold

$$H_0^{1,2}(\Omega) = H_0^1(\Omega) = W_0^{1,2}$$

$$H^{1,2}(\Omega) = H^1(\Omega) = W^{1,2}(\Omega)$$

In this space, the following definitions hold:

1. **The inner product in** $W^{1,2}(\Omega)$:

$$(u,v)_{W^{1,2}(\Omega} = \int_\Omega (uv + \nabla u \cdot \nabla v)dx \tag{2.13}$$

where $uv + \nabla u \cdot \nabla v = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}$ If we denote the L_2-inner product by

$$(u,v)_{L_2(\Omega} = \int_\Omega (uv)dx$$

then equation(2.13) reads as

$$(u,v)_{W^{1,2}(\Omega)} = (u,v)_{L_2(\Omega)} + (\nabla u, \nabla v)_{L_2(\Omega)} \tag{2.14}$$

2. **The Norm**

$$\begin{aligned} \|u\|^2_{W^{1,2}(\Omega)} &= (u,u)_{W^{1,2}(\Omega)} = (u,u)_{L_2(\Omega)} + (\nabla u, \nabla u)_{L_2(\Omega)} \\ &= \|u\|^2_{L_2(\Omega)} + \|\nabla u\|^2_{L_2(\Omega)} & (2.15) \\ \|u\|^2_{W^{1,2}(\Omega)} &= \int_\Omega \left(|u|^2 + |\nabla u|^2\right) dx & (2.16) \end{aligned}$$

where $\|\nabla u\|_{L_2(\Omega)} = \left(\sum_{i=1}^{n} \left|\frac{\partial u}{\partial x_i}\right|^2\right)$

3. **The Distance or Metric**

$$d(u,v)_{W^{1,2}(\Omega} = \|u - v\|_{W^{1,2}(\Omega)}$$

Remark 4

1. (a) For the general Sobolev space, $W^{k,2}(\Omega)$ the inner product is

$$(u,v)_{W^{k,2}(\Omega)} = (u,v)_{L_2(\Omega)} + (u^l, v^l)_{L_2(\Omega)} + \cdots + (u^k, v^k)_{L_2(\Omega)}$$

where $u^l = \nabla u$ and $v^l = \nabla v$

(b) The norm is given by

$$\|u\|^2_{W^{k,2}(\Omega)} = (u, u)_{W^{k,2}(\Omega)}$$

and

$$d(u, v)_{W^{k,2}(\Omega)} = \|u - v\|_{W^{k,2}(\Omega)}$$

2. The space $W_0^{k,2}(\Omega)$ is the subspace of $W^{k,2}(\Omega)$ for which the following also holds

$$u(a) = u^l(a) = \cdots = u^{k-1}(a) = 0$$

$$u(b) = u^l(b) = \cdots = u^{k-1}(b) = 0$$

i.e. u and its $(k-1)$ derivatives are zero on the boundary $\partial\Omega$.

3. If $\Omega \subset \mathbb{R}^n$, we always let $|\Omega|$ denote the value

$$|\Omega| = \int_\Omega 1 dx$$

(which equals the length, area or volume if $n = 1, 2$ or 3, respectively).

Linear Operator

A linear operator (or a linear map, linear mapping, linear transformation) is a function between two vector spaces that preserve the operations of vector addition and scalar multiplication.

Definition 8

Let X and Y be vector spaces over the same field K. A function $F : X \to Y$ is said to be a linear map if for any two vectors x and y in X and any scalar and in K, the following two conditions are satisfied:

$$\begin{cases} F(x+y) = F(x) + F(y) \text{ additivity} \\ F(\alpha x + \beta y) = \alpha F(x) + \beta F(y) \text{ homogeniety of degree one} \end{cases} \tag{2.17}$$

More generally, for any vectors $x_1, \ldots, x_m \in X$ and scalars $\alpha_1, \ldots, \alpha_m \in K$, the following equality holds:

$$F(\alpha_1 x_1 + \cdots + \alpha_m x_m) = \alpha_1 F(x_1) + \cdots + \alpha_m F(x_m)$$

Remark 5

A linear operator $F : X \to Y$ is continuous if and only if it is bounded. Thus for linear maps, continuity and boundedness are equivalent. Here X and Y are normed linear spaces.

Bounded Linear Operator B(X,Y):

A linear operator $F : X \to Y$ is bounded if there is $C > 0$ such that

$$\|Fx\|_Y \leq C\|x\|_X \quad \forall x \in X$$

Definition 9

An operator F which maps its domain D_F into the set of real, or complex, numbers is called a functional (real, or complex, respectively). In other words a functional is an operator whose range lies in K (scalars $K = \mathbb{R}$ or $\mathbb{C}$).

$$F : X \to K$$

A functional F assigns to every element $u \in D_F$ a certain number Fu (also written as $F(u)$), real or complex.

Definition 10

A functional F is called bounded in D_F if there exist a number K such that for all elements $u \in D_F$ the relation

$$\|Fu\| \leq K\|u\| \tag{2.18}$$

is valid. The least of the numbers K for which (2.26) is satisfied is called the norm of the functional F; written as $\|F\|$.

Definition 11

A functional F is called continuous at the point $u_0 \in D_F$ if, for every sequence of elements $u_n \in D_F$ for which

$$\lim_{n\to\infty} u_n = u_0 \text{ in } H$$

we have

$$\lim_{n\to\infty} Fu_n = Fu_0.(\text{ i.e. } \lim_{n\to\infty} \|Fu_n - Fu_0\| = 0).$$

In other words, a functional F is continuous if the following holds,

$$u_n \to u_0 \implies Fu_n \to Fu_0 \text{ as } n \to \infty.$$

If a functional F is continuous at every point in its domain then it is continuous throughout its domain.

Definition 12

A real functional F is called linear if its D_F is a linear set and if, for arbitrary real numbers $\alpha_1, \dots, \alpha_n$ and for arbitrary elements $\alpha_1, \dots, \alpha_n$ from D_F , the relation

$$F(\alpha_1 u_1 + \cdots + \alpha_n u_n) = \alpha_1 F(x_1) + \cdots + \alpha_n F(x_n)$$

holds in general. To prove that F is linear, it suffices to show that

$$F(\alpha u) = \alpha F(u)$$

$$F(u_1 + u_2) = F(u_1) + F(u_2)$$

Here Fu_n and Fu are real numbers and that the symbol

$$\lim_{n \to \infty} Fu_n = Fu$$

denotes the convergence of sequence of real numbers Fu_n to the number Fu. For values of the functional F at the point $u \in D_F$ we often use the notation

$$\langle F, u \rangle$$

instead of the symbol Fu or $F(u)$. A linear functional is a special kind of linear operator, the concept of continuity for linear functionals is the same as that for linear operators in a normed space.

Example 2

Let $F : X \to R$. Then F is a functional if

a. $\langle F, \alpha x_1 + \beta x_2 \rangle = \alpha \langle F, x_1 \rangle + \beta \langle F, x_2 \rangle$

b. $| \langle F, x \rangle | \leq C||x||_X$ boundedness of f

Moreover $f \in X^*$ and $\langle f, x \rangle \leq ||f||_X^* ||x||_X$

Remark 6

1. The use of the word bounded is different from its use in the phrase bounded function. The function $x \to 2x$ (same as $f(x) = 2x$) is not a bounded function on R, it tends to infinity as $x \to \infty$. But regarded as a functional on R, it satisfies (2.26) with $K = 2$, and is therefore a bounded functional. It is the

ratio

$$\frac{|fx|}{||fx||} = \frac{|2x|}{||x||} = \frac{2||x||}{||x||} = 2$$

that must be bounded. This ratio satisfies (2.26).

2. With ordinary functions the values of the independent variables are usually numbers. However with functionals, the values of the independent variables are mostly functions.

In an example we let $g : C[0,1] \to R$ be defined by

$$g(x) = \int_0^1 x(t)dt$$

If $x_1(t) = t$ (which is a continuous function) on [0,1] , then the value of g at x_1 is given by

$$\langle g, x_1 \rangle = \int_0^1 tdt = \frac{1}{2}$$

while if $x_2(t) = \cos \pi t$ on [0,1] then

$$\langle g, x_2 \rangle = g(x_2) = \int_0^1 x_2(t)dt = \int_0^1 \cos \pi tdt = \left[\frac{1}{\pi} \sin tdt\right]_0^1 = 0$$

Here x can be replaced by any function that is continuous on [0,1]. Once x is chosen, the corresponding value of the dependent variable $y = g(x)$ is given by the integral of x from 0 to 1. Note that the independent variable x stands for a function and not a number.

Functionals defined on an inner product space

We demonstrate this with an example below.

Let v_0 be a fixed element in an inner product space. Consider the operator F defined by

$$Fu = (u, v_0) \quad \text{for every} \quad u \in V. \tag{2.19}$$

then

1. F is clearly a functional since, for every $u \in V$ the inner product (u, v_0) is a real number.

2. It is linear because

$$F(\alpha_1 u_1 + ... + \alpha_n u_n) = (\alpha_1 u_1 + ... + \alpha_n u_n, v_0) = (\alpha_1 u_1, v_0) + ... + (\alpha_n u_n, v_0)$$

$$= \alpha_1 (u_1, v_0) + ... + \alpha_n (u_n, v_0)$$

$$= \alpha_1 F u_1 + ... + \alpha_n F u_n$$

3. F is bounded because by the Cauchy Schwarz inequality we have

$$|Fu| = |u, v_0| \leq ||u|| ||v_0|| = K||u||$$

where $K = ||v_0||$. Hence F is bounded.

4. By definition

$$||F|| = \sup_{||u||=1} |Fu| = \sup_{||u||=1} |(u, v_0)| \leq \sup_{||u||=1} ||u|| ||v_0|| = ||v_0||$$

thus

$$||F|| \leq ||v_0|| \tag{2.20}$$

However, for the fixed element v_0 we have

$$|Fv_0| = |(v_0, v_0)| = |||v_0|||v_0|| = ||v_0||^2 \geq ||v_0||$$

Again by definition

$$||F|| = \sup_{||v_0|| \leq 1} |Fv_0| \geq \sup_{||v_0|| \leq 1} ||v_0|| = ||v_0||$$

thus

$$||F|| \geq ||v_0|| \tag{2.21}$$

Hence from (2.28) and (2.29) we conclude that the operator norm of F is given by

$$||F|| = ||v_0||.$$

Dual Space X^*

Any vector space, X, has a corresponding dual vector space X^* (or just dual space for short) consisting of all linear functionals on X. Dual spaces are employed for defining and studying concepts like measures, distributions, and Hilbert spaces. Consequently, the dual space is an important concept in the study of functional analysis. The fundamental principle of functional analysis is that investigations of spaces are often combined with those of the dual spaces. There are two types of dual spaces, the algebraic dual space and the continuous dual space. The algebraic dual space is defined for all vector spaces.

Definition 13

Let X be a vector space over a field K. Then we define the dual space (or conjugate space) of X to be

$$X^* = \{f : X \to K \backslash f \text{ is linear}\}.$$

Thus, the vector space X is a space consisting of bounded linear functionals (i.e., $B(X, K)$) with the range in scalars K. If $f, g \in X^*$ then f, g satisfy the following operations

$$(f + g)(x) = f(x) + g(x) \quad \forall x \in X \text{ additivity} \tag{2.22}$$

$$(\lambda f)(x) = \lambda f(x) \quad \forall x \in X, \lambda \in \text{K scalar multiplication}$$

A linear functional $f : X \to K$ is bounded if for some $C > 0$

$$|fx| \leq C\|x\|_X \quad \forall x \in X$$

The space X^* is again a normed vector space if the norm of the functional F is defined by the formula

$$\|f\|_{X^*} = \sup_{0 \neq x \in X} \frac{|f(x)|}{\|x\|_X} = \sup_{\|x\| \leq 1} |f(x)| = \sup_{\|x\| = 1} |f(x)|$$

where $f(x) = \langle f, x \rangle_{X^*,X}$. Moreover

$$|\langle f, x \rangle_{X^*,X}| \leq \|f\|_{X^*}\|x\|_X$$

The bracket $\langle ., . \rangle_{X^*,X}$ is called the duality pairing between X^* and X. The two spaces X^* and X are isomorphic (identical) if and only if the dimension of X is finite. If not, then X^* has a larger (infinite) dimension than X. The two spaces are situated symmetrically, dual to each other (provided the dimension is finite). For instance, dual to the space of row n-vectors is the space of column n-vectors. The dual space X^* of a normed space X is a Banach space (whether or not X is). The dual space of

1. $(l_1)^* = l_\infty$ but $(l_\infty)^* \neq l_1$

2. $1 < p < \infty \implies (l_p)^* = l_q$ where $\frac{1}{p} + \frac{1}{q} = 1$, $(c)^* = (c_0)^* = l_1$

Bidual Space X^{**}

The Bidual space is denoted by $X^{**} = (X^*)^*$. It can be shown that X may be identified with a certain subspace $\bar{X}$ of X^{**}. The space X can then be thought of as being embedded in its bidual X^{**}.

