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Optimal scientific production over the life cycle

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Abstract

The paper develops an optimization model of the career of a scientist. Recognizing that research efforts and networking get more efficient if the scientist is more knowledgeable, history dependent solutions are developed. We give a theoretical underpinning of the four different research patterns detected in Way et al. (2017, Proceedings of the National Academy of Sciences). If the scientist does not bother about his reputation at the end of his career, we show that a sufficient education level is needed for the scientist to develop a typical research pattern where productivity increases in the beginning of his career, while it declines towards retirement. If the education level is not sufficient, a fading research pattern will result where productivity declines over time. On the other hand, when the scientist appreciates to have a good reputation at the end of his career, sufficient education will result in increasing productivity over the career lifetime, preventing a midlife slump.

Keywords: Optimal control, research patterns, Skiba, age-dependent scientific production, networking: initial knowledge

1. Introduction

The publications of a scientist over his/her career are usually not evenly spaced in time. The main idea is that productivity patterns quite often show an intuitively plausible time course: scientific creativity tends to rise rapidly to a peak and then gradually declines. There are many studies of career paths of creative people since the famous statistician Quételet (1835) started pertinent research almost 200 years ago. Typical life cycle patterns are not only observed in academia, but also in artistic production, in criminal behavior and other fields.

Way et al. (2017) provide a readable introductory survey into age patterns of scientific production. Using a large data set originating in computer science departments in the U.S. and Canada, these authors show that the conventional narrative of a steeply increasing and then gradually declining production trajectory is not generally true. They identify a much richer diversity of production patterns. One important purpose of the present paper is to explain how such a diversity might come about. While virtually all models dealing with the dynamics of scientific production are descriptive (see, e.g., Rinaldi et al., 2000; Rinaldi and Amigoni, 2000), in what follows we propose a normative approach.

We assume that productivity of a scientist depends on the scientist's knowledge and his reputation. A scientist invests in his human capital, or knowledge, by working behind the desk, reading books and papers, developing new ideas, etc. Once a certain stock of knowledge has been built up, the scholar

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can fruitfully work on his reputation. Investing in the stock of reputation consists of presenting at conferences, contacting colleagues, networking at receptions, etc. These efforts can be summarized as networking investment. The output of a scientist is publishing papers. Necessary for that is the accumulation of a sufficient stock of knowledge. On the other hand, a scientist's productivity also depends on his reputation in the sense that reputation helps to find better coauthors and to get more informative about to which journals one should submit his papers in order to get more successful.

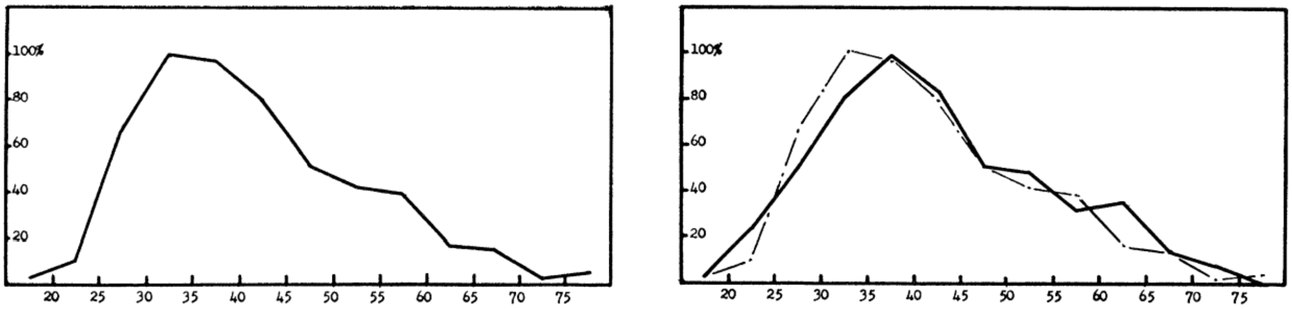
The aim of the scientist is to maximize the discounted stream of publications over time taking into account the costly investments both in knowledge as well as in reputation. The dynamics of the system is given by two ordinary differential equations describing the impact of investment in knowledge and networking on the stocks of knowledge and reputation, respectively. An important feature of the model is that the efficiency of both investment in human capital and networking investment depends on the stock of knowledge. The larger the scientist's knowledge the more effective his investment in research is. Also the networking investment will be more effective if a scientist performs networking while exposing more knowledge about relevant research topics.

Solving the resulting optimal control model by using Pontryagin's maximum principle shows that the shape of the optimal paths depends crucially on the initial situation. If the stock of knowledge is initially too small, it turns out that the researcher's career will not be very productive. If, however, a certain human capital endowment (the so-called Skiba threshold) is exceeded initially, the career will flourish. A large stock of knowledge fosters the investment in networking, making the leverage effect of the stock of reputation work. Besides the case of an infinite planning horizon, the case of a given finite end time is considered. Various terminal conditions (ranging from 'there are no pockets in a shroud' until 'the reputation in the posterity is quite important') will lead to different scientific career patterns.

Literature review

In an old, but still readable survey paper on the economics of science, Stephan (1996) referred to the extreme inequality of scientific productivity. Temporal patterns of scientific productivity vary substantially not only across individuals but also for each researcher over time. Some individuals remain productive throughout their career. Others who show early promise become deadwood after some time: Some authors publish papers like a well-oiled machine, while others produce at an erratic rate (Goodwin and Sauer, 1995).

Typically, an initial rise of creative productivity is followed by an eventual decline. Both the aging process as well as human capital models predict a hump-shaped productivity pattern over the life cycle. There is however, an obvious asymmetry in the distribution of research productivity over the life cycle. While for most of the researchers productivity rises sharply in the early stages of the career, its decline is rather slowly, of which Figure 1 forms a nice illustration. Almost one century ago the famous demographer and mathematician Lotka (1926) stressed the highly skewed nature of scientific publications. Studying nineteenth century physics journals, he observed that approximately six percent of publishing scientists produced half of all papers. Inequality in creative research between scientists at a given time can be explained by different motivation and abilities. There is, however a second kind of inequality in productivity between scientists, namely those of a cohort of scientists over time.



(a) Age versus creative production rate for Russians only, in science and mathematics

(b) Solid line: age versus creative production rate for Englishmen only, in science and mathematics. Broken line, same as in panel (a).

Figure 1: Age vs. creative production rate for Russians only, in science and mathematics. (Source: H.C. Lehman, Men's creative production rate at different ages and in different countries, The Scientific Monthly, May 1954, 321-326).

Stephan (1996) mentions a few studies illustrating this dynamic inequality. Besides different capacity and efficiency of scientists, she refers to some form of feedback mechanism denoted by 'the winner takes all'. Past success in research usually acts as leverage for future productivity. Scientists of considerable reputation have a 'cumulative advantage' in the sense that they will accrue greater increments of recognition for their contributions. This peculiar fact has also been denoted as Matthew Effect in science. Essentially it says 'Once a Nobel Laureate, always a Nobel Laureate'. The Gospel According to St. Matthew puts it this way: 'the Matthew effect consists in the accruing of greater increments of recognition for particular scientific contributions to scientists of considerable repute and the withholding of such recognition from scientists who have not yet made their mark' (compare Lehman, 1954; Merton, 1968). Our model accounts for this effect due to the fact that the stock of reputation has a positive effect on the researcher's output.

Faria and McAdam (2015) show that tenure and promotion of academic specialists are characterized by a bang-bang solution in which the scholar shifts from maximum to minimum effort levels and productivity depending on incentives and impatience. Yegorov et al. (2016) explain the differences in payoffs to talent by analyzing the impact of the initial stock of human capital (which is a productive capital) as well as his or her initial market access (or bargaining powers) within an optimal control framework. McDowell (1982) considers varying learning behavior of professors over their career which may be explained as response to the durability of knowledge. An optimal control model is analyzed but also evidence of obsolescence of knowledge by providing literature decay rates is given. Symonds et al. (2006) show discrepancies in the publication rate between women and men appearing early in their careers.