Remark 7

1. To each $x \in X$ there corresponds an $F_x \in X^{**}$ i.e., there is a natural continuous linear transformation

$$\Pi : X \to X^{**}$$

$$x \to F_x \text{ i.e. } \Pi(x) = F_x$$

defined by

$$F_x(f) = f(x)$$

where

$$f : X \to \mathbb{R} \quad \forall x \in X, f(x) \in \mathbb{R}$$

$$F_x : X \to \mathbb{R} \quad \forall x \in X, F_x \in \mathbb{R}$$

for every $x \in X$, $f \in X^*$

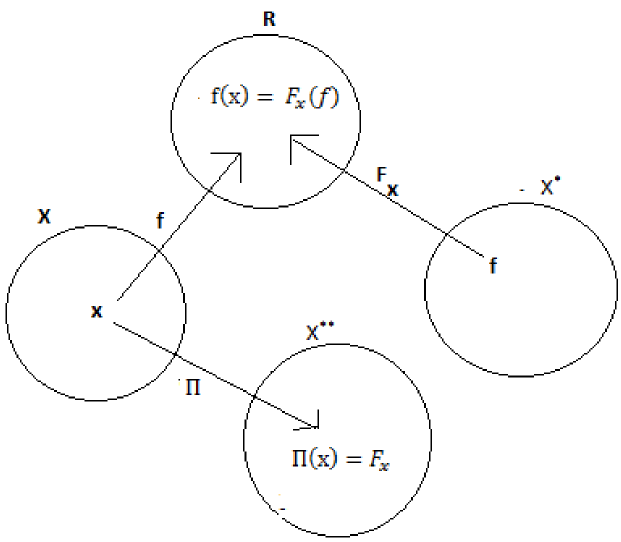

Figure 3: Relationships between X, X^* and X^{**}

F_x **is a linear operator**

1 The functional $f \in X^*$ acts on elements $x \in X$ and produces a number $f(x) \in \mathbb{R}$

2 The functional $F_x \in X^{**}$ acts on element $f \in X^*$ and produces a number $F_x(f) \in \mathbb{R}$

3 The subscript x in F_x indicates that F was obtained by the use of a certain $x \in X$

Proof a. Additive Let $f, g \in X^*$ then

$$F_x(f+g) = (f+g)(x) = f(x) + g(x) = F_x(f) + F_x(g) \quad (2.23)$$

b. Homogeinity

$$F_x(\alpha f) = (\alpha f)(x) = \alpha f(x) = \alpha F_x(f) \quad (2.24)$$

From (2.31) and (2.32) we conclude that F_x is an element of X^{**} i.e. $F_x(f)$ is a linear functional of X^* . Furthermore by definition,

$$\|F_x\|_{X^{**}} = \sup_{\|f\|\neq 0} \frac{|\langle F_x, f\rangle|_{X^{**},X}}{\|f\|_{X^*}} = \frac{|\langle f, x\rangle|_{X^*,X}}{\|f\|_{X^*}} \leq \frac{\|f\|_{X^*}\|x\|_X}{\|f\|_{X^*}}$$

$$= \|x\|_X$$

Thus

$$\|F_x\|_{X^{**}} \leq \|x\|_X$$

since $x \to F_x$(i.e. $\Pi(x) = F_x$) and $\big\|\Pi(x)\big\| = \|F_x\| \leq \|x\|_X$ Moreover

$$\big\|\Pi(x)\big\| = \sup_{\|f\|\neq 0} \frac{\big\|\Pi(x)\big\|}{\|x\|_X} = \sup_{\|f\|\neq 0} \frac{\|F_x\|}{\|x\|_X} \leq \sup_{\|f\|\neq 0} \frac{\|x\|_X}{\|x\|_X} = 1$$

It follows that Π is a bounded linear transformation from X into X^{**} with $\|\Pi\| \leq 1$. Π is linear since its domain is a vector space. Moreover from $(\Pi(x))(f) = F_x(f) = f(x)$ we have

$$\begin{aligned}
\Pi\big(\alpha(x)+\beta(y)\big)(f) &= F_{\alpha x+\beta y}(f) \\
&= f(\alpha x+\beta y) \\
&= \alpha f(x)+\beta f(y) \\
&= \alpha F_x(f)+\beta F_y(f) \\
&= \alpha(\Pi x)(f)+\beta(\Pi y)(f)
\end{aligned}$$

Π is also called the canonical embedding of X into X^{**}.

Proposition 1

Let $x \in X$ be fixed and introduce the map

$$F_x : f \in X^* \to \langle F_x, f\rangle_{X^{**},X^*} \in \mathbb{R}$$

then $F_x \in X^{**}$ and the map $\Pi : x \in X \to F_x \in X^{**}$ is an isometry, i.e.

$$\|x\|_X = \|F_x\|_X **$$

Functional on Hilbert Space

A functional $f \in H^* = X^*$ can be represented uniquely by a function $g_f \in H = X$ (i.e. $L_2(\Omega)$).

Thus the number of elements in X^* is equal to the number of elements in $L_2(\Omega)$. In other words X^* and $L_2(\Omega)$ are equivalent but not the same. What is important here is that each element in X^* has a unique representation in $L_2(\Omega)$.

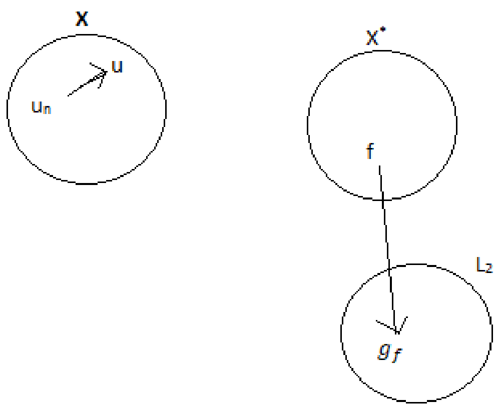

Figure 4: Riesz Representation

Riesz Representation Theorem

Let H be a Hilbert space and $f \in H^*$. Then there exists a unique $g_f \in H$ such that the following inner product relation holds

$$\langle f, x \rangle_{H^*,H} = (g_f, x) = \int_\Omega g_f dx \qquad \forall x \in H$$

Moreover the one to one mapping

$$g : f \in H^* \to g_f \in H$$

is an isometry (called the Riesz isometry) i.e. it satisfies

$$\|g_f\|_H = \|f\|_{H^*}$$

Reflexive Spaces

Definition 14

Let $\Pi : X \to X^{**}$. A normed linear space X is said to be reflexive if Π is onto. i.e. $R(\Pi) = X^{**}$.

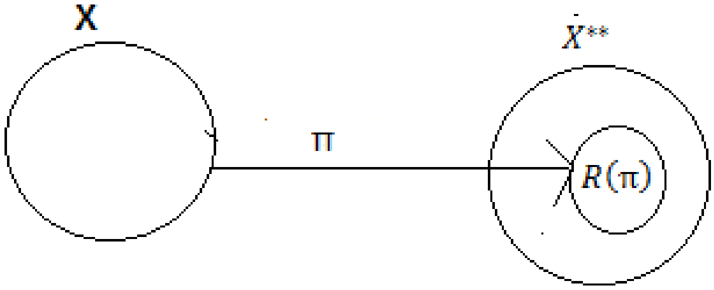

Figure 5: Non-reflexive space

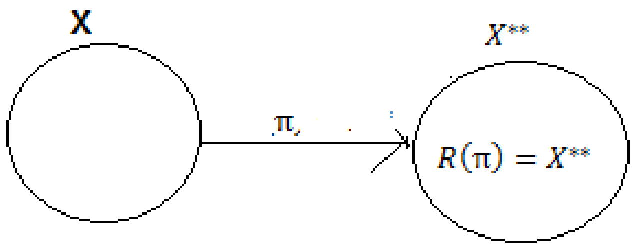

Figure 6: Reflexive space

In other words X is reflexive if X and X^{**} are indistinguishable as a normed linear spaces.

Remark 8

1. If Π acts on X to obtain $\Pi(X)$ in X^{**}(i.e., $\Pi(X) \subset X^{**}$) as in Figure 5 then X in canonically embedded in X^{**} but if the range of X is X^{**}(i.e.$R(\Pi) = X^{**}$) as in Figure 6, then $X = X^{**}$ and X is said to be reflexive. Accordingly by Eberlein Smuljan theorem, there exists a subsequence $x_{n_k} \in X$ which converges weakly to x i.e. $x_{n_k} \to x$

2. The space X is called reflexive if Π is bijective.

3. $F_x \in X^{**}$ means F_x is a bounded linear functional on X^* i.e. $F_x : X^* \to \mathbb{R}$. Thus for every $f \in X^*, F_x(f) \in \mathbb{R}$

4. If X is reflexive, it is isomorphic (hence isometric) with X^{**}.

5. If a normed space X is reflexive, it is complete (hence a Banach space) but completeness does not imply reflexivity. Thus all reflexive normed spaces are Banach spaces, since X must be isometric to the complete space X^{**}.

6. Complete spaces are more useful than incomplete spaces because a typical strategy for solving equations is to construct a sequence of approximations to a solution and then prove that it is a Cauchy sequence. In a complete space, we can deduce that the sequence converges to a member of the space under consideration.

7. Every finite-dimensional normed space is reflexive, simply because in this case, the space, its dual and bidual all have the same linear dimension, hence the linear injection Π from the definition is bijective, by the rank-nullity theorem.

8. The Banach space C_0 of scalar sequences tending to 0 at infinity, equipped with the supremum norm, is not reflexive. The Banach space $C([0,1])$ of continuous functions on $[0,1]$ is not reflexive. The l_1 and l_∞ spaces are not reflexive.

9. All Hilbert spaces are reflexive, as are the L_p space for $1 < p < \infty$. More generally, all uniformly convex Banach spaces are reflexive according to the Milman-Pettis theorem.

10. Every Hilbert space is a Banach space but the converse need not hold. Therefore it is possible for a Banach space not to have a norm given by an inner product. For example, the sup-norm cannot be given by an inner product.

Eberlein-Smuljan theorem for reflexive spaces

Assume that X is reflexive and let (x_n) be a bounded sequence in X. Then

1. There exist a sequence (x_{n_k}) of (x_n) and $x \in X$ such that as $k \to \infty$, $(x_{n_k}) \to x$ weakly in X

2. If each weakly convergent subsequence of (x_{n_k}) has the same limit as x, then the whole sequence (x_n) weakly converges to x i.e.$(x_n) \to x$ weakly in X.

The proposition below enables us to pass to the limit in the products of weak-strong convergent sequences.

Proposition 2

Let $(x_n) \subset X$ and $(y_n) \subset X^*$ such that

$$\begin{cases} x_n \to x \text{ weakly in X} \\ y_n \to y \text{ strongly in } X^* \end{cases}$$

Then

$$\lim_{n\to\infty} \langle y_n, x_n \rangle_{X^*,X} = \langle y, x \rangle_{X^*,X}$$

This proposition tells us that the product of two sequences, one converging strongly in $L^p(\Omega)$ and the other converging weakly in $L^p(\Omega)$ (the dual of $L^p(\Omega)$ converges weakly to the product of their limits). Note that the dual of $L_2(\Omega) = L_2(\Omega)$.

Rapidly Oscillating Periodic Functions

Definition 15

A function f is periodic with period p greater than zero if

$$f(x+p) = f(x)$$

for all values of x in the domain of f. Moreover, if a function f is periodic with period p, then for all x in the domain of f and all integers n,

$$f(x+np) = f(x)$$

If $f(x)$ is a function with period p, then $f(ax)$, where a is a positive constant, is periodic with period $\frac{p}{a}$. For example $f(x) = \sin x$ has a period 2π, therefore $\sin 5x$ will have a period $\frac{2\pi}{5}$. Here $a = 5$.

An **aperiodic function** (non-periodic function) is one that has no such period p (not to be confused with an antiperiodic function, below, for which $f(x+p) = -f(x)$ for some p. A periodic oscillating function plays an essential role in homogenization theory. Consider a periodic function of the form

$$a_\varepsilon(x) = a\left(\frac{x}{\varepsilon}\right)$$

For example a periodic function $a(x) = \sin x$ whose period changes rapidly according to a sequence which tends to zero is represented as

$$a_\varepsilon(x) = a\left(\frac{x}{\varepsilon}\right) = \sin\left(\frac{x}{\varepsilon}\right)$$

Where ε which can be defined by a sequence $\frac{1}{2^n}, n = 0, 1, \dots$ depicts the rapidly oscillating nature of the function. If a is $Y-$periodic, then $a_\varepsilon(x)$ $\varepsilon Y-$ periodic. Moreover the smaller ε is, the more rapid are the oscillations. Therefore, a natural question is to describe the behavior of the sequence a_ε as $\varepsilon \to 0$. In general rapidly oscillating periodic functions converge weakly to their mean value $M_Y(v)$.

Theorem 4 (Periodicity)

Let $1 \leq p \leq \infty$ and f be a $Y-$periodic function in $L^p(Y)$ set

$$f_\varepsilon(x) = f\left(\frac{x}{\varepsilon}\right) \text{ a.e. on } \mathbb{R}^N$$

then if $p < +\infty$ as $\varepsilon \to 0$, $f_\varepsilon \to M_Y(f) = \frac{1}{|Y|}\int_Y f(y)dy$ weakly in $L^p(\omega)$, for any bounded open subset $\omega \in \mathbb{R}^N$.

Theorem 5

If $p = +\infty$, one has

$$f_\varepsilon \to M_Y(f) = \frac{1}{|Y|}\int_Y f(y)dy \text{ weakly in } L^\infty(\mathbb{R}^N)$$

Remark 9

The mean value of a periodic function is essential when studying periodic oscillating functions.

Definition 16

Let Ω be a bounded open set of $\mathbb{R}^N$ and f a function in $L^1(\Omega)$. The mean value of f over Ω is real number $M_\Omega(f)$ given by

$$M_\Omega(f) = \frac{1}{|\Omega|}\int_\Omega f(y)dy$$

Proposition 3 (Poincare's Inequality)

There exists a constant C_Ω such that

$$\|u\|_{L^2(\Omega)} \le C_\Omega \|Du\|_{L(\Omega;\mathbb{R}^n)} \quad \forall u \in H_0^{1,2}$$

where the constant C_Ω is a constant depending on the diameter of Ω.

Lemma 2

Let Ω be a bounded open set and let $1 \le p \le +\infty$. Then there exists a constant $C > 0$ such that

Proof

The sobolev norm is defined by

$$\|u\|_{W_0^{1,p}} = \left(\|u\|^p_{L^p(\Omega)} + \|Du\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}}$$

Applying the Poincare's inequality above to $\|u\|^p_{W^p(\Omega)}$ for $p = 2$ we get

$$\begin{aligned}
\|u\|^p_{W_0^{1,2}} &\leq \left(C_\Omega^2 \|Du\|^p_{L^2(\Omega)} + \|Du\|^p_{L^2(\Omega)}\right)^{\frac{1}{2}} \\
&= \left((C_\Omega^2 + 1)\|Du\|^p_{L^p(\Omega)}\right)^{\frac{1}{2}} = \underbrace{(C_\Omega^2 + 1)^{\frac{1}{2}}}_{\text{C}} \|Du\|_{L^2(\Omega)} \\
&= C\|Du\|_{L^2(\Omega;\mathbb{R}^n)} \qquad (2.25)
\end{aligned}$$

for the one dimensional case. This actually proves that the norm $\|u\|_{W_0^{1,2}}$ and $\|Du\|_{L^2(\Omega;\mathbb{R}^n)}$ are equivalent in $W_0^{1,2}$ in that

$$\|u\|_{W_0^{1,p}} = \left(\|u\|^p_{L^p(\Omega)} + \|Du\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}} \geq \left(\|Du\|^p_{L^p(\Omega;\mathbb{R}^n)}\right)^{\frac{1}{p}}$$

i.e.,

$$\|Du\|_{L^2(\Omega;\mathbb{R}^n)} \leq \|u\|_{W_0^{1,p}} \qquad (2.26)$$

Thus from (2.25) and (2.26) we see that the Poincare's inequality implies that

$$\|u\|_{W_0^{1,2}(\Omega)} = \|Du\|_{L^2(\Omega;\mathbb{R}^n)} \qquad (2.27)$$

This equivalence holds for the subspace of functions with mean zero value. It also holds on the quotient space $H^1(\Omega)/\mathbb{R}$.

Definition 17

Suppose that Ω is connected. The quotient space

$$H^1(\Omega)/\mathbb{R}$$

is defined as the space of classes of equivalence with respect to the relation $u \simeq v \iff u - v$ is a constant, $\forall u, v \in H^1(\Omega)$.

Definition 18 (Weak Convergence)

A sequence x_n in a normed space X is said to be weakly convergent if there is an $x \in X$ such that for every $f \in X^*$,

$$\lim_{n \in \infty} f(x_n) = f(x) \text{ or } \langle f, x_n \rangle_{X^*,X} \to \langle f, x \rangle_{X^*,X} \text{ as } n \to \infty \tag{2.28}$$

This is written

$$x_n \rightharpoonup x \text{ or } x_n \underrightarrow{w} x \text{ or } x_n \to x \text{ weakly in } X$$

The element x is called the weak limit of (x_n), and we say that (x_n) converges weakly to x.

This convergence takes place in X while weak convergence occurs in the dual space X^*.

Remark 10

1. Using the Reisz representation theorem we define weak convergence in $L_2(\Omega)$ as

$$\lim_{n\to\infty} \int_\Omega g_f x_n dx = \int_\Omega g_f dx \iff x_n \to x \text{ weakly in } L_2(\Omega)$$

This is because If $x, x_n \in X$ and $f \in X^*$ then there exists $g_f \in L_2(\Omega)$ such that the following definitions holds

$$\begin{aligned} \langle f, x \rangle_{X^*,X} &= \int_\Omega g_f x dx \quad \forall x \in X \\ \langle f, x_n \rangle_{X^*,X} &= \int_\Omega g_f x_n dx \quad \forall x \in X \end{aligned}$$

i.e.

$$\int_\Omega g_f x_n dx = \int_\Omega g_f dx \iff \lim_{n\to\infty} \int_\Omega g_f x_n dx = \int_\Omega g_f x dx$$

or

$$\langle f, x_n \rangle \to \langle f, x \rangle \text{ as } n \to \infty$$

2. In Sobolev space we write $u^\varepsilon \to u^0$ in $H_0^1(\Omega)$ as $\varepsilon \to 0$ if

$$\lim_{\varepsilon \to 0}(f, u^\epsilon) = (f, u^0) \quad \forall f \in H^{-1}(\Omega)$$

The uniqueness of the weak limit is a consequence of the Hahn-Banach theorem (see Yosida (1964))

3. Weak convergence in $H_0^{1,2}(\Omega)$ implies strong convergence in $L_2(\Omega)$.

4. Weak convergence does not play a role in calculus because in a finite dimensional normed spaces the distinction between strong and weak convergence disappears completely.

Lemma 3 (Weak Convergence)

In a normed space X we have $x_n \to x$ if and only:

i. The sequence $(\|x_n\|)$ is bounded, i.e. there exists a constant C independent of n such that

$$\|x_n\| \leq C \quad \forall n \in \mathbb{N}$$

The above result is a particular case of the Banach-Stenhaus theorem (see Yosida(1964) for a proof).

ii. For every element f of a subset $M \subset X^*$ we have $f(x_n) \to f(x)$ (see proof on pg 261 of Kreysig).

iii. The weak limit x of (x_n) is unique.

iv. Every subsequence of (x_n) converges weakly to x.

Theorem 6 (Strong and Weak Convergence)

Let (x_n) be a sequence in a normed space X. Then:

(a) Strong convergence implies weak convergence with the same limit.

(b) The converse of (a) is not generally true.

(c) If $\dim X < +\infty$,then weak convergence implies strong convergence. A weakly convergent sequence is necessarily bounded in the norm.

Definition 19 (Weak Convergence)

Let (x_n) be a sequence in $L^p(\Omega)$ with $1 < p < \infty$. The weak convergence

$$x_n \to x \text{ weakly in } L^p(\Omega)$$

signifies that

$$\int_{\Omega} x_n \phi dx \to \int_{\Omega} x \phi dx \quad \forall \phi \in L^q(\Omega)$$

with

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Proposition 4

Let $1 < p < \infty$ and (x_n) be a sequence in $L^p(\Omega)$.Then the following equivalence holds (a) $x_n \to x$ weakly in $L^p(\Omega) \iff$ (b)

$$\begin{cases} (i) \|x_n\|_{L^p(\Omega)} \leq C \text{ (independently of x)} \\ (ii) \int_I x_n dx \to \int_I x dx \text{ for any interval } I \in \Omega \end{cases}$$

Weak* Convergence

Definition 20

Let X be a Banach space. A sequence (x_n) in X^* is said to converge weakly* to x iff

$$\langle x_n, x' \rangle_{X^*,X} \to \langle x, x' \rangle_{X^*,X} \quad \forall x' \in X$$

i.e. if $x' \in X$ then the action of x_n on x' converges to the action of x on x'. This weak* convergence is denoted

$$x_n \to x \text{ weakly* in } X^*$$

Note: Any weakly convergent sequence in X is also weakly* convergent.

Lemma 4

Let $\omega^\varepsilon, \omega^0 \in L^1(\Omega)$. We write $\omega^\varepsilon \to \omega^0$, if the sequence ω^ε is bounded in $L^1(\Omega)$ and the relation

$$\lim_{\varepsilon \to 0} \int_\Omega \omega^\varepsilon \phi dx = \int_\Omega \omega^0 \phi dx$$

holds for any $\phi \in C_0^\infty$. A weak limit is uniquely determined. We note that weak convergence in $L^1(\Omega)$ implies weak* convergence, but the converse statement in general does not hold. One often has to find the limit of the scalar product $p^\varepsilon \cdot v^\varepsilon$ with $p^\varepsilon \cdot v^\varepsilon \to 0$ in $L^2(\Omega)$. In trivial case, when either p or v strongly converges in $L^2(\Omega)$ we have $p^\varepsilon \cdot v^\varepsilon$ in $L^1(\Omega)$ and, hence it follows that $p^\varepsilon \cdot v^\varepsilon \to 0$. If both these sequences are only known to be weakly convergent in $L^2(\Omega)$, one cannot so easily pass to the limit in the scalar product, unless the factors possess some additional properties which compensate for the lack of strong convergence. The possibility of passing to the limit in the latter case is provided by the compensated compactness lemma.

The Operator Equation $Au = f$

Problems in engineering can be cast in terms of boundary conditions. Not all problems of the type $Au = f$, with the appropriate $Au = f$ have solutions. We consider three types of problems associated with $Au = f$.

1. A is a linear algebraic operator. An example of A is provided by the set of

linear algebraic equations

$$\begin{aligned}
a_{11}a_1 + a_{12}a_2 + \cdots + a_{1n}a_n &= f_1 \\
a_{21}a_1 + a_{22}a_2 + \cdots + a_{2n}a_n &= f_2 \\
\vdots \;+\; \vdots \;+\cdots\; + \;\vdots &= \vdots \\
a_{n1}a_1 + a_{n2}a_2 + \cdots + a_{nn}a_n &= f_n
\end{aligned}$$

where $[A] = [a_{i,j}]$ is called the coefficient matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \vdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}$$

Such equations arise, for example, in the solution of the operator equation $Au = f$ by variational methods.

2. A is a differential operator. Example $Au = \triangle u = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2} = \nabla \cdot \nabla u = \operatorname{div}(\nabla u)$

3. The variational problem of the operator equation:

$$B(u, v) = l(v)$$

where $B(u, v)$ and $l(v)$ are bilinear and linear forms associated with the equation $Au = f$

Remark 11

The existence and uniqueness of solutions to a differential operator equation depend on the data as well as the positivity of the operator.

Lipshitz Boundary

These are bounded regions with smooth or piecewise smooth boundaries without cupsidal points, e.g. a circle, an annulus, etc.

Test Function

A test function is a smooth function with compact support $\phi \in C_0^\infty[a, b]$. They are needed for theoretical purposes only.

Smooth Function

A function $\phi : \mathbb{R}^N \to \mathbb{R}$ is said to be smooth or infinitely differentiable if its derivatives of all orders exist and are continuous. The set of all smooth functions with compact support in $[a, b]$ is denoted by $C_0^\infty[a, b]$. These are functions that are identically equal to zero in a certain neighborhood of the endpoints $x = a$ and $x = b$ of the given interval. The subscript "0" on $[a, b]$ indicates that functions $\phi \in C_0^\infty[a, b]$ vanish near the boundary of Ω. The space $C_0^\infty[a, b]$ is dense in $L_2[a, b]$.

Support of a Function

The support of a function $f(x)$ where $x \in [a, b]$ is defined to be the closure of the set of points in $[a, b]$ at which f is non zero. Thus for a function $\mathbb{R}$(or $\mathbb{C}$) the support of ϕ is the set

$$K = \{\overline{x \in \mathbb{R} | \phi(x) \notin 0}\}$$

which is always closed by definition.

Integration by parts formula

By definition

$$\int_a^b \frac{dy}{dx} dx = [y(x)]_a^b = y(b) - y(a) \tag{2.29}$$

Put $y(x) = u(x)v(x)$ and differentiate using the product rule to obtain

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \tag{2.30}$$

substituting (2.30) in (2.29) gives

$$\int_a^b \left(u\frac{dv}{dx} + v\frac{du}{dx} \right) dx = [u(x)v(x)]_a^b$$

or

$$\begin{aligned} \int_a^b u\frac{dv}{dx}dx &= [u(x)v(x)]_a^b - \int_a^b v\frac{du}{dx}dx \\ \int_a^b uv'dx &= [u(x)v(x)]_a^b - \int_a^b vu'dx \end{aligned} \tag{2.31}$$

Another form is

$$\int_a^b vdu = [u(x)v(x)]_a^b - \int_a^b udv$$

In the one dimensional case, where we deal with two points on the real line, it is sufficient to use integration by parts.

Variational (or weak) formulation and weak solution for a B.V.P.

We consider the variational formulations of operator equations of the form

$$Au = f \text{ in } \Omega \tag{2.32}$$

where A is a linear or nonlinear operator from an inner product space U into another inner product space V.

Consider the partial differential equation

$$-\nabla^2 u = \left(\frac{d^2u}{dx^2} + \frac{d^2u}{dx^2} \right) = f \quad \text{in } \Omega \tag{2.33}$$

$$u = 0 \quad \text{on } \partial\Omega \tag{2.34}$$

Where $\Omega \in \mathbb{R}^2$ is a plane and $\partial\Omega$ is the boundary of Ω. Equations (2.33)-(2.33) are known as the Dirichlet problem for the Poisson equation. Assume that $f \in C(\bar{\Omega}), (\bar{\Omega}) = \Omega + \partial\Omega$ The classical solution of problem (2.33)and (2.34) means the function $u(x, y)$ satisfies (2.33) in the open domain which is continuous in the closed domain $\bar{\Omega}$ and is equal to zero on the boundary $\partial\Omega$. By assumption $f \in C(\bar{\Omega})$ (i.e., the space of continuous functions with solution u belonging to $C^2(\bar{\Omega})$ continuous partial derivatives up to the second-order inclusive), and equals zero on $\partial\Omega$. The set D_A of these admissible functions, $D_A = \{u(x) \in C^2(\bar{\Omega}), \quad x \in \Omega \subset \mathbb{R}^2, u = 0 \quad \text{on } \partial\Omega\}$ forms a linear space, because if u_1 and u_2 are arbitrary functions in D_A, then the linear combination $\alpha u_1 + \beta u_2$ for any scalars α and β also belongs to D_A. Note that if the boundary condition in (2.34) is nonhomogeneous (for example, $u = g$ on $\partial\Omega$ then the set D_A is not a linear space). The given problem can be stated as follows: Find $u \in D_A$ such that (2.33) is satisfied. The operator $A = -\nabla^2$ assigns to every function $u \in D_A$ a function $v = -\nabla^2 u$ continuous in Ω. The set of all functions $v = -\nabla^2 u$, is called the range of $A = -\nabla^2$, and is denoted by $\mathbb{R}(A)$.

Definition 21 (Bilinear form)

Let H be a Hilbert space. A mapping

$$a(.,.) : H \times H \to \mathbb{R}$$

is called a bilinear form on H if it is linear in both arguments.

Definition 22

Let H be a Hilbert space. A bilinear form a on H is called continuous (or bounded) if there exists a positive constant K such that

$$|a(u, v)| \leq K\|u\|\|v\| \quad \forall u, v \in H$$

and coercive if there exists a positive constant α such that

$$|a(u,u)| \geq \alpha \|u\|^2 \quad \forall u \in H$$

Lemma 5 [Lax-Milgram]

Let a be a bounded, coercive bilinear form on a Hilbert space H. Then for every bounded linear functional f in H^* there exists a unique element $u \in H$ such that

$$a(u,v) = \langle f, v \rangle \quad \forall v \in H$$

Symmetric, Positive, and Positive-Definite Operator

An operator A, linear in its domain D_A is called symmetric in $D_A \subset H$ if for every pair of elements u, v from D_A we have

$$(Au, v) = (u, Av)$$

A symmetric operator is said to be positive in its domain D_A if for all u in D_A the following relation holds:

$$(Au, u) \geq 0 \text{ and } (Au, u) = 0 \implies u = 0 \text{ in } D_A$$

Further we can find a constant $\alpha > 0$ such that for all u in D_A the relation

$$(Au, u) \geq \alpha \|u\|^2$$

holds, then the operator A is called positive-definite in D_A .

For example let $H \in L^2(\Omega)$ where $\mathbb{R}^2$ a plane domain is. Let D_A be the set of functions from $C^2(\Omega)$ which vanish on the boundary $\partial\Omega$. Since $C^2(\Omega)$ is dense in $L^2(\Omega)$, D_A is dense in $L_2(\Omega)$. Let A be the differential operator,

$$A = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = -\nabla^2$$

1. First we show that A is symmetric on D_A Consider

$$
\begin{aligned}
(Au, v) &= \int_\Omega -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) v dxdy \\
&= \int_\Omega (-\nabla^2 u) v dxdy \\
&= -\int_\Omega [\nabla \cdot (\nabla u) v] dxdy \\
&= -\left(\int_\Omega -(\nabla u \cdot \nabla v) dxdy + \int_\Omega (n \cdot \nabla u) v ds\right) \quad \text{Green's rule}
\end{aligned}
$$

Since $v = 0$ on the $\partial\Omega$, the boundary term vanishes. Due to the symmetry of the right side in u and v, we immediately have $(Au, v) = (Av, u) = (u, Av)$.

2. Next we prove that A is positive. For u in D_A, we have

$$
(Au, u) = \int_\Omega [\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2] dxdy \geq 0
$$

if $(Au, u) = 0$, then it follows that $\left(\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0\right)$ $u =$ constant. Since $u = 0$ on $\partial\Omega$ it follows that this constant is zero, and $u = 0$. Thus $(Au, u) = 0$ implies $u = 0$. This proves that A is positive.

3. To prove that A is positive-definite on D_A, we invoke the Friedrichs inequality: For u in $C^1(\Omega)$ the following inequality holds:

$$
\int_\Omega u^2(x) dx \leq c_1 \sum_{k=1}^{2} \int_\Omega \left(\frac{\partial u}{\partial x_k}\right)^2 dx + c_2 \int_\Omega u^2(s) ds \tag{2.35}
$$

where c_1 and c_2 are nonnegative constants dependent on Ω but independent of u. For u in D_A we have $u = 0$ on $\partial\Omega$ hence the second term on the right side of (2.35) is zero. Consequently,

$$
(Au, u) \geq \frac{1}{c_1} \int_\Omega u^2(x) dx = \frac{1}{c_1} \|u\|
$$

Thus A is positive definite on D_A .