El Ouardighi et al. (2013) study by means of an optimal control model how individual investments into research and teaching skills subject to a fixed time budget can affect academic careers. Seidl et al. (2016) extend their approach by considering the option to leave academia. To do so, they compare, similar to the approach analyzed in Caulkins et al. (2015), solutions with a free end time (i.e. solutions, where it is optimal to leave academia) to infinite time horizon solutions (i.e. solutions, where it is optimal to stay in academia). They find that the optimal long-run career strategy can be history-dependent, i.e. it depends on the initial skills in research and teaching.

Since the early work of Becker (1962), economists have studied the question how behavior varies over the life cycle in occupations where human capital plays an important role. These models typically produce a peaked profile. First, investment in human capital increases, but at some point, due to finiteness of life, it starts declining. While all life-cycle models for scientists deal with efficient allocation of research time, they differ in the objective functionals of the scientists. Usually it is not the income stream alone which is maximized, but a utility function including among other things, the research output. Levin and Stephan (1991) provide an interesting ansatz in that direction, where our model can be seen as an extension of the Levin-Stephan model. Simonton (1989, 1997) develops a simple three-state model explaining both longitudinal and cross-sectional variations in the scientific output.

Reviewing life-cycle models until mid of the nineties, Stephan (1996) comes to the conclusion that

> *"the human capital approach does not provide the cornerstone on which we should model the behavior of scientists."*

It does not provide an explanation of why the productivity of a cohort of scientists is unequal over time. The author concludes further

> *"that the production of scientific knowledge is far more complex than the human capital model assumes and that these complexities have a great deal to say about patterns that evolve over the life cycle."*

Our model tries just to take this aspect into account. In particular, by explicitly considering the reputation of a scientist we are able to describe unequal age patterns of scientific production.

Kanazawa (2003) refers to the similarity of the age-pattern and scientific creativity and those of crime. While this might be seen as curiosity we mention the fact that many human behavioral aspects exhibit a one-peaked right-skew age-pattern: both the age-specific first marriage intensity as well as fertility provide typical demographic examples. In his Handbook of Genius, Simonton (2014) provides a rich collection of interesting contributions to creativity in a broader sense. Specifically, see Jones et al. (2014).

The paper is organized as follows. In Section 2 the model is presented including some basic analysis by application of Pontryagin's maximum principle. In Section 3 we show among other things the existence of multiple equilibria whose basins of attraction are separated by Skiba curves (Grass et al., 2008), i.e. the effect of initial knowledge is considered by developing history-dependent optimal trajectories. Section 4 shows how the four career patterns recently identified by Way et al. (2017) can be generated as optimal trajectories for appropriate parameter values. In particular, the slump of life satisfaction ('mid-life crisis') is discussed in this framework (see Schwandt, 2016a,b). Section 5 concludes the paper.

2. Model

For a scientist investing in knowledge, K, is the major activity. Such knowledge investments, I, consists of, e.g., working behind the desk, including developing new ideas, making calculations, and reading. In addition, the scientist has the option to establish a reputation, R, by networking, N, which one does, e.g., by making presentations at conferences, talking to colleagues, visiting receptions, inviting colleagues, and writing emails.

The output of a scientist is publishing papers, P. A necessary condition to do so is having built up a stock of knowledge being strictly positive. Building up reputation can work as a leverage with respect to productivity. To model this we introduce the scientific production function

$$P = P(K, R) = K^{\alpha}(R + 1)^{\beta}, \tag{1}$$

with α and β denoting positive constants smaller than one. The functional form reflects that one can be productive without working on reputation.

We consider a representative scientist, who has the aim to maximize the discounted stream of his scientific publications plus the utility arising from investing in knowledge and networking. An important feature of our model is the fact that doing research and networking usually create utility for a scientist as long as it is done 'to a reasonable extent.' Only if I and N exceed some thresholds, these activities are connected with disutilities, i.e. they must be seen as costly. In mathematical terms the following optimization model results:

$$\max_{I(\cdot),N(\cdot)} \int_0^T \mathrm{e}^{-rt} \left(P(K(t), R(t)) - C_1(I(t)) - C_2(N(t))\right) \mathrm{d}t + \mathrm{e}^{-rT} \left(\kappa_1 K(T) + \kappa_2 R(T)\right) \tag{2a}$$

$$\text{s.t.} \quad \dot{K}(t) = g(K(t))I(t) - \delta_1 K(t), \quad t \in [0, T\rangle \tag{2b}$$

$$\dot{R}(t) = h(K(t))N(t) - \delta_2 R(t), \quad t \in [0, T\rangle \tag{2c}$$

$$K(0) = K_0 \geq 0, R(0) = R_0 \geq 0 \tag{2d}$$

$$I(t) \geq 0, N(t)) \geq 0, \quad t \in [0, T\rangle. \tag{2e}$$

with

$$[0, T\rangle := \begin{cases} [0, T] & \text{if } T < \infty \\ [0, \infty) & \text{otherwise.} \end{cases}$$

Eqs. (2b) and (2c) describe the accumulation of knowledge and reputation dependent on research and network activities, respectively. The functions $g(\cdot), h(\cdot)$ are assumed to be convex-concave (S-shaped) and $C_{1,2}(\cdot)$ are linear-quadratic. Specifically we choose

$$g(K) := \frac{a(l + K^\theta)}{1 + K^\theta} \tag{2f}$$

$$h(K) := \frac{eK^\sigma}{1 + K^\sigma} \tag{2g}$$

$$C_1(I) := d_1 I^2 - c_1 I \quad \text{and} \quad C_2(N) := d_2 N^2 - c_2 N, \tag{2h}$$

where $g(K)$ is an increasing function reflecting that investing in knowledge is more fruitful if one has already built up some knowledge. Also, working on reputation is much more effective if one is knowledgeable, so that $h(K)$ is an increasing function as well. This is because when being more knowledgeable the scientist makes a good impression when presenting his research, talking to other researchers, writing emails and so on. Note that the functional forms $C_1(I)$ and $C_2(N)$ are such that the scientist enjoys positive marginal utility from knowledge investment and networking as long as I and N are not too large.

Concerning the parameter κ_2, one possibility is to take the salvage value to be equal to zero ($\kappa_2 = 0$). One could also argue that taking a salvage value into account is reasonable to reflect that the scientist's reputation at the end of his career should be valued positively. That would mean formulating a salvage value being increasing in R, which is achieved by assuming κ_2 to be positive. To ease the full appreciation of the model generality, Table 1 gives an exhaustive overview of the model parameters and their meaning.

The model as such is interpretable in a more general sense. It does not only hold for the career of a scientist, but also for an artist, an office worker, a football player, and so on. It also holds for a startup like some "Silicon Valley" firm that first develops an idea, then works to extend it (I and K), but then starts the networking process to find venture capitalists, attracting promising people, etc. (N and R).

We solve the model by applying Pontryagin's maximum principle (see, e.g. Grass et al., 2008). This method starts out with setting up the current-value Hamiltonian Eq. (3a) and its derivatives with respect to the states Eqs. (3b) and (3c) and controls Eqs. (3d) and (3e)

$$\begin{aligned} \mathcal{H} = K^\alpha (R+1)^\beta - d_1 I^2 + c_1 I - d_2 N^2 + c_2 N + \lambda_1 \left(\frac{a(l + K^\theta)}{1 + K^\theta} I - \delta_1 K \right) + \\ \lambda_2 \left(\frac{eK^\sigma}{1 + K^\sigma} N - \delta_2 R \right), \end{aligned} \tag{3a}$$

$r > 0$	discount rate
$a > 0$	parameter governing the speed of knowledge accumulation
$e > 0$	parameter governing the speed of reputation accumulation
$0 \leq l \leq 1$	parameter influencing the speed of knowledge accumulation
$\theta \geq 1$	parameter determining the shape of the knowledge accumulation function
$\sigma \geq 1$	parameter determining the shape of the reputation accumulation function
$\delta_1 > 0$	rate of forgetfulness of knowledge
$\delta_2 > 0$	depreciation rate of reputation
$0 \leq \alpha \leq 1$	elasticity coefficient of knowledge investment
$0 \leq \beta \leq 1$	elasticity coefficient of reputation by networking
$c_1 > 0$	parameter describing how much knowledge investment adds to utility
$c_2 > 0$	parameter describing how much networking adds to utility
$d_1 > 0$	parameter describing how much knowledge investment adds to disutility
$d_2 > 0$	parameter describing how much networking adds to disutility
$\kappa_1 > 0$	salvage value parameter measuring the intellectual ambition
$\kappa_2 > 0$	salvage value parameter measuring the reputational ambition