Lemma 6

Let $\xi \in \mathbb{R}^n$. Then

$$|a^{\varepsilon}(x)\xi| = \left|a\left(\frac{x}{\varepsilon}\right)\right|\xi = |a(y)\xi| \leq K|\xi|$$

for some positive constant K.

Proof Let $A = \max[a_{ij}]$, Then

$$\begin{aligned}
|a^{\epsilon}(x)\xi|^2 &= (a_{11}\xi_1 + \cdots + a_{1n}\xi_n)^2 + \cdots + (a_{n1}\xi_n + \cdots + a_{nn}\xi_n)^2 \\
&\leq n(a_{11}^2\xi_1^2 + \cdots + a_{1n}^2\xi_n^2) + \cdots + n(a_{n1}^2\xi_1^2 + \cdots + a_{nn}^2\xi_n^2) \\
&\leq n\left((A^2\xi_1^2 + \cdots + A^2\xi_n^2) + \cdots + (A^2\xi_1^2 + \cdots + A^2\xi_n^2)\right) \\
&= n^2A^2(\xi_1^2 + \cdots + \xi_n^2) = n^2A^2|\xi|^2 = K^2|\xi|^2
\end{aligned}$$

where $K^2 = n^2A^2$. Thus

$$|a^{\varepsilon}(x)\xi| \leq K|\xi|$$

This proves boundedness (or continuity) of a^{ϵ}

Existence and uniqueness of the Dirichlet problem

As we know a given problem may or may not have a real solution (e.g. the problem: Find $x \in \mathbb{R}$ such that $x^2 + 1 = 0$). However the existence of a solution of the weak formulation follows by Lax-Milgram Lemma. Moreover the solution of the weak formulation is unique. This fact is proved as follows. The weak formulation of the Dirichlet problem

$$\begin{aligned}
-\triangle u &= f \text{ in } \Omega \\
u &= 0 \text{ on } \partial\Omega
\end{aligned} \tag{2.36}$$

is given by the equation below

$$-\int_{\Omega} \phi \triangle u dx = -\int_{\Omega} \phi(\nabla \cdot \nabla u)dx \underbrace{\overset{=}{Green's formula}} \int_{\Omega} \phi(\nabla \cdot \nabla u)dx$$

$$= \int_\Omega f\phi dx \; \forall \phi \in W_0^{1,2}(\Omega) \tag{2.37}$$

We prove uniqueness of the solution u by assuming that both u_1 and u_2 are solutions of (**??**) and show that $u_1 = u_2$. Now if u_1 is a solution of (2.36) then the following holds

$$\int_\Omega \nabla u_1 \cdot \nabla \phi dx = \int_\Omega f\phi dx \; \forall \phi \in W_0^{1,2}(\Omega) \tag{2.38}$$

and if u_2 is a solution of (2.36) then

$$\int_\Omega \nabla u_2 \cdot \nabla \phi dx = \int_\Omega f\phi dx \; \forall \phi \in W_0^{1,2}(\Omega) \tag{2.39}$$

Subtracting (2.38) from (2.39) we get

$$\int_\Omega \nabla(u_1 - u_2) \cdot \nabla \phi dx = 0 \; \forall \phi \in W_0^{1,2}(\Omega) \tag{2.40}$$

Since this holds $\forall \phi \in W_0^{1,2}(\Omega)$, it particularly holds for $\phi = u_1 - u_2$. Replacing ϕ with $u_1 - u_2$ in (2.40) we get

$$\int_\Omega |\nabla(u_1 - u_2)|^2 dx = 0 \; \forall u_1, u_2 \in W_0^{1,2}(\Omega)$$

This integral means

$$|\nabla(u_1 - u_2)| = 0 \quad \forall x \in \Omega$$

which implies that

$$(u_1 - u_2) = \text{ constant in } \Omega$$

Because $u_1 = u_2 = 0$ on the boundary $\partial\Omega$ the constant is 0, i.e. $u_1 = u_2$ for all $x \in \Omega$. This shows that the solution u in (2.36) is unique for all $\phi \in W_0^{1,2}(\Omega)$. In other words

$$a(u, \phi) = L(\phi) \quad \forall \phi \in W_0^{1,2}(\Omega)$$

where

$$a(u,\phi) = \int_\Omega \nabla u \cdot \nabla \phi dx$$

and

$$L(\phi) = \int_\Omega f\phi dx$$

It is possible to prove that $a(\cdot,\cdot)$ is a bilinear form on $W_0^{1,2}(\Omega)$ i.e., for all functions u, v, w in $W_0^{1,2}(\Omega)$ and $k \in \mathbb{R}$, it holds that $a(u,v)$ is a real number and

$$\begin{aligned} a(u+v,w) &= a(u,v) + a(u,w) \\ a(ku,w) &= ka(u,w) \\ a(w,u+v) &= a(w,u) + a(w,v) \\ a(u,kw) &= ka(u,w) \end{aligned}$$

By way of example, we seek to prove that the bilinear form $a(u,\phi)$ and $L(\phi)$ defined for the Dirichlet problem

$$\begin{aligned} -\triangle u &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned}$$

satisfies the following conditions:

1. $a(\cdot,\cdot)$ is symmetric, i.e.

$$a(\phi,v) = a(v,\phi)\ \forall \phi, v \in W_0^{1,2}(\Omega)$$

2. $a(\cdot,\cdot)$ is continuous (or bounded) i.e. there is a constant $\alpha > 0$ such that

$$|a(\phi,v)| \leq \alpha \|\phi\|_{W_0^{1,2}(\Omega)} \|v\|_{W_0^{1,2}(\Omega)} \quad \forall \phi, v \in W_0^{1,2}(\Omega)$$

3. $a(\cdot,\cdot)$ is elliptic (or coercive) i.e. there is a constant $\alpha > 0$ such that

$$a(\phi,\phi) \geq \alpha \|\phi\|^2_{W_0^{1,2}(\Omega)} \quad \forall \phi \in W_0^{1,2}(\Omega)$$

4. L is continuous, i.e. there is a constant $\alpha > 0$ such that

$$|L(\phi)| \leq \beta \|\phi\|_{W_0^{1,2}(\Omega)}$$

Solution

1.

$$a(u,\phi) = \int_\Omega \nabla u . \nabla u dx = a(\phi, u) \ \phi, v \in W_0^{1,2}(\Omega)$$

This proves 1.

2. By Cauchy Schwarz inequality

$$\begin{aligned} |(\phi, v)| &= \left| \int_\Omega \nabla \phi . \nabla u dx \right| \leq \left(\int_\Omega |\nabla \phi|^2 dx \right)^{\frac{1}{2}} \left(\int_\Omega |\nabla \phi|^2 dx \right)^{\frac{1}{2}} \\ &\leq \left(\int_\Omega |\nabla \phi|^2 + |\phi|^2 d \right)^{\frac{1}{2}} \left(\int_\Omega |\nabla u|^2 + |u|^2 \right)^{\frac{1}{2}} \\ &= \|\phi\|_{W_0^{1,2}(\Omega)} \|u\|_{W_0^{1,2}(\Omega)} \ \ \forall \phi, v \in W_0^{1,2}(\Omega) \end{aligned}$$

This proves 2

3.

$$a(\phi,\phi) = \int_\Omega |\nabla \phi|^2 dx$$

By Poincare inequality we have that

$$\int_\Omega \phi^2 dx \leq C_0 \int_\Omega |\nabla \phi|^2 dx$$

Thus

$$\int_\Omega \phi^2 dx + \int_\Omega |\nabla \phi|^2 dx \leq C_0 \int_\Omega |\nabla \phi|^2 dx + \int_\Omega |\nabla \phi|^2 dx$$

$$= (C_0 + 1) \int_\Omega |\nabla \phi|^2 dx$$

i.e,

$$\frac{1}{(C_0+1)} \left(\int_\Omega (\phi^2 + |\nabla \phi|^2) dx \right) \leq \int_\Omega |\nabla \phi|^2 dx$$

Thus

$$= \frac{1}{(C_0+1)}\|\phi\|_{W_0^{1,2}(\Omega)}$$

Put

$$\alpha = \frac{1}{(C_0+1)}$$

to obtain

$$a(\phi,\phi) = \alpha\|\phi\|_{W_0^{1,2}(\Omega)}$$

Which proves 3.

4.

$$\begin{aligned} L(\phi) &= \int_\Omega f\phi dx \\ |L(\phi)| &= \left|\int_\Omega f\phi dx\right| \leq \left(\int_\Omega |f|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega |\phi|^2 dx\right)^{\frac{1}{2}} \\ &\leq \left(\int_\Omega |f|^2 dx\right)^{\frac{1}{2}}\left(\int_\Omega |\phi|^2 + |\nabla\phi|^2 dx\right)^{\frac{1}{2}} = \beta\|\phi\|_{W_0^{1,2}(\Omega)} \end{aligned}$$

where

$$\beta = \left(\int_\Omega |f|^2 dx\right)^{\frac{1}{2}}$$

Thus

$$|L(\phi)| = |\beta|\|\phi\|_{W_0^{1,2}(\Omega)}$$

which proves 4.

Chapter Summary

In the first part of this chapter, some methods of homogenization were mentioned and studied. The second part gives a demonstration of significant theorems, propositions, definitions and lemmas used in the theory of homogenization.

CHAPTER THREE

METHODOLOGY

Introduction

In this chapter, we look at some mathematical techniques that can be employed in homogenizing partial differential equations.

Homogenization Techniques

Two Scale (Multi-scale) Convergence

This was specially introduced for studying homogenization problems. It makes the formal two scale asymptotic analysis mathematically rigorous. The two scale limit captures the oscillations involved in a weakly convergence sequence. Here, it is very important to note that rapid oscillations and concentrations are the main cause which prevents the weakly convergent sequence to be strongly convergent. In this direction, we state the following theorem by Nguetseng (1989).

Lemma 1 (Two-scale Convergence)

Let $\{u_\varepsilon\}$ be a uniformly bounded sequence in $L^2(\Omega)$. Then there is a subsequence of ε denoted again by ε, and
u_0=$u_0(x,y) \in L^2(\Omega, L^2_p(Y))$ such that
$\int_\Omega u_\varepsilon(x)\Psi\left(x,\frac{x}{\varepsilon}\right) \to \int_{\Omega\times Y} u_0(x,y)\Psi(x,y)dxdy$
as $\varepsilon \to 0$, for all $\Psi \in C_c(\bar{\Omega}), C_p(Y))$. Moreover,

$$\int_\Omega u_\varepsilon(x)\nu(x)\omega\left(\frac{x}{\varepsilon}\right) \to \int_{\Omega\times Y} u_0(x,y)\nu(x)\omega(y)dydx \tag{3.1}$$

as $\varepsilon \to 0, \quad \forall \nu \in C_c(\bar{\Omega}) \quad \text{and} \quad \forall \omega \in L^2_p(Y)$.

Further, if u is the L^2 weak limit of u_ε then by taking $\Omega \equiv 1$ in the above equation we get

$$u(x) = \int_Y u_0 \, (x, y) dy. \tag{3.2}$$

Here L_p^2 (Y) denotes the space of L^2-periodic function and C_p (Y) denotes the space of continuous periodic functions on Y.

Method of Asymptotic Expansion (Multiple-scale Expansion Method)

In any asymptotic problem, the first step is to look for a suitable asymptotic expansion and try to guess the correct limit from the formal analysis. The normal expansion like in any other asymptotic problem is as follows:

$u_\varepsilon(x)$=$u_0(x, y)$+$\varepsilon u_1(x, y)$+$\varepsilon^2 u_2(x, y) + \cdots$

Indeed this expansion leads to the anticipated but incorrect answer

$$\begin{cases} -\frac{d}{dx}\left(a^* \frac{du}{dx}\right) = f \quad \text{in} \quad (0, 1) \\ u(0) = u(1) = 0 \end{cases}$$

Keeping the particular problem in mind, one looks for:

$u_\varepsilon(x) = u_0(x, y) + \varepsilon u_1(x, y) + \varepsilon^2 u_2(x, y) + \cdots$

where x is the slow variable and $y = \frac{x}{\varepsilon}$ is the fast or rapidly oscillating variable. Then if possible, we see that u_0 is independent of y and obtain the equation satisfied by u_0.

The Γ-convergence Method

Introduced by De Giorgi in the early 1970's, the Γ-convergence is an abstract notion of functional convergence aiming at describing the asymptotic behavior of families of minimum problems usually depending on some parameters whose nature may be geometric or constitutive, deriving from a discretization argument, an approximation procedure, etc. The Γ-convergence has several applications in many research areas including for example the calculus of variation and the homogenization of partial differential equations. It should be noted that the epiconvergence introduced by Attouch in 1984 is a functional convergence notion close to the Γ-convergence. We now give the definition of the Γ-convergence, its fundamental theorem and hint how it is utilized to handle the homogenization of partial differential equations.

Definition 1

Let W be a metric space endowed with a distance d. Let $(F_\varepsilon)_{(\varepsilon \in E)}$ be a sequence of real functions defined on W. The sequence $(F_\varepsilon)_{(\varepsilon \in E)}$ is said to Γ-converge to a limit function F_0 if for any $x \in W$, the following hold:

1. (lim inf inequality) any sequence $(x_\varepsilon)_{(\varepsilon \in E)}$ converging to x in W as $\varepsilon \to 0$ satisfies $F_0(x) \leq \liminf_{\varepsilon \to 0} F_\varepsilon(x_\varepsilon)$,

2. (existence of a recovering sequence) there exists a sequence $(x_\varepsilon)_{(\varepsilon \in E)}$ converging to x as $\varepsilon \to 0$ and such that $F_0(x) \geq \liminf_{\varepsilon \to 0} F_\varepsilon(x_\varepsilon)$, Indeed, the Γ-convergence and the Γ-limit depend on the choice of the distance. A Γ-limit F_0 is a lower semi continuous function on W, that is, $F_0(x) \leq \liminf_{\varepsilon \to 0} F_\varepsilon(x_\varepsilon)$, for any sequence $x_\varepsilon \to x \quad as \quad \varepsilon \to 0$. The Γ-limit of the constant sequence F is the lower semicontinuous envelope (relaxation) of F. To formulate the main (as regards the homogenization theory) theorems of the Γ-convergence, we recall that a sequence $(F_\varepsilon)_{(\varepsilon \in E)}$ of real functions defined on W is said to be equi-mildly coercive on W if there exists a compact set K (independent of ε) such that

$$\inf_{x \in W} F_\varepsilon(x) = \lim_{x \in K} F_\varepsilon(x)$$

Theorem 1

Let $(F_\varepsilon)_{(\varepsilon \in E)}$ be an equi-mildly coercive sequence on W which Γ-converges to a limit F_0. Then,

1. The minima of F_ε converges to that of F_0, that is,
$\min_{x \in W} F_0(x) = \lim_{\varepsilon \to 0} \left(inf_{x \in W} F_\varepsilon(x) \right)$

2. The minimizers of F_ε converge to those of F_0, that is,

if $x_\varepsilon \to x$ in W and $\lim_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon) = \lim_{\varepsilon\to 0}(\inf_{\varepsilon\in W} F_\epsilon(x))$, then, x is a minimizer of F_0.

Theorem 2

Assume that the metric space W endowed with the distance d is separable. For any sequence of functions $(F_\varepsilon)_{(\varepsilon\in E)}$ defined on W there exists a subsequence E' of E and a Γ-limit F_0 such that $(F_\varepsilon)_{(\varepsilon\in E')}$ Γ-converges to F_0 as $E' \ni \varepsilon \to 0$. Loosely speaking, to utilize the Γ-convergence in homogenization, one usually transforms the partial differential equation into a minimization problem. To illustrate this, we consider the model problem

$$\begin{cases} -\nabla(A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon) = f \quad \text{in} \quad \Omega \\ u_\varepsilon = 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{3.3}$$

where Ω is a bounded domain in R^N, and the matrix A is coercive, bounded and symmetric. For $\varepsilon \in E$, we set

$$F_\varepsilon(x) = \frac{1}{2}\int_\Omega A\left(\frac{x}{\varepsilon}\right)\nabla u.\nabla dx - \int_\Omega f u dx \quad u \in H_0^1(\Omega) \tag{3.4}$$

It is well known that, when the matrix A is symmetric, the problem (2.7) is equivalent to the following minimization problem:

$$\begin{cases} \text{Find} \quad u_\varepsilon \in H_0^1(\Omega) \quad \text{such that} \\ F_\varepsilon(u_\varepsilon) \leq F_\varepsilon(v) \\ \text{for all} \quad v \in H_0^1 \end{cases} \tag{3.5}$$

Hence, the Γ-convergence of the sequences of functionals $(F_\varepsilon)_{(\varepsilon\in E)}$ (defined by (2.7)) in L^2 (Ω)-strong is equivalent to the homogenization of the partial differential equation (2.6). The Γ-convergence method applied to homogenization theory is neither restricted to linear equation nor to periodic structure. Albeit the

Γ-convergence method is one of the most utilized homogenization technique, it is sometimes blamed for being of limited interest to real-life problems of continuum physics. For example, Tartar (1993) believes that the energy minimization approaches to homogenization problems (or to continuum mechanics in general) is fake mechanics, since from the first principle of thermodynamics, nature conserves energy rather than minimizing it. However, Benamou and Brenier (2000) were able to formulate evolution problems of continuum physics as minimization problem by means of the optimal transportation theory.

The H-convergence and the G-Convergence Methods

The H-convergence of Tartar and Murat (1997) is a generalization to non-symmetric problems of the G-convergence of Spagnolo (1960). The letters G and H stand for 'Green' and 'Homogenization', respectively. These convergence methods are equivalent to the convergence of the associated Green kernel. We briefly present the H-convergence method for specific simple example of operators. Let Ω be a bounded open set in R^N, and let $0 < \alpha \leq \beta$ be two positive constants. We define $M(\alpha, \beta, \Omega)$ to be the set of all $N \times N$-matrices defined on Ω with uniform coercivity constant α and $L^\infty(\Omega)$-bound β. $M(\alpha, \beta, \Omega)$=$A \in L^\infty(\Omega; R^{N\times N}) : \alpha|\xi|^2 \leq A(x)\xi.\xi \leq \beta|\xi|^2 \quad \forall \xi \in R^N$and a.e. $x \in \Omega$. We consider a sequence $(A_\varepsilon)_{(\varepsilon \in E)} \subset M(\alpha, \beta, \Omega)$ without any periodicity assumption or any symmetric hypothesis either. Given $f \in L^2(\Omega)$, there exists, by the Lax-Milgram lemma, a unique solution $u_\epsilon \in H_0^1(\Omega)$ to

$$\begin{cases} -\nabla(A_\varepsilon \nabla u_\varepsilon) = f \quad \text{in} \quad \Omega \\ u_\varepsilon = 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{3.6}$$

Definition 2

The sequence $(A_\varepsilon)_{(\varepsilon\in E)}$ is said to H-convergence to a limit A^* as $E \ni \varepsilon \to 0$, if, for any right-hand side $f \in L^2(\Omega)$ in (2.9), we have

$$u_\varepsilon \to u_0 \quad \text{weakly in} \quad H_0^1(\Omega)$$

$$A_\varepsilon \nabla u_\varepsilon \to A^* \nabla u_0 \quad \text{weakly in} \quad L^2(\Omega)^N$$

as $E \ni \varepsilon \to 0$, where u_0 is the solution to the homogenized equation associated with A:

$$\begin{cases} -\nabla(A^*\nabla u_0) = f \quad \text{in} \quad \Omega \\ u_0 = 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{3.7}$$

Among useful properties of the H-convergence, we have the following:

1. if $(A_\varepsilon)_{(\varepsilon\in E)} \subset M(\alpha,\beta,\Omega)$ H-converges, its H-limit is unique,

2. let $(A_\varepsilon)_{(\varepsilon\in E)}$ and $(B_\varepsilon)_{(\varepsilon\in E)}$ be two sequences in $M(\alpha,\beta,\Omega)$ which H-converge to A^* and B^*, respectively. If $A_\varepsilon = B_\varepsilon$ in $\omega \subsetneqq \Omega$, then $A^* = B^*$ in ω,

3. the H-limit does neither depend on the source term nor the boundary condition on $\partial\Omega$,

4. if $(A_\varepsilon)_{(\varepsilon\in E)} \subset M(\alpha,\beta,\Omega)$ H-converges to A^*, then the associated density of energy also converges, that is , $A_\varepsilon\nabla u_\varepsilon \cdot u_\varepsilon$ converges to $A^*\nabla u \cdot \nabla u$ in the sense of distributions in Ω. Indeed, the H-convergence is a local property. Note that the definition of the H-convergence differs from that of the G-convergence by requiring the convergence of the flux $(A_\varepsilon u_\varepsilon)_{(\varepsilon\in E)}$ in addition to that of the sequence $(u_\varepsilon)_{(\varepsilon\in E)}$. This additional requirement is essential when removing the symmetry hypothesis (say, when passing from G-convergence to H-convergence) since it ensures the uniqueness of the H-limit. Without the following compactness result, the concept of H-convergence would be useless.

Theorem 3

For any sequence $(A_\varepsilon)_{(\varepsilon\in E)} \subset M(\alpha,\beta,\Omega)$, there exists a subsequence E' of E and a homogenized limit A^*, belonging to $M(\alpha,\frac{\beta^2}{\varepsilon},\Omega)$, such that A_ε H-converges to A^* as $E' \ni \varepsilon \to 0$.

The G-convergence version of Theorem 3 was proved for the first time by Spagnolo (1976) by means of the convergence of the Green functions. It can also be proved using the Γ-convergence. Tartar (1997), proposed a simpler proof in the general framework of H-convergence. We present the idea (in the periodic setting) of tartar's proof which has been later called the energy method, and sometimes, the oscillating test function method. We assume that the matrix A_ε in (2.9) is given by $A_\varepsilon(x) = A(\frac{x}{\varepsilon})$, where A is 1-periodic in all coordinate directions. The variational formulation of (2.9) reads

$$\int A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon.\nabla\varphi dx = \int f(x)\varphi(x)dx \quad \forall\varphi\in H_0^1 \tag{3.8}$$

The coercivity of the matrix A implies the boundedness of the sequence $(u_\varepsilon)_(\varepsilon \in E)$ in $H_0^1(\Omega)$. Hence, it weakly converges (up to a subsequence) to some $u_0 \in H_0^1(\Omega)$. We recall that, with the hypothesis on A, we have

$$A_\varepsilon \to \int_\Omega A(y)dy \text{ weakly in } L^2(\Omega)$$

so that the left-hand side of (11) involves a product of two weakly converging sequences in $L^2(\Omega)$:$A\left(\frac{\cdot}{\varepsilon}\right)_(\varepsilon\in E)$ and $(\nabla u_\varepsilon)_\varepsilon$. Therefore we cannot pass to the limit by means of classical arguments. Tartar's idea is to use the oscillating test function defined by

$$\varphi_\varepsilon(x) = \varphi(x) + \varepsilon\sum_{i=1}^N \frac{\partial\varphi}{\partial x_i}(x)\omega_i^*\left(\frac{x}{\varepsilon}\right) \quad (\varepsilon\in E)$$

where $\varphi \in D(\Omega)$, and where $\varphi_i = (i = 1,\ldots,N)$ solve the so-called dual cell problem

$$\begin{cases} -\nabla_y \left(A^t(y)(e_i \nabla_y \varphi_i(y))\right) = 0 \quad \text{in} \quad (0,1)^N \\ y \to \varphi_i(y) \quad (0,1)^N - \text{periodic} \end{cases}$$

with $e_i = (\delta_{ij})_{j=1}^N$. Utilizing this test function, Tartar was able to eliminate the 'bad terms' in (2.11), and to pass to the limit and get a macroscopic equation. However, constructing the oscillating test functions is not always as easy as in this simple periodic problem. We conclude this subsection by mentioning that Donato et al.(1994) have extended the H-convergence method (under the label H_0-convergence) to the case of perforated domains with a Neumann condition on the holes.

The method of asymptotic expansion also known as the multiple scale expansion method is a technique of homogenization which is commonly used in mathematics and physics. This method was initially introduced by mechanical scientists and engineers. Subsequently this approach was employed in the study of problems with periodic structures. The fundamental idea governing the method of asymptotic expansion is to assume that the solution to the classical homogenization problem is of the form

$$u^\varepsilon = u_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \cdots$$

where the terms in the above expansion depends on both x and $y = \frac{x}{\varepsilon}$ with x representing the macroscopic or global scale while $y = \frac{x}{\varepsilon}$ represents the microscopic or local scale. Since the coefficients of the problem of which u^ε is the presumed solution is Y-periodic i.e. each u_i is Y-periodic in the second variable $y = \frac{x}{\varepsilon}$. By using this method, we obtain both the homogenized problem and the homogenized solution. However, the results are heuristically obtained and the calculations involved in this method are very lengthy and cumbersome which makes it prone to errors.

In the past few years, more mathematically rigorous methods and procedures have been discovered to obtain the limit problem. In all these methods, the two-scale convergence by Nguetseng(1989) is so far one of the most powerful techniques and has been designed to go beyond periodic homogenization. The main aim of this chapter is to throw light on the method of asymptotic expansion also known as the multiple scale expansion method by using it to find the effective equation of an elliptic partial differential equation with rapidly oscillating coefficients.

Derivation of the Homogenized problem

Let us consider the one dimensional Dirichlet Boundary Value Problem of the form

$$\begin{cases} A^{\varepsilon}u_{\varepsilon} = \frac{\partial}{\partial x}\left(a^{\varepsilon}(x)\frac{\partial u_{\varepsilon}}{\partial x}\right) = f \quad \text{in} \quad \Omega \\ u_{\varepsilon} = 0 \quad\quad\quad\quad \text{on} \quad \partial\Omega \end{cases} \tag{3.9}$$

where Ω is a domain with periodic heterogeneities of length scale $\varepsilon(\varepsilon > 0), f \in L^2(\Omega), u_0 \in L^2(\Omega)$.

We study the asymptotic behavior of the solution u_{ε} by assuming that u_{ε} has two scale expansion of the form

$$u^{\varepsilon}(x) = u_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \cdots = \sum_{i=0}^{\infty} \varepsilon^i u_i\left(x, \frac{x}{\varepsilon}\right) \tag{3.10}$$

with $u_i(x, y), y = \frac{x}{\varepsilon}$ for $i = 0, 1, 2, 3 \ldots$ such that

$$\begin{cases} u_i(x, y) \text{ is defined for } x \in \Omega \text{ and } y \in Y \\ u_i(., y) \text{ is Y-periodic} \end{cases}$$

Next we let $\psi = \psi(x, y)$ be a function depending on two variables of $\mathbb{R}^N$ and denote ψ^ε by the following:

$$\psi^\varepsilon(x) = \psi(x, \frac{x}{\varepsilon}) = \psi(x, y), \text{ where } y = \frac{x}{\varepsilon}$$

Now by applying the chain rule, we have

$$\frac{\partial \psi^\varepsilon(x)}{\partial x_i} = \frac{\partial \psi}{\partial x_i} + \frac{\partial \psi}{\partial y_i} \cdot \frac{\partial y}{\partial x} = \frac{\partial \psi}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial \psi}{\partial y_i}$$

$$\frac{\partial \psi^\varepsilon(x)}{\partial x_i} = \left(\frac{\partial}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_i}\right)\psi(x, y) \tag{3.11}$$

For the N-dimensional case. In gradient notation, we write (3.3) above as

$$\nabla_x \psi^\varepsilon(x) = \left(\nabla_x + \frac{1}{\varepsilon}\nabla_y\right)\psi(x, y) \tag{3.12}$$

thus in the one dimensional case we have

$$\frac{d\psi^\varepsilon(x)}{dx} = \left(\frac{\partial}{\partial x} + \frac{1}{\varepsilon}\frac{\partial}{\partial y}\right)\psi(x, y) \tag{3.13}$$

Define $a^\varepsilon(x) = a\left(\frac{x}{\varepsilon}\right) = a(y)$. Then from (3.1)

$$\begin{aligned}
A^\epsilon = -\frac{d}{dx}\left\{a(y)\left(\frac{\partial}{\partial x} + \frac{1}{\varepsilon}\frac{\partial}{\partial y}\right)\right\} &= -\left(\frac{\partial}{\partial x} + \frac{1}{\varepsilon}\frac{\partial}{\partial y}\right)\left\{a(y)\left(\frac{\partial}{\partial x} + \frac{1}{\varepsilon}\frac{\partial}{\partial y}\right)\right\} \\
&= -\frac{\partial}{\partial x}\left(a(y)\frac{\partial}{\partial x}\right) - \frac{\partial}{\partial x}\left(a(y)\frac{1}{\epsilon}\frac{\partial}{\partial y}\right) - \frac{1}{\varepsilon}\frac{\partial}{\partial y}\left(a(y)\frac{\partial}{\partial x}\right) \\
&\quad - \frac{1}{\varepsilon^2}\frac{\partial}{\partial y}\left(a(y)\frac{\partial}{\partial y}\right) \\
&= -\frac{\partial}{\partial x}\left(a(y)\frac{\partial}{\partial x}\right) + \frac{1}{\varepsilon}\left\{-\frac{\partial}{\partial x}\left(a(y)\frac{\partial}{\partial y}\right) - \frac{\partial}{\partial y}\left(a(y)\frac{\partial}{\partial x}\right)\right\} \\
&\quad + \frac{1}{\varepsilon^2}\left\{-\frac{\partial}{\partial y}\left(a(y)\frac{\partial}{\partial y}\right)\right\}
\end{aligned}$$

$$A^\varepsilon = A_2 + \varepsilon^{-1}A_1 + \varepsilon^{-2}A_0 \tag{3.14}$$

Where,

$$A_0 = -\frac{\partial}{\partial y}\left(a(y)\frac{\partial}{\partial y}\right) \tag{3.15}$$

$$A_1 = -\frac{\partial}{\partial x}\left(a(y)\frac{\partial}{\partial y}\right) - \frac{\partial}{\partial y}\left(a(y)\frac{\partial}{\partial x}\right) \tag{3.16}$$

$$A_2 = -\frac{\partial}{\partial y}\left(a(y)\frac{\partial}{\partial y}\right) \tag{3.17}$$