Table 1: Overview of model parameters and their meaning.

from which we obtain that

$$\partial_K \mathcal{H} = \alpha K^{\alpha-1}(R+1)^{\beta} + \lambda_1 \left(\frac{a\theta K^{\theta-1}(1-l)}{(1+K^{\theta})^2} I - \delta_1 \right) + \lambda_2 \frac{e\sigma K^{\sigma-1}}{(1+K^{\sigma})^2} N \tag{3b}$$

$$\partial_R \mathcal{H} = \beta K^{\alpha}(R+1)^{\beta-1} - \lambda_2 \delta_2 \tag{3c}$$

$$\partial_I \mathcal{H} = c_1 - 2d_1 I + \lambda_1 \frac{a(l+K^{\theta})}{1+K^{\theta}} \tag{3d}$$

$$\partial_N \mathcal{H} = c_2 - 2d_2 N + \lambda_2 \frac{eK^{\sigma}}{1+K^{\sigma}}. \tag{3e}$$

We only have to consider the interior control solutions since boundary solutions, i.e. $I = 0$ or $N = 0$, will not occur due to the linear-quadratic cost functions Eq. (2h). Therefore, application of the maximum principle generates the following necessary optimality conditions by setting Eqs. (3d) and (3e) to zero:

$$I^* = \frac{1}{2d_1} \left(c_1 + \lambda_1 \frac{a(l+K^{\theta})}{1+K^{\theta}} \right) \tag{4a}$$

$$N^* = \frac{1}{2d_2} \left(c_2 + \lambda_2 \frac{eK^{\sigma}}{1+K^{\sigma}} \right). \tag{4b}$$

Plugging the expressions for the controls Eq. (4) into the state equations Eqs. (2b) and (2c) and the derivatives Eqs. (3b) and (3c) into the costate equations yields the canonical system:

$$\dot{K}(t) = \frac{aK(t)^{\theta}}{1+K(t)^{\theta}} I^*(t) - \delta_1 K(t) \tag{5a}$$

$$\dot{R}(t) = \frac{eK(t)^{\sigma}}{1+K(t)^{\sigma}} N^*(t) - \delta_2 R(t) \tag{5b}$$

$$\dot{\lambda}_1(t) = r\lambda_1(t) - \partial_K \mathcal{H}(t) \tag{5c}$$

$$\dot{\lambda}_2(t) = r\lambda_2(t) - \partial_R \mathcal{H}(t). \tag{5d}$$

This dynamical system cannot be analytically treated and thus allows only a numerical analysis, the outcome of which we present in the next two sections.

3. When does the initial knowledge level matter?

As argued in the Introduction, knowledge investments and networking are more effective if the knowledge level of the scientist is high. For this reason his initial knowledge level, which the scientist developed during his studies preceding his career, can be a crucial factor for career development. So, the question in particular is whether it is realistic to pursue a scientific career in case of a too low initial knowledge level. In this light the next proposition provides information, because it defines scenarios under which a scientific career will not be fruitful in the sense that the long run equilibrium prescribes that both the level of knowledge as well as scientific reputation will vanish.

Proposition 1. *The equilibrium at the origin exists for all $\theta > 1$ and $\sigma \geq 1$. For $\theta = 1$ this equilibrium exists if and only if*

$$\delta_1 - \frac{ac_1}{2d_1} > 0. \tag{6}$$

Proof. Appendix A. □

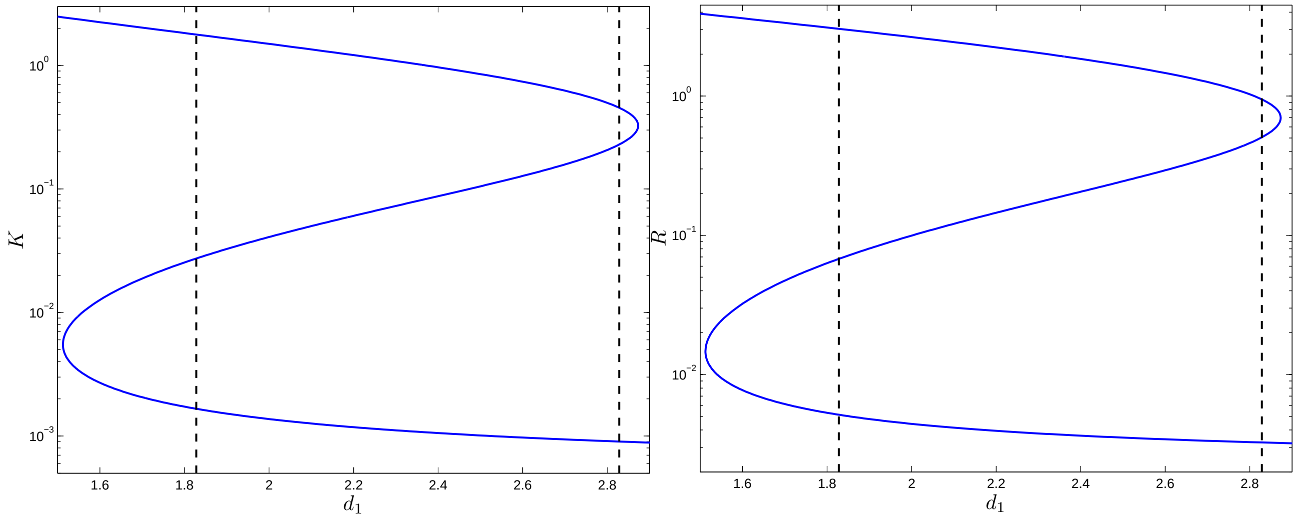
We have chosen the parameter values specified in Table 2 as the base case. Specifically we set $\theta = \sigma = 1$, and therefore the equilibrium at the origin only exists if expression (6) is satisfied.

r	a	e	l	θ	σ	δ_1	δ_2	α	β	c_1	c_2	d_1	d_2	κ_1	κ_2
0.04	0.5	0.5	0	1	1	0.2	0.1	0.8	0.5	1	1	$*$	1	0	$*$

Table 2: The specified parameter values for the (Skiba) base case.

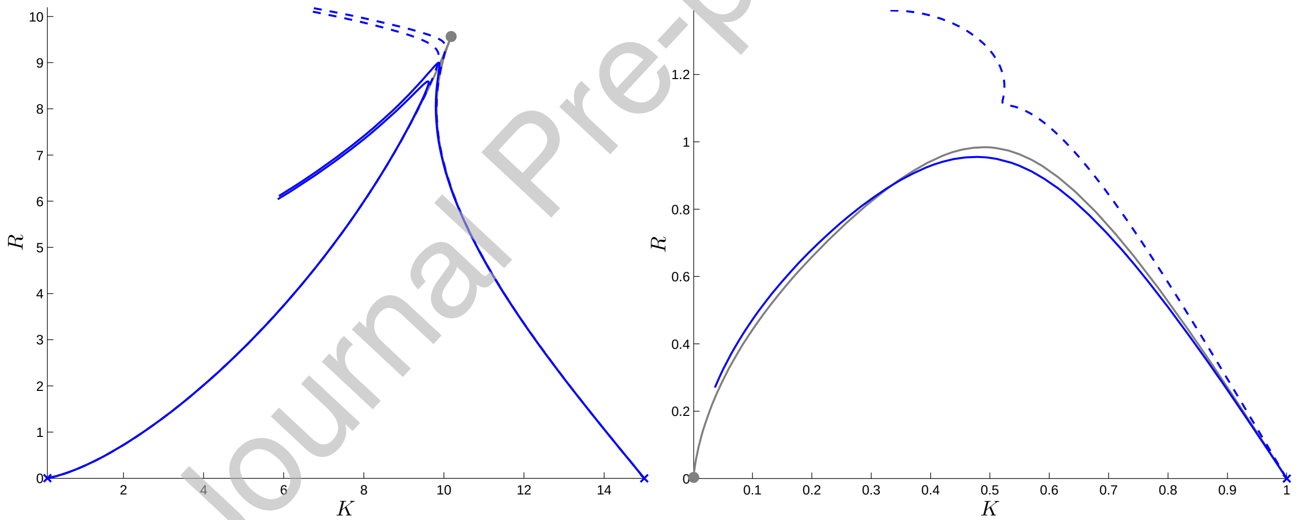
For the numerical analysis of the canonical system Eq. (5) we applied a boundary value problem (BVP) approach combined with a continuation strategy, as presented in Grass (2012). For the actual calculations `OCMat` was used, a MATLAB package that is in constant development by one of the authors.[1]

[1]The actual version can be received from the authors. An older version can be downloaded from http://orcos.tuwien.ac.at/research/ocmat_software.



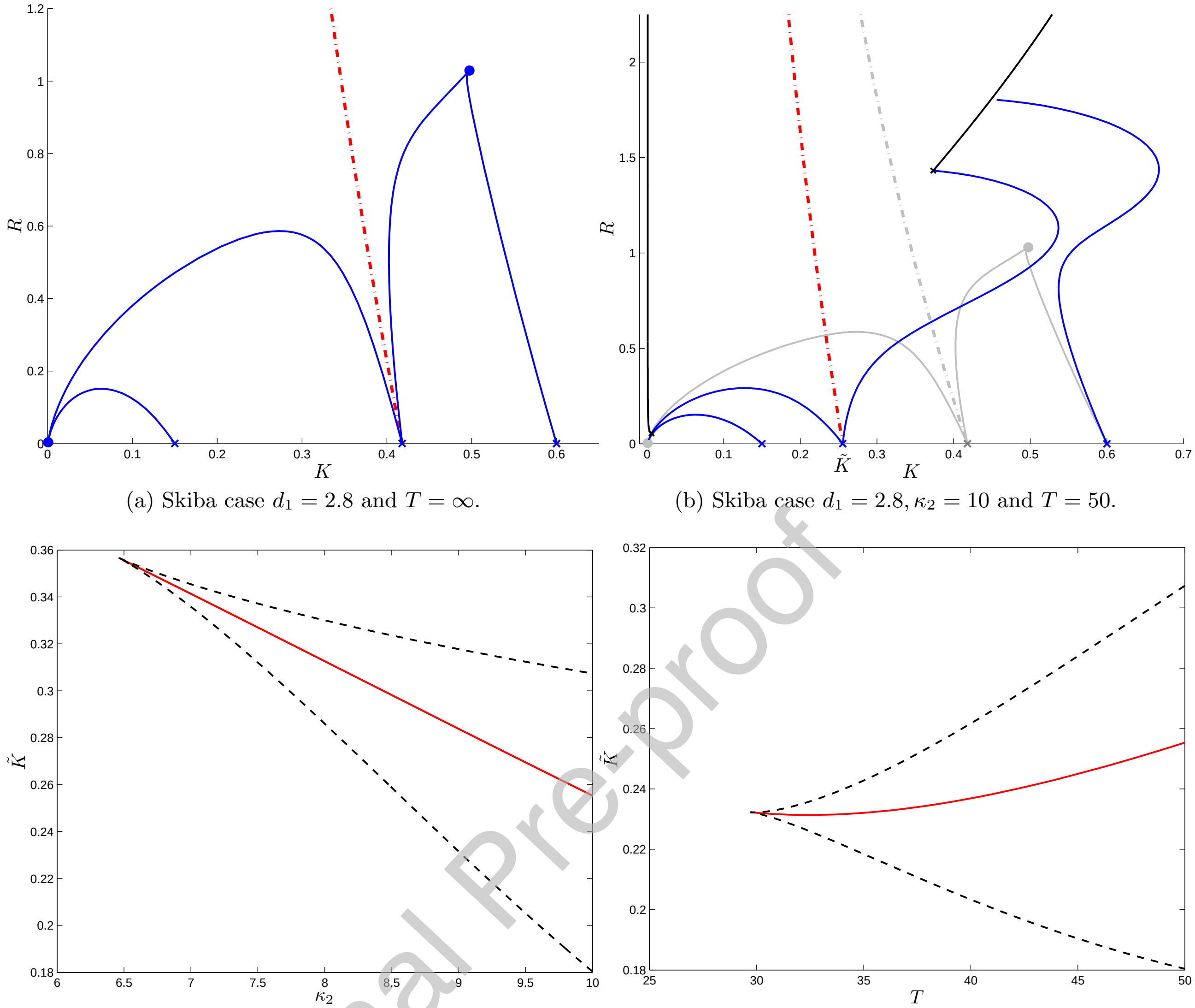
(a) Bifurcation diagram (d_1) (b) Bifurcation diagram (d_1)

Figure 2: Bifurcation diagram for parameter d_1. For the region in-between the dashed lines Skiba solutions exist, whereas outside this region the long run equilibrium is unique. For the numerical treatment of the equilibrium at the origin we introduce a small value τ (see Appendix A.1). This explains the discrepancy in the positive values shown in the bifurcation diagram and the exact value zero.



(a) Base case $d_1 = 0.5, \kappa_2 = 0$ and 10 (b) $d_1 = 3, \kappa_2 = 0$ and 10

Figure 3: Solution for the base case with different parameter values d_1. The blue curves correspond to the finite time horizon solutions with $T = 50$. The solid curves correspond to $\kappa_2 = 0$ and the dashed curves to $\kappa_2 = 10$. The gray curves depict the infinite time horizon solutions. The crosses denote the initial states of the solution paths. An equilibrium (of the infinite time horizon problem) is depicted by a dot.



(a) Skiba case $d_1 = 2.8$ and $T = \infty$.

(b) Skiba case $d_1 = 2.8, \kappa_2 = 10$ and $T = 50$.

(c) Skiba point $(\tilde{K}, 0)$ for varying κ_2.

(d) Skiba point $(\tilde{K}, 0)$ for varying T.

Figure 4: In (a) the Skiba case for the infinite time horizon problem is depicted. The crosses denote the initial states, an equilibrium is represented by a dot. The Skiba curve is given by the red curve. In panel (b) the Skiba solutions for the finite (colored) and infinite time horizon (gray) are depicted. The dashed-dotted curves show the Skiba-curves. Panel (c) shows the change of the Skiba value for K with fixed $R = 0$ and varying κ_2 (red) curve. The dashed lines enclose the region of initial K values, where two different solutions exist. Panel (d) shows that a Skiba solution for the finite time horizon only exists for a large enough time horizon, in the actual case $T \geq 29.73$. Moreover, the according Skiba point increases for larger values of T, which is consistent with (b), where the Skiba point of the infinite time horizon problem lies to the right of $\tilde{K}$.

Figure 2 shows bifurcation diagrams regarding the possible steady state values of knowledge, K, and reputation, R. The inverted S-shaped curves represent the location of the steady state. The meaning of the two vertically shaped dashed lines is that for values of d_1 that fall in between these dashed lines optimal trajectories exist that either converge to the upper or the lower steady state. Here the basins of attraction of the two steady states is separated by a Skiba curve, which is further explained after presenting Figure 4. To the right of this area the optimal trajectory converges to the origin, and to the left always convergence to the upper steady state occurs. Hence, the bifurcation parameter is d_1, where an increase of this value raises the cost of investment in knowledge. We first observe that Figure 2 confirms Eq. (6), from which we obtain that for $d_1 \leq 1.25$ the equilibrium at the origin does not exist, whereas for $d_1 > 1.25$ this equilibrium exists. In particular, the parameter d_1

specifies how fast marginal utility of the scientist declines, once he decides to invest more in knowledge. A large value of d_1 thus implies it is costly to increase knowledge fast. Therefore, only convergence to the small steady state takes place in such a case, as can be inferred from Figure 2.

An important implication of Figure 2 is that the parameter d_1 is crucial for the long run outcome of the model. What academic institutions could learn from this is that it pays off when it finds methods to reduce the disutility of performing knowledge investments, or, in other words, doing research. In particular, if the academic institution succeeds in reducing d_1 that much that its value is lower than the value corresponding to the location of the left dashed vertical line, the scientist will keep on increasing knowledge and will have an active research career throughout his academic life. Methods to reduce d_1 could be, for instance, providing enough office space, or reducing teaching load and administrative duties so that the researcher can do research without feeling the stress of having too many deadlines.