From the above, we have;

$A^{\varepsilon}u_{\varepsilon} = (A_2 + \varepsilon^{-1}A_1 + \varepsilon^{-2}A_0)(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots)$

$$= (A_2u_0 + \varepsilon A_2u_1 + \varepsilon^2 A_2u_2 + \cdots) + (\varepsilon^{-1}A_1u_0 + A_1u_1 + \varepsilon A_1u_2 + \cdots)$$

$$+(\varepsilon^{-2}A_0u_0 + \varepsilon^{-1}A_0u_1 + A_0u_2 + \dots)$$

$$= \varepsilon^{-2}A_0u_0 + \varepsilon^{-1}(A_1u_0 + A_0u_1) + \varepsilon^0(A_2u_0 + A_1u_1 + A_0u_0)$$

$$+\varepsilon(A_2u_1 + A_1u_2) + \cdots = f(x)$$

Equating the three lowest powers of ε(i.e. $\varepsilon^0, \varepsilon^{-1}$ and ε^{-2}), we obtain the following systems of equations:

$$A_0u_0 = 0 \tag{3.18}$$

$$A_1u_0 + A_0u_1 = 0 \tag{3.19}$$

$$A_2u_0 + A_1u_1 + A_0u_2 = f(x) \tag{3.20}$$

To solve the above equations, we need the Lemma below i.e.:

Lemma 7

Consider the boundary value problem

$$A^{\varepsilon}\Phi = F \text{ in a unit Y - cell}$$

where $\Phi(y)$ is Y- periodic and $F \in L^2(Y)$. Then the following holds:

i. There exists a weak Y-periodic Φ if and only if $\frac{1}{|y|}\int_Y F dy = 0$

ii. If there exists a weak Y-periodic solution Φ, then it is unique up to an additive constant, that is if we find one solution $\Phi_0(y)$, every solution is of the form $\Phi(y) = \Phi_0(y) + c$, where c is a constant independent of y.

We now solve the three systems of equations.

Now from (3.10) i.e.

$$A_0 u_0(x, y) = 0.$$

Since the operator $A_0 = -\frac{\partial}{\partial y}\left(a(y)\frac{\partial}{\partial y}\right)$ contains only derivatives with respect to y. This means that if A_0 operates on any function which is a function of x only, the results will be zero. Thus we conclude that for $u_0(x, y)$ to be a solution of (3.18), it must be a function of x only, i.e.

$$u_0(x, y) = u(x).$$

From (3.19)

$$A_1 u_0 + A_0 u_1 = 0$$

so that

$$\begin{aligned}
A_0 u_1 &= -A_1 u_0 \\
A_0 u_1 &= -\left[-\frac{\partial}{\partial x}\left(a(y)\frac{\partial u_0}{\partial y}\right) - \frac{\partial}{\partial y}\left(a(y)\frac{\partial u_0}{\partial x}\right)\right] \qquad (3.21)\\
&= -\left[\frac{\partial}{\partial x}\left(a(y)\frac{\partial u(x)}{\partial y}\right) - \frac{\partial}{\partial x}\left(a(y)\frac{\partial u(x)}{\partial y}\right)\right] \\
&= -\left[-\frac{\partial}{\partial x}\left(a(y)\frac{\partial u(x)}{\partial y}\right)\right] \\
&= \left(\frac{\partial a(y)}{\partial y}\right)\left(\frac{\partial u(x)}{\partial x}\right) \qquad (3.22)
\end{aligned}$$

Note that the first term in (3.13) is zero because $\frac{\partial u(x)}{\partial y} = 0$. Moreover integrating (3.14) over Y we have

$$\int_Y \frac{\partial}{\partial y}\left(a(y)\frac{\partial u_1(x, y)}{\partial x}\right) dy = \frac{\partial u(x)}{\partial x}\int_Y \frac{\partial a(y)}{\partial y} dy = 0 \qquad (3.23)$$

because $a(y)$ is y-periodic. Since the integral on the right hand side of (3.15) is equal to zero we conclude that by Lemma 7, u_1 admits a solution up to an additive constant which is independent of y. Plugging $A_0 = -\frac{\partial}{\partial y}\left(a(y)\frac{\partial}{\partial y}\right)$ into (3.14) and separating variables on the right hand side, we can solve for u_1 i.e.

$$-\frac{\partial}{\partial y}\left(a(y)\frac{\partial u_1(x,y)}{\partial x}\right) = \left(\frac{\partial a(y)}{\partial y}\right)\left(\frac{\partial u(x)}{\partial x}\right) \tag{3.24}$$

Since the right hand side of (3.24) can be separated into a function of x and y only, and $\left(\frac{\partial a(y)}{\partial y}\right)$ is a periodic function, it can be represented as say $\omega(y)$. By linearity, $u_1(x,y)$ must have a solution of the form,

$$u_1(x,y) = \omega(y)\frac{\partial u(x)}{\partial x}. \tag{3.25}$$

According to Lemma 7(ii) any solution $u_1(x,y,t)$ must be unique up to an additive constant, i.e. a constant independent of y and so

$$u_1(x,y) = \omega(y)\frac{\partial u(x)}{\partial x} + \widetilde{u_1}(x).$$

By substituting (3.17) into (3.16) we have

$$-\frac{\partial}{\partial y}\left[a(y)\frac{\partial}{\partial y}\left(\omega(y)\frac{\partial u(x)}{\partial x} + \widetilde{u_1}(x)\right)\right] = \left(\frac{\partial a(y)}{\partial y}\right)\left(\frac{\partial u(x)}{\partial x}\right)$$

and since $\frac{\partial \widetilde{u_1}(x)}{\partial y} = 0$ re-arrangement yields

$$-\left(\frac{\partial u(x)}{\partial x}\right)\left[\frac{\partial}{\partial y}\left(a(y)\frac{\partial \omega(y)}{\partial y}\right)\right] = \left(\frac{\partial a(y)}{\partial y}\right)\left(\frac{\partial u(x)}{\partial x}\right).$$

Further simplification yields the following cell problem

$$-\frac{\partial}{\partial y}\left(a(y)\frac{\partial \omega(y)}{\partial y}\right) = \frac{\partial a(y)}{\partial y} \tag{3.26}$$

.

From (3.18) we have that

$$\frac{\partial}{\partial y}\left[a(y)\left(1+\frac{\partial\omega(y)}{\partial y}\right)\right]=0. \tag{3.27}$$

For $\phi \in C^{\infty}_{per}(Y)$ we see that the weak formulation of (3.19) is given by

$$\int_Y \left[a(y)\left(1+\frac{\partial\omega(y)}{\partial y}\right)\frac{\partial\phi}{\partial y}\right]dy=0. \tag{3.28}$$

By expanding and sending one term to the right hand side, we have

$$\int_Y a(y)\frac{\partial\omega(y)}{\partial y}\frac{\partial\phi}{\partial y}dy=-\int_Y a(y)\frac{\partial\phi}{\partial y}dy. \tag{3.29}$$

Equation (3.21) can be written as

$$\int_Y (a(y)D\omega(y)D\phi)dy=-\int_Y (a(y)D(\phi))dy \quad , \text{ where } D=\frac{\partial}{\partial y}. \tag{3.30}$$

Finally we solve for u_2(i.e. from (3.12) by re-arranging it as follows

$$A_0u_2=f-(A_1u_1+A_0u_2)$$

. In order for this equation to have a unique solution, it is necessary and sufficient for the right hand side of the equation to average to zero. Since we have assumed that the $f(x)$ in independent of y, then from the Lemma 1, there exists a unique solution u_2 provided

$$\int_Y (f-(A_1u_1+A_0u_2)dy)=0 \tag{3.31}$$

Now simplifying the term involving A_1 , we have

$$\begin{aligned}\int_Y A_1u_1dy &= -\int_Y \frac{\partial}{\partial x}\left(a(y)\frac{\partial u_1(x,y)}{\partial y}\right)dy-\int_Y \frac{\partial}{\partial y}\left(a(y)\frac{\partial u_1(x,y)}{\partial x}\right)dy\\ &= I_1+I_2 \end{aligned} \tag{3.32}$$

Now

$$I_2 = -\int_Y \frac{\partial}{\partial y}\left(a(y)\frac{\partial u_1(x,y)}{\partial x}\right) dy = 0 \tag{3.33}$$

Since $a(y)$ and $\frac{\partial u_1}{\partial x}$ are both y-periodic, it implies that the product $a(y)\frac{\partial u_1}{\partial x}$ is also periodic and so by Y-periodicity (3.25) holds. Using (3.17) and taking into consideration the fact that $\frac{\partial \tilde{u}(x)}{\partial y} = 0$ we find that

$$\begin{aligned} I_1 &= \int_Y -\frac{\partial}{\partial x}\left(a(y)\frac{\partial u_1(x,y)}{\partial y}\right) dy \\ &= \int_Y -\frac{\partial}{\partial x}\left[a(y)\frac{\partial}{\partial y}\left(\omega(y)\frac{\partial u(x)}{\partial x} + \tilde{u_1}(x)\right)\right] dy \\ &= \int_Y -a(y)\frac{\partial}{\partial x}\left[\left(\frac{\partial \omega(y)}{\partial y}\right)\left(\frac{\partial u(x)}{\partial x}\right)\right] dy \\ &= \left[\int_Y -a(y)\left(\frac{\partial \omega(y)}{\partial y}\right) dy\right]\frac{\partial^2 u(x)}{\partial x^2} \end{aligned} \tag{3.34}$$

Substituting (3.24) and (3.25) into (3.26) we obtain

$$\int_Y A_1 u_1 dy = \left[\int_Y -a(y)\left(\frac{\partial \omega(y)}{\partial y}\right) dy\right]\frac{\partial^2 u(x)}{\partial x^2}. \tag{3.35}$$

Finally we consider the term $A_2 u_0$ in (3.23) we obtain

$$\int_Y A_2 u_0 dy = \int_Y -\frac{\partial}{\partial x}\left(a(y)\frac{\partial u(x)}{\partial x}\right) dy = -\left(\int_Y a(y)dy\right)\frac{\partial^2 u(x)}{\partial x^2} \tag{3.36}$$

Substituting (3.28) and (3.27) in (3.23), we have

$$\int_Y \left(f dy - \left[\int_Y -a(y)\left(\frac{\partial \omega(y)}{\partial y}\right) dy\right]\frac{\partial^2 u(x,t)}{\partial x^2} - \left(\int_Y a(y)dy\right)\frac{\partial^2 u(x)}{\partial x^2}\right) = 0$$

$$\int_Y \left[\left(a(y) + a(y)\frac{\partial \omega(y)}{\partial y}\right) dy\right]\frac{\partial^2 u(x)}{\partial x^2} \quad = \quad -\int_Y f dy.$$

.

Rearranging we have,

$$\begin{aligned}
\int_Y \left[a(y)\left(1+\frac{\partial\omega(y)}{\partial y}\right)dy\right]\frac{\partial^2 u(x)}{\partial x^2} &= -\int_Y f dy \\
&= -f\int_Y dy = -f|Y| \\
-\frac{1}{|Y|}\left[\int_Y \left[a(y)\left(1+\frac{\partial\omega(y)}{\partial y}\right)\right]dy\right]\frac{\partial^2 u(x)}{\partial x^2} &= f
\end{aligned}$$

$$-B\frac{\partial^2(x,t)}{\partial x^2} = f \tag{3.37}$$

where

$$B = \frac{1}{|Y|}\int_Y \left[a(y)\left(1+\frac{\partial w(y)}{\partial y}\right)\right]dy. \tag{3.38}$$

The equation (3.29) is therefore the homogenized equation and the coefficient B (3.30) called the Homogenized coefficient.

Chapter Summary

The method of asymptotic expansion is not just a powerful tool but also a formal technique used to homogenize partial differential equations. This method can be used without fore knowledge about specified properties of the solution to the micro structured problem. As a consequence, this method is only used to guess the nature and form of the homogenized problem.

Here in this chapter, the multiple scale expansion technique was used to homogenize an elliptic partial equation with rapidly varying coefficient.

CHAPTER FOUR

RESULTS AND DISCUSSION

Introduction

In Chapter 3 we reviewed the elliptic equation of the form $A^{\varepsilon}u_{\varepsilon} = f$, where ε describes the wavelength of the different stages involved in the process of getting to the homogenized state. We discussed how the multiple scale expansion method also known as the method of asymptotic expansion can be used in estimating the average properties of composite materials which are periodic in nature.

In this chapter, we shall use the method discussed in Chapter 3, that is, the multiple scale expansion method to find time dependent properties of composite materials or to homogenize parabolic partial differential equations of the form (4.2).

Homogenization of parabolic partial differential equation

A partial differential equation with rapidly oscillating coefficient can be described by the equation

$$A^{\varepsilon}u_{\varepsilon} = f \tag{4.1}$$

where the parameter ε, depicts the oscillating or changing nature of the coefficients. For each ε, a solution u_{ε} can be obtained for (4.1). The main idea behind the method of multiple scales is to take the presence of two or more characteristic scales explicitly into account, and to incorporate this information into the structure of the power series expansion.

In this chapter, we will consider parabolic partial differential equation of the form (4.2) where $\varepsilon > 0$ is a small parameter,$f \in L^2(\Omega \times T)$ and $u_0 \in L^2(\Omega)$. Our main aim here is to describe the asymptotic behavior of (4.2) as $\varepsilon \to 0$.

To this end, we consider the one dimensional Dirichlet boundary value problem of the form

$$\begin{cases} \frac{\partial u_\varepsilon(x,t)}{\partial t} - \text{div}(a^\varepsilon(x,t)\nabla u_\varepsilon(x,t) = f(x,t) \quad \text{in} \quad \Omega \times (0,T) \\ u_\varepsilon(x,t) = 0 \quad \text{on} \quad \partial\Omega \times (0,T) \\ u_\varepsilon(x,0) = u_0(x) \quad \text{in} \quad \Omega \end{cases} \tag{4.2}$$

where $a^\varepsilon(x,t)$ satisfies the coercivity assumption , i.e., there exist two positive constants $0 < \alpha \leq \beta$ such that, for any vector $\xi \in R^N$ and at any point $y \in Y$, $\alpha|\xi|^2 \leq \Sigma_{i,j=1}^N a_i j(y)\xi_i\xi_j \leq \beta|\xi|^2$ and remains in the set of $L^\infty(\Omega \times (0,T))$.