Figure 3b shows optimal trajectories for a large value of d_1, and indeed they end up at the origin. A small d_1 means that the scientist is so clever that it is relatively easy to create knowledge. Figure 3a depicts trajectories ending up at the larger steady state, where, indeed, the value for d_1 is relatively low. A positive κ_2 indicates that at the horizon date reputation is positively valued. The corresponding trajectories therefore end up at larger values of reputation in both Figure 3, where in Figure 3a this happens at the expense of the final value of knowledge that is lower there.

For intermediate levels of d_1 we have a situation with multiple equilibria as Figure 2 shows. Figure 4a depicts the solution of the model if we take an infinite horizon date. Different trajectories are considered starting out from different initial situations. The most reasonable initial situation is such that the reputation of a researcher is low initially, so then $R(0)$ should be small, i.e. close to zero. Before the career starts, it is very seldom the researcher has build up a network by then. The initial knowledge level $K(0)$ should also be relatively small, but note that the scientist starts his scientific career after his studies. During his studies the scientist has built up some knowledge, so $K(0)$ need not be too close to zero.

In Figure 4a a Skiba curve occurs, separating the regions of attraction of the smaller and the larger equilibrium. To the left of the curve we have initial states from which the optimal trajectory ends up at the equilibrium at the origin, and to the right the optimal trajectories converge to the equilibrium with positive knowledge and reputation values. The Skiba phenomenon is caused by the control-state interaction term $Ig(K)$ in the state equation for knowledge. This term makes that investment in knowledge especially pays off when knowledge is large. Hence, therefore the scientist does not invest enough in knowledge to let it increase when the knowledge level is low.

Since, as argued above, typically $R(0)$ is close to zero, the point where the Skiba curve intersects with the K-axis is very relevant, since it determines a threshold value for $K(0)$, denoted by $\tilde{K}$. The meaning of the threshold value is that for initial knowledge levels above $\tilde{K}$ the scientist goes for a scientific career that let him end at the upper steady state. If, however, the initial knowledge level is such that it falls below $\tilde{K}$, the researcher under consideration will invest more time in other activities than research like, for instance, teaching or administration. The result is that he ends up in the equilibrium at the origin.

Considering the different trajectories we observe that when knowledge K is low relative to R, the scientist does not invest too much in networking so that it loses reputation value. On the other hand, the scientist substantially invests in reputation in situations where the knowledge is relatively large. We conclude that R runs after K: R increases (decreases) when it is small (large) compared to K. The intuition for such behavior is that it especially pays to invest in networking when the knowledge is large, i.e. the reputation investment effectivity function $h(K)$ is increasing in knowledge K.

In reality a scientist has no infinite life so it makes most sense to consider a finite time model. Figure 4b shows trajectories where the scientific career encompasses 50 years. The objective function includes a salvage value, which is such that at the end of the scientific life reputation is positively valued. Like with the infinite time solution, also here a Skiba curve can be detected, implying that a sufficiently large initial knowledge level is required to build up a scientific career. Figure 4b shows that the finite time Skiba curve lies to the left of the infinite time one, where having finite time reduces $\tilde{K}$ from about 0.42 to 0.26. Hence, in more cases it is optimal to build up a scientific career in finite time. This is because of the relatively high valuation of reputation at the end of the scientific lifetime. The latter also causes that the two trajectories to the right of the Skiba curve have in common that at a final time interval reputation increases while knowledge decreases.

The phase portrait in Figure 4b contains two manifolds of end-states that are depicted as black curves. These manifolds are the finite time counterparts to the two equilibria of the infinite time problem in Figure 4a. The manifold corresponding to the low equilibrium is very closely located to the R-axis. The second manifold has substantially positive knowledge values. The location on such a manifold where a particular trajectory will end, depends on the initial values of knowledge and reputation. The shape of the Skiba curve is such that it intersects the K-axis, while it approaches the R-axis asymptotically. This implies that it is not optimal to build up a scientific career if initially the scientist has zero knowledge and very high reputation, which is quite unlikely of course. The reason is that investments in both knowledge and reputation are not effective with zero knowledge.

Figure 4d shows that the threshold $\tilde{K}$ increases with the horizon date. This implies that if the scientific career lasts for a longer time, a higher initial knowledge level is needed for building up a fruitful scientific career. The reason is that under this parametrization, the scientist very much appreciates to have a high scientific reputation at the end of his career. The earlier this event takes place the higher the incentive to increase reputation and thus the more attractive it is to invest in a scientific career. The history dependent equilibria separated by the Skiba curve, only exist for values of T larger than 30 years. For smaller T building up a scientific career is more attractive, which explains that for T small enough only convergence to the larger steady state will occur. We conclude that there could be a negative effect of delaying the pension date; if the horizon date lies to far into the future, the scientist could be demotivated to keep on being an active researcher.

That, if the scientist has a higher reputational ambition (i.e. κ_2), the scientist has a larger incentive to build up a scientific career, is confirmed in Figure 4c. We see that for a larger value of κ_2 the threshold is lower, implying that a lower initial knowledge level is needed for starting a trajectory that converges to the larger equilibrium. Moreover, for κ_2 small enough only the smaller steady state exists so that the scientist will not invest too much in a scientific career resulting in convergence to the origin, whatever his initial knowledge level is.

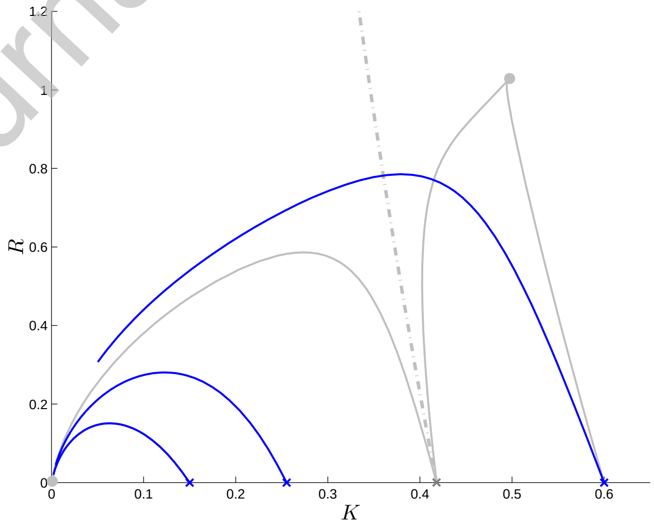


Figure 5: Finite time solution with zero salvage value and $d_1 = 2.8$.

Figure 5 shows what happens if the scientist does not care about his reputation at the end of his career, thus when there is no salvage value. Here the scientist is of the opinion that 'there are no pockets in a shroud', so that he assigns no positive value to his reputation at the end of the scientific life. We see that this results in trajectories where the scientist is not interested in building up knowledge, whereas investments in reputation, i.e. networking, will be stopped when the scientist reaches the end of his career.

Figure 5 also confirms that the firm invests in networking when the scientist's knowledge level is high. This makes sense because, since $h(K)$ is an increasing function, the marginal effect of network investment on reputation increases in knowledge. The implication is that at the start of a typical academic career, where the scientist has a relatively limited initial knowledge level, he will begin with investing in knowledge. Later on, when the scientist's knowledge level has become sufficiently high, a new phase of the career will start in which the scientist will also invest substantially in networking.

4. Optimal patterns of scientific production

Using data of computer science departments in North-America, Way et al. (2017) identify four patterns of individuals' scientific productivity, see Figure 6. For simplicity, the authors restrict themselves to two stages in the productivity cycle. They use a piecewise linear model

$$f(t) = \begin{cases} b + m_1 t & 0 \leq t \leq t^* \\ b + m_1 t^* + m_2(t - t^*) & t > t^* \end{cases} \tag{7}$$

to describe two phases of scientific research. By denoting the slopes of the two phases by m_1 and m_2, they are able to depict the four cases in the four quadrants Q_i of the $(m_1\ m_2)$-plane $(i = 1, \ldots, 4)$.

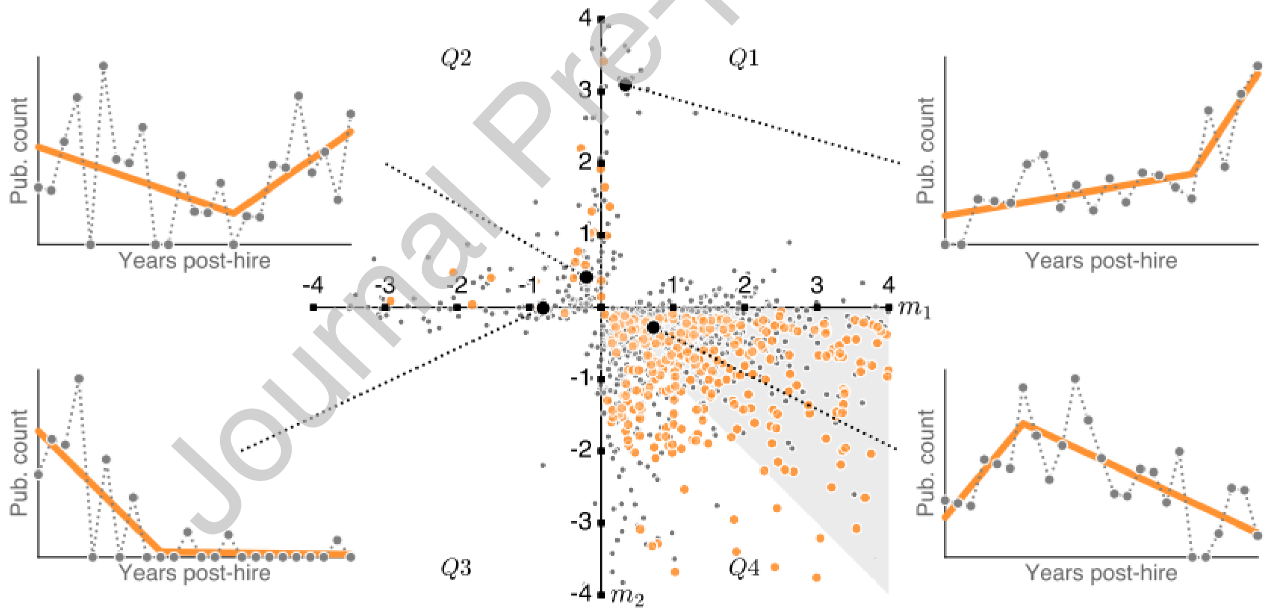


Figure 6: Distribution of individuals' productivity trajectory parameters. Diverse trends in the individual productivity fall into four quadrants based on their slopes m_1 and m_2 in the piecewise linear model Eq. (7). Plots show example publication trajectories to illustrate general characteristics of each quadrant. (Source: Way et al., 2017).

Clearly, among the four patterns sketched in Figure 6, the most frequent productivity pattern corresponds to Q_4 (positive m_1, negative m_2). However, as Way et al. (2017) claim, rising productivity followed by a declining publication activity is by no means universal. The aim of this section is to illustrate that for each of the four patterns shown in Figure 6, a parameter scenario can be developed for which our model generates a solution characterizing the associated pattern. All regimes are based on the parameter values depicted in Table 3. To develop the different productivity patterns we vary three different parameters, namely d_1, indicating the disutility part of knowledge investments, κ_1, being the unit value of knowledge at the end of the scientist's career, and κ_2, being the unit value of reputation at the horizon date. Table 4 gives an overview of their values in the different regimes.

l	θ	d_1	κ_1	κ_2
0.25	2	*	*	*

Table 3: The parameter values that changed compared with Table 2.

Regime	d_1	κ_1	κ_2
Q_4(typical)	3	0	0
Q_3(fading)	6	0	0
Q_2(slump)	5	20	5
Q_1(busy)	5	30	5

Table 4: d_1: how much knowledge investment adds to disutility, κ_1: measure of intellectual ambition, κ_2: measure of reputational ambition, in the four different regimes.

In what follows we treat the four regimes of Figure 6 in a clockwise manner, starting with the most common life cycle pattern Q_4. Figure 7 illustrates the case of a first increasing and then decreasing path of scientific productivity, called the *typical* case, see also Figure 1 in the Introduction. In the typical case the scientist starts off investing in his career, resulting in a steady increase of his productivity. Furthermore a typical scientist's intellectual and reputational ambition is relatively low, or, to put in other words the salvage value is zero. This is the reason that at the end of his career he gradually builds off his activities, continuously approaching retirement.

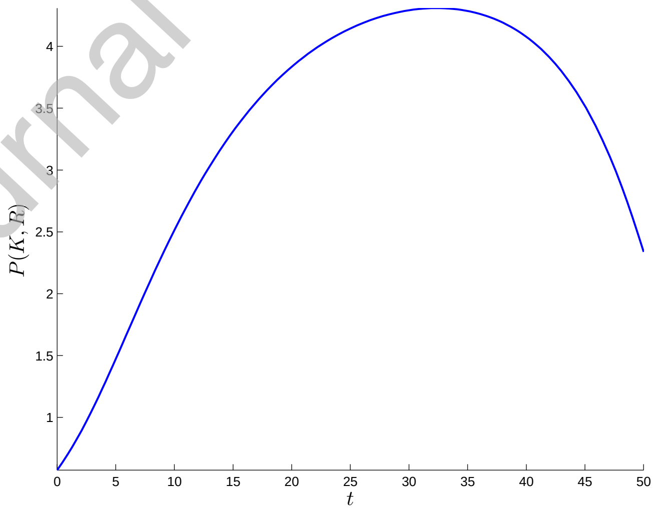


Figure 7: Typical case. Scientific productivity over age. Case of first increasing and then decreasing productivity. The basic parameters are given in Table 3, the varying parameters are specified as $\kappa_1 = 0$ and $d_1 = 3$. The initial state values are $K(0) = 0.5$ and $R(0) = 0$.

The pattern Q_3 depicted in Figure 8 is another distribution which is familiar, called the *fading* case. In the beginning of his career the scientist produces still at a reasonable level, based on the knowledge he obtained during his studies. After getting tenure, the publication rate decreases, converging to zero productivity at the end of his career. Besides having a relatively low intellectual and reputational ambition, the fading scientist also experiences heavy disutilities from investing in knowledge (d_1 relatively large), so he gradually reduces his research activities over time.

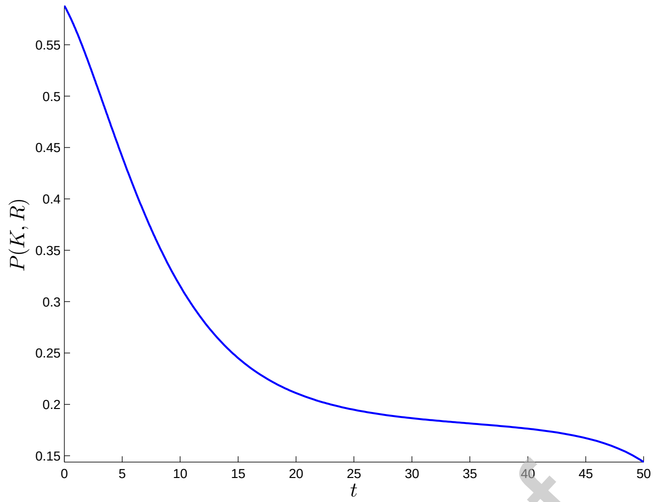


Figure 8: Fading case. Scientific productivity over age. Case of decreasing productivity. The basic parameters are given in Table 3, the varying parameters are specified as $\kappa_1 = 0$ and $d_1 = 6$. The initial state values are $K(0) = 0.5$ and $R(0) = 0$.

The next productivity distribution, Q_2, shows a *slump* somewhere in the middle of the career (Figure 9). This can typically be caused by a slump of life satisfaction ('midlife crisis'), as discussed in Schwandt (2016a). Towards the end of his career, the "slump" scientist resumes research activities. The reason is that, due to his rather large intellectual and reputational ambition, he wants to be remembered because of his knowledge and being a researcher with high reputation. This is reflected in a positive salvage value being increasing in both the final values of knowledge and reputation.