A natural way to introduce the periodicities $a^\varepsilon(x,t)$ is to suppose that they have the form:

$$a^\varepsilon(x,t) = a\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right) = a\left(\frac{x}{\varepsilon}, \tau\right) \tag{4.3}$$

Thus we have:

$$\begin{cases} A^\varepsilon u_\varepsilon = \frac{\partial u_\varepsilon(x,t)}{\partial t} - \text{div}(a\left(\frac{x}{\varepsilon}, \tau\right)\nabla u_\varepsilon(x,t) = f(x,t) \quad \text{in} \quad \Omega \times (0,T) \\ u_\varepsilon(x,t) = 0 \quad \text{on} \quad \partial\Omega \times (0,T) \\ u_\varepsilon(x,0) = u_0(x) \quad \text{in} \quad \Omega \end{cases} \tag{4.4}$$

We study the asymptotic behavior of the solution u_ε by assuming that u_ε has two scale expansion of the form

$$u^\varepsilon(x,t) = u_0\left(x, \frac{x}{\varepsilon}, t, \frac{t}{\varepsilon}\right) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}, t, \frac{t}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}, t, \frac{t}{\varepsilon}\right) + \cdots$$

$$= \sum_0^\infty \varepsilon^i u_i\left(x, \frac{x}{\varepsilon}, t, \frac{t}{\varepsilon}\right) \tag{4.5}$$

with $u_i(x,y,t,\tau)$, $y = \frac{x}{\varepsilon}$ for $i = 0, 1, 2, 3 \ldots$

such that

$$\begin{cases} u_i(x,y,t,\tau) & \text{is defined for} \quad x \in \Omega, \quad t,\tau \in (0,T) \quad \text{and} \quad y \in Y, \\ u_i(x,y,t,\tau) & \text{is Y-periodic} \end{cases} \tag{4.6}$$

Next we let $\psi = \psi(x,y,t,\tau)$ be a function depending on two variables of R^N and denote ψ_ε by the following:

$\psi^\varepsilon(x,t) = \psi(x, \frac{x}{\varepsilon}, t, \tau) = \psi(x,y,t,\tau)$, where $y = \frac{x}{\varepsilon}$.

Now by applying the chain rule, we have

$$\frac{d\psi^\varepsilon(x,t)}{dx_i} = \frac{\partial\psi}{\partial x_i} + \frac{\partial\psi}{\partial y_i}.\frac{\partial y}{\partial x} = \frac{\partial\psi}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial\psi}{\partial y_i}.$$

$$\frac{d\psi^\varepsilon(x,t)}{dx_i} = \left(\frac{\partial}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_i}\right)\psi(x,y,t,\tau). \tag{4.7}$$

for the N-dimensional case.

In gradient notation, we write (4.7) above as

$$\nabla_x\psi^\varepsilon(x,t) = \left(\nabla_x + \frac{1}{\varepsilon}\nabla_y\right)\psi(x,y,t,\tau). \tag{4.8}$$

Thus in the one dimensional case we have

$$\frac{d\psi^\varepsilon(x,t)}{dx} = \left(\frac{\partial}{\partial x} + \frac{1}{\varepsilon}\frac{\partial}{\partial y}\right)\psi(x,y,t,\tau). \tag{4.9}$$

Similarly, from the chain rule,

$$\frac{\partial\psi^\varepsilon(x,t)}{\partial t_i} = \frac{\partial\psi}{\partial t_i} + \frac{\partial\psi}{\partial \tau_i}.\frac{\partial y}{\partial t} = \frac{\partial\psi}{\partial t_i} + \frac{1}{\varepsilon}\frac{\partial\psi}{\partial \tau_i}.$$

$$\frac{\partial\psi^\varepsilon(x,t)}{\partial t_i} = \left(\frac{\partial}{\partial t_i} + \frac{1}{\varepsilon}\frac{\partial}{\partial \tau_i}\right)\psi(x,y,t,\tau). \tag{4.10}$$

for the N-dimensional case.

In gradient notation, we write (4.10) above as

$$\nabla_t\psi^\varepsilon(x,t) = \left(\nabla_t + \frac{1}{\varepsilon}\nabla_\tau\right)\psi(x,y,t,\tau). \tag{4.11}$$

Thus in the one dimensional case we have

$$\frac{\partial \psi^{\varepsilon}(x,t)}{\partial t} = \left(\frac{\partial}{\partial t} + \frac{1}{\varepsilon}\frac{\partial}{\partial \tau}\right)\psi(x,y,t,\tau). \tag{4.12}$$

Define $a^{\varepsilon}(x,t) = a(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}) = a(y,\tau)$. Then from (4.4)

$$\begin{cases} A^{\varepsilon}u_{\varepsilon} = \frac{\partial u_{\varepsilon}(x,t)}{\partial t} - \operatorname{div}(a(y,\tau)\nabla u_{\varepsilon}(x,t)) = f(x,t) \quad \text{in} \quad \Omega\times(0,T) \\ u_{\varepsilon}(x,t) = 0 \quad \text{on} \quad \partial\Omega\times(0,T) \\ u_{\varepsilon}(x,0) = u_0(x) \quad \text{in} \quad \Omega \end{cases}$$

In this work, we analyze the one dimensional version of (4.4) i.e.

$$\frac{\partial u_{\varepsilon}}{\partial t} - \frac{d}{dx}\left(a^{\varepsilon}(x,t)\frac{du_{\varepsilon}}{dx}\right) = f \tag{4.13}$$

$$\text{Let} \quad A^{\varepsilon} = \frac{\partial}{\partial t} - \frac{d}{dx}\left(a^{\varepsilon}(x,t)\frac{d}{dx}\right) \tag{4.14}$$

Applying the chain rule (4.9) and (4.12) on (4.14), we obtain

$$A^{\varepsilon} = \frac{\partial}{\partial t} + \frac{1}{\varepsilon}\frac{\partial}{\partial \tau} - \left(\frac{\partial}{\partial x} + \frac{1}{\varepsilon}\frac{\partial}{y}\right)\left(a(y,\tau)\left(\frac{\partial}{\partial x} + \frac{1}{\varepsilon}\frac{\partial}{y}\right)\right) \tag{4.15}$$

$$= \frac{\partial}{\partial t} + \frac{1}{\varepsilon}\frac{\partial}{\partial \tau} - \frac{\partial}{\partial x}\left(a(y,\tau)\frac{\partial}{\partial x}\right) - \frac{\partial}{\partial x}\left(a(y,\tau)\frac{1}{\varepsilon}\frac{\partial}{\partial y}\right)$$
$$-\frac{1}{\varepsilon}\frac{\partial}{\partial y}\left(a(y,\tau)\frac{\partial}{\partial x}\right) - \frac{1}{\varepsilon^2}\frac{\partial}{\partial y}\left(a(y,\tau)\frac{\partial}{\partial y}\right)$$

$$= \varepsilon^0\left\{\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\left(a(y,\tau)\frac{\partial}{\partial x}\right)\right\}$$
$$+\frac{1}{\varepsilon}\left\{\frac{\partial}{\partial \tau} - \frac{\partial}{\partial x}\left(a(y,\tau)\frac{\partial}{\partial y}\right) - \frac{\partial}{\partial y}\left(a(y,\tau)\frac{\partial}{\partial x}\right)\right\}$$
$$+\frac{1}{\varepsilon^2}\frac{\partial}{\partial y}\left(a(y,\tau)\frac{\partial}{\partial y}\right).$$

All these operators depend on τ parametrically, that is

$$A^{\varepsilon} = \varepsilon^0 A_2 + \varepsilon^{-1}A_1 + \varepsilon^{-2}A_0 \tag{4.16}$$

where

$$A_0 = \frac{\partial}{\partial y}\left(a(y,\tau)\frac{\partial}{\partial y}\right) \tag{4.17}$$

$$A_1 = \left\{\frac{\partial}{\partial \tau} - \frac{\partial}{\partial x}\left(a(y,\tau)\frac{\partial}{\partial y}\right) - \frac{\partial}{\partial y}\left(a(y,\tau)\frac{\partial}{\partial x}\right)\right\} \tag{4.18}$$

$$A_2 = \left\{\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\left(a(y,\tau)\frac{\partial}{\partial x}\right)\right\}. \tag{4.19}$$

From (4.5) and (4.16) we have

$$A^{\varepsilon}u_{\varepsilon} = (\varepsilon^0 A_2 + \varepsilon^{-1}A_1 + \varepsilon^{-2}A_0)(\varepsilon^0 u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots)$$

$$= (\varepsilon^0 A_2 u_0 + \varepsilon A_2 u_1 + \varepsilon^2 A_2 u_2 + \cdots) + (\varepsilon^{-1}A_1 u_0 + \varepsilon^0 A_1 u_1 + \varepsilon A_1 u_2 + \cdots)$$

$$+(\varepsilon^{-2}A_0 u_0 + \varepsilon^{-1}A_0 u_1 + \varepsilon^0 A_0 u_2 + \cdots)$$

$$= \varepsilon^{-2}A_0 u_0 + \varepsilon^{-1}(A_1 u_0 + A_0 u_1) + \varepsilon^0(A_2 u_0 + A_1 u_1 + A_0 u_2) + \cdots = f(x,t)$$

Equating the three lowest powers of ε (i.e. ε^0,ε^{-1} and ε^{-2}) we obtain the following systems of equations:

$$A_0 u_0 = 0 \tag{4.20}$$

$$A_1 u_0 + A_0 u_1 = 0 \tag{4.21}$$

$$A_2 u_0 + A_1 u_1 + A_0 u_2 = f(x,t) \tag{4.22}$$

To solve the above equations, we make use of the Lemma 7 i.e.:

Lemma 7

Consider the boundary value problem

$$A^{\varepsilon}\Phi = F \text{ in a unit Y - cell}$$

where $\Phi(y)$ is Y- periodic and $F \in L^2(Y)$. Then the following holds:

i. There exists a weak Y-periodic solution Φ if and only if $\frac{1}{|y|}\int_Y F dy = 0$

ii. If there exists a weak Y-periodic solution Φ, then it is unique up to an additive constant, that is if we find one solution $\Phi_0(y)$, every solution is of the form $\Phi(y) = \Phi_0(y) + c$, where c is a constant independent of y.

We now solve the three systems of equations.

Now from (4.20)

$$A_0 u_0(x, y, t, \tau) = 0,$$

Since the operator $A_0 = -\frac{\partial}{\partial y}(a(y,\tau)\frac{\partial}{\partial y}$ contains only derivatives with respect to y it implies that if A_0 operates on any function which does not depend on y, the results will be zero. Thus we conclude that for $u_0(x, y, t, \tau)$ to be a solution of (4.17), it must be independent of y, i.e.

$$u_0(x, y, t, \tau) = u(x, t, \tau) = u(x, t, \tau)$$

From (4.21)

$$A_1 u_0 + A_0 u_1 = 0$$

so that

$$A_0 u_1 = -A_1 u_0$$

$$A_0 u_1 = -\left[\frac{\partial u_0}{\partial \tau} - \left[\frac{\partial}{\partial x}\left(a(y,\tau)\frac{\partial u_0}{\partial y}\right) + \frac{\partial}{\partial y}\left(a(y,\tau)\frac{\partial u_0}{\partial x}\right)\right]\right] \tag{4.23}$$

From lemma 7, u_1 is solution to (4.23) if and only if

$$\int_Y -A_1 u_0 dy = 0 \tag{4.24}$$

i.e.

$$\int_Y \left[-\frac{\partial u_0}{\partial \tau} + \frac{\partial}{\partial y}\left(a(y,\tau)\frac{\partial u_0}{\partial x}\right)\right] dy = \int_Y -\frac{\partial u_0}{\partial \tau} dy + \frac{\partial u_0}{\partial x}\int_Y \frac{\partial a(y,\tau)}{\partial y} dy \tag{4.25}$$

we see that

$$\int_Y \frac{\partial a(y,\tau)}{\partial y} dy = 0 \quad \text{since} \quad a(y,\tau) \quad \text{is} \quad y-\text{periodic}$$

Also,

$$\int_Y -\frac{\partial u_0}{\partial \tau} dy = 0 \quad \text{if and only if} \quad \frac{\partial u_0(x,t,\tau)}{\partial \tau} = 0$$

$$\text{now} \quad \frac{\partial u_0(x,t,\tau)}{\partial \tau} = 0 \Longrightarrow u_0(x,t,\tau) = u_0(x,t)$$

thus

$$u_0(x,y,t,\tau) = u(x,t)$$

Substituting $A_0 = -\frac{\partial}{\partial y}\left(a(y,\tau)\frac{\partial}{\partial y}\right)$ and $\frac{\partial u_0}{\partial \tau} = 0$ into (4.23) and separating variables on the right hand side, we can solve for u_1, we obtain

$$-\frac{\partial}{\partial y}\left(a(y,\tau)\frac{\partial u_1(x,y,t,\tau)}{\partial y}\right) = \left(\frac{\partial a(y,\tau)}{\partial y}\right)\left(\frac{\partial u(x,t)}{\partial x}\right) \tag{4.26}$$

Since the right hand side of (4.26) can be separated into a function of x and y only, and $\frac{\partial a(y,\tau)}{\partial y}$ is a periodic function, it can be represented as say $\omega(y,\tau)$. By linearity, $u_1(x,y,t,\tau)$ must have a solution of the form:

$$u_1(x,y,t,\tau) = \omega(y,\tau)\frac{\partial u(x,t)}{\partial x} + \tilde{u}_1(x,t,\tau) \tag{4.27}$$

By substituting (4.27) into (4.26) we have

$$-\frac{\partial}{\partial y}\left[a(y,\tau)\frac{\partial}{\partial y}\left(\omega(y,\tau)\frac{\partial u(x,t)}{\partial x} + \tilde{u}_1(x,t,\tau)\right)\right] = \left(\frac{\partial a(y,\tau)}{\partial y}\right)\left(\frac{\partial u(x,t)}{\partial x}\right)$$

and since

$$\frac{\partial \tilde{u}_1(x,t,\tau)}{\partial y} = 0,$$

re-arrangement yields

$$-\left(\frac{\partial u(x,t)}{\partial x}\right)\left[\frac{\partial}{\partial y}\left(a(y,\tau)\frac{\partial \omega(y,\tau)}{\partial y}\right)\right] = \left(\frac{\partial a(y,\tau)}{\partial y}\right)\left(\frac{\partial u(x,t)}{\partial x}\right)$$

Further simplification yields the following cell problem

$$-\frac{\partial}{\partial y}\left(a(y,\tau)\frac{\partial \omega(y,\tau)}{\partial y}\right) = \frac{\partial a(y,\tau)}{\partial y} \tag{4.28}$$

From (4.28) we have that

$$\frac{\partial}{\partial y}\left[a(y,\tau)\left(1 + \frac{\partial \omega(y,\tau)}{\partial y}\right)\right] = 0 \tag{4.29}$$

which is the cell problem of (4.2) with ω as solution.