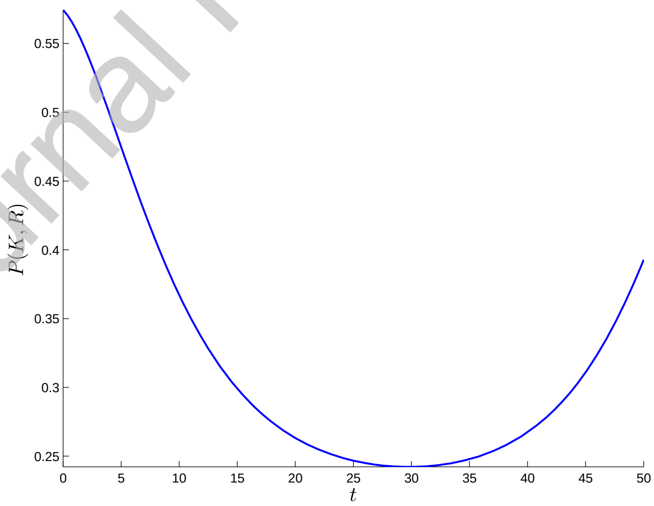


Figure 9: Slump case. Scientific productivity over age. Case of first decreasing and then increasing productivity. The basic parameters are given in Table 3, the varying parameters are specified as $\kappa_1 = 20$ and $\kappa_2 = 5$ and $d_1 = 5$. The initial state values are $K(0) = 0.5$ and $R(0) = 0$.

The last regime, Q_1, describes the 'late researcher', called the *busy* case. While at the beginning he produces few papers, he gradually gets his career going, as illustrated in Figure 10. Towards the end he accelerates research intensity because he highly values being knowledgeable and having a high reputation at the end of his career. This is reflected in the relatively high values of the intellectual ambition parameter κ_1 and the reputational ambition parameter κ_2.

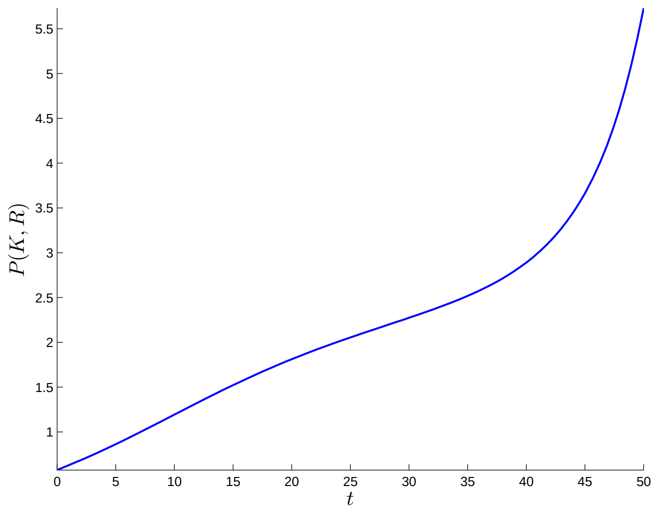


Figure 10: Busy case. Scientific productivity over age. Case of increasing productivity. The basic parameters are given in Table 3, the varying parameters are specified as $\kappa_1 = 30$ and $\kappa_2 = 5$ and $d_1 = 5$. The initial state values are $K(0) = 0.5$ and $R(0) = 0$.

The importance of the initial knowledge level regarding research patterns.

Education matters, especially in how a researcher starts a career. This already became apparent in Section 3 where in several cases a Skiba curve separated career patterns starting out with a high initial knowledge level resulting in an active career and a relatively low initial knowledge level being not enough to become a productive researcher. At the end of Section 4 we show how the "Skiba approach" sheds light on how a researcher chooses between the different research patterns.

First, consider the two patterns where the researcher has a limited intellectual and reputational ambition, namely the *typical* and the *fading* pattern. Just a little bit more knowledge can result in a *typical* career, rather than a *fading* one, where the *typical* researcher is far more productive. Based on the parameter values of Table 5, Figure 11 illustrates this fact where the two depicted trajectories start out from the threshold value $\tilde{K} = 0.5$. Here the *typical* pattern results for any $K(0) > \tilde{K}$, whereas a *fading* career is the result of having $K(0) < \tilde{K}$.

r	a	e	l	θ	σ	δ_1	δ_2	α	β	c_1	c_2	d_1	d_2	κ_1	κ_2
0.04	0.5	0.5	0.25	2	1	0.2	0.1	0.8	0.5	1	1	4.6198	1	0	0

Table 5: The specified parameter values for Figure 11.

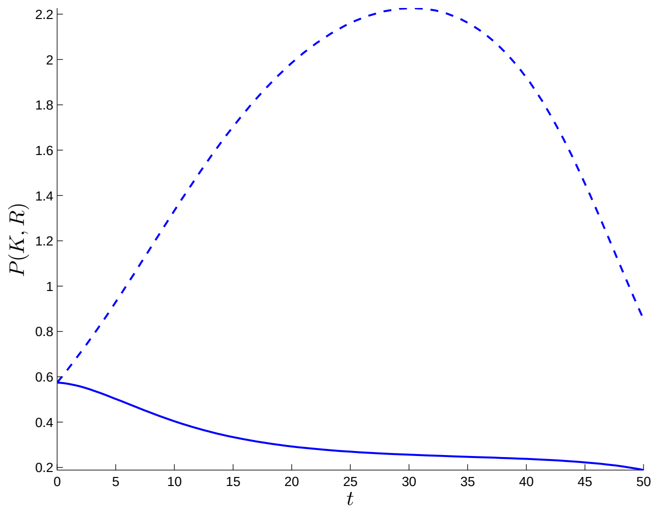


Figure 11: A Skiba solution exhibiting the two regimes Q_3 and Q_4 for $K(0) = 0.5$ and $R(0) = 0$. For the parameter values see Table 5.

Second, we compare the patterns where a high value is assigned to being knowledgeable with high reputation at the end of the career, namely *slump* and *busy*. For such a researcher a *slump* can be avoided if he is well educated. In particular, Figure 12, being based on the parameter scenario of Table 6, again has a threshold value of $\tilde{K} = 0.5$, which implies that for $K(0) > 0.5$ it is optimal to start a *busy* career with increasing research productivity. Knowledge investments and networking are highly effective if the researcher possesses a lot of knowledge, which explains increasing activities during the life of the *busy* researcher. However, in all other cases, thus where education did not result in a high enough initial knowledge level, a *slump* in the career cannot be avoided.

r	a	e	l	θ	σ	δ_1	δ_2	α	β	c_1	c_2	d_1	d_2	κ_1	κ_2
0.04	0.5	0.5	0.25	2	1	0.2	0.1	0.8	0.5	1	1	5.254	1	20	5

Table 6: The specified parameter values for Figure 12.

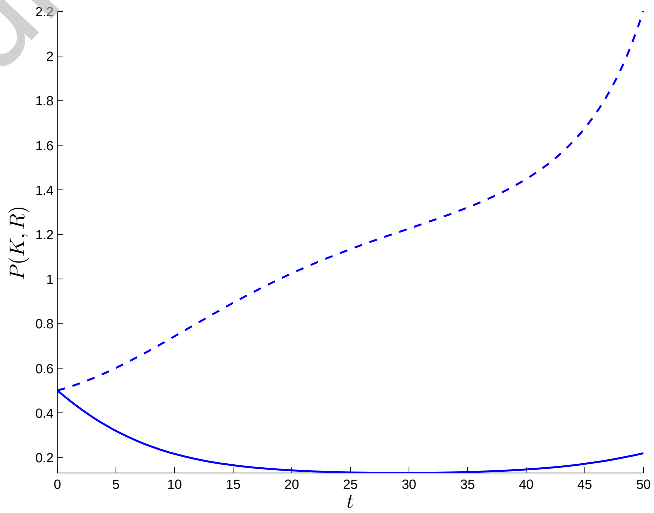


Figure 12: A Skiba solution exhibiting the two regimes Q_1 and Q_2 for $K(0) = 0.5$ and $R(0) = 0$. For the parameter values see Table 6.

5. Conclusion

While there exist many papers describing the life cycle behavior of scientific production, in our normative approach we focus on the optimal way in which different age-patterns of publication output comes about. Since the empirical research in Way et al. (2017) we know that among others four different patterns of scientific productivity can be discerned. In particular, regarding the productivity of the scientist during his scientific life, we can identify patterns called *typical* (increasing and then decreasing productivity), *fading* (decreasing productivity), *slump* (decreasing and then increasing productivity), and *busy* (increasing productivity). The present paper aims to provide a theoretical foundation for this empirical research. To do so, we design an optimization model of the representative scientist. The scientist derives positive utility from publishing papers and from activities like performing research, which we denote by 'investing in knowledge', and networking. However, working too hard causes a disutility, i.e. too large investments in knowledge and networking are costly.

We show that *typical* and *fading* patterns typically arise in scenarios where the scientist has a limited intellectual and reputational ambition. In such a case the scientist will choose for a *typical* pattern if the disutility for hard working is not too high. Here it could help that during his studies the scientist obtained a lot of knowledge. This is because, if the scientist starts his career being knowledgeable, investments in knowledge and networking become more efficient.

If the scientist does assign a substantial positive value to being regarded as knowledgeable with a high reputation at the end of his career, the patterns *slump* and *busy* come into the picture. We show that a *slump* pattern, where the scientist is not very productive halfway his career, can be avoided by high quality education. Again, starting the career with a lot of knowledge make investments in knowledge and networking efficient. This raises productivity along the lifetime, resulting in the *busy* pattern.

Besides the knowledge at the beginning of the career, also the long term intellectual and reputational ambition of the scientist turns out to be of crucial importance. If the academic institution is able to feed these ambitions of its scientists, then patterns like *slump* and *busy* will prevail in its institute instead of scientific careers emphasizing *typical* and *fading* patterns. Feeding intellectual and reputational ambitions could be achieved by, for instance, introducing rewards for scientific publications, clear promotion criteria based on scientific publications, providing interesting seminar programs, and reserving enough funds for visiting conferences.

Trying to capture a problem in a model by definition means some real life aspects are left out. We did capture that an increase of the scientist's knowledge results in more efficient research activities. What also could have been included is that performing research benefits from the scientist's reputation. If the scientist get stuck somewhere, it is easier to ask a distinguished colleague for help if the scientist himself is of high reputation. Moreover, competition between scientists could affect the age patterns, which was also not taken into account in the present paper.

Also, as we have it now the model is autonomous. Another realistic extension would be that the efficiency of accumulating knowledge and reputation explicitly depends on timing, so that effects of aging and learning can be taken into account.

Acknowledgements

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A. Proof of Proposition 1

For the following analysis we assume $l = 0$. For $K = 0$ the term $K^{\alpha-1}(R+1)^{\beta}$ becomes singular and hence we have to handle a diverging costate λ_1. Hence, to prove that the equilibrium at the origin satisfies the necessary optimality conditions we need to check for which conditions the transversality condition

$$\lim_{t\to\infty} \mathrm{e}^{-rt} K(t)\lambda_1(t) = 0 \tag{A.1}$$

holds. The transversality condition

$$\lim_{t\to\infty} \mathrm{e}^{-rt} R(t)\lambda_2(t) = 0$$

is trivially satisfied, since the equilibrium value for $\hat{\lambda}_2 = 0$ at $\hat{R}$.

Near the origin and for $\theta > 1$ the state dynamics is mainly driven by the linear term and we approximately find

$$K(t) = \mathrm{e}^{-\delta_1 t} K(0) \tag{A.2a}$$
$$R(t) = \mathrm{e}^{-\delta_2 t} R(0). \tag{A.2b}$$

Thus, for $(K(0), R(0))$ small enough the solution converges to the origin. Next we prove Eq. (A.1). Plugging Eq. (A.2a) into the costate equation yields

$$\dot{\lambda}_1(t) = (r+\delta_1)\lambda_1(t) - \varphi_1 \mathrm{e}^{-(\delta_1(\alpha-1)+\delta_2\beta)t}$$

with

$$\varphi_1 := \alpha K(0)^{\alpha-1}(R(0)+1)^{\beta}.$$

Solving the ODE we find

$$\begin{aligned}\lambda_1(t) &= \mathrm{e}^{(r+\delta_1)t}\lambda_1(0) - \varphi_1 \mathrm{e}^{(r+\delta_1)t}\int_0^t \mathrm{e}^{-(r+\delta_1+\delta_1(\alpha-1)+\delta_2\beta)s}\,\mathrm{d}s \\ &= \mathrm{e}^{(r+\delta_1)t}\lambda_1(0) + \frac{\varphi_1 \mathrm{e}^{(r+\delta_1)t}}{r+\delta_1\alpha+\delta_2\beta}\left(\mathrm{e}^{-(r+\delta_1\alpha+\delta_2\beta)t} - 1\right)\end{aligned}$$

Hence the transversality condition is satisfied if the following condition holds

$$\begin{aligned}\lim_{t\to\infty} \mathrm{e}^{-rt} K(t)\lambda_1(t) &= \lim_{t\to\infty} \mathrm{e}^{-rt}\mathrm{e}^{-\delta_1 t}\mathrm{e}^{(r+\delta_1)t}\left(\lambda_1(0) + \frac{\varphi_1\left(\mathrm{e}^{-(r+\delta_1\alpha+\delta_2\beta)t} - 1\right)}{r+\delta_1\alpha+\delta_2\beta}\right) \\ &= \lambda_1(0) - \frac{\varphi_1}{r+\delta_1\alpha+\delta_2\beta} = 0.\end{aligned}$$

Finally the corresponding time path is

$$\lambda_1(t) = \frac{\varphi_1 \mathrm{e}^{-(\delta_1(\alpha-1)+\delta_2\beta)t}}{r+\delta_1\alpha+\delta_2\beta}.$$

Plugging the state and costate paths into the expressions for the controls yield

$$\lim_{t\to\infty} I^*(t) = \frac{c_1}{2d_1} \quad \text{and} \quad \lim_{t\to\infty} N^*(t) = \frac{c_2}{2d_2} \tag{A.2c}$$

since

$$\begin{aligned}&\lim_{t\to\infty} K(t)^{\theta}\lambda_1(t) = \lim_{t\to\infty} \mathrm{e}^{-(\delta_1\theta+\delta_1(\alpha-1)+\delta_2\beta)t} = 0 \\ &\lim_{t\to\infty} R(t)^{\sigma}\lambda_2(t) = 0.\end{aligned}$$

Finally we consider the case $\theta = \sigma = 1$. We note that the controls are continuous and for $d_1 > 0$ and $d_2 > 0$ bounded. Thus, the limits $\lim_{t\to\infty} N^*(t) = \bar{N}$ and $\lim_{t\to\infty} I^*(t) = \bar{I}$ exist. Then for t large enough the state dynamics for K can be written as

$$\dot{K}(t) = a\bar{I}K(t) - \delta_1 K(t) = \left(a\bar{I} - \delta_1\right)K(t) = -\xi K(t)$$

with

$$\xi := \delta_1 - a\bar{I}K(t)$$

where the higher order terms of the Taylor Series of

$$\frac{1}{1+x} = 1 - x + \frac{x^2}{2} + \ldots$$

were omitted. Subsequently we assume $\xi > 0$. Then we find structurally the same equation as Eq. (A.2a), namely

$$K(t) = \mathrm{e}^{-\xi t} K(0) \tag{A.3}$$

and proceed analogously. Moreover from Eq. (A.2c) we find

$$\bar{I} = \frac{c_1}{2d_1} \quad \text{and} \quad \bar{N} = \frac{c_2}{2d_2}.$$

Thus, we proved that the equilibrium at the origin exists for all $\theta > 1$ and $\sigma \geq 1$. For $\theta = 1$ this equilibrium exists if and only if the parameters satisfy

$$\xi = \delta_1 - a\bar{I} = \delta_1 - \frac{ac_1}{2d_1} > 0. \tag{A.4}$$

A.1. Numerical treatment

For the numerical treatment of the equilibrium at the origin we introduce a parameter $\tau \ll 1$ in the first state dynamics, yielding

$$\dot{K}(t) = \tau + g(K(t))I(t) - \delta_1 K(t), \quad t \in [0, T\rangle. \tag{A.5a}$$

Due to this adaptation the equilibrium at the origin is shifted away from zero in K and therefore avoids the divergence of the λ_1 costate.

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