Finally we solve for u_2 (i.e. from (4.22)) for the homogenized equation by re-arranging it as follows

$$A_0 u_2 = f - (A_1 u_1 + A_2 u_0)$$

In order for this equation to have a unique solution, it is necessary and sufficient for the right hand side of the equation to average to zero. Since we have assumed that the $f(x,t)$ is independent of y, then from Lemma 7, there exists a unique solution u_2 provided

$$\int_Y (f - (A_1 u_1 + A_2 u_0))\, dy = 0$$

$$\int_Y f dy - \int_Y A_1 u_1 dy - \int_Y A_2 u_0 dy = 0$$

$$\int_Y f dy = \int_Y A_1 u_1 dy + \int_Y A_2 u_0 dy \tag{4.30}$$

Using (4.27)

$A_1 u_1 = A_1\left(\omega(y,\tau)\frac{\partial u(x,t)}{\partial x} + \tilde{u}_1(x,t,\tau)\right) = A_1\left(\omega(y,\tau)\frac{\partial u(x,t)}{\partial x}\right) + A_1 \tilde{u}_1(x,t,\tau).$

Integrating over Y we have

$$\int_Y A_1 u_1 dy = \int_Y A_1 \tilde{u}_1(x,t,\tau) dy + \int_Y A_1\left(\omega(y,\tau)\frac{\partial u(x,t)}{\partial x}\right) dy.$$

Now

$$\int_Y A_1\tilde{u}_1(x,t,\tau)dy = \int_Y \frac{\partial \tilde{u}_1(x,t,\tau)}{\partial \tau}dy - \int_Y \frac{\partial}{\partial y}\left(a(y,\tau)\frac{\partial \tilde{u}_1(x,t,\tau)}{\partial x}\right)dy$$
$$- \int_Y \frac{\partial}{\partial x}\left(a(y,\tau)\frac{\partial \tilde{u}_1(x,t,\tau)}{\partial y}\right)dy$$
$$= \int_Y \frac{\partial \tilde{u}_1(x,t,\tau)}{\partial \tau}dy$$

since

$$\int_Y \frac{\partial}{\partial y}\left(a(y,\tau)\frac{\partial \tilde{u}_1(x,t,\tau)}{\partial x}\right)dy = 0, \quad \text{by Y-periodicity}$$

and

$$\int_Y \frac{\partial}{\partial x}\left(a(y,\tau)\frac{\partial \tilde{u}_1(x,t,\tau)}{\partial y}\right)dy = 0, \quad \text{because} \quad \tilde{u}_1(x,t,\tau)$$

is independent of y

thus

$$\int_Y A_1\tilde{u}_1(x,t,\tau)dy = \frac{\partial \tilde{u}_1(x,t,\tau)}{\partial \tau}\int_Y dy = \frac{\partial \tilde{u}_1(x,t,\tau)}{\partial \tau}|Y| = \frac{\partial \tilde{u}_1(x,t,\tau)}{\partial \tau}$$

Therefore

$$\int_Y A_1u_1(x,t,\tau)dy = \frac{\partial \tilde{u}_1(x,y,\tau)}{\partial \tau} + \int_Y A_1\left(\omega(y,\tau)\frac{\partial u(x,t)}{\partial x}\right)dy \quad (4.31)$$

Also A_0u_2 in (4.30) can be simplified as follows

$$\int_Y A_2u_0dy = \int_Y \left(\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(a(y,\tau)\frac{\partial u(x,t)}{\partial x}\right)\right)dy \quad (4.32)$$

Substituting (4.32), (4.31) into (4.30) we obtain

$$\frac{\partial \tilde{u}_1(x,y,\tau)}{\partial \tau} + \int_Y A_1\left(\omega(y,\tau)\frac{\partial u(x,t)}{\partial x}\right)dy$$

$$+ \int_Y \left(\frac{\partial u(x,t)}{\partial t} - \frac{\partial}{\partial x}\left(a(y,\tau)\frac{\partial u(x,t)}{\partial x}\right)\right)dy$$

$$= \int_Y f dy \tag{4.33}$$

Integrating with respect to τ and making use τ-periodicity we see that

$$\int_\tau \frac{\partial \tilde{u}_1(x,t,\tau)}{\partial \tau} d\tau = 0$$

this means we can solve for $\tilde{u}_1(x,t,\tau)$ if and only if

$$\int_\tau \int_Y A_1 \left(\omega(y,\tau) \frac{\partial u(x,t)}{\partial x} \right) dy d\tau$$

$$+ \int_\tau \int_Y \left(\frac{\partial u(x,t)}{\partial t} - \frac{\partial}{\partial x} \left(a(y,\tau) \frac{\partial u(x,t)}{\partial x} \right) \right) dy d\tau$$

$$= \int_\tau \int_Y f(x,t) dy d\tau = f(x,t) \int_\tau \int_Y dy d\tau = f(x,t)|Y||\tau|$$

$$= f(x,t) \tag{4.34}$$

Since $|Y| = 1$ and $|\tau| = 1$

But

$$A_1 = \frac{\partial}{\partial \tau} - \frac{\partial}{\partial x} \left(a(y,\tau) \frac{\partial}{\partial y} \right) - \frac{\partial}{\partial y} \left(a(y,\tau) \frac{\partial}{\partial x} \right)$$

Therefore

$$A_1 \left(\omega(y,\tau) \frac{\partial u}{\partial x} \right) = \frac{\partial u}{\partial \tau} \left(\omega \frac{\partial u}{\partial x} \right) - \frac{\partial}{\partial x} \left(a(y,\tau) \frac{\partial}{\partial y} \left(\omega \frac{\partial u}{\partial x} \right) \right)$$
$$- \frac{\partial}{\partial y} \left(a(y,\tau) \frac{\partial}{\partial x} \left(\omega \frac{\partial u}{\partial x} \right) \right)$$

Integrating both sides over Y and τ we obtain

$$\int_\tau \int_Y A_1 \left(\omega(y,\tau) \frac{\partial u}{\partial x} \right) dy d\tau = \int_\tau \int_Y \frac{\partial u}{\partial \tau} \left(\omega \frac{\partial u}{\partial x} \right) dy d\tau$$

$$- \int_\tau \int_Y \frac{\partial}{\partial x} \left(a(y,\tau) \frac{\partial}{\partial y} \left(\omega \frac{\partial u}{\partial x} \right) \right) dy d\tau - \int_\tau \int_Y \frac{\partial}{\partial y} \left(a(y,\tau) \frac{\partial}{\partial x} \left(\omega \frac{\partial u}{\partial x} \right) \right) dy d\tau$$

therefore

$$\int_\tau \int_Y A_1 \left(\omega(y,\tau) \frac{\partial u}{\partial x} \right) dy d\tau = -\int_\tau \int_Y \frac{\partial}{\partial x} \left(a(y,\tau) \frac{\partial}{\partial y} \left(\omega \frac{\partial u}{\partial x} \right) \right) dy d\tau \tag{4.35}$$

since

$$\int_\tau \int_Y \frac{\partial}{\partial \tau} \left(\omega(y,\tau) \frac{\partial u}{\partial x} \right) dy d\tau = 0 \quad \text{by } \tau\text{-periodicity}$$

and

$$\int_\tau \int_Y \frac{\partial}{\partial y} \left(a(y,\tau) \frac{\partial}{\partial x} \left(\omega \frac{\partial u}{\partial x} \right) \right) dy d\tau = 0 \quad \text{by Y-periodicity}$$

Substituting (4.35) into (4.34) we find that

$$-\int_\tau \int_Y \frac{\partial}{\partial x} \left(a(y,\tau) \frac{\partial}{\partial y} \left(\omega(y,\tau) \frac{\partial u(x,t)}{\partial x} \right) \right) dy d\tau + \int_\tau \int_Y \frac{\partial u(x,t)}{\partial t} dy d\tau$$
$$- \int_\tau \int_Y \frac{\partial}{\partial x} \left(a(y,\tau) \frac{\partial u(x,t)}{\partial x} \right) dy d\tau = f(x,t)$$

but

$$\int_\tau \int_Y \frac{\partial u}{\partial t} dy d\tau = \frac{\partial u}{\partial t} \int_\tau \int_Y dy d\tau = \frac{\partial u}{\partial t}, \quad \text{since y and } \tau \text{ are 1-periodic}$$

Thus

$$-\int_{\tau}\int_{Y}\frac{\partial}{\partial x}\left(a(y,\tau)\frac{\partial}{\partial y}\left(\omega(y,\tau)\frac{\partial u(x,t)}{\partial x}\right)\right)dyd\tau+\frac{\partial u(x,t)}{\partial t}$$

$$-\int_{\tau}\int_{Y}\frac{\partial}{\partial x}\left(a(y,\tau)\frac{\partial u(x,t)}{\partial x}\right)dyd\tau=f(x,t)$$

rearranging

$$\frac{\partial u(x,t)}{\partial t}-\int_{\tau}\int_{Y}\left(a(y,\tau)+a(y,\tau)\frac{\partial \omega(y,\tau)}{\partial y}\right)\frac{\partial^2 u}{\partial x^2}dyd\tau=f(x,t)$$

$$\frac{\partial u(x,t)}{\partial t}-B\frac{\partial^2 u}{\partial x^2}=f(x,t) \tag{4.36}$$

where

$$B=\int_{\tau}\int_{Y}\left(a(y,\tau)+a(y,\tau)\frac{\partial \omega(y,\tau)}{\partial y}\right)dyd\tau \tag{4.37}$$

The equation (4.36) is therefore the homogenized equation and the coefficient B (4.37) is the homogenized coefficient.

Chapter Summary

The results presented here confirms that homogenization is very useful in the analysis of problems involving partial differential equations with rapidly oscillating coefficients. In this thesis, the multiple scale expansion technique also known as the method of asymptotic expansion was used to homogenize a parabolic partial differential equation with rapidly oscillating coefficient.
The original equation with the rapidly oscillating coefficient was then replaced by the homogenized or effective equation with a stable coefficient known as the homogenized coefficient.

CHAPTER FIVE

SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

Overview

In this chapter the major findings and outcomes of this thesis are given in terms of a brief summary and conclusion. We proceed to make recommendations to enhance further research work in future.

Summary

Homogenization theory with regards to parabolic partial differential equations has been proposed and developed in this thesis based on a multiple scale expansion technique also known as the method of asymptotic expansion. The method of asymptotic expansion was used to find the average property or characteristic of a composite material where the effect of time is taken into account. This was done by finding the effective or homogenized equation of a parabolic partial differential equation originally with a rapidly oscillating coefficient.

A parabolic partial differential equation of the form (4.2) was homogenized using the multiple scale expansion technique to obtain (4.36).

Conclusion

Specifically, the multiple scale expansion technique also known as the method of asymptotic expansion is shown to yield accurate estimates on the homogenized solution. A parabolic partial differential equation with rapidly oscillating coefficient was homogenized and replaced by an auxiliary partial differential equation with a stable coefficient called the effective coefficient.

Recommendation

Composite materials that are periodic in nature is the main focus of this work. Future works may examine how to homogenize non-periodic composite materials using the multiple scale expansion technique.

REFERENCES

Allaire, G. (1992). Homogenization and two-scale convergence. *Society for industrial and applied mathematics journal of Mathematical Analysis,* 23 , 1482-1518.

Allaire,G. (2010). *Introduction to homogenization theory.*Palaiseau, France:CMAP.

Almaraz, H.B. (2012). *Numerical modelling based on the multiscale homogenization theory. Application in composite materials and structures.* Unpublished doctoral thesis. Department of Strength of Materials and Structural Engineering, Technical University of Catalonia. Barcelona, Spain.

Almqvist, A., Essel, E.K., Persson, L.E. & Wall P. (2007). Homogenization of the unstationary incompressible Reynolds equation. *Tribology International,* 40(1), 1344-1350.

Almqvist, A. (2006). *On the Effects of surface Roughness in Lubrication.* Unpublished doctoral dissertation, Department of Mathematics, Luleå University of Technology, Luleå.

Almqvist, A., Larsson, R. & Wall, P. (2006). *The homogenization process of the time dependent Reynolds equation describing compressible liquid flow, Research Report No. 4* . Luleå,Sweden :Department of Mathematics, Luleå University of Technology.

Alouges, F. (2017). *Introduction to periodic homogenization*. CMAP, Ecole polytechnique, Universite Paris-Saclay, Palaisaeu, France.

Bayada, G. & Faure, J.B. (1989). A Double scale analysis approach of the Reynolds roughness comments and application to the journal bearing. *Journal of Tribology,* 111 (2), 323-330.

Bensoussan, A., Boccardo, L. & Murat, F. (1986). Homogenization of elliptic equations with principal part not in divergence form and hamiltonian with quadratic growth. *Communications of pure and applied mathematics,* 39(6).

Bensoussan, A., Lions, J.L. & Papanicolaou, G. (1978). *Asymptotic analysis for periodic structures.* Amsterdam, North-Holland.

Bourgeat, A. & Pankratov, L. (1996). Homogenization of semilinear parabolic equations in domains with spherical traps. *An international journal,* 64 (3-4), 303-317, DOI:10.1080/00036819708840538.

Bystrom, J. (2002). *Some mathematical and engineering aspects of the homogenization theory* Lulea University of Technology, Lulea, Sweden.

Cioranescu, D. & Donato, P.(1999). *An Introduction to homogenization.* New York, USA : Oxford University Press Inc.

Douanla, H. (2013). *Two-scale Convergence and Homogenization of Some partial differential equations.* Chalmers University of Technology and University of Gothenburg.Gothenburg, Sweden.

Effendiev, Y. & Pankov, A. (2004). Numerical Homogenization of Nonlinear Random Parabolic Operators. *Siam journal on multiplescale modeling and simulation,* 2(2), 237-268.

Essel, E.K. (2007). *Homogenization of Reynolds equations.* Luleå, Sweden : University Printing Office, Luleå.

Emereuwa,C.A. (2015). *Homogenization of partial differential equation: from multiple scale expansion to Tartar's H-Measures.* University of Pretoria.Pretoria, South Africa.

Lobkova, T. (2017). *Homogenization Results for Parabolic and Hyperbolic-Parabolic Problems and Further Results on Homogenization in Perforated Domains.* Mid-Sweden University, Sweden.

Nguetseng, G. (2003). Homogenization Structures and Applications I.*Journal for Analysis and its Applications,* 22(1), 73-107. Nguestseng, G. (1989).A general convergence result for a functional related to the theory of homogenization. *SIAM J. Math. Anal.*

Sackitey, A. (2019). *Homogenization of elliptic equations in periodic domains. The case of elliptic equations of the curl type.* University of Cape Coast, Cape Coast, Ghana.

Jimenez, S. (2009). *Introduction to homogenization.* Louisiana State University, Baton Rouge, Louisiana, USA.

Tatar, L. & Murat, F. (1977). *H-Convergence.* Birkhauser, Boston, MA. 5(31), (21-43